

**A Mathematical
Reformulation of the
General Theory
of International Trade**

THEODORE OTTE YNTEMA

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BY ELEMENTARY methods of mathematical analysis, the author re-examines and extends the conclusions which follow from the classical and neo-classical theory of international trade.

New contributions include:

- (1) a more accurate definition of assumptions;
- (2) a simpler and more elegant statement of equilibrium conditions;
- (3) a technique for analyzing—subject to the assumptions made—the effects of a disturbance in conditions of demand or supply in trade balances;
- (4) a method for generalizing many of these conclusions to any number of commodities;
- (5) a possible basis for a statistical approach to some problems in international trade.

**MATERIALS FOR THE STUDY
OF BUSINESS**

**A MATHEMATICAL REFORMULATION
OF THE GENERAL THEORY OF
INTERNATIONAL TRADE**

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OF THE GENERAL THEORY OF
INTERNATIONAL TRADE

By

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**TO
K.**

PREFACE

The idea from which this dissertation grew was conceived under the stimulus of lectures on "International Economic Policies" delivered by Professor Jacob Viner. I am indebted to him for painstaking criticism, for constructive suggestions, for references—in fact, for most generous assistance at every turn. From Professor Henry Schultz I have received constant encouragement and, out of his wide knowledge of mathematical economics, technical aid which has been invaluable. My other major obligations are to Professors Frank Knight, Harold Hotelling, and John Canning for reading and commenting on the manuscript; to Professor F. W. Taussig and to the late Professor Allyn Young for instruction and inspiration; to Mr. Alexander Oppenheim for the solutions of Appendix I; and, not least, to a wife who "suffereth long and is kind." My debts to other teachers and friends are legion.

I am not so sanguine as to expect that I have avoided entirely errors of exposition or of theory. Responsibility for them is, of course, mine alone.

Chapter ii is the foundation upon which the three following chapters are separately based. If the reader prefers to omit the somewhat lengthy symbolic translations of chapter iii, he may proceed directly to chapters iv and v.

This monograph was originally submitted in partial fulfilment of the requirements for the doctorate in the University of Chicago, August, 1929. Except for the addition of section 4 in chapter v, no important revisions have been made in the original thesis.

THEODORE O. YNTEMA

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APPENDIX I

SOME MATHEMATICAL THEOREMS

I. TWO THEOREMS IN DETERMINANTS

Lemma 1.— $m \geq 1$. $n \geq 1$. m and n are integers. $\xi_i, \eta_j, a_{ij}, b_{ij}$, ($i=1, \dots, m; j=1, \dots, n$), are greater than or equal to zero. But at least one of the ξ_i 's or η_j 's is not zero, and at least one of the a_{ij} 's or b_{ij} 's in each row is not zero. The determinant $D' =$

$$\begin{vmatrix}
 \left(\xi_1 + \sum_1^n a_{1j} \right) & \circ & \dots & \circ & a_{11} & a_{12} & \dots & a_{1n} \\
 \circ & \left(\xi_2 + \sum_1^n a_{2j} \right) & \dots & \circ & a_{21} & a_{22} & \dots & a_{2n} \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 \circ & \circ & \dots & \left(\xi_m + \sum_1^n a_{mj} \right) & a_{m1} & a_{m2} & \dots & a_{mn} \\
 b_{11} & b_{21} & \dots & b_{m1} & \left(\eta_1 + \sum_1^m b_{i1} \right) & \circ & \dots & \circ \\
 b_{12} & b_{22} & \dots & b_{m2} & \circ & \left(\eta_2 + \sum_1^m b_{i2} \right) & \dots & \circ \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 b_{1n} & b_{2n} & \dots & b_{mn} & \circ & \circ & \dots & \left(\eta_n + \sum_1^m b_{in} \right)
 \end{vmatrix}$$

is positive.

