

**The Mathematical Problem  
of the  
Price Index**

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of the  
Price Index

BY

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## CHAPTER I

### THE CONCEPTION OF THE PRICE INDEX OF A GROUP OF COMMODITIES

ONE of the reasons why there has been so much controversy on the subject of the price index of a group of commodities is that writers have never agreed on a definition. Certainly it is not easy to define the price index in a few words nor to form a precise conception of it.

The following statement of what the writer puts forward, without claiming any originality for it, as the most satisfactory conception of the price index can hardly be called a definition, as he proceeds by first expressing the conception in vague terms and afterwards trying to give a more precise meaning to those terms.

In accordance with the usual notation, let the prices of the commodities in the base year be  $p_0', p_0'', p_0'''$ , etc., and the prices in the given year  $p_1', p_1'', p_1'''$ , etc. Let the quantities of the commodities in the base year be  $q_0', q_0'', q_0'''$ , etc., and the quantities in the given year  $q_1', q_1'', q_1'''$ , etc.

Then the values of the commodities are  $p_0'q_0'$ ,  $p_0''q_0''$ ,  $p_0'''q_0'''$ , etc., in the base year and  $p_1'q_1'$ ,  $p_1''q_1''$ ,  $p_1'''q_1'''$ , etc., in the given year.

The aggregate value of the commodities is  $\Sigma p_0q_0$  in the base year and  $\Sigma p_1q_1$  in the given year.

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Comparing the commodities in the given year with the commodities in the base year; we note that both prices and quantities have changed. The aggregate value has also changed, and the change in aggregate value is partly due to the changes in price and partly to the changes in quantity.

Here we are using vague phrases: "due to the changes in price" and "due to the changes in quantity." We shall hope to make clearer, presently, what we mean by these phrases.

Let us now suppose that, instead of varying with the commodity, the change of price had been uniform for all commodities in the group and that, in like manner, there had been a uniform change in the quantities. Let the proportionality factor for price be  $P_{01}$  and the proportionality factor for quantity  $Q_{01}$ .

Then, in this case (which we will call the hypothetical case, to distinguish it from the actual case), we have a group of commodities of which the prices in the base year are the same as in the actual case, but the prices in the given year are  $P_{01}p_0'$ ,  $P_{01}p_0''$ ,  $P_{01}p_0'''$ , etc., while the quantities in the base year are the same as in the actual case, but the quantities in the given year are  $Q_{01}q_0'$ ,  $Q_{01}q_0''$ ,  $Q_{01}q_0'''$ , etc.

The aggregate value, in the hypothetical case, is  $\Sigma p_0 q_0$  in the base year and  $\Sigma P_{01} p_0 Q_{01} q_0$  or (since  $P_{01}$  and  $Q_{01}$  are the same for all the commodities)  $P_{01} Q_{01} \Sigma p_0 q_0$  in the given year.

So far, it will be observed, we may attach any values we like to  $P_{01}$  and  $Q_{01}$ .

Now, a fundamental conception of the price index and the quantity index is that, if they are applied as proportionality factors to the prices and quantities of

the commodities in the base year, the resulting aggregate value of the commodities is the same as the aggregate value in the given year in the actual case. In other words, if  $P_{01}$  and  $Q_{01}$  are the price index and the quantity index, they must fulfil the condition

$$\Sigma P_{01} p_0 Q_{01} q_0 = \Sigma p_1 q_1$$

or, since  $P_{01}$  and  $Q_{01}$  are the same for all the commodities,

$$P_{01} Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$$

If it were not so,  $P_{01}$  and  $Q_{01}$  would have no relation to the prices and quantities in the given year, and could in no way serve to establish a comparison between the prices in the given year and the prices in the base year and between the quantities in the given year and the quantities in the base year.

Up to this point there is general agreement. It is usually admitted that the price index,  $P_{01}$ , and the quantity index,  $Q_{01}$ , must have the property indicated by the above equation.

But this does not carry us very far towards the determination of  $P_{01}$  and  $Q_{01}$ . Obviously there is an infinite series of pairs of values for  $P_{01}$  and  $Q_{01}$  which fulfil the condition. They take the forms  $\left(\frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}\right)^a$  and  $\left(\frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}\right)^{1-a}$ .

Hence it is clear that  $P_{01}$  and  $Q_{01}$  must fulfil other conditions. The conception (the writer submits) must be extended by saying that the price index must correspond to the change in the aggregate value due to the changes in price and the quantity index must correspond

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to the change in the aggregate value due to the changes  
in quantity.

But here again we are using vague terms. What do we mean by "correspond to" and by "due to the changes in price"? And how is the change in aggregate value measured?

Let us answer the last question first.

The change in aggregate value may be measured in three ways :

(1) As the ratio of the aggregate value in the given year to the aggregate value in the base year, or  $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$ ;

(2) As the difference between the aggregate value in the given year and the aggregate value in the base year, or  $\Sigma p_1 q_1 - \Sigma p_0 q_0$ ;

(3) As the ratio of the difference between the aggregate value in the given year and the aggregate value in the base year to the aggregate value in the base year, or  $\frac{\Sigma p_1 q_1 - \Sigma p_0 q_0}{\Sigma p_0 q_0}$ .

Reverting to the conception of  $P_{01}$  and  $Q_{01}$  as proportionality factors and recalling what we have called the hypothetical case, we may note that the change of aggregate value, in whichever way it may be measured, is the same in the hypothetical case as in the actual case, since

$$\Sigma P_{01} p_0 Q_{01} q_0 = \Sigma p_1 q_1$$

$$\Sigma P_{01} p_0 Q_{01} q_0 - \Sigma p_0 q_0 = \Sigma p_1 q_1 - \Sigma p_0 q_0$$

and  $\frac{\Sigma P_{01} p_0 Q_{01} q_0 - \Sigma p_0 q_0}{\Sigma p_0 q_0} = \frac{\Sigma p_1 q_1 - \Sigma p_0 q_0}{\Sigma p_0 q_0}$ .



The first measure of the change of aggregate value,  $\frac{\sum p_1 q_1}{\sum p_0 q_0}$ , is a ratio. Let us suppose that it is transformed into the product of two ratios, one (which we will call  $R_p$ ) representing the change in the aggregate value due to the changes in price, and the other (which we will call  $R_q$ ) representing the change in the aggregate value due to the changes in quantity. We have still to explain what we mean by "due to the changes in price" and "due to the changes in quantity," but let us postpone the explanation for a moment.

In the hypothetical case, let us suppose that the change in aggregate value  $\frac{\sum P_{01} p_0 Q_{01} q_0}{\sum p_0 q_0}$  (which is equal to  $\frac{\sum p_1 q_1}{\sum p_0 q_0}$ ) is transformed into the product of two ratios, one (which we will call  $[R_p]$ ) representing the change in the aggregate value due to the changes in price and the other (which we will call  $[R_q]$ ) representing the change in the aggregate value due to the changes in quantity.

Then  $[R_p]$  must be equal to  $R_p$  and  $[R_q]$  must be equal to  $R_q$ . This is what we mean when we say that "the price index must correspond to the change in the aggregate value due to the changes in price, and the quantity index must correspond to the change in the aggregate value due to the changes in quantity."

In the hypothetical case, it is obvious that  $[R_p]$  is  $P_{01}$  and that  $[R_q]$  is  $Q_{01}$ . Hence, if we can determine  $R_p$  and  $R_q$  our problem is solved.

There must be a similar correspondence if we take the other measures of the change in the aggregate value. If the change in the aggregate value is

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$\Sigma p_1q_1 - \Sigma p_0q_0$ , we may speak of it as the increase in the aggregate value and divide it into the sum of two values, one (which we will call  $V_p$ ) being the increase in the aggregate value due to the changes in price, and the other (which we will call  $V_q$ ) being the increase in the aggregate value due to the changes in quantity.

In the hypothetical case the increase in the aggregate value is  $\Sigma P_{01}p_0Q_{01}q_0 - \Sigma p_0q_0$  (which is equal to  $\Sigma p_1q_1 - \Sigma p_0q_0$ ). Let us divide it into two values, one (which we will call  $[V_p]$ ) being the increase in the aggregate value due to the changes in price, and the other (which we will call  $[V_q]$ ) being the increase in the aggregate value due to the changes in quantity.

Then  $[V_p]$  must be equal to  $V_p$  and  $[V_q]$  to  $V_q$ .

Again, if the change in the aggregate value is expressed by  $\frac{\Sigma p_1q_1 - \Sigma p_0q_0}{\Sigma p_0q_0}$  (the proportional increase)

we may divide it into the sum of two ratios, one (which we will call  $\dot{R}_p$ , to distinguish it from  $R_p$ ) representing the proportional increase in the aggregate value due to the changes in price, and the other (which we will call  $\dot{R}_q$ ) representing the proportional increase in the aggregate value due to the changes in quantity.

In the hypothetical case, the proportional increase in the aggregate value is expressed by  $\frac{\Sigma P_{01}p_0Q_{01}q_0 - \Sigma p_0q_0}{\Sigma p_0q_0}$

(which is equal to  $\frac{\Sigma p_1q_1 - \Sigma p_0q_0}{\Sigma p_0q_0}$ ), and we may divide it into the sum of two ratios, one (which we will call  $[\dot{R}_p]$ ) representing the proportional increase in the aggregate value due to the changes in price, and the other (which we will call  $[\dot{R}_q]$ ) representing the pro-

portional increase in the aggregate value due to the changes in quantity.

Then  $[\dot{R}_P]$  must be equal to  $\dot{R}_P$  and  $[\dot{R}_Q]$  to  $\dot{R}_Q$ .

But obviously  $\dot{R}_P = \frac{V_P}{\Sigma p_0 q_0}$ ,  $\dot{R}_Q = \frac{V_Q}{\Sigma p_0 q_0}$ ,  $[\dot{R}_P] = \frac{[V_P]}{\Sigma p_0 q_0}$  and  $[\dot{R}_Q] = \frac{[V_Q]}{\Sigma p_0 q_0}$ .

Hence, if  $[V_P] = V_P$  and  $[V_Q] = V_Q$ , we also have  $[\dot{R}_P] = \dot{R}_P$  and  $[\dot{R}_Q] = \dot{R}_Q$ , so that the last two equations do not give us any new conditions for determining  $P_{01}$  and  $Q_{01}$ .

It should be noted that all the conditions above indicated must be fulfilled. We must have  $[R_P] = R_P$ ,  $[R_Q] = R_Q$ ,  $[V_P] = V_P$ ,  $[V_Q] = V_Q$ ,  $[\dot{R}_P] = \dot{R}_P$  and  $[\dot{R}_Q] = \dot{R}_Q$ . It is not that we arbitrarily choose one method of expressing the change in aggregate value. It must be indifferent which method we choose.

## CHAPTER II

### THE TRANSFORMATION OF MEASURES OF CHANGE

We have now to make clear what we mean by "the change in the aggregate value due to the changes in price," and "the change in the aggregate value due to the changes in quantity." We shall obtain a clearer idea of this by studying the following series of propositions relating, not to prices, quantities and values, but to pure numbers.

1.—Let  $A + a + b + c + \dots = BR_aR_bR_c \dots$

where (speaking in vague terms)  $R_a$  corresponds to  $a$ ,  $R_b$  to  $b$ ,  $R_c$  to  $c$ , and so on. Let us further assume that  $A$  is positive and that, although any of the terms  $a$ ,  $b$ ,  $c$ , etc., may be negative,  $A + a + b + c + \dots$  is also positive.

Speaking more precisely, let  $A + a + b + c + \dots$  be equal to  $B$  multiplied by factors  $R_a$ ,  $R_b$ ,  $R_c$ , etc., such that the operation of multiplying  $B$  by  $R_a$  (simultaneously with the other factors) contributes as much to the product  $BR_aR_bR_c \dots$  as the operation of adding  $a$  to  $A$  contributes to the sum  $A + a + b + c + \dots$ , and similarly for  $R_b$ ,  $R_c$ , etc. (We need not say "adding  $a$  to  $A$  simultaneously with the other terms," for the effect of adding  $a$  is the same whether it is added with the other terms or is added separately.)

If  $a = 0$ ,  $R_a = 1$ , since adding 0 leaves the sum

unchanged and multiplying by 1 leaves the product unchanged. Hence, putting  $a = 0$ ,  $b = 0$ ,  $c = 0$ , and so on, we have  $A = B$ .

Thus we must say :

$$\text{Let } A + a + b + c + \dots = AR_aR_bR_c \dots$$

We must consider closely what we are doing when we endeavour to determine  $R_a$ ,  $R_b$ ,  $R_c$ , etc. We know precisely what are the changes, *measured as differences*, produced by adding  $a$ ,  $b$ ,  $c$ , etc., to  $A$ ; we want to know what are those changes *measured as ratios*.

If only  $a$  is added to  $A$ , there is no difficulty. We have

$$A + a = AR_a$$

which gives 
$$R_a = \frac{A + a}{A}$$

But if there are even two terms added,  $a$  and  $b$ , the problem becomes more complicated.

$$\text{Let } A + a + b = AR_aR_b$$

Now suppose that  $a + b$  is constant and equal to  $K$ , but that  $a$  and  $b$  vary.

If  $a = 0$ , it is clear, as we have seen above, that  $R_a = 1$ . In this case,  $b = K$  and  $R_b = \frac{A + K}{A}$ .

Now let  $a$  increase gradually, and  $b$  decrease gradually,  $a + b$  always remaining equal to  $K$ . Then, since the change produced by adding  $a$  gradually increases and the change produced by adding  $b$  gradually diminishes,  $R_a$  must gradually increase and  $R_b$  must gradually diminish.

