

**INTRODUCTION TO  
MATHEMATICAL PROBABILITY**

## PREFACE

This book is an outgrowth of lectures on the theory of probability which the author has given at Stanford University for a number of years. At first a short mimeographed text covering only the elementary parts of the subject was used for the guidance of students. As time went on and the scope of the course was gradually enlarged, the necessity arose of putting into the hands of students a more elaborate exposition of the most important parts of the theory of probability. Accordingly a rather large manuscript was prepared for this purpose. The author did not plan at first to publish it, but students and other persons who had opportunity to peruse the manuscript were so persuasive that publication was finally arranged.

The book is arranged in such a way that the first part of it, consisting of Chapters I to XII inclusive, is accessible to a person without advanced mathematical knowledge. Chapters VII and VIII are, perhaps, exceptions. The analysis in Chapter VII is rather involved and a better way to arrive at the same results would be very desirable. At any rate, a reader who does not have time or inclination to go through all the intricacies of this analysis may skip it and retain only the final results, found in Section 11. Chapter VIII, though dealing with interesting and historically important problems, is not important in itself and may without loss be omitted by readers. Chapters XIII to XVI incorporate the results of modern investigations. Naturally they are more complex and require more mature mathematical preparation.

Three appendices are added to the book. Of these the second is by far the most important. It gives an outline of the famous Tshebysheff-Markoff method of moments applied to the proof of the fundamental theorem previously established by another method in Chapter XIV.

No one will dispute Newton's assertion: "*In scientiis addiscendis exempla magis presunt quam praecepta.*" But especially is it so in the theory of probability. Accordingly, not only are a large number of illustrative problems discussed in the text, but at the end of each chapter a selection of problems is added for the benefit of students. Some of them are mere examples. Others are more difficult problems, or even important theorems which did not find a place in the main text. In all such cases sufficiently explicit indications of solution (or proofs) are given.

The book does not go into applications of probability to other sciences. To present these applications adequately another volume of perhaps larger size would be required.

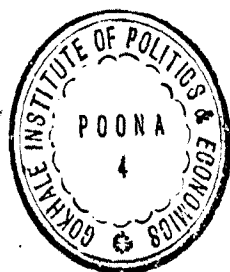
No one is more aware than the author of the many imperfections in the plan of this book and its execution. To present an entirely satisfactory book on probability is, indeed, a difficult task. But even with all these imperfections we hope that the book will prove useful, especially since it contains much material not to be found in other books on the same subject in the English language.

J. V. USPENSKY.

STANFORD UNIVERSITY,  
*September, 1937.*

INTRODUCTION  
TO  
MATHEMATICAL PROBABILITY

BY  
J. V. USPENSKY  
*Professor of Mathematics, Stanford University*



FIRST EDITION  
SEVENTH IMPRESSION

McGRAW-HILL BOOK COMPANY, Inc.  
NEW YORK AND LONDON  
1937

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