Let $k = m + n$. Suppose the result is true of all determinants of this type of order $< k$, i.e., of order $1, 2, \dots, k - 1$. Then it is true for those of order k . For if each column is written as

$$\begin{array}{l}
 \xi_1 + \sum_1^n a_{1j} \dots\dots\dots \\
 0 + 0 \dots\dots\dots \\
 \dots\dots\dots \\
 0 + 0 \dots\dots\dots \\
 0 + b_{11} \dots\dots\dots \\
 0 + b_{12} \dots\dots\dots \\
 \dots\dots\dots \\
 0 + b_{1m} \dots\dots\dots
 \end{array}$$

then D' may be split up into the sum of positive multiples of 2^{m+n} determinants of the same type, all of which save one are of order $< k$ and so are positive.¹ The exception is obtained by putting all the ξ 's and η 's equal to zero. But plainly this determinant is zero, since the sum of the first m columns is equal to the sum of the last n columns.

It is sufficient, therefore, to show that the determinant of order 1 is positive. Every such determinant will be merely a positive ξ or η . Each such order 1 determinant obtained by successively splitting up the original determinant and its resulting parts would have, as multipliers, the product of $(k - 1)$ ξ 's and η 's (see footnote 1) drawn from determinants of orders k to 2 .

¹ In order to preserve the D' type of the lower order determinants obtained by splitting up the D' determinant of order k , it is necessary with each loss in order to redefine the ξ 's and η 's and the Σ terms with which they are associated. In each such lower order determinant the Σ must cover only the terms appearing in the same row of that determinant, and the ξ or η must include all the other terms which appeared in the row of the original determinant of order k but have disappeared in the lower order determinant.

The lemma is therefore proved. Every D' type determinant is positive.

Lemma 2.—The determinant $D'' =$

$$\begin{array}{cccccccc}
 \circ & \circ & \dots & \circ & a_{11} & a_{12} & \dots & a_{1n} \\
 \circ & \left(\xi_1 + \sum_1^n a_{1j} \right) & \dots & \circ & a_{21} & a_{22} & \dots & a_{2n} \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 \circ & \circ & \dots & \left(\xi_m + \sum_1^n a_{mj} \right) & a_{m1} & a_{m2} & \dots & a_{mn} \\
 b_{11} & b_{21} & \dots & b_{m1} & \left(\eta_1 + \sum_1^m b_{i1} \right) & \circ & \dots & \circ \\
 b_{12} & b_{22} & \dots & b_{m2} & \circ & \left(\eta_2 + \sum_1^m b_{i2} \right) & \dots & \circ \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 b_{1n} & b_{2n} & \dots & b_{mn} & \circ & \circ & \dots & \left(\eta_n + \sum_1^m b_{in} \right)
 \end{array}$$

is negative.

This determinant is of order $m+n=k$. Suppose the result were true for all determinants of the type D'' of order $<k$, i.e., $(2, \dots, k-1)$. Then it can be shown to be true for D'' of order k . As before, we can split up D'' of order k into the sum of positive

multiples of $2^{(m+n-1)}$ determinants, all save one of order $< k$, of type D'' , and therefore negative. The exception is of order k and is obtained by putting all the ξ 's and η 's equal to zero. If for this exception, the columns 2 to m be added to column 1 and the last n columns be subtracted from column 1, the determinant will be

a_{11}	○	○	a_{11}	a_{12}	a_{1n}
$\left(\xi_1 + \sum_{j=1}^n a_{1j}\right)$	○	a_{21}	a_{22}	a_{2n}
.....
○	$\left(\xi_m + \sum_{j=1}^n a_{mj}\right)$	a_{m1}	a_{m2}	a_{mn}
b_{21}	b_{m1}	$\left(\eta_1 + \sum_{i=1}^m b_{i1}\right)$	○	○
b_{22}	b_{m2}	○	$\left(\eta_2 + \sum_{i=1}^m b_{i2}\right)$	○
.....
b_{2n}	b_{mn}	○	○	$\left(\eta_n + \sum_{i=1}^m b_{in}\right)$

By expansion this exception becomes $-\sum_{j=1}^{j=n} a_{1j}$ times a minor of type D' , which is positive. This exceptional determinant is there-

fore negative. To prove the lemma, it only remains to show that the lowest order determinant of type D'' is negative. All such determinants of order 1 are zero. The order 2 determinants of type D'' which do not vanish are

$$\begin{vmatrix} 0 & a \\ b & \eta + b \end{vmatrix}$$

and are obviously negative.