When  $a = b$ , adding  $b$  increases the sum by the same amount as adding  $a$ , and multiplying by  $R_b$  must,

therefore, increase the product by the same amount as multiplying by  $R_a$ . Hence  $R_a = R_b = \left(\frac{A+K}{A}\right)^{\frac{1}{2}}$ .

As  $a$  continues to increase and becomes larger than  $b$ ,  $R_a$  becomes greater than  $R_b$ . When  $a$  becomes equal to  $K$  and  $b$  becomes 0,  $R_a = \frac{A+K}{A}$  and  $R_b = 1$ .

If  $a$  still continues to increase  $b$  becomes negative. As the change produced in  $A$  by adding a negative quantity is to decrease it,  $R_b$  must now be less than 1. As  $a$  continues to increase until it reaches  $\infty$ ,  $R_a$  also reaches  $\infty$ , and  $R_b$  decreases until it becomes 0.  $R_b$  does not become negative since, by hypothesis,  $A + a + b$  is positive.

A special case, which does not occur when we suppose  $a + b = K$ , is when  $a + b = 0$ , or  $b = -a$ . In this case, since the total change produced is nil,  $R_a R_b = 1$ , or  $R_b = \frac{1}{R_a}$ .

Similar reasoning may be applied to any number of terms and to cases where  $a, b, c$ , etc., vary independently and we obtain the following general properties, which apply not only to  $R_a$  and  $R_b$ , but to any two of the factors  $R_a, R_b, R_c$ , etc.

If  $a = 0$ ,  $R_a = 1$ ; if  $b = 0$ ,  $R_b = 1$ , and so on.

If  $a \geq b$ ,  $R_a \geq R_b$ .

If  $a = -b$ ,  $R_a = \frac{1}{R_b}$ .

Again, if we treat  $a + b$  as a single term of the sum and let  $R_{a+b}$  be the factor corresponding to  $a + b$ , then  $R_{a+b} = R_a R_b$ , since multiplying by  $R_a R_b$  increases the product by the same amount as adding both  $a$  and  $b$  (that is,  $a + b$ ) to the sum.

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If  $b = a$ , we have  $R_{2a} = (R_a)^2$ , and this may be extended so that, where  $x$  is a positive integer,  $R_{xa} = (R_a)^x$ .

Again, from the fact that when  $b = -a$ ,  $R_b = \frac{1}{R_a} = (R_a)^{-1}$  we can deduce that  $R_{-2a} = (R_a)^{-2}$ , and this can be extended, so that where  $x$  is a negative integer it is also true that  $R_{xa} = (R_a)^x$ .

Now let  $a = xr$ ,  $b = yr$ ,  $c = zr$ , and so on, where  $x, y, z$ , etc., are integers. This can always be done by making  $r$  small enough.

$$\begin{aligned} \text{Then } A + a + b + c + \dots &= AR_aR_bR_c \dots \\ &= AR_{xr}R_{yr}R_{zr} \dots \\ &= A(R_r)^x(R_r)^y(R_r)^z \dots \\ &= A(R_r)^{x+y+z+\dots} \end{aligned}$$

$$\text{Hence } R_r = \left( \frac{A + a + b + c + \dots}{A} \right)^{\frac{1}{x+y+z+\dots}}$$

$$\text{But } R_a = R_{xr} = (R_r)^x$$

$$\begin{aligned} \text{Therefore } R_a &= \left( \frac{A + a + b + c + \dots}{A} \right)^{\frac{x}{x+y+z+\dots}} \\ &= \left( \frac{A + a + b + c + \dots}{A} \right)^{\frac{a}{a+b+c+\dots}} \end{aligned}$$

$$\text{Similarly } R_b = \left( \frac{A + a + b + c + \dots}{A} \right)^{\frac{b}{a+b+c+\dots}}$$

$$R_c = \left( \frac{A + a + b + c + \dots}{A} \right)^{\frac{c}{a+b+c+\dots}}$$

and so on.

In the special case in which  $A = 1$  we have

$$1 + a + b + c + \dots = R_aR_bR_c \dots$$

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$$R_a = (1 + a + b + c + \dots)^{\frac{a}{a+b+c+\dots}}$$

$$R_b = (1 + a + b + c + \dots)^{\frac{b}{a+b+c+\dots}}$$

$$R_c = (1 + a + b + c + \dots)^{\frac{c}{a+b+c+\dots}}$$

and so on.

It will be noted that we are not transforming  $a + b + c + \dots$  into the product of factors  $R_a$ ,  $R_b$ ,  $R_c$ , etc., of which  $R_a$  corresponds to  $a$ ,  $R_b$  to  $b$ ,  $R_c$  to  $c$ , and so on, but  $1 + a + b + c + \dots$ . It would seem that it is not possible to convert  $a + b + c + \dots$  into the product of such factors.

2.—In the equation

$$A + a + b + c + \dots = AR_a R_b R_c \dots$$

let  $A = a_0 + b_0 + c_0 + \dots$ ,  $a = a_1 - a_0$ ,  $b = b_1 - b_0$ ,  $c = c_1 - c_0$ , and so on. We must further assume that  $a_0 + b_0 + c_0 + \dots$  and  $a_1 + b_1 + c_1 + \dots$  are both positive, though any of the terms  $(a_1 - a_0)$ ,  $(b_1 - b_0)$ ,  $(c_1 - c_0)$ , etc., may be negative.

Then we can say:

$$\begin{aligned} \text{Let } & a_0 + b_0 + c_0 + \dots + (a_1 - a_0) + (b_1 - b_0) \\ & \qquad \qquad \qquad + (c_1 - c_0) + \dots \\ & = (a_0 + b_0 + c_0 + \dots) R_{a_1-a_0} R_{b_1-b_0} R_{c_1-c_0} \dots \end{aligned}$$

In this case  $R_{a_1-a_0}$  is such that multiplying  $(a_0 + b_0 + c_0 + \dots)$  by  $R_{a_1-a_0}$  (simultaneously with the other factors  $R_{b_1-b_0}$ ,  $R_{c_1-c_0}$ , etc.) contributes as much to the product  $(a_0 + b_0 + c_0 + \dots) R_{a_1-a_0} R_{b_1-b_0} R_{c_1-c_0} \dots$  (which is equal to  $a_1 + b_1 + c_1 + \dots$ ) as changing  $a_0$  to  $a_1$  contributes to the sum  $a_1 + b_1 + c_1 + \dots$  (or, in other words, adds to the sum  $a_0 + b_0 + c_0 + \dots$ ),



and similarly for  $R_{b_1-b_0}$ ,  $R_{c_1-c_0}$ , etc. Hence we may say that  $R_{a_1-a_0}$  represents the change (measured as a ratio) in  $a_0 + b_0 + c_0 + \dots$  due to the change of  $a_0$  to  $a_1$ ,  $R_{b_1-b_0}$  represents the change (measured as a ratio) due to the change of  $b_0$  to  $b_1$ , and so on.

On the analogy of the formula in section 1, we have

$$\begin{aligned}
 R_{a_1-a_0} &= \left( \frac{a_0 + b_0 + \dots + (a_1 - a_0)}{a_0 + b_0 + c_0 + \dots} \right)^{\frac{a_1 - a_0}{(a_1 - a_0) + (b_1 - b_0) + \dots}} \\
 &= \left( \frac{a_1 + b_1 + c_1 + \dots}{a_0 + b_0 + c_0 + \dots} \right)^{\frac{a_1 - a_0}{(a_1 - a_0) + (b_1 - b_0) + (c_1 - c_0) + \dots}}
 \end{aligned}$$

and similarly for  $R_{b_1-b_0}$ ,  $R_{c_1-c_0}$ , etc.

Hence we can transform the ratio  $\frac{a_1 + b_1 + c_1 + \dots}{a_0 + b_0 + c_0 + \dots}$  into the product of factors corresponding to  $a_1 - a_0$ ,  $b_1 - b_0$ ,  $c_1 - c_0$ , etc., that is corresponding to the changes from  $a_0$  to  $a_1$ , from  $b_0$  to  $b_1$ , from  $c_0$  to  $c_1$ , etc.

3.—If we say :

$$\text{Let } A + \frac{a + b + c + \dots}{B} = ARaRbRc \dots$$

defining  $Ra$ ,  $Rb$ ,  $Rc$ , etc., in a similar manner to the way in which they are defined in section 1, we find that

$$Ra = \left( 1 + \frac{a + b + c + \dots}{AB} \right)^{\frac{a}{a+b+c+\dots}}$$

By putting  $A = 1$ ,  $B = a_0 + b_0 + c_0 + \dots$ ,  $a = a_1 - a_0$ ,  $b = b_1 - b_0$ ,  $c = c_1 - c_0$ , etc., we arrive at the same result as in section 2.

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4.—If we say :

$$\text{Let } A + a + b + c + \dots = B + R_a + R_b + R_c + \dots,$$

where  $R_a$  contributes as much to the sum  $B + R_a + R_b + R_c + \dots$  as  $a$  contributes to the sum  $A + a + b + c + \dots$ , and similarly for  $R_b$ ,  $R_c$ , etc., then it is obvious that  $B = A$ ,  $R_a = a$ ,  $R_b = b$ ,  $R_c = c$ , and so on, but if we shut our eyes to the obvious fact and calculate  $R_a$ ,  $R_b$ ,  $R_c$ , etc., by similar methods we find that

$$R_a = (a + b + c + \dots) \frac{a}{a + b + c + \dots} = a$$

$$R_b = (a + b + c + \dots) \frac{b}{a + b + c + \dots} = b$$

$$R_c = (a + b + c + \dots) \frac{c}{a + b + c + \dots} = c$$

and so on. This serves as a check on the soundness of our method.

5.—Let  $Aabc \dots = B + R_a + R_b + R_c + \dots$

where  $R_a$  corresponds to  $a$ ,  $R_b$  to  $b$ ,  $R_c$  to  $c$ , and so on.

Speaking more precisely, let  $R_a$ ,  $R_b$ ,  $R_c$ , etc., be such that adding  $R_a$  to  $B$  contributes as much to the sum  $B + R_a + R_b + R_c + \dots$  as multiplying  $A$  by  $a$  (simultaneously with the other factors) contributes to the product  $Aabc \dots$ , and similarly for  $R_b$ ,  $R_c$ , etc.

If  $a = 1$ ,  $R_a = 0$ , since multiplying by 1 leaves the product unchanged and adding 0 leaves the sum unchanged. Hence, putting  $a = 1$ ,  $b = 1$ ,  $c = 1$ , and so on, we have  $A = B$ .

Thus we must say :

$$\text{Let } Aabc \dots = A + R_a + R_b + R_c + \dots$$

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Let us consider, in this case also, what we are doing when we endeavour to determine  $R_a$ ,  $R_b$ ,  $R_c$ , etc. We know what are the changes, *measured as ratios*, produced in  $A$  by multiplying it by  $a$ ,  $b$ ,  $c$ , etc. We wish to know what those changes are *measured as differences*.

If  $A$  is only multiplied by one factor,  $a$ , we have  $Aa = A + R_a$ , which gives  $R_a = Aa - A$ .

If, however,  $A$  is multiplied even by only two factors,  $a$  and  $b$ , the problem is already more complicated. We have  $Aab = A + R_a + R_b$ .

Let us suppose that  $ab$  remains constant and equal to  $K$  but that  $a$  and  $b$  vary.

If  $a = 1$ ,  $R_a = 0$ , as we have seen above. In this case  $b = K$  and  $R_b = AK - A$ .

Now let  $a$  increase gradually and  $b$  gradually decrease. Then  $R_a$  gradually increases and  $R_b$  gradually decreases.

When  $a = b$ ,  $R_a = R_b$ , since in this case multiplying by  $a$  increases the product by the same amount as multiplying by  $b$  and, therefore, adding  $R_a$  must increase the sum by the same amount as adding  $R_b$ .

As  $a$  continues to increase and  $b$  to decrease,  $a$  becomes greater than  $b$ , and  $R_a$  becomes greater than  $R_b$ . When  $a$  becomes equal to  $K$  and  $b$  to  $1$ ,  $R_a = AK - A$  and  $R_b = 0$ . If  $a$  increases still further, so that  $b$  becomes less than  $1$ , the change produced in  $A$  by multiplying it by  $b$  is to reduce it and therefore  $R_b$  becomes negative. When  $a$  becomes  $\infty$  and  $b$  becomes  $0$ ,  $R_a = \infty$  and  $R_b = -\infty$ .

A case that does not occur when we suppose that  $ab = K$  is the case in which  $ab = 1$  or  $b = \frac{1}{a}$ . In this case,

the result of multiplying by  $ab$  is to leave  $A$  unchanged. Hence the result of adding  $R_a$  and  $R_b$  must be to leave  $A$  unchanged. Thus  $R_a + R_b = 0$  or  $R_b = -R_a$ .

The same reasoning may be extended to other factors and applied in cases in which  $a$ ,  $b$ ,  $c$ , vary independently, and thus we obtain the following general properties, which apply not only to  $R_a$  but to any two of the terms  $R_a$ ,  $R_b$ ,  $R_c$ , etc.

If  $a = 1$ ,  $R_a = 0$ ; if  $b = 1$ ,  $R_b = 0$ , and so on.

If  $a \geq b$ ,  $R_a \geq R_b$ .

If  $b = \frac{1}{a}$ ,  $R_b = -R_a$ .

Again, if we treat  $ab$  as a single factor and let  $R_{ab}$  be the term corresponding to  $ab$ , then  $R_{ab} = R_a + R_b$  since multiplying by both  $a$  and  $b$  (that is,  $ab$ ) increases the product by the same amount as multiplying by both  $R_a$  and  $R_b$  (that is,  $R_a + R_b$ ) to the sum.

From this it also results that  $R_{a^n} = nR_a$ , where  $n$  is a positive integer, and from the property that  $b = a^{-1}$ ,  $R_b = -R_a$  it can also be shown that  $R_{a^{-n}} = -nR_a$ , where  $n$  is a negative integer.

Now let  $a = r^x$ ,  $b = r^y$ ,  $c = r^z$ , and so on, where  $x$ ,  $y$ ,  $z$ , etc., are integers. This can always be done, not absolutely, but to any desired degree of approximation provided  $a$ ,  $b$ ,  $c$ , etc., are positive, and in what follows we must suppose this to be the case.

For example, let  $a = 1.36$ ,  $b = .84$ ,  $c = 15.27$ . We have :

$$\text{Log } 1.36 = .1335389$$

$$\text{Log } .84 = \bar{1}.9242793 = -.0757207$$

$$\text{Log } 15.27 = 1.1838390$$

1·36 is approximately equal to  $(10^{\frac{x}{10^7}})^{1335889}$

·84 is approximately equal to  $(10^{\frac{x}{10^7}})^{-757207}$

15·27 is approximately equal to  $(10^{\frac{x}{10^7}})^{11838390}$

The larger the number of places of decimals to which the logarithms are calculated the closer the approximation becomes.

Hence, if  $a, b, c$ , etc., are positive, we may say let  $a = r^x, b = r^y, c = r^z$ , and so on, where  $x, y, z$ , etc., are integers.

$$\begin{aligned} \text{Then } Aabc \dots &= Ar^x r^y r^z \dots \\ &= A + Rr^x + Rr^y + Rr^z + \dots \\ &= A + xRr + yRr + zRr + \dots \end{aligned}$$

where  $Rr$  is the term corresponding to the factor  $r$ .

$$\text{Hence } Aabc \dots - A = (x + y + z + \dots)Rr$$

$$Rr = A(abc \dots - 1) \frac{1}{x + y + z + \dots}$$

$$\text{But } Ra = Rr^x = xRr$$

$$\text{Therefore } Ra = A(abc \dots - 1) \frac{x}{x + y + z + \dots}$$

$$\text{But } x : y : z \dots = \log a : \log b : \log c \dots$$

since  $\log a = x \log r, \log b = y \log r, \log c = z \log r$ , and so on.

Therefore

$$\begin{aligned} Ra &= A(abc \dots - 1) \frac{\log a}{\log a + \log b + \log c + \dots} \\ &= A(abc \dots - 1) \frac{\log a}{\log abc \dots} \end{aligned}$$

Similarly

$$R_b = A(abc \dots - 1) \frac{\log b}{\log abc \dots}$$

$$R_c = A(abc \dots - 1) \frac{\log c}{\log abc \dots}$$

and so on.

In the special case in which  $A = 1$ , we have

$$abc \dots = 1 + R_a + R_b + R_c + \dots$$

$$R_a = (abc \dots - 1) \frac{\log a}{\log abc \dots}$$

$$R_b = (abc \dots - 1) \frac{\log b}{\log abc \dots}$$

$$R_c = (abc \dots - 1) \frac{\log c}{\log abc \dots}$$

and so on.

This only applies where  $a, b, c$ , etc., are all positive.

6.—In the equation

$$Aabc \dots = A + R_a + R_b + R_c + \dots$$

let  $A = a_0 b_0 c_0 \dots$ ,  $a = \frac{a_1}{a_0}$ ,  $b = \frac{b_1}{b_0}$ ,  $c = \frac{c_1}{c_0}$ , and so on,

where  $a_0, b_0, c_0$ , etc., and  $a_1, b_1, c_1$ , etc., are all positive.

Then

$$\begin{aligned} (a_0 b_0 c_0 \dots) \frac{a_1}{a_0} \cdot \frac{b_1}{b_0} \cdot \frac{c_1}{c_0} \dots \\ = a_0 b_0 c_0 \dots + R_{a_1} + R_{b_1} + R_{c_1} + \dots \end{aligned}$$

where  $R_{a_1}$  is such that adding  $R_{a_1}$  to  $a_0 b_0 c_0 \dots$  contributes as much to the sum  $a_0 b_0 c_0 \dots + R_{a_1} + R_{b_1} + R_{c_1} + \dots$  (which is equal to  $a_1 b_1 c_1 \dots$ ) as multi-

plying  $a_0 b_0 c_0 \dots$  by  $\frac{a_1}{a_0}$  (simultaneously with the other factors  $\frac{b_1}{b_0}, \frac{c_1}{c_0}$ , etc.) contributes to the product  $(a_0 b_0 c_0 \dots) \frac{a_1}{a_0} \cdot \frac{b_1}{b_0} \cdot \frac{c_1}{c_0} \dots$  (which is also equal to  $a_1 b_1 c_1 \dots$ ) and similarly for  $R_{a_0}^{b_1}, R_{b_0}^{c_1}$ , etc. Hence we may say that  $R_{a_0}^{a_1}$  represents the change (measured as a difference) in  $a_0 b_0 c_0 \dots$  due to the change from  $a_0$  to  $a_1$ , that  $R_{b_0}^{b_1}$  represents the change (measured as a difference) in  $a_0 b_0 c_0 \dots$  due to the change from  $b_0$  to  $b_1$ , and so on.

We have

$$R_{a_0}^{a_1} = (a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots) \frac{\log \frac{a_1}{a_0}}{\log \frac{a_1 b_1 c_1 \dots}{a_0 b_0 c_0 \dots}}$$

$$R_{b_0}^{b_1} = (a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots) \frac{\log \frac{b_1}{b_0}}{\log \frac{a_1 b_1 c_1 \dots}{a_0 b_0 c_0 \dots}}$$

$$R_{c_0}^{c_1} = (a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots) \frac{\log \frac{c_1}{c_0}}{\log \frac{a_1 b_1 c_1 \dots}{a_0 b_0 c_0 \dots}}$$

and so on.

Thus we can transform the difference  $a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots$ , which is the increase of  $a_0 b_0 c_0 \dots$  due to the change of  $a_0$  to  $a_1$ ,  $b_0$  to  $b_1$ ,  $c_0$  to  $c_1$ , and so on, into the sum of a series of terms  $R_{a_0}^{a_1}, R_{b_0}^{b_1}, R_{c_0}^{c_1}$ , etc., of which  $R_{a_0}^{a_1}$  represents the increase of  $a_0 b_0 c_0 \dots$  due to the change of  $a_0$  to  $a_1$  (simultaneously with the

change of  $b_0$  to  $b_1$ ,  $c_0$  to  $c_1$ , and so on),  $R_{b_0}^{b_1}$  represents the increase of  $a_0 b_0 c_0 \dots$  due to the change of  $b_0$  to  $b_1$  (simultaneously with the other changes), and so on.

It should be noted that in this case it is not necessary that  $a_0, b_0, c_0$ , etc., and  $a_1, b_1, c_1$ , etc., should be pure numbers, since the reasoning turns on the supposition that  $\frac{a_1}{a_0} = r^x, \frac{b_1}{b_0} = r^y, \frac{c_1}{c_0} = r^z$ , and so on, and  $\frac{a_1}{a_0}, \frac{b_1}{b_0}, \frac{c_1}{c_0}$ , etc., are ratios and therefore pure numbers. It is, of course, essential that  $a_1$  and  $a_0$  should be measured in the same unit, that  $b_1$  and  $b_0$  should be measured in the same unit, and so on, and that it should be possible to attach a meaning to  $a_1 b_1 c_1 \dots$  and  $a_0 b_0 c_0 \dots$ .

7.—If we say :

Let  $Aabc \dots = BRaRbRc \dots$

where  $Ra$  is such that it contributes as much to the product  $BRaRbRc \dots$  as  $a$  contributes to the product  $Aabc \dots$ , and similarly for  $Rb, Rc$ , etc., then it is obvious that  $B = A, Ra = a, Rb = b, Rc = c$ , and so on. But if we shut our eyes to this and calculate  $Ra, Rb, Rc$ , etc., by similar methods, we get the result :

$$Ra = (abc \dots)^{\frac{\log a}{\log abc \dots}} = e^{(\log abc \dots) \frac{\log a}{\log abc \dots}} = e^{\log a} = a$$

$$Rb = (abc \dots)^{\frac{\log b}{\log abc \dots}} = e^{(\log abc \dots) \frac{\log b}{\log abc \dots}} = e^{\log b} = b$$

$$Rc = (abc \dots)^{\frac{\log c}{\log abc \dots}} = e^{(\log abc \dots) \frac{\log c}{\log abc \dots}} = e^{\log c} = c$$

and so on. This is a further check on the soundness of our method.

8.—Applying the method to other functions, giving in each case the appropriate definitions of  $Ra, Rb, Rc$ ,



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 etc., and making the necessary limitations, we obtain  
 the following results :—

$$\text{If } f^{abc\dots} = tRaRbRc\dots$$

$$Ra = (f^{abc\dots} - 1) \frac{\log a}{\log abc\dots}$$

$$\text{If } f^{a_1 b_1 c_1 \dots} = f^{a_0 b_0 c_0 \dots} R_{a_0} R_{b_0} R_{c_0} \dots$$

$$R_{a_0} = \left( \frac{f^{a_1 b_1 c_1 \dots}}{f^{a_0 b_0 c_0 \dots}} \right) \frac{\log \frac{a_1}{a_0}}{\log \frac{a_1 b_1 c_1 \dots}{a_0 b_0 c_0 \dots}}$$

$$\text{If } f^{abc\dots} = t + Ra + Rb + Rc + \dots$$

$$Ra = (f^{abc\dots} - t) \frac{a}{a + b + c + \dots}$$

$$\text{If } f^{a_1 b_1 c_1 \dots} = f^{a_0 b_0 c_0 \dots} + R_{a_0} + R_{b_0} + R_{c_0} + \dots$$

$$R_{a_0} = (f^{a_1 b_1 c_1 \dots} - f^{a_0 b_0 c_0 \dots}) \frac{\frac{a_1}{a_0}}{\frac{a_1}{a_0} + \frac{b_1}{b_0} + \frac{c_1}{c_0} + \dots}$$

$$\text{If } f^{a+b+c+\dots} = 1 + Ra + Rb + Rc + \dots$$

$$Ra = (f^{a+b+c+\dots} - 1) \frac{a}{a + b + c + \dots}$$

$$\text{If } f^{a_1 + b_1 + c_1 + \dots} = f^{a_0 + b_0 + c_0 + \dots} + R_{a_0 - a_0} + R_{b_0 - b_0} + R_{c_0 - c_0} + \dots$$

$$R_{a_0 - a_0} = (f^{a_1 + b_1 + c_1 + \dots} - f^{a_0 + b_0 + c_0 + \dots}) \frac{a_1 - a_0}{(a_1 - a_0) + (b_1 - b_0) + (c_1 - c_0) + \dots}$$

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If  $x^{a+b+c+\dots} = R_a R_b R_c \dots$

$$R_a = (x^{a+b+c+\dots})^{\frac{a}{a+b+c+\dots}} = x^a$$

as it obviously must be.

If  $\log(x + a + b + c + \dots) = R_a + R_b + R_c + \dots$

$$R_a = \left\{ \log(x + a + b + c + \dots) \right\} \frac{a}{a + b + c + \dots}$$

If  $\log(a_1 + b_1 + c_1 + \dots) =$   
 $\log(a_0 + b_0 + c_0 + \dots) + R_{a_1 - a_0} + R_{b_1 - b_0} + R_{c_1 - c_0} + \dots$

$$R_{a_1 - a_0} = \left\{ \frac{\log(a_1 + b_1 + \dots)}{\log(a_0 + b_0 + \dots)} \right\} \frac{a_1 - a_0}{(a_1 - a_0) + (b_1 - b_0) + \dots}$$

If  $\log Aabc \dots = (\log A)R_a R_b R_c \dots$

$$R_a = \left( \frac{\log Aabc \dots}{\log A} \right)^{\frac{\log a}{\log abc \dots}}$$

In this case if  $A = 1$ ,  $R_a$ ,  $R_b$ ,  $R_c$ , etc., all become equal to infinity.

If  $\log a_1 b_1 c_1 \dots = (\log a_0 b_0 c_0 \dots) R_{\frac{a_1}{a_0}} R_{\frac{b_1}{b_0}} R_{\frac{c_1}{c_0}} \dots$

$$R_{\frac{a_1}{a_0}} = \left( \frac{\log a_1 b_1 c_1 \dots}{\log a_0 b_0 c_0 \dots} \right)^{\frac{\log \frac{a_1}{a_0}}{\log \frac{a_1 b_1 c_1 \dots}{a_0 b_0 c_0 \dots}}}$$

If  $\log abc \dots = R_a + R_b + R_c + \dots$

$$R_a = (\log abc \dots) \frac{\log a}{\log abc \dots} = \log a$$

as it obviously must be.

9.—We must now endeavour to apply the method to a more complex function.

In the equation

$$A + a + b + c + \dots = ARaRbRc \dots$$

let  $A = a_0'b_0'c_0' \dots + a_0''b_0''c_0'' \dots + \dots$ , let  $a = a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots$ ,  $b = a_1''b_1''c_1'' \dots - a_0''b_0''c_0'' \dots$ , and so on. Let us further assume that all the numbers of the form  $a_0, a_1, b_0, b_1, c_0, c_1$ , etc., are positive.