Lemma 2 is therefore proved.

2. A THEOREM IN LINEAR EQUATIONS

In the determinant given below, the elements in the rows represent the coefficients of the unknowns in $(m+n-1)$ linear equations. The b 's are coefficients of the y 's, (y_1 to y_m), and the a 's are the coefficients of the x 's, (x_1 to x_n) (see footnote 1). The k 's are the constants on the right-hand side of the equations and are assumed to be all of the same sign. The a 's and b 's are ≥ 0 , but at least one of the a 's or b 's in each row does not vanish.

* The x 's and y 's are used here as general symbols for unknowns. For purposes of chapter v the x of the Appendix corresponds to $\frac{dy_{1i}}{y_{1i}}$ of the text, and y to $\frac{dz_1}{z_1}$.

$\sum_1^n a_{2i}$	○	○	a_{21}	a_{22}	a_{2n}	k_2
○	$\sum_1^n a_{3i}$	○	a_{31}	a_{32}	a_{3n}	k_3
...	
○	○	$\sum_1^n a_{mi}$	a_{m1}	a_{m2}	a_{mn}	k_m
b_{21}	b_{31}	b_{m1}	$\left(b_{11} + \sum_2^m b_{i1} \right)$	○	○	○
b_{22}	b_{32}	b_{m2}	○	$\left(b_{12} + \sum_2^m b_{i2} \right)$	○	○
...	
b_{2n}	b_{3n}	b_{mn}	○	○	$\left(b_{1n} + \sum_2^m b_{in} \right)$	○

From the last n equations it is evident that if all the y 's are of the same sign, the x 's are all of the opposite sign. It is necessary, therefore, only to show that all the y 's are of the same sign, which is the sign of the k 's. Let us call the determinant above Δ_{11} since the rows and columns are both numbered beginning with 2.

Obviously, Δ_{11} is of the type D' and positive. The minor, $\Delta_{11,22}$, is also of type D' and positive. $\Delta_{11,23}$ is, however, of type D'' and therefore negative. The minor, $\Delta_{11,24}$, is, after one interchange of rows, of type D'' and therefore has the sign $-(-)$ or positive.

The minor, $\Delta_{11,23}$, is, after two interchanges of rows, of type D'' and therefore negative. Thus the minors alternate in sign and, in general $\Delta_{11,2q}$, ($q \leq m$), has the sign $-(-1)^{q-1}$ or $(-1)^q$.

Now

$$y_2 = \frac{k_2 \Delta_{11, 22} - k_3 \Delta_{11, 23} + k_4 \Delta_{11, 24} - \dots + k_m (-1)^m \Delta_{11, 2m}}{\Delta_{11}} .$$

But since the $\Delta_{11,2q}$'s alternate in sign, beginning with a positive, it follows that the numerator is of the same sign as the sign of the k 's (which is assumed to be uniform). Since Δ_{11} has been shown positive, y_2 is therefore of the same sign as the k 's. But every y , (y_2 to y_m), is consequently of the sign of the k 's, for it is quite immaterial which of the y 's is y_2 , and any other y may be interchanged in position with y_2 by simply shifting an even number of rows and columns.

It should be noticed that the demonstration is unaffected if some, not all, of the k 's are equal to zero.

Thus we have shown (a) that all the y 's, (y_2 to y_m), have the same sign, which is the uniform sign of the k 's, and (b) that all the x 's, (x_1 to x_n), have the opposite sign from that of the k 's.