Then  $a_0'b_0'c_0' \dots + a_0''b_0''c_0'' \dots + \dots + (a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots) + (a_1''b_1''c_1'' \dots - a_0''b_0''c_0'' \dots) + \dots = (a_0'b_0'c_0' \dots + a_0''b_0''c_0'' \dots + \dots)R(a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots)R(a_1''b_1''c_1'' \dots - a_0''b_0''c_0'' \dots) \dots$  where  $R(a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots)$  is such that multiplying  $(a_0'b_0'c_0' \dots + a_0''b_0''c_0'' \dots + \dots)$  by  $R(a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots)$  simultaneously with the other factors  $R(a_1''b_1''c_1'' \dots - a_0''b_0''c_0'' \dots)$ , etc., contributes as much to the product  $(a_0'b_0'c_0' \dots + a_0''b_0''c_0'' \dots + \dots)R(a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots)R(a_1''b_1''c_1'' \dots - a_0''b_0''c_0'' \dots) \dots$  (which is equal to  $a_1'b_1'c_1' \dots + a_1''b_1''c_1'' \dots + \dots$ ) as adding  $(a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots)$  to  $(a_0'b_0'c_0' \dots + a_0''b_0''c_0'' \dots + \dots)$  contributes to the sum  $a_0'b_0'c_0' \dots + a_0''b_0''c_0'' \dots + \dots + (a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots) + (a_1''b_1''c_1'' \dots - a_0''b_0''c_0'' \dots) + \dots$  (which is also equal to  $a_1'b_1'c_1' \dots + a_1''b_1''c_1'' \dots + \dots$ ) and similarly for the other factors.

Then  $R(a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots)$

$$= \left( \frac{a_1'b_1'c_1' \dots + a_0'b_0'c_0' \dots}{a_1''b_1''c_1'' \dots + a_0''b_0''c_0'' \dots + \dots} \right) \frac{a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots}{(a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots) + \dots}$$

We may write this more simply

$$R(a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots) = \left( \frac{\sum a_1'b_1'c_1' \dots}{\sum a_0'b_0'c_0' \dots} \right)^{\frac{a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots}{\sum (a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots)}}$$

$$\text{Now } a_1'b_1'c_1' \dots = \frac{a_1'}{a_0'} a_0' \frac{b_1'}{b_0'} b_0' \frac{c_1'}{c_0'} c_0' \dots$$

$$\text{and } a_0''b_0''c_0'' \dots + a_0''b_0''c_0'' \dots + \dots + \\ (a_1'b_1'c_1' \dots - a_0'b_0'c_0' \dots) + (a_1''b_1''c_1'' \dots \\ - a_0''b_0''c_0'' \dots) + \dots$$

$$= a_1'b_1'c_1' \dots + a_1''b_1''c_1'' \dots + \dots \\ = \frac{a_1'}{a_0'} a_0' \frac{b_1'}{b_0'} b_0' \frac{c_1'}{c_0'} c_0' \dots + \frac{a_1''}{a_0''} a_0'' \frac{b_1''}{b_0''} b_0'' \frac{c_1''}{c_0''} c_0'' \dots + \dots$$

Let

$$\frac{a_1'}{a_0'} a_0' \frac{b_1'}{b_0'} b_0' \frac{c_1'}{c_0'} c_0' \dots + \frac{a_1''}{a_0''} a_0'' \frac{b_1''}{b_0''} b_0'' \frac{c_1''}{c_0''} c_0'' \dots + \dots \\ = (a_0'b_0'c_0' \dots + a_0''b_0''c_0'' \dots \\ + \dots) R_{\frac{a_1'}{a_0'}} R_{\frac{b_1'}{b_0'}} R_{\frac{c_1'}{c_0'}} \dots R_{\frac{a_1''}{a_0''}} R_{\frac{b_1''}{b_0''}} R_{\frac{c_1''}{c_0''}} \dots$$

where multiplying  $(a_0'b_0'c_0' \dots + a_0''b_0''c_0'' \dots + \dots)$  by  $R_{\frac{a_1'}{a_0'}}$  (simultaneously with the other factors)

contributes as much to the product  $(a_0'b_0'c_0' \dots + a_0''b_0''c_0'' \dots + \dots) R_{\frac{a_1'}{a_0'}} R_{\frac{b_1'}{b_0'}} R_{\frac{c_1'}{c_0'}} \dots R_{\frac{a_1''}{a_0''}} R_{\frac{b_1''}{b_0''}} R_{\frac{c_1''}{c_0''}} \dots$

(which is equal to  $a_1'b_1'c_1' \dots + a_1''b_1''c_1'' \dots + \dots$ ) as multiplying  $a_0'b_0'c_0' \dots$  by  $\frac{a_1'}{a_0'}$  (simulta-

neously with  $\frac{b_1'}{b_0'}$ ,  $\frac{c_1'}{c_0'}$ , etc., and at the same time that

$a_0''b_0''c_0'' \dots$  is multiplied by  $\frac{a_1''}{a_0''}$ ,  $\frac{b_1''}{b_0''}$ ,  $\frac{c_1''}{c_0''}$ , etc., and

so on) contributes to the sum  $\frac{a_1'}{a_0'} a_0' \frac{b_1'}{b_0'} b_0' \frac{c_1'}{c_0'} c_0' \dots +$

$\frac{a_1''}{a_0''} \frac{b_1''}{b_0''} \frac{c_1''}{c_0''} \dots + \dots$  (which is also equal to  $a_1' b_1' c_1' \dots + a_1'' b_1'' c_1'' \dots + \dots$ ), and similarly for the other factors.

Now, in the second case, multiplying  $a_0' b_0' c_0' \dots$  by  $\frac{a_1'}{a_0'} \cdot \frac{b_1'}{b_0'} \cdot \frac{c_1'}{c_0'} \dots$  produces precisely the same result as adding  $(a_1' b_1' c_1' \dots - a_0' b_0' c_0' \dots)$  to  $a_0' b_0' c_0' \dots + a_0'' b_0'' c_0'' \dots + \dots$  in the first case.

Hence  $R_{a_0'}^{a_1'} R_{b_0'}^{b_1'} R_{c_0'}^{c_1'} \dots = R_{(a_1' b_1' c_1' \dots - a_0' b_0' c_0' \dots)}$

Reasoning on the same lines as in sections I and 5, it is clear that  $R_{a_0'}^{a_1'} \cdot \frac{b_1'}{b_0'} = R_{a_0'}^{a_1'} \cdot R_{b_0'}^{b_1'}$ , and that if  $\frac{b_1'}{b_0'} = \frac{a_1'}{a_0'}$ ,  $R_{b_0'}^{b_1'} = R_{a_0'}^{a_1'}$ . Hence it can be shown that if  $x$  is a positive integer,  $R_{(a_0')}^x = (R_{a_0'}^{a_1'})^x$ . Again if  $\frac{b_1'}{b_0'} = \frac{1}{\frac{a_1'}{a_0'}} \cdot \frac{a_1'}{a_0'} \cdot \frac{b_1'}{b_0'} = 1$

and  $R_{a_0'}^{a_1'} \cdot R_{b_0'}^{b_1'} = 1$ , whence  $R_{b_0'}^{b_1'} = \frac{1}{R_{a_0'}^{a_1'}} = (R_{a_0'}^{a_1'})^{-1}$ . From

this it can be shown that, if  $x$  is a negative integer, it is also true that  $R_{(a_0')}^x = (R_{a_0'}^{a_1'})^x$ . (It must be noted

that if  $\frac{b_1''}{b_0''} = \frac{a_1'}{a_0'}$ ,  $R_{b_0''}^{b_1''}$  is *not* equal to  $R_{a_0'}^{a_1'}$ ).

Now let  $\frac{a_1'}{a_0'} = (r')^x$ ,  $\frac{b_1'}{b_0'} = (r')^y$ ,  $\frac{c_1'}{c_0'} = (r')^z$ , and so on, where  $x'$ ,  $y'$ ,  $z'$ , etc., are integers.

Then  $R_{a_0'}^{a_1'} = (Rr')^x$ ,  $R_{b_0'}^{b_1'} = (Rr')^y$ ,  $R_{c_0'}^{c_1'} = (Rr')^z$ , and so on.

Thus  $R_{a_0'}^{a_1'} R_{b_0'}^{b_1'} R_{c_0'}^{c_1'} \dots = (Rr')^{x+y+z+\dots}$

But  $R_{a_0'}^{a_1'} R_{b_0'}^{b_1'} R_{c_0'}^{c_1'} \dots = R_{(a_1' b_1' c_1' \dots - a_0' b_0' c_0' \dots)}$

Therefore

$$(R_r)^{x+y+z+\dots} = \left( \frac{\Sigma a_1 b_1 c_1 \dots}{\Sigma a_0 b_0 c_0 \dots} \right)^{\frac{a_1' b_1' c_1' \dots - a_0' b_0' c_0' \dots}{\Sigma (a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots)}}$$

$$R_r = \left( \frac{\Sigma a_1 b_1 c_1 \dots}{\Sigma a_0 b_0 c_0 \dots} \right)^{\frac{a_1' b_1' c_1' \dots - a_0' b_0' c_0' \dots}{\Sigma (a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots)} \cdot \frac{x}{x+y+z+\dots}}$$

But  $R_{a_1'} = (R_r)^x$

Hence  $R_{a_1'}$

$$= \left( \frac{\Sigma a_1 b_1 c_1 \dots}{\Sigma a_0 b_0 c_0 \dots} \right)^{\frac{a_1' b_1' c_1' \dots - a_0' b_0' c_0' \dots}{\Sigma (a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots)} \cdot \frac{x}{x+y+z+\dots}}$$

$$= \left( \frac{\Sigma a_1 b_1 c_1 \dots}{\Sigma a_0 b_0 c_0 \dots} \right)^{\frac{a_1' b_1' c_1' \dots - a_0' b_0' c_0' \dots}{\Sigma (a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots)} \cdot \frac{\log \frac{a_1'}{a_0'}}{\log \frac{a_1' b_1' c_1' \dots}{a_0' b_0' c_0' \dots}}}$$

Similarly  $R_{b_1'}$

$$= \left( \frac{\Sigma a_1 b_1 c_1 \dots}{\Sigma a_0 b_0 c_0 \dots} \right)^{\frac{a_1' b_1' c_1' \dots - a_0' b_0' c_0' \dots}{\Sigma (a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots)} \cdot \frac{\log \frac{b_1'}{b_0'}}{\log \frac{a_1' b_1' c_1' \dots}{a_0' b_0' c_0' \dots}}}$$

and so on.

If we now say :

$$\begin{aligned} \text{Let } \Sigma \frac{a_1}{a_0} \frac{b_1}{b_0} \frac{c_1}{c_0} \dots \\ = (\Sigma a_0 b_0 c_0 \dots) R \left( \frac{a_1'}{a_0'}, \frac{a_1''}{a_0''}, \frac{a_1'''}{a_0'''} \dots \right) R \left( \frac{b_1'}{b_0'}, \frac{b_1''}{b_0''}, \frac{b_1'''}{b_0'''} \dots \right) \dots \end{aligned}$$

where  $R \left( \frac{a_1'}{a_0'}, \frac{a_1''}{a_0''}, \frac{a_1'''}{a_0'''} \dots \right)$  is such that multiplying  $\Sigma a_0 b_0 c_0 \dots$  by  $R \left( \frac{a_1'}{a_0'}, \frac{a_1''}{a_0''}, \frac{a_1'''}{a_0'''} \dots \right)$ , simultaneously with the other factors, contributes as much to the product  $(\Sigma a_0 b_0 c_0 \dots) R \left( \frac{a_1'}{a_0'}, \frac{a_1''}{a_0''}, \frac{a_1'''}{a_0'''} \dots \right) R \left( \frac{b_1'}{b_0'}, \frac{b_1''}{b_0''}, \frac{b_1'''}{b_0'''} \dots \right) \dots$  (which is equal to  $\Sigma a_1 b_1 c_1 \dots$ ) as multiplying  $a_0' b_0' c_0' \dots$  by

$\frac{a_1'}{a_0'}$ ,  $a_0''b_0''c_0'' \dots$  by  $\frac{a_1''}{a_0''}$ ,  $a_0'''b_0'''c_0''' \dots$  by  $\frac{a_1'''}{a_0'''}$ ,  
 and so on (at the same time that  $a_0'b_0'c_0' \dots$  is  
 multiplied by  $\frac{b_1'}{b_0'}$ ,  $\frac{c_1'}{c_0'}$ , etc.,  $a_0''b_0''c_0'' \dots$  is multiplied  
 by  $\frac{b_1''}{b_0''}$ ,  $\frac{c_1''}{c_0''}$ , etc., and so on), and similarly for the other  
 factors, then it is evident that

$$R\left(\frac{a_1'}{a_0'}, \frac{a_1''}{a_0''}, \frac{a_1'''}{a_0'''} \dots\right) = R_{\frac{a_1'}{a_0'}} R_{\frac{a_1''}{a_0''}} R_{\frac{a_1'''}{a_0'''}} \dots$$

Hence

$$R\left(\frac{a_1'}{a_0'}, \frac{a_1''}{a_0''}, \frac{a_1'''}{a_0'''} \dots\right) = \left(\frac{\sum a_1 b_1 c_1 \dots}{\sum a_0 b_0 c_0 \dots}\right) \frac{\Sigma(a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots) \frac{\log \frac{a_1}{a_0}}{\log \frac{a_1 b_1 c_1 \dots}{a_0 b_0 c_0 \dots}}}{\Sigma(a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots)}$$

Similarly

$$R\left(\frac{b_1'}{b_0'}, \frac{b_1''}{b_0''}, \frac{b_1'''}{b_0'''} \dots\right) = \left(\frac{\sum a_1 b_1 c_1 \dots}{\sum a_0 b_0 c_0 \dots}\right) \frac{\Sigma(a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots) \frac{\log \frac{b_1}{b_0}}{\log \frac{a_1 b_1 c_1 \dots}{a_0 b_0 c_0 \dots}}}{\Sigma(a_1 b_1 c_1 \dots - a_0 b_0 c_0 \dots)}$$

and so on.

It will be seen that it is not necessary that the  
 symbols of the form  $a_0$ ,  $b_0$ ,  $c_0$ , etc., and  $a_1$ ,  $b_1$ ,  $c_1$ , etc.,  
 should represent pure numbers. It is, however, essen-  
 tial that each pair of the form  $a_0$  and  $a_1$ ,  $b_0$  and  $b_1$ ,  $c_0$   
 and  $c_1$ , and so on, should be measured in the same  
 unit, so that  $\frac{a_1}{a_0}$ ,  $\frac{b_1}{b_0}$ ,  $\frac{c_1}{c_0}$ , etc., are pure numbers. It is  
 also essential that all products of the form  $a_0 b_0 c_0 \dots$   
 or  $a_1 b_1 c_1 \dots$  should represent the same thing and be  
 measured in the same unit.

But  $R\left(\frac{a_1'}{a_0'}, \frac{a_1''}{a_0''}, \frac{a_1'''}{a_0'''} \dots\right)$  is the change, measured as a  
 ratio, of  $\sum a_0 b_0 c_0 \dots$  due to the changes of  $a_0'$  to  $a_1'$ ,

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of  $a_0''$  to  $a_1''$ , of  $a_0'''$  to  $a_1'''$ , and so on, and similarly for  $R(\frac{b_1'}{b_0'}, \frac{b_1''}{b_0''}, \frac{b_1'''}{b_0'''} \dots)$ , etc. Thus in transforming

$\frac{\Sigma a_1 b_1 c_1 \dots}{\Sigma a_0 b_0 c_0 \dots}$  into the product of ratios  $R(\frac{a_1'}{a_0'}, \frac{a_1''}{a_0''}, \frac{a_1'''}{a_0'''} \dots)$ ,  $R(\frac{b_1'}{b_0'}, \frac{b_1''}{b_0''}, \frac{b_1'''}{b_0'''} \dots)$ , etc., we have solved a problem of which the problem of the price index and the quantity index is only a special case.



## CHAPTER III

### THE DETERMINATION OF THE FORMULA FOR THE PRICE INDEX

We are now in a position to determine the formulæ for  $P_{01}$  and  $Q_{01}$  in accordance with the conception that we have put forward. Of course, if a different conception is adopted, the mathematical problem will be different.

1.—Taking the measure of the change of aggregate value as  $\frac{\sum p_1 q_1}{\sum p_0 q_0}$ , we have said: Let us suppose that it is transformed into the product of two ratios, one (which we will call  $R_p$ ) representing the change in the aggregate value due to the changes in price, and the other (which we will call  $R_q$ ) representing the change in the aggregate value due to the changes in quantity.

In the light of the illustrations given, we may expand this by saying:

$$\text{Let } \frac{\sum p_1 q_1}{\sum p_0 q_0} = (\sum p_0 q_0) R_p R_q,$$

where  $R_p$  and  $R_q$  are such that multiplying  $\sum p_0 q_0$  by  $R_p$  (simultaneously with  $R_q$ ) contributes as much to the product  $(\sum p_0 q_0) R_p R_q$  (which is equal to  $\sum p_1 q_1$ ) as multiplying  $p_0' q_0'$  by  $\frac{p_1'}{p_0'}$ ,  $p_0'' q_0''$  by  $\frac{p_1''}{p_0''}$ ,  $p_0''' q_0'''$

by  $\frac{p_1'''}{p_0''}$ , and so on (simultaneously with the multiplication of  $p_0'q_0'$  by  $\frac{q_1'}{q_0'}$ ,  $p_0''q_0''$  by  $\frac{q_1''}{q_0''}$ ,  $p_0'''q_0'''$  by  $\frac{q_1'''}{q_0'''}$ , and so on) contributes to the sum  $\sum \frac{p_1}{p_0} p_0 \frac{q_1}{q_0} q_0$  (which is also equal to  $\sum p_1 q_1$ ) and multiplying  $\sum p_0 q_0$  by  $R_q$  (simultaneously with  $R_p$ ) contributes as much to the product  $(\sum p_0 q_0) R_p R_q$  as multiplying  $p_0'q_0'$  by  $\frac{q_1'}{q_0'}$ ,  $p_0''q_0''$  by  $\frac{q_1''}{q_0''}$ ,  $p_0'''q_0'''$  by  $\frac{q_1'''}{q_0'''}$ , and so on (simultaneously with the multiplication of  $p_0'q_0'$  by  $\frac{p_1'}{p_0'}$ ,  $p_0''q_0''$  by  $\frac{p_1''}{p_0''}$ ,  $p_0'''q_0'''$  by  $\frac{p_1'''}{p_0'''}$ , and so on) contributes to the sum  $\sum \frac{p_1}{p_0} p_0 \frac{q_1}{q_0} q_0$ .

If now we say :

$$\text{Let } \sum \frac{p_1}{p_0} p_0 \frac{q_1}{q_0} q_0 = (\sum p_0 q_0) R_{p_0} R_{q_0} R_{p_1} R_{q_1} R_{p_2} R_{q_2} R_{p_3} R_{q_3} \dots$$

where  $R_{p_0}$  is such that multiplying  $\sum p_0 q_0$  by  $R_{p_0}$  (simultaneously with the other factors) contributes as much to the product  $(\sum p_0 q_0) R_{p_0} R_{q_0} R_{p_1} R_{q_1} R_{p_2} R_{q_2} R_{p_3} R_{q_3} \dots$

(which is equal to  $\sum p_1 q_1$ ) as multiplying  $p_0'$  by  $\frac{p_1'}{p_0'}$

(at the same time that  $q_0'$  is multiplied by  $\frac{q_1'}{q_0'}$ ,  $p_0''$  by  $\frac{p_1''}{p_0''}$ ,  $q_0''$  by  $\frac{q_1''}{q_0''}$ , and so on) contributes to the sum

$\sum \frac{p_1}{p_0} p_0 \frac{q_1}{q_0} q_0$  (which is also equal to  $\sum p_1 q_1$ ) and similarly

for the other factors, then it is clear that

$$R_{\rho_0'} R_{\rho_0''} R_{\rho_0'''} \dots = R_P$$

and

$$R_{\rho_0'} R_{\rho_0''} R_{\rho_0'''} \dots = R_Q$$

But the equation is of the form

$$\Sigma a_1 b_1 c_1 \dots = (\Sigma a_0 b_0 c_0 \dots) R_{\rho_0'} R_{\rho_0''} R_{\rho_0'''} \dots R_{\rho_0^{(n)}} R_{\rho_0^{(n+1)}} R_{\rho_0^{(n+2)}} \dots$$

Hence on the analogy of

$$R_{\rho_0'} = \left( \frac{\Sigma a_1 b_1 c_1 \dots}{\Sigma a_0 b_0 c_0 \dots} \right)^{\frac{\log \frac{a_1'}{a_0'}}{\log \frac{a_1' b_1' c_1' \dots}{a_0' b_0' c_0' \dots}}}$$

we have

$$R_{\rho_0'} = \left( \frac{\Sigma \rho_1 q_1}{\Sigma \rho_0 q_0} \right)^{\frac{(\rho_1' q_1' - \rho_0' q_0') \frac{\log \frac{\rho_1'}{\rho_0'}}{\log \frac{\rho_1' q_1'}{\rho_0' q_0'}}}{\Sigma (\rho_1 q_1 - \rho_0 q_0)}}$$

and since

$$R_P = R_{\rho_0'} R_{\rho_0''} R_{\rho_0'''} \dots$$

we obtain

$$R_P = \left( \frac{\Sigma \rho_1 q_1}{\Sigma \rho_0 q_0} \right)^{\frac{\Sigma (\rho_1 q_1 - \rho_0 q_0) \frac{\log \frac{\rho_1}{\rho_0}}{\log \frac{\rho_1 q_1}{\rho_0 q_0}}}{\Sigma (\rho_1 q_1 - \rho_0 q_0)}}$$

Similarly

$$R_Q = \left( \frac{\Sigma \rho_1 q_1}{\Sigma \rho_0 q_0} \right)^{\frac{\Sigma (\rho_1 q_1 - \rho_0 q_0) \frac{\log \frac{q_1}{q_0}}{\log \frac{\rho_1 q_1}{\rho_0 q_0}}}{\Sigma (\rho_1 q_1 - \rho_0 q_0)}}$$

It is obvious that  $R_P = P_{01}$  and  $R_Q = Q_{01}$ , but we need not assume this.

In the hypothetical case :

$$\text{Let } \Sigma P_{01} \rho_0 Q_{01} q_0 = (\Sigma \rho_0 q_0) [R_P] [R_Q]$$

where  $[R_P]$  and  $[R_Q]$  are defined in an analogous way.

By substituting  $P_{01}$  for  $\frac{p_1}{p_0}$  and  $Q_{01}$  for  $\frac{q_1}{q_0}$  in the formula for  $R_p$ , we have

$$\begin{aligned} [R_p] &= \left( \frac{\sum P_{01} p_0 Q_{01} q_0}{\sum p_0 q_0} \right)^{\frac{\sum (P_{01} p_0 Q_{01} q_0 - p_0 q_0) \frac{\log P_{01}}{\log P_{01} Q_{01}}}{\sum (P_{01} p_0 Q_{01} q_0 - p_0 q_0)}} \\ &= \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{\log P_{01}}{\log \frac{\sum p_1 q_1}{\sum p_0 q_0}}} \\ &= \left( e^{\log \frac{\sum p_1 q_1}{\sum p_0 q_0}} \right)^{\frac{\log P_{01}}{\log \frac{\sum p_1 q_1}{\sum p_0 q_0}}} \\ &= e^{\log P_{01}} \\ &= P_{01} \end{aligned}$$

Similarly  $[R_q] = Q_{01}$ .

As, in accordance with the conception  $[R_p] = R_p$  and  $[R_q] = R_q$ , we have  $P_{01} = R_p$  and  $Q_{01} = R_q$ ,

hence

$$P_{01} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{\sum (p_1 q_1 - p_0 q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}{\sum (p_1 q_1 - p_0 q_0)}}$$

and

$$Q_{01} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{\sum (p_1 q_1 - p_0 q_0) \frac{\log \frac{q_1}{q_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}{\sum (p_1 q_1 - p_0 q_0)}}$$

2.—If we now take the second measure of the change in the aggregate value, that is,  $\sum p_1 q_1 - \sum p_0 q_0$ , which we may speak of as the increase in aggregate value, we notice first that the increase due to the change of  $p_0' q_0'$  to  $p_1' q_1'$  is clearly  $p_1' q_1' - p_0' q_0'$ . Hence if we can transform  $p_1' q_1' - p_0' q_0'$  into the sum of two

values, of which one,  $V_{\frac{p_1'}{p_0}}$ , represents the increase of  $p_0'q_0'$  due to multiplying  $p_0'q_0'$  by  $\frac{p_1'}{p_0}$ , (simultaneously with  $\frac{q_1'}{q_0}$ ) and the other,  $V_{\frac{q_1'}{q_0}}$ , represents the increase of  $p_0'q_0'$  due to multiplying  $p_0'q_0'$  by  $\frac{q_1'}{q_0}$ , (simultaneously with  $\frac{p_1'}{p_0}$ ), then  $\Sigma V_{\frac{p_1'}{p_0}}$  will represent the increase in  $\Sigma p_0'q_0'$  due to the changes in price and  $\Sigma V_{\frac{q_1'}{q_0}}$  will represent the increase in  $\Sigma p_0'q_0'$  due to the changes in quantity.

Hence  $V_P$ , as we defined it on page 6, is equal to  $\Sigma V_{\frac{p_1'}{p_0}}$  and  $V_Q$  is equal to  $\Sigma V_{\frac{q_1'}{q_0}}$ .

We have

$$p_1'q_1' - p_0'q_0' = V_{\frac{p_1'}{p_0}} + V_{\frac{q_1'}{q_0}}$$

or

$$p_1'q_1' = p_0'q_0' + V_{\frac{p_1'}{p_0}} + V_{\frac{q_1'}{q_0}}$$

This last equation is of the form

$$a_1b_1c_1 \dots = a_0b_0c_0 \dots + R_{\frac{a_1}{a_0}} + R_{\frac{b_1}{b_0}} + R_{\frac{c_1}{c_0}} + \dots$$

in which case we have found that

$$R_{\frac{a_1}{a_0}} = (a_1b_1c_1 \dots - a_0b_0c_0 \dots) \frac{\log \frac{a_1}{a_0}}{\log \frac{a_1b_1c_1 \dots}{a_0b_0c_0 \dots}}$$

Hence  $V_{\frac{p_1'}{p_0}} = (p_1'q_1' - p_0'q_0') \frac{\log \frac{p_1'}{p_0}}{\log \frac{p_1'q_1'}{p_0'q_0'}}$

and  $V_{\frac{q_1'}{q_0}} = (p_1'q_1' - p_0'q_0') \frac{\log \frac{q_1'}{q_0}}{\log \frac{p_1'q_1'}{p_0'q_0'}}$

We have therefore

$$V_p = \Sigma V_{\frac{p_1}{p_0}} = \Sigma (p_1 q_1 - p_0 q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}}$$

$$\text{and } V_q = \Sigma V_{\frac{q_1}{q_0}} = \Sigma (p_1 q_1 - p_0 q_0) \frac{\log \frac{q_1}{q_0}}{\log \frac{p_1 q_1}{p_0 q_0}}$$

Turning to the hypothetical case we may say :

$$\text{Let } P_{01} p_0' Q_{01} q_0' = p_0' q_0' + V_{\frac{P_{01} p_0'}{p_0'}} + V_{\frac{Q_{01} q_0'}{q_0'}}$$

where  $V_{\frac{P_{01} p_0'}{p_0'}}$  and  $V_{\frac{Q_{01} q_0'}{q_0'}}$  are defined in an analogous way.