APPENDIX II

NOTES ON OTHER FORMULATIONS OF INTERNATIONAL TRADE THEORY

Classical Theory.

As was pointed out in chapter i, this monograph is merely a mathematical exposition and development of the classical theory of international trade. Classical theory in its later stages incorporated, *inter alia*, the following concepts: various forms of the quantity theory of money; demand and supply schedules or functions (although these were not used definitely in international trade theory, but instead the rather vague term "reciprocal demand"); the uniformity of competitive prices of internationally traded commodities, with allowances for barriers; J. S. Mill's "Equation of International Demand," stipulating the equality of the values of exports and imports for a country; and the maintenance of this equality through the shifts of price levels, brought about by changes in the quantity of money in circulation in the various countries or by changes in the exchange rates. These concepts find expression in our equations.

Certain other concepts and methods of approach employed by the classical school are much less prominent or are omitted in our treatment. The most striking contrast appears in our use of "deflated" money supply schedules instead of labor or real sacrifice costs. Although the doctrine of comparative costs is not explicitly recognized in our equilibrium equations, it is obvious that it could readily be stated in terms of relative supply prices. For this purpose the fixed-schedule or deflated prices, i.e., the y_{pk} 's, may be used. Or, if the ratio of the supply prices of the two commodities in one country be compared with the corresponding ratio in the other country, the actual money supply prices may be utilized.

An exposition of the theory of comparative costs in terms of supply prices instead of real costs was given by Pareto¹ in his early writings and, less satisfactorily, by J. S. Nicholson in his *Principles of Political Economy*. Professor Taussig, subsequent to the publication of his book, *International Trade*, indicated to the writer the desirability of approaching the problem in this way and then moving on to a discussion in terms of sacrifice costs.

The advantages of using prices instead of sacrifice costs and utilities in approaching the theory of international trade are readily appreciated. If the price and quantity phenomena are dealt with by a simple extension of the ordinary treatment of supply and demand, the difficulties of defining sacrifice costs and relating them to money costs do not confuse the introductory analysis. Then, after the price-quantity interactions have been adequately described, the subjective elements may be taken into consideration. On the other hand, an introduction to the theory of international trade in terms of sacrifice costs or in terms of costs measured in quantities of productive factors does avoid some difficulties arising out of shifting price levels. But the perplexing relations between real costs and money costs must then be considered immediately or some simplifying assumption must be made. Furthermore, this approach has not led to a clear and accurate description of the immediately observable price-quantity phenomena.

Although the analysis of the theory of international trade in this monograph has been almost entirely related to such observable phenomena, it is, of course, true that the really important elements in any economic situation are subjective. Thus, while Angell's argument for an analysis in terms of prices is well taken,² his unwillingness to plumb beneath them is ill considered.³ Still,

¹ "Teoria matematica del commercio internazionale," *Giornale degli Economisti*, X (1895), 476-98; *Cours d'économie politique*, pp. 209 ff.; *Manuel d'économie politique*, pp. 505 ff.

² J. W. Angell, *The Theory of International Prices*, pp. 461 ff.

³ Jacob Viner, "Angell's 'Theory of International Prices,'" *Journal of Political Economy*, XXXIV (1926), 622-23.

the outright refusal to deal with the transition from prices to human values is less dangerous than the failure to appreciate the difficulties involved. The classical economists were, in general, too prone to assume the proportionality of the quantity of labor or of composite resources expended to sacrifice cost and to money cost, respectively.

Without any pretense at tracing the history of the theory of international trade, we shall note briefly the principal contributions to which our algebraic and graphic methods are related.

Alfred Marshall

Among the major accomplishments of Marshall in the theory of international trade were his invention and application of a graphic technique. The curves of international exchange, the concepts of elasticities referring to these curves, the identification of stable and unstable equilibria, and the elaborate analysis of disturbances and their effects upon the terms of trade are the features of his work of greatest interest to us. The relation of our treatment to Marshall's methods was set forth at some length in chapter iv, sections 4-6, and we need not repeat it here.

F. Y. Edgeworth

In his series of articles on "The Theory of International Values" in the *Economic Journal*,¹ Edgeworth made illuminating criticisms on the contributions of other writers. The constructive portions of his papers were devoted to a graphic analysis of a variety of problems and an algebraic description of equilibrium.

In the course of an otherwise brilliant application of the Marshallian graphics, he fell into a curious error. While discussing the effects of taxes and other impediments to imports or exports,² he affirmed that, whereas a tax or impediment to exports always shortens the export co-ordinate (in Figure 11, the abscissa

¹ *Op. cit.*, IV (1894), 35-50, 424-43, 606-38. Reproduced with some modifications as "The Pure Theory of International Values" in Edgeworth's *Papers Relating to Political Economy*, II, 3-60.

² *Economic Journal*, IV (1894), 429-35.

of *OE*), a tax or impediment to imports always *lengthens the import co-ordinate* of the curve (the ordinate of *OE*). This asymmetry and its consequences he regarded as the most distinctive contribution of the mathematical method. As was shown in chapter iv, section 5, an impediment to imports also *shortens the export co-ordinate* of the Marshallian curve.

Although the effects of import duties and export duties on the Marshallian curve, and therefore on the commodity terms of trade, are symmetrical, other results of these disturbances are unlike. In the latter part of section 3, chapter v, it was shown that an import duty levied by country 1 raises the height of its schedules and hence its domestic price level relative to those of country 2 (z_1 increases). It was also pointed out that an export duty would have these same effects if the foreign demand for the imports of country 1 were inelastic. But if, as is more likely, the foreign demand is elastic, the results would be reversed: the schedules and domestic price level of country 1 would fall relative to those of country 2.

Edgeworth's algebraic description of equilibrium starts with composite utility functions for each of the two countries and a constant amount of work (measured in disutility) expended in each country on the production of internationally traded commodities. The utility function for each country is then maximized, subject to the conditions of constant costs of production, fixed quantity of resources, and the equality of values of imports and exports. Edgeworth's equations are adequate to indicate the status at equilibrium; but in his illustration, at least, they are limited to constant costs and are not as closely related as are our equations to the demand and supply functions and the prices with which economists are most familiar. They involve, moreover, the vague and questionable concept of utility or disutility for a country as a whole.

In passing, we should call attention to Edgeworth's symbol v , "the number of units of work in the first country, of which the

product is equivalent in value to the product of a unit of work in the second country." If the disutility of production, in Edgeworth's sense, were proportional to the deflated supply price, y_{pk} , in each country, this v would be proportional to the reciprocal of our $z_{\frac{1}{2}}$.

H. Cunynghame

The curves of import demand and export supply, together with their derivation from the ordinary domestic demand and supply

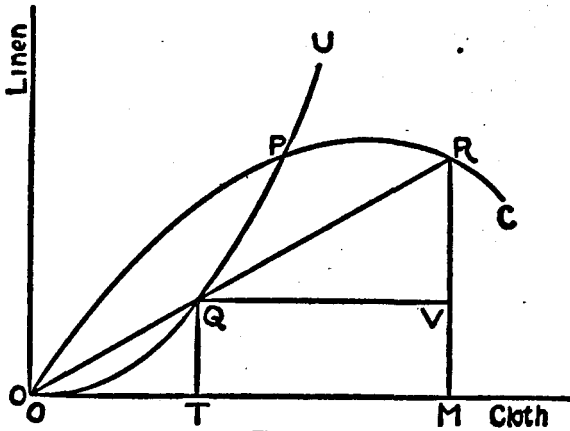


FIG. 13

curves, are to be found in Cunynghame's *A Geometrical Political Economy*, chapter x. But the analysis of the "exceptional" case of import demand when the domestic supply curve for the commodity would be negatively sloped is not differentiated from that for the ordinary case. Consequently, the author is led into a somewhat fantastic explanation of equilibrium for the exceptional case.¹