Then  $[V_p]$ , as we defined it on page 6, is equal to  $\Sigma V_{\frac{P_{01} p_0'}{p_0'}}$  and  $[V_q]$  is equal to  $\Sigma V_{\frac{Q_{01} q_0'}{q_0'}}$ .

Substituting  $P_{01}$  for  $\frac{p_1'}{p_0'}$  and  $Q_{01}$  for  $\frac{q_1'}{q_0'}$  in the formulæ for  $V_{\frac{p_1'}{p_0'}}$  and  $V_{\frac{q_1'}{q_0'}}$  we have

$$V_{\frac{P_{01} p_0'}{p_0'}} = (P_{01} p_0' Q_{01} q_0' - p_0' q_0') \frac{\log P_{01}}{\log P_{01} Q_{01}}$$

$$V_{\frac{Q_{01} q_0'}{q_0'}} = (P_{01} p_0' Q_{01} q_0' - p_0' q_0') \frac{\log Q_{01}}{\log P_{01} Q_{01}}$$

These give

$$\begin{aligned} [V_p] &= \Sigma V_{\frac{P_{01} p_0'}{p_0'}} = \Sigma (P_{01} p_0' Q_{01} q_0' - p_0' q_0') \frac{\log P_{01}}{\log P_{01} Q_{01}} \\ &= \Sigma (p_1 q_1 - p_0 q_0) \frac{\log P_{01}}{\log \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}} \end{aligned}$$

and

$$[V_e] = \Sigma V_{P_{01}q_0} = \Sigma (p_1q_1 - p_0q_0) \frac{\log Q_{01}}{\log \frac{\Sigma p_1q_1}{\Sigma p_0q_0}}$$

But, in accordance with the conception,  $[V_p] = V_p$  and  $[V_e] = V_e$ .

Hence

$$\Sigma (p_1q_1 - p_0q_0) \frac{\log P_{01}}{\log \frac{\Sigma p_1q_1}{\Sigma p_0q_0}} = \Sigma (p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}$$

$$\log P_{01} = \left( \log \frac{\Sigma p_1q_1}{\Sigma p_0q_0} \right) \frac{\Sigma (p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma (p_1q_1 - p_0q_0)}$$

which gives

$$P_{01} = \left( \frac{\Sigma p_1q_1}{\Sigma p_0q_0} \right)^{\frac{\Sigma (p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma (p_1q_1 - p_0q_0) \frac{p_1}{p_0q_0}}}$$

Similarly

$$Q_{01} = \left( \frac{\Sigma p_1q_1}{\Sigma p_0q_0} \right)^{\frac{\Sigma (p_1q_1 - p_0q_0) \frac{\log \frac{q_1}{q_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma (p_1q_1 - p_0q_0) \frac{p_1}{p_0q_0}}}$$

These are the same formulæ that we obtained by taking  $\frac{\Sigma p_1q_1}{\Sigma p_0q_0}$  as the measure of the change in aggregate value.

3.—Still taking the change in aggregate value as expressed by  $\Sigma p_1q_1 - \Sigma p_0q_0$ , we can obtain the same result without introducing the hypothetical case.

$$\text{Let } \Sigma p_0 q_0 + \Sigma V_{p_0}^{p_1} + \Sigma V_{q_0}^{q_1} = (\Sigma p_0 q_0) R_p R_q$$

where  $R_p$  and  $R_q$  are such that multiplying  $\Sigma p_0 q_0$  by  $R_p$  (simultaneously with  $R_q$ ) contributes as much to the product  $(\Sigma p_0 q_0) R_p R_q$  (which is equal to  $\Sigma p_1 q_1$ ) as adding  $\Sigma V_{p_0}^{p_1}$  to  $\Sigma p_0 q_0$  contributes to the sum  $\Sigma p_0 q_0 + \Sigma V_{p_0}^{p_1} + \Sigma V_{q_0}^{q_1}$  (which is also equal to  $\Sigma p_1 q_1$ ), and similarly for  $R_q$ . Then it is evident that  $R_p = P_{01}$  and  $R_q = Q_{01}$ , since  $\Sigma V_{p_0}^{p_1}$  represents the change (measured as an increase) in the aggregate value due to the changes in price and  $\Sigma V_{q_0}^{q_1}$  represents the change (measured as an increase) in the aggregate value due to the changes in quantity.

But the equation

$$\Sigma p_0 q_0 + \Sigma V_{p_0}^{p_1} + \Sigma V_{q_0}^{q_1} = (\Sigma p_0 q_0) R_p R_q$$

is analogous to the equation

$$A + a + b + c + \dots = A R_a R_b R_c \dots$$

In this case

$$R_a = \left( \frac{A + a + b + c + \dots}{A} \right)^{\frac{a}{a+b+c+\dots}}$$

Hence by analogy

$$\begin{aligned} R_p &= \left( \frac{\Sigma p_0 q_0 + \Sigma V_{p_0}^{p_1} + \Sigma V_{q_0}^{q_1}}{\Sigma p_0 q_0} \right)^{\frac{\Sigma V_{p_0}^{p_1}}{\Sigma V_{p_0}^{p_1} + \Sigma V_{q_0}^{q_1}}} \\ &= \left( \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \right)^{\frac{\Sigma (\rho_1 q_1 - p_0 q_1) \frac{\log \frac{p_1}{p_0}}{p_1}}{\Sigma (\rho_1 q_1 - p_0 q_1)}} \end{aligned}$$



$$\text{Hence } P_{01} = \frac{\left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right) \frac{\frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}{\frac{\sum (p_1 q_1 - p_0 q_0)}{\sum p_0 q_0}}}{\frac{\sum (p_1 q_1 - p_0 q_0)}{\sum p_0 q_0}}$$

$$\text{Similarly } Q_{01} = \frac{\left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right) \frac{\frac{\log \frac{q_1}{q_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}{\frac{\sum (p_1 q_1 - p_0 q_0)}{\sum p_0 q_0}}}{\frac{\sum (p_1 q_1 - p_0 q_0)}{\sum p_0 q_0}}$$

4.—We have already seen (page 7) that the third measure of the change in aggregate value,  $\frac{\sum p_1 q_1 - \sum p_0 q_0}{\sum p_0 q_0}$ , does not give us any further conditions for determining  $P_{01}$  and  $Q_{01}$ . But by three different methods we have arrived at the result that

$$P_{01} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right) \frac{\frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}{\frac{\sum (p_1 q_1 - p_0 q_0)}{\sum p_0 q_0}}$$

$$\text{and } Q_{01} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right) \frac{\frac{\log \frac{q_1}{q_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}{\frac{\sum (p_1 q_1 - p_0 q_0)}{\sum p_0 q_0}}$$

5.—In the case that  $\frac{p_1}{p_0} = \frac{1}{\frac{q_1}{q_0}}$ ,  $(p_1 q_1 - p_0 q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}}$  takes the form  $\frac{0}{0}$ , but it does not become indeterminate.

$$\text{We have } \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}} = (p_1 q_1 - p_0 q_0) \frac{\log \frac{p_1}{p_0}}{\log \left( 1 + \frac{p_1 q_1 - p_0 q_0}{p_0 q_0} \right)}$$

But when  $\frac{p_1q_1 - p_0q_0}{p_0q_0}$  is infinitesimal

$$\log_e \left( 1 + \frac{p_1q_1 - p_0q_0}{p_0q_0} \right) = \frac{p_1q_1 - p_0q_0}{p_0q_0}$$

Hence when  $\frac{p_1}{p_0} = \frac{q_1}{q_0}$

$$(p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}} = p_0q_0 \log_e \frac{p_1}{p_0}$$

It must be noted that in this case we must use logarithms to the base  $e$ .

Again if  $\Sigma p_1q_1 = \Sigma p_0q_0$ ,  $\log P_{01}$  takes the form  $\frac{0}{0}$ , but it is not indeterminate.

$$\log P_{01} = \frac{\left( \log \frac{\Sigma p_1q_1}{\Sigma p_0q_0} \right) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma(p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}}$$

$$= \left\{ \log \left( 1 + \frac{\Sigma(p_1q_1 - p_0q_0)}{\Sigma p_0q_0} \right) \right\} \frac{\log \frac{p_1}{p_0}}{\Sigma(p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}}$$

Hence when  $\frac{\Sigma(p_1q_1 - p_0q_0)}{\Sigma p_0q_0}$  is infinitesimal

$$\log_e P_{01} = \frac{\Sigma(p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma p_0q_0}$$

In the case in which, for all commodities,  $\frac{p_1}{p_0} = \frac{I}{Q_1}$   
 $Q_0$

$$\log_e P_{01} = \frac{\sum p_{0f_0} \log_e \frac{p_1}{p_0}}{\sum p_{0f_0}}$$

6.—We must emphasize that our formulæ have been arrived at on the basis of the conception that we have outlined at the beginning of these notes. If a different conception is adopted, the mathematical problem is different and the resulting formulæ will be different.

Assuming, however, that our conception is admitted, we may venture, in the light of the foregoing discussion of the problem, to give the following definition of the price index and the quantity index as here conceived:—

The price index is the change, *measured as a ratio*, in the aggregate value produced by the changes in price (when occurring simultaneously with the changes in quantity), and the quantity index is the change, *measured as a ratio*, in the aggregate value produced by the changes in quantity (when occurring simultaneously with the changes in price).

This definition is perfectly consistent with the conception put forward and avoids the notion of proportionality factors and the introduction of the hypothetical case, but it is hardly intelligible without the explanations and illustrations we have given.

## CHAPTER IV

### THE PRICE INDICES OF SEPARATE COMMODITIES IN THE GROUP.

1.—We have not only determined the formula for the general price index of a group of commodities, but incidentally we have determined the formula for the price index of each separate commodity in the group, when treated as part of the group and not considered independently. The factors  $R_{p_0}^{p_1'}$ ,  $R_{p_0}^{p_1''}$ ,  $R_{p_0}^{p_1'''}$ , etc., as defined on page 30, represent, in fact, the change, measured as a ratio, in the aggregate value produced by the changes in price of the separate commodities (when occurring simultaneously with the changes in price of the other commodities and the changes in quantity of all the commodities).

These factors,  $R_{p_0}^{p_1'}$ ,  $R_{p_0}^{p_1''}$ ,  $R_{p_0}^{p_1'''}$ , etc., are, then, the price indices of the separate commodities, when regarded as part of the group. They must be carefully distinguished from the price relatives,  $\frac{p_1'}{p_0'}$ ,  $\frac{p_1''}{p_0''}$ ,  $\frac{p_1'''}{p_0'''}$ , etc., of the commodities when considered separately and apart from the group.

But we have :

$$R_{p_0}^{p_1'} = \frac{\left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{\log \frac{p_1'}{p_0'}}{\log \frac{p_1' q_1}{p_0' q_0}}}{\sum (p_1 q_1 - p_0 q_0)}$$

and similarly for  $R_{p_0}^{p_1''}$ ,  $R_{p_0}^{p_1'''}$ , etc.

If, then, we call the price index of any commodity in the group  $P_{p_n}$  (instead of  $R_{p_n}$ ) we have

$$P_{p_n} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}$$

and  $P_{p_n} \cdot P_{p_n} \cdot P_{p_n} \dots = P_{01}$ .

The indices of separate commodities may be of considerable practical value as furnishing a measure of the change in aggregate value due to the change in price of each commodity of the group. Where, for example, there is a general rise in prices, it is of interest to inquire which commodities have contributed most to the rise. The price indices of the separate commodities will give precise indications of this, and will enable the commodities to be arranged in order according to the extent to which their rise in price has contributed to the increase in aggregate value.

2.—Price indices can also be calculated for sub-groups of commodities, treated, of course, as forming part of the whole group. These must not be confused with the price indices of the sub-groups when taken separately.

Let the prices of a sub-group of commodities be  $p_{m_0}'$ ,  $p_{m_0}''$ ,  $p_{m_0}'''$ , etc., in the base year and  $p_{m_1}'$ ,  $p_{m_1}''$ ,  $p_{m_1}'''$ , etc., in the given year, and the corresponding quantities be  $q_{m_0}'$ ,  $q_{m_0}''$ ,  $q_{m_0}'''$ , etc., in the base year and  $q_{m_1}'$ ,  $q_{m_1}''$ ,  $q_{m_1}'''$ , etc., in the given year. Then if  $P_{p_{m_n}}$  is the price index of any commodity in the sub-group and  $P_{P_{m_n}}$  is the price index of the sub-group treated as part of the whole group

$$P_{P_{m_1}} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{\frac{\log \frac{p_{m_1}}{p_{m_0}}}{\log \frac{p_{m_1} q_{m_1}}{p_{m_0} q_{m_0}}}}{\frac{\log \frac{p_{m_1} q_{m_1}}{p_{m_0} q_{m_0}}}{\log \frac{p_{m_1} q_{m_1}}{p_{m_0} q_{m_0}}}}}$$

$$P_{P_{M_{01}}} = P_{P_{m_1}} \cdot P_{P_{m_2}} \cdot P_{P_{m_3}} \dots$$

$$= \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{\frac{\log \frac{p_{m_1}}{p_{m_0}}}{\log \frac{p_{m_1} q_{m_1}}{p_{m_0} q_{m_0}}}}{\frac{\log \frac{p_{m_1} q_{m_1}}{p_{m_0} q_{m_0}}}{\log \frac{p_{m_1} q_{m_1}}{p_{m_0} q_{m_0}}}}}$$

On the other hand, if  $PM_{01}$  is the price index of the sub-group independently of the rest of the whole group

$$PM_{01} = \left( \frac{\sum p_{m_1} q_{m_1}}{\sum p_{m_0} q_{m_0}} \right)^{\frac{\frac{\log \frac{p_{m_1}}{p_{m_0}}}{\log \frac{p_{m_1} q_{m_1}}{p_{m_0} q_{m_0}}}}{\frac{\log \frac{p_{m_1} q_{m_1}}{p_{m_0} q_{m_0}}}{\log \frac{p_{m_1} q_{m_1}}{p_{m_0} q_{m_0}}}}}$$

It may also at times be useful to calculate  $PM_{01}$ , but care must be taken to distinguish it from  $P_{P_{M_{01}}}$ .