Cunynghame's attempt to explain the derivation of Marshall's curves is confused and erroneous. In Figure 13, a reproduction of Figure 56 in *A Geometrical Political Economy*, OU is the locus of a point whose co-ordinates are quantities of cloth and linen having equal total effective utilities, and OC is the locus of a point whose co-ordinates are quantities of cloth and linen having equal total

¹ *A Geometrical Political Economy*, pp. 87-89.

costs of production.¹ As regards exclusively domestic trade, the co-ordinates of *P* would indicate the quantities of cloth and linen produced and exchanged against each other. But the problem is more complex if a portion of one commodity is exported and a portion of the other commodity is imported. If *OM* cloth were produced, its total cost would be equal to that of *MR* linen. And since *OT* and *TQ* bear, respectively, the same proportion to *OM* and *MR*, it would be true that *OT* cloth and *TQ* linen would have equal costs of production *assuming the scale of output of cloth to be OM and the scale of output of linen to be MR*. Further, *OT* cloth and *TQ* linen would have equal total effective utilities *if they were the whole quantities on the market*. But it does not follow that, when the scale of output of cloth is *OM*, the portion *QT* of cloth would exchange for the whole domestic production *TQ* of linen, leaving *TM* (or *QV*) cloth to be exported for *VR* linen. For this violates the two italicized assumptions. The scale of production of linen is no longer *MR* but *TQ*, and the quantity of linen on the market is no longer *TQ* but *MR*. Consequently, *QV* and *VR* cannot be taken to represent the co-ordinates of a point on a Marshallian international trade curve of demand for linen in terms of cloth. The attempt to derive a Marshallian curve of import of cloth and export of linen from Figure 13 by a similar procedure would likewise fail. These procedures which would be correct (though not applicable in this problem) if the ordinates were total money value or cost fail when the ordinates are quantities of second commodity of varying value and cost of production. It is not difficult to understand that Marshall should have raised objections to Cunyngame's interpretations of his curves.

J. W. Angell

In the last chapter of his *Theory of International Prices* Angell obtained some novel effects with the Marshallian curves. Al-

¹ Cunyngame's definition of these curves is not at all clear. This seemed the most reasonable interpretation of his statements.

though the exposition is not easy to interpret, it would seem that each Marshallian curve is to be constructed out of the demand and supply curves of each country on the assumption of perfect competition. But since competition often is not keen, producers and traders frequently get something more than a competitive return. In such cases the point whose co-ordinates indicate the quantities exchanged lies somewhere within the area bounded by the curves. This treatment, however, neglects the opposite cases of excessive competition in which producers cannot recoup their costs because of a decline in demand or an overexpansion of the industry.

Aside from the effects of imperfect competition, Angell's arguments on the indeterminateness of the terms of trade are not impressive. The treatment of transportation costs is quite unsatisfactory. If the conditions of supply and demand for transportation services are competitive and are known, they do not bring into the situation an indeterminate element, as we have demonstrated in chapter iii, section 5, and suggested in chapter iv, section 5.

Nor is Angell's analysis of the multi-commodity barter case valid. Surely the assumption that barter reduces to a "series of pairs of independent operations" is a gratuitous and unreasonable assumption. On the contrary, we should expect the most intimate relations between the barter exchanges of different commodities as traders sought out the best bargains. Further, the important balance of values would not be that of the individual transactions but that for the whole value of imports and exports of a country. This is the only reasonable sense in which Mill's "Equation of International Demand" is to be interpreted.

As regards the effects of money on the terms of trade, Angell points out that dynamic monetary changes are particularly important in the short run and leave their impress more or less permanently on an economic system.

A. A. Cournot

Cournot's most important contributions appeared in his first book, *Researches into the Mathematical Principles of the Theory of Wealth*. But they relate to a single commodity which forms so small a portion of the total value of imports or exports that the repercussions of disturbances upon price levels may be neglected. We shall not stop to review Cournot's conclusions since they fall outside the scope of the general theory of international trade.