3.—In the same way as we have determined a price index for each commodity separately, we can also determine a quantity index. This we may call  $Q_{q_m}$ , and we have

$$Q_{q_m} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{\frac{\log \frac{q_1}{q_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}{\frac{\log \frac{p_1 q_1}{p_0 q_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}}$$

and  $Q_{q_m} \cdot Q_{q_n} \cdot Q_{q_m} \dots = Q_{01}$ .

But there is also a value index for each commodity, that is, the increase, measured as a ratio, in the aggre-

gate value due to the change in value (or to the change in price and to the change in quantity jointly) of each commodity. If we call this  $V_{v_{01}}$  we have

$$V_{v_{01}} = P_{p_{01}} Q_{q_{01}} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{p_1 q_1 - p_0 q_0}{\sum (p_1 q_1 - p_0 q_0)}}$$

and 
$$V_{v_{01}} V_{v_{02}} V_{v_{03}} \dots = \frac{\sum p_1 q_1}{\sum p_0 q_0}$$

We need not speak of a general value index, as this would be merely  $\frac{\sum p_1 q_1}{\sum p_0 q_0}$ .

We can thus present a complete picture of the changes in aggregate value produced by the changes in price and the changes in quantity by giving for each commodity the following indices :

$$P_{p_{01}} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{(\log \frac{p_1}{p_0}) \frac{p_1 q_1}{p_0 q_0}}{\sum (\log \frac{p_1}{p_0}) \frac{p_1 q_1}{p_0 q_0}}}$$

$$Q_{q_{01}} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{(\log \frac{q_1}{q_0}) \frac{p_1 q_1}{p_0 q_0}}{\sum (\log \frac{q_1}{q_0}) \frac{p_1 q_1}{p_0 q_0}}}$$

$$V_{v_{01}} = \left( \frac{\sum p_1 q_1}{\sum p_0 q_0} \right)^{\frac{p_1 q_1 - p_0 q_0}{\sum (p_1 q_1 - p_0 q_0)}}$$

as well as the general price index,  $P_{01}$ , and the general quantity index,  $Q_{01}$ .

4.—If we wish to express the increase in aggregate value due to the change in the price of a single commodity as a percentage of the increase in aggregate

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value due to all the changes in price, we can convert the product  $P_{p_a'} P_{p_a''} P_{p_a'''} \dots$  (which is equal to  $P_{01}$ ) into a sum of terms.

$$\text{Let } P_{p_a'} P_{p_a''} P_{p_a'''} \dots = 1 + R_{r p_a'} + R_{r p_a''} + R_{r p_a'''} + \dots$$

Then  $R_{r p_a'}$

$$= (P_{p_a'} P_{p_a''} P_{p_a'''} \dots - 1) \frac{\log P_{p_a'}}{\log P_{p_a'} P_{p_a''} P_{p_a'''} \dots}$$

$$= (P_{01} - 1) \frac{\log P_{p_a'}}{\log P_{01}}$$

$$\frac{R_{r p_a'}}{P_{01} - 1} = \frac{\log P_{p_a'}}{\log P_{01}}$$

Hence the increase in the aggregate value due to the increase in the price of a single commodity, expressed as a percentage of the increase in aggregate value due to all the changes in price, is  $\frac{100 \log P_{p_a'}}{\log P_{01}}$ .

Again, if we wish to express the increase in the aggregate value due to the change in the price of a single commodity as a percentage of the total increase in the aggregate value we can convert  $P_{p_a'} P_{p_a''} P_{p_a'''} \dots Q_{q_a'} Q_{q_a''} Q_{q_a'''} \dots$  (which is equal to  $P_{01} Q_{01}$ ) into a sum of terms.

$$\begin{aligned} \text{Let } & P_{p_a'} P_{p_a''} P_{p_a'''} \dots Q_{q_a'} Q_{q_a''} Q_{q_a'''} \dots \\ = & 1 + \dot{R}_{r p_a'} + \dot{R}_{r p_a''} + \dot{R}_{r p_a'''} + \dots + \\ & \dot{R}_{q_a'} + \dot{R}_{q_a''} + \dot{R}_{q_a'''} + \dots \end{aligned}$$

This gives

$$\frac{\dot{R}_{r p_a'}}{P_{01} Q_{01} - 1} = \frac{\log P_{p_a'}}{\log P_{01} Q_{01}}$$



Hence the increase in the aggregate value due to the increase in the price of a single commodity expressed as a percentage of the total increase in aggregate value is  $\frac{100 \log P_{p_{11}}}{\log P_{01} Q_{01}}$ .

$$\begin{aligned} \text{But } \frac{\log P_{p_{11}}}{\log P_{01}} &= \frac{(p_1' q_1' - p_0' q_0') \frac{\log \frac{p_1'}{p_0'}}{\log \frac{p_1' q_1'}{p_0' q_0'}}}{\Sigma(p_1 q_1 - p_0 q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}}} \\ &= \frac{V_{\frac{p_1'}{p_0}}}{\Sigma V_{\frac{p_1}{p_0}}} \end{aligned}$$

$$\begin{aligned} \text{and } \frac{\log P_{p_{11}}}{\log P_{01} Q_{01}} &= \frac{\log P_{p_{11}}}{\log \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}} \\ &= \frac{(p_1' q_1' - p_0' q_0') \frac{\log \frac{p_1'}{p_0'}}{\log \frac{p_1' q_1'}{p_0' q_0'}}}{\Sigma(p_1 q_1 - p_0 q_0)} \\ &= \frac{V_{\frac{p_1'}{p_0}}}{\Sigma(p_1 q_1 - p_0 q_0)} \end{aligned}$$

Hence the increase in aggregate value due to the change in the price of a single commodity expressed as a percentage of the increase in aggregate value due

to all the changes in price is  $\frac{100V_{\rho_1}}{\sum V_{\rho_0}}$  and expressed as a percentage of the total increase in aggregate value is

$$\frac{100V_{\rho_1}}{\sum(\rho_1q_1 - \rho_0q_0)}$$

But these results could have been obtained directly. We have determined them indirectly in order to give a further demonstration of the soundness and flexibility of our method of transforming measures of change.

5.—In like manner we can express the increase in the aggregate value due to the change in quantity of any one commodity as a percentage either of the increase in aggregate value due to all the changes in quantity, or as a percentage of the total increase in aggregate value.

We can further express the change in aggregate value due to the change in the value of any one commodity (that is, to the change in price and in quantity jointly) as a percentage of the total change in aggregate value. This is obviously  $\frac{100(\rho_1q_1 - \rho_0q_0)}{\sum(\rho_1q_1 - \rho_0q_0)}$ .

We can present another complete picture of the changes in the aggregate value due to the changes in price, in quantity and in value, in the form of percentages of the total increase in the aggregate value. To do so we must give for each commodity :

The increase in the aggregate value due to the increase in price of the commodity expressed as a percentage of the total increase in the aggregate value. The formula for this is

$$\frac{100(p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma(p_1q_1 - p_0q_0)}$$

The increase in the aggregate value due to the increase in quantity of the commodity expressed as a percentage of the total increase in aggregate value :

$$\frac{100(p_1q_1 - p_0q_0) \frac{\log \frac{q_1}{q_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma(p_1q_1 - p_0q_0)}$$

The increase in the value of the commodity expressed as a percentage of the total increase in aggregate value :

$$\frac{100(p_1q_1 - p_0q_0)}{\Sigma(p_1q_1 - p_0q_0)}$$

To these must be added :

The increase in aggregate value due to the changes in price in all the commodities, expressed as a percentage of the total increase in aggregate value :

$$\frac{100 \Sigma(p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma(p_1q_1 - p_0q_0)}$$

The increase in aggregate value due to the changes in quantity in all the commodities, expressed as a percentage of the total increase in aggregate value :

$$\frac{100 \Sigma(p_1 q_1 - p_0 q_0) \frac{\log \frac{q_1}{q_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}{\Sigma(p_1 q_1 - p_0 q_0)}$$

By substituting  $\Sigma p_0 q_0$  for  $\Sigma(p_1 q_1 - p_0 q_0)$  in the denominator of the foregoing formulæ we can express the increases in aggregate value due to the changes in price, quantity and value of the separate commodities as percentages of the aggregate value in the base year, and these are possibly even more useful figures.

## CHAPTER V

### THE DIFFERENTIAL METHOD OF DETERMINING THE PRICE INDEX

THE writer first obtained his formulæ for  $P_{01}$  and  $Q_{01}$  by applying differential methods,<sup>1</sup> and though the reasoning is not entirely convincing in itself, the fact that it results in precisely the same formulæ as the methods set out in the preceding chapters may serve as a confirmation of the soundness of those methods.

In applying differential methods we suppose that the changes in price from  $p_0'$  to  $p_0''$ , from  $p_0''$  to  $p_1''$ , from  $p_0'''$  to  $p_1'''$ , and so on, and the changes in quantity from  $q_0'$  to  $q_1'$ , from  $q_0''$  to  $q_1''$ , from  $q_0'''$  to  $q_1'''$ , and so on, take place by infinitesimal increments. This is the weak point in the method, as it obviously does not correspond to reality. However, there is nothing to prevent us from making the assumption and analysing the changes on the basis of that supposition.

Taking the equation

$$p_1 q_1 = p_0 q_0 + V_{p_0} + V_{q_0}$$

where  $V_{p_0}$  and  $V_{q_0}$  are defined as on page 33, we may write it in the form

<sup>1</sup> The first to apply differential methods to the price index was Professor F. DIVISIA. See: "Economie rationnelle," Paris, 1928, Chapter XIV.

$$V_{\frac{p_1}{p_0}} + V_{\frac{q_1}{q_0}} = p_1 q_1 - p_0 q_0$$

Treating  $p_0$  and  $q_0$  as constant, and making the equation general for any value of  $p_1$  and  $q_1$ , we have

$$V_{\frac{p}{p_0}} + V_{\frac{q}{q_0}} = pq - p_0 q_0$$

Differentiating, we have

$$\delta V_{\frac{p}{p_0}} + \delta V_{\frac{q}{q_0}} = q \delta p + p \delta q$$

But  $q \delta p$  is clearly the infinitesimal increment of value due to the increment of price  $\delta p$ , and  $p \delta q$  is the increment of value due to the increment of quantity  $\delta q$ .

Hence 
$$\delta V_{\frac{p}{p_0}} = q \delta p$$

and 
$$\delta V_{\frac{q}{q_0}} = p \delta q$$

These equations cannot be integrated to determine  $V_{\frac{p}{p_0}}$  and  $V_{\frac{q}{q_0}}$  unless we assume a relation between  $p$  and  $q$ , because the differential forms are such that the integral depends on the path. What assumption are we justified in making? Is there any assumption that we are compelled by the conditions of the problem to make?

Here we fall back on the principle of symmetry;  $\Sigma p_1 q_1$  and  $\Sigma p_0 q_0$  are symmetrical in relation to  $p$  and  $q$ . At every stage in the determination of  $P_{01}$  and  $Q_{01}$  there must be symmetry as between  $p$  and  $q$ ; they must, in fact, be interchangeable without altering the equations obtained at any point. Hence the assumption we make must be symmetrical. If we assume that  $\frac{q}{q_0} = f\left(\frac{p}{p_0}, \frac{p_1}{p_0}, \frac{q_1}{q_0}\right)$ , then the function must be such

that the same equation can be written  $\frac{p}{p_0} = f\left(\frac{q}{q_0}, \frac{q_1}{q_0}, \frac{p_1}{p_0}\right)$ .

Moreover, the symmetry must be continued along the path in either direction beyond the tract that lies between the points  $(p_0, q_0)$  and  $(p_1, q_1)$ . We must imagine  $p$  and  $q$  increasing from 0 to  $\infty$  or decreasing from  $\infty$  to 0 with symmetry at every point. Hence the path must pass through the points  $(0, 0)$  and  $(\infty, \infty)$  or through the points  $(\infty, 0)$  and  $(0, \infty)$ .

An assumption that fulfils these conditions is

$$\frac{q}{q_0} = \left(\frac{p}{p_0}\right)^a$$

$a$  being determined by the fact that the path passes through  $(p_0, q_0)$  and  $(p_1, q_1)$ , and it does not appear that any other assumption fulfils them.