We are, however, indebted to Cournot for his independent development of demand and supply functions and for his use of the Taylor expansion in studying the effects of small disturbances upon an equilibrium situation.

Leon Walras

The name of Leon Walras is included here, not because of any direct contributions to the theory of international trade, but because of his development of the algebraic description of equilibrium for a single country. The two sets of equations (Sets I and II) which are the basis of our exposition are analogous, respectively, to those which express the equality of the values of sales and purchases for each individual and to those which express the equality of the sum of the quantities of a commodity sold to the sum of the quantities bought.

Auspitz and Lieben

Auspitz and Lieben derived and used curves of import demand and export supply in total form. But, since they consistently adopted total *money* value or cost as ordinates, their curves are to be distinguished from Marshall's curves of international exchange.

Their conclusions, like Cournot's, are applicable to a single commodity which forms so small a portion of the total value of imports and exports that the effects on price levels may be neglected.

Vilfred Pareto

The most pretentious mathematical analysis of the theory of international trade known to the writer is that by Vilfred Pareto.¹ In abbreviated form his procedure is as follows: The equations describing the economic system of two countries as separate entities are first written. If these countries now engage in international trade, i.e., import and export certain products (but not productive services), new unknowns are introduced. These are the n quantities of the n commodities exchanged between the two markets and the rate of exchange between the money units of the two countries—in all $(n+1)$ unknowns. But there are $(n+1)$ additional equations. The value of exports is equal to the value of imports (one equation serves for either country). And n equations state that the prices of the n international commodities are equivalent in the two countries, allowing for the (known) costs of surmounting any barriers to trade.

This treatment seems extraordinarily simple. It does not involve explicitly the quantity theory of money or make any reference to shifting price levels in the respective countries. How, then, can the equations be adequate to describe equilibrium in international trade?

Among the imports or exports of each country there ordinarily appears the commodity gold. When equilibrium is achieved, it must be true, for gold as for other commodities, that the demand in the two countries combined is equal to the supply. If the two countries are on the gold standard, the equilibrium which is reached after any disturbance (such as an indemnity) is introduced, would be characterized by *domestic* prices which had shifted in response to changes in the utility of gold and in its cost

¹ An extensive treatment appears in his two articles in the *Giornale degli Economisti*: "Teoria matematica dei cambi forestieri," VIII (1894), 142-73; and "Teoria matematica del commercio internazionale," X (1895), 476-98. Much of this is reproduced in the *Cours d'économie politique*. In the *Manuel d'économie politique* the general mathematical analysis is much abbreviated.

of production and by prices of *internationally traded commodities* which had altered in response to these changes for gold and also because the quantities of these commodities produced and consumed had changed. The equilibrium depicted is analogous to that which is obtained in chapter ii, section 9, when the demand for and supply of gold for all countries combined is stipulated in the equations.

Such an equilibrium, as we have said, is approached but very slowly. It would be highly desirable to be able to consider the incomplete equilibrium which is more quickly achieved through a redistribution of the gold stock of the world, even though the demand for gold in the arts plus the demand to maintain the monetary stock were not exactly equated to the supply. This could be done approximately in Pareto's equations, as in ours, by omitting the commodity gold or by assuming the value of gold imports and exports to be known. The use of an ordinary commodity as a *numéraire* in Pareto's system would not involve any inconvenience. But in any case, Pareto's equations do not explicitly recognize—in fact, they mask rather effectively—the shifts of price levels which are so prominent in the classical theory. It is interesting to observe that in the context of these equations the discussion of disturbances to equilibrium devotes considerable attention to the quantity theory of money and shifts of price levels and rather neglects the long-run adjustment of prices to the utility and cost of production of gold as required in the algebraic exposition.

In summary, the major points which distinguish Pareto's equations from ours are the following:

1. The former involve the utility functions of individuals, the factors of production, and, in fact, all the elements in Pareto's description of internal equilibrium.

2. They are limited to two countries, but they could be extended to the general case.

3. They are not clearly related to monetary mechanisms and shifts in price levels.

4. The effects of disturbances to equilibrium cannot, through them, be readily associated with particular elasticities of demand and supply.

5. If all the equations of both countries are included, the system is very much more complicated than ours and less amenable to the analysis of the effects of disturbances to equilibrium.

There are other features of Pareto's treatment which should be noted. It includes a demonstration that the Marshallian curves may be derived from his equations by the elimination of all the unknowns except the quantities of the two commodities respectively imported and exported. In order to investigate the effects of disturbances, Pareto used small finite differences; but his procedure was much less general and systematic than ours. He also developed a somewhat complicated measure of the gain or loss resulting from disturbances to international trade. Not least significant of his accomplishments, though among the simplest from a mathematical viewpoint, was his exposition of the theory of comparative costs in terms of relative prices. It is unfortunate that this treatment suffered from a needless complexity of symbols.

We cannot help deploring that the unfinished treatise on international trade for which Pareto is reported to have accumulated extensive notes should never have reached fruition.

Pietri-Tonelli

This writer has taken Pareto's contributions and expounded them with added clarity.

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¹ For a more extensive bibliography on the theory of international trade the reader is referred to J. W. Angell, *The Theory of International Prices*, pp. 535-56.

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GLOSSARY OF SYMBOLS

This glossary includes only symbols which are used outside the immediate context of their definition.

Subscripts

d —Refers to demand; always followed by a comma.

s —Refers to supply; always followed by a comma.

First subscript (or first after a comma) denotes the country.

Second subscript (or second after a comma) denotes the commodity.

p —General subscript for a country.

k —General subscript for a commodity.

i —General subscript in summations.

Quantities of Commodities

$q_{d,pk}$ —Quantity demanded.

$q_{s,pk}$ —Quantity supplied.

x_{pk} —Quantity exported (+) or imported (-).

Prices of Commodities

Y_{pk} —Actual money price.

y_{pk} —Deflated or fixed-height schedule price.

Functional Relations between Prices and Quantities

$$q_{d,pk} = \psi_{d,pk}(y_{pk}) .$$

$$q_{s,pk} = \psi_{s,pk}(y_{pk}) .$$

$$x_{pk} = f_{pk}(y_{pk}) .$$

$$y_{pk} = \phi_{pk}(x_{pk}) .$$

Exchange Rate

$\frac{c_i}{p}$ —The number of units of country p 's currency obtainable for one unit of the currency of country i .

Index of Height of Actual Money Schedules

$r_p = \frac{Y_{ph}}{y_{ph}}$, i.e., the ratio for country p of the height of its actual demand and supply schedules to the height of its fixed-height or deflated schedules.

Net Monetary Factor

$z_1 = c_1 \frac{r_1}{p} = \frac{y_{ph}}{y_{ih}}$. If barriers exist, z_1 is equal to the ratio of y_{ph} to y_{ih} after either is appropriately modified to compensate for the barriers.

Elasticities

e_{ph} —Elasticity of supply.

η_{pk} —Elasticity of demand.

h_{pk} —Elasticity of export supply or import demand.

$h_{s,pk}$ —Elasticity of export supply.

$h_{d,pk}$ —Elasticity of import demand.

$E_{d,p}^k$ —Marshall's elasticity of demand for his curves, i.e., the elasticity of the demand of country p for its imports, k , in terms of its exports, k' .

Other symbols

T —An ad valorem duty.

T' —A specific duty.

B' —A specific bounty.

$I_{(p \leftarrow p')}$ —An indemnity payable by country p' to country p , measured in the currency of country p .

$$a_{pk} = x_{pk} y_{pk} (1 + h_{pk}) .$$

$$b_{pk} = x_{pk} h_{pk} .$$

Δ_{11} —A determinant of the type (99) in chapter v, section 2.

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