This assumption gives the relation

$$\frac{q}{q_0} = \left(\frac{p}{p_0}\right)^{\frac{\log \frac{q_1}{q_0}}{\log \frac{p_1}{p_0}}}$$

which may also be written

$$\frac{p}{p_0} = \left(\frac{q}{q_0}\right)^{\frac{\log \frac{p_1}{p_0}}{\log \frac{q_1}{q_0}}}$$

Continuing, for the sake of simplicity, to write  $a$

for  $\frac{\log \frac{q_1}{q_0}}{\log \frac{p_1}{p_0}}$ , we have

$$\begin{aligned} \delta V_{\frac{p}{p_0}} &= q \delta p \\ &= \frac{q_0}{p_0^a} p^a \delta p \end{aligned}$$

$$\int_0^{V_{\frac{p_1}{p_0}}} dV_{\frac{p_1}{p_0}} = \frac{q_0}{p_0^a} \int_{p_0}^{p_1} p^a dp$$

$$V_{\frac{p_1}{p_0}} = \frac{q_0}{p_0^a} \cdot \frac{1}{a+1} (p_1^{a+1} - p_0^{a+1})$$

$$= \frac{1}{a+1} \left( p_1 \frac{p_1^a}{p_0^a} q_0 - p_0 q_0 \right)$$

$$= \frac{1}{a+1} (p_1 q_1 - p_0 q_0)$$

But  $\frac{1}{a+1} = \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}}$

Hence  $V_{\frac{p_1}{p_0}} = (p_1 q_1 - p_0 q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}}$

Similarly,  $V_{\frac{q_1}{q_0}} = (p_1 q_1 - p_0 q_0) \frac{\log \frac{q_1}{q_0}}{\log \frac{p_1 q_1}{p_0 q_0}}$

In the hypothetical case :

$$V_{\frac{P_{01} p_0}{p_0}} = (P_{01} p_0 Q_{01} q_0 - p_0 q_0) \frac{\log P_{01}}{\log P_{01} Q_{01}}$$

and  $V_{\frac{Q_{01} q_0}{q_0}} = (P_{01} p_0 Q_{01} q_0 - p_0 q_0) \frac{\log Q_{01}}{\log P_{01} Q_{01}}$

Equating  $\Sigma V_{\frac{p_1}{p_0}}$  with  $\Sigma V_{\frac{P_{01} p_0}{p_0}}$  and  $\Sigma V_{\frac{q_1}{q_0}}$  with  $\Sigma V_{\frac{Q_{01} q_0}{q_0}}$  we obtain the same formulæ for  $P_{01}$  and  $Q_{01}$  as we obtained in Chapter III.

The formulæ for  $P_{01}$  and  $Q_{01}$  have been arrived at by differential methods, but without directly obtaining



differential equations for  $P$  and  $Q$ . These may, however, be obtained.

By the conception of the price index, the increase in aggregate value due to the increases in price when the prices are increased from  $p_0$  to  $p$  and the quantities from  $q_0$  to  $q$  is equal to the increase in aggregate value due to the increases in price when the prices are increased from  $p_0$  to  $Pp_0$  and the quantities from  $Q_0$  to  $Qq_0$ .

Similarly, the increase in the aggregate value due to the increases in price when the prices increase from  $p_0$  to  $p + \delta p$  and the quantities increase from  $q_0$  to  $q + \delta q$  is equal to the increase in the aggregate value when the prices increase from  $p_0$  to  $(P + \delta P)p_0$  and the quantities from  $q_0$  to  $(Q + \delta Q)q_0$ .

By subtraction, we get that the increase in aggregate value due to the increases in price when the prices increase from  $p$  to  $p + \delta p$  and the quantities from  $q$  to  $q + \delta q$  is equal to the increase in aggregate value when the prices increase from  $Pp_0$  to  $(P + \delta P)p_0$  and the quantities from  $Qq_0$  to  $(Q + \delta Q)q_0$ .

$$\begin{aligned} \text{Hence} \quad \Sigma Qq_0 \delta P p_0 &= \Sigma q \delta p \\ Q \delta P \cdot \Sigma p_0 q_0 &= \Sigma q \delta p \\ \frac{\delta P}{P} &= \frac{\Sigma q \delta p}{P Q \Sigma p_0 q_0} \\ &= \frac{\Sigma q \delta p}{\Sigma p q} \end{aligned}$$

$$\text{Similarly,} \quad \frac{\delta Q}{Q} = \frac{\Sigma p \delta q}{\Sigma p q}$$

The differential equation

$$\frac{\delta P}{P} = \frac{\Sigma q \delta p}{\Sigma p q}$$

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 is analogous to the differential equation for the monet-  
 ary index worked out by Professor Divisia<sup>1</sup>:

$$\frac{\delta I}{I} = \frac{\Sigma q \delta p}{\Sigma p q}$$

and could have been arrived at by precisely the same  
 reasoning as that used by Professor Divisia.

In the form

$$\frac{\delta P}{P} = \frac{\Sigma q \delta p}{\Sigma p q}$$

the differential equation for  $P$  cannot be integrated  
 even by making the assumption

$$\frac{q}{q_0} = \left(\frac{p}{p_0}\right)^a$$

But in the form

$$Q \delta P \Sigma p_0 q_0 = \Sigma q \delta p$$

it can be integrated by making this assumption and  
 the further assumption that, by analogy,  $Q = P^A$ .  
 This assumption is justified, as we must assume that  
 whatever path is followed by the point  $(p, q)$ , an  
 analogous path must, in the hypothetical case, be  
 followed by the point  $(Pp_0, Qq_0)$ .

In fact, it will be observed that, in the calculations  
 by which we arrived at the formulæ for  $P_{01}$  and  $Q_{01}$ , if,  
 instead of deducing  $V_{\frac{P_0 p_0}{P_1 p_1}}$  from  $V_{\frac{P_1}{P_0}}$ , we had calculated  
 it by integration, we should have had to make the  
 assumption

$$Q = P^{\frac{\log Q_{01}}{\log P_{01}}}$$

<sup>1</sup> F. DIVISIA, *op. cit.*, page 269.

analogous to the assumption

$$\frac{q}{q_0} = \left(\frac{p}{p_0}\right)^{\frac{\log \frac{q_1}{q_0}}{\log \frac{p_1}{p_0}}}$$

The result of integrating the differential equations with the aid of the two assumptions is to give precisely the same formulæ as we have obtained by other methods.

## CHAPTER VI

### SOME PROPERTIES OF THE PRICE INDEX

WITHOUT actually determining the formulæ for  $P_{01}$  and  $Q_{01}$ , some properties of the price index and of the quantity index may be deduced from the conception here adopted.

1.—It is possible, for example, to determine the limits between which  $P_{01}$  and  $Q_{01}$  must lie.

We have

$$\begin{aligned} V_{\frac{p_1}{p_0}} + V_{\frac{q_1}{q_0}} &= p_1 q_1 - p_0 q_0 \\ &= (p_1 - p_0) q_0 + (q_1 - q_0) p_0 + (p_1 - p_0)(q_1 - q_0) \end{aligned}$$

Now  $(p_1 - p_0)q_0$  is clearly part of the increase in value due to the increase of price, while  $(q_1 - q_0)p_0$  is part of the increase in value due to the increase of quantity.

Hence  $V_{\frac{p_1}{p_0}}$  lies between  $(p_1 - p_0)q_0$  and  $(p_1 - p_0)q_0 + (p_1 - p_0)(q_1 - q_0)$ , that is, between  $(p_1 - p_0)q_0$  and  $(p_1 - p_0)q_1$ .

Similarly  $V_{\frac{q_1}{q_0}}$  lies between  $(q_1 - q_0)p_0$  and  $(q_1 - q_0)p_1$ .

Note that, since

$$V_{\frac{p_1}{p_0}} + V_{\frac{q_1}{q_0}} = p_1 q_1 - p_0 q_0$$

if  $V_{\frac{p_1}{p_0}}$  were at the limit  $(p_1 - p_0)q_0$ , then  $V_{\frac{q_1}{q_0}}$  would be

$(q_1 - q_0)p_1$ , and if  $V_{p_0}$  were at the limit  $(p_1 - p_0)q_1$ , then  $V_{q_0}$  would be  $(q_1 - q_0)p_0$ .

Hence it is more correct to say :

$V_{p_0}$  lies between  $(p_1 - p_0)q_0$  and  $(p_1 - p_0)q_1$ , and  $V_{q_0}$  lies between  $(q_1 - q_0)p_1$  and  $(q_1 - q_0)p_0$ .

Taking the whole group of commodities,  $\Sigma V_{p_0}$  lies between  $\Sigma(p_1 - p_0)q_0$  and  $\Sigma(p_1 - p_0)q_1$ , and  $\Sigma V_{q_0}$  lies between  $\Sigma(q_1 - q_0)p_1$  and  $\Sigma(q_1 - q_0)p_0$ .

Substituting  $P_{01}p_0$  for  $p_1$  and  $Q_{01}q_0$  for  $q_1$  in these statements, we have that, in the hypothetical case,  $\Sigma V_{P_{01}p_0}$  lies between  $\Sigma(P_{01} - 1)p_0q_0$  and  $\Sigma(P_{01} - 1)p_0Q_{01}q_0$  and  $\Sigma V_{Q_{01}q_0}$  lies between  $\Sigma(Q_{01} - 1)q_0P_{01}p_0$  and  $\Sigma(Q_{01} - 1)q_0q_0$ .

Equating the limits of  $\Sigma V_{P_{01}p_0}$  and  $\Sigma V_{Q_{01}q_0}$  and those of  $\Sigma V_{q_0}$  and  $\Sigma V_{P_{01}p_0}$ , we have at the one limit

$$(P_{01} - 1)\Sigma p_0q_0 = \Sigma(p_1 - p_0)q_0$$

and  $P_{01}(Q_{01} - 1)\Sigma p_0q_0 = \Sigma(q_1 - q_0)p_1$

and at the other limit

$$Q_{01}(P_{01} - 1)\Sigma p_0q_0 = \Sigma(p_1 - p_0)q_1$$

and  $(Q_{01} - 1)\Sigma p_0q_0 = \Sigma(q_1 - q_0)p_0$

These equations give at the one limit

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \quad \text{and} \quad Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$$

and at the other limit

$$P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \quad \text{and} \quad Q_{01} = \frac{\Sigma p_0 q_1}{\Sigma p_0 q_0}$$

Hence  $P_{01}$  lies between  $\frac{\sum p_1 q_0}{\sum p_0 q_0}$  and  $\frac{\sum p_1 q_1}{\sum p_0 q_1}$  and  $Q_{01}$  lies between  $\frac{\sum p_1 q_1}{\sum p_1 q_0}$  and  $\frac{\sum p_0 q_1}{\sum p_0 q_0}$ .

This explains why Professor Irving Fisher's "ideal" formulæ<sup>1</sup>

$$P_{01} = \sqrt{\frac{\sum p_1 q_0 \cdot \sum p_1 q_1}{\sum p_0 q_0 \cdot \sum p_0 q_1}}$$

and

$$Q_{01} = \sqrt{\frac{\sum p_0 q_1 \cdot \sum p_1 q_1}{\sum p_0 q_0 \cdot \sum p_1 q_0}}$$

give a very good approximation for the price index and the quantity index, since they are the geometrical means of the limits between which  $P_{01}$  and  $Q_{01}$  must lie.

2.—In the special case in which  $q_1 = q_0$ , that is, in which the quantities remain unchanged, the limits coincide, so that  $P_{01} = \frac{\sum p_1 q_0}{\sum p_0 q_0}$  and  $Q_{01} = 1$ . Similarly, if the prices remain unchanged,  $P_{01} = 1$  and  $Q_{01} = \frac{\sum p_0 q_1}{\sum p_0 q_0}$ .

3.—We are now in a position to prove that, if the conception of the price index here put forward is adopted, the price index *cannot* have the property

$$P_{02} = P_{01} P_{12}$$

one case of which would be that, if the base were changed, the price index in the given year relative to the new base would be equal to the price index in

<sup>1</sup> Irving FISHER, "The Making of Index Numbers," Boston and New York, 1922.

the given year relative to the old base, multiplied by the price index in the old base year relative to the new base.

If the equation  $P_{02} = P_{01}P_{12}$  were *generally* true, it would be true whatever were the prices and quantities in the year indicated by the figure 1. Let us suppose that the prices in that year were the same as in the year 0, but that the quantities were the same as in the year 2. In other words, for each commodity, let  $p_1 = p_0$  and  $q_1 = q_2$ . In this case  $P_{01} = 1$  and  $P_{12} = \frac{\sum p_2 q_2}{\sum p_0 q_2}$ , so that

$$P_{01}P_{12} = \frac{\sum p_2 q_2}{\sum p_0 q_2}$$

Now let us make the supposition that in the year 1 the prices were the same as in the year 2, but the quantities were the same as in the year 0. In other words, for each commodity, let  $p_1 = p_2$  and  $q_1 = q_0$ .

In this case  $P_{01} = \frac{\sum p_2 q_0}{\sum p_0 q_0}$  and  $P_{12} = 1$ , so that

$$P_{01}P_{12} = \frac{\sum p_2 q_0}{\sum p_0 q_0}$$

But  $P_{02}$  cannot be equal both to  $\frac{\sum p_2 q_2}{\sum p_0 q_2}$  and to  $\frac{\sum p_2 q_0}{\sum p_0 q_0}$ , so that it cannot be *generally* true that  $P_{02} = P_{01}P_{12}$ .

Professor Fisher recognised that the formula for the price index ought not to have this property, and that his so-called "circular test" is not strictly applicable.<sup>1</sup> The foregoing is a formal proof that, if the price index be as we here conceive it,  $P_{02}$  cannot generally be equal to  $P_{01}P_{12}$ .

<sup>1</sup> Irving FISHER, *op. cit.*, page 270.

4.—Another special case in which the price index can be calculated without determining the general formulæ for  $P_{01}$  and  $Q_{01}$  is the case in which the increases in price and quantity are all infinitesimal.

In the formulæ representing the limits between which  $P_{01}$  must lie, substitute  $p_i$  for  $p_0$ ,  $p_i + \delta p_i$  for  $p_1$ ,  $q_i$  for  $q_0$ ,  $q_i + \delta q_i$  for  $q_1$ .

Then  $P_{i..i+\delta}$  lies between  $\frac{\Sigma(p_i + \delta p_i)q_i}{\Sigma p_i q_i}$  and  $\frac{\Sigma(p_i + \delta p_i)(q_i + \delta q_i)}{\Sigma p_i(q_i + \delta q_i)}$ .

$$\begin{aligned} \text{But } \frac{\Sigma(p_i + \delta p_i)q_i}{\Sigma p_i q_i} &= 1 + \frac{\Sigma q_i \delta p_i}{\Sigma p_i q_i} \\ \text{and } \frac{\Sigma(p_i + \delta p_i)(q_i + \delta q_i)}{\Sigma p_i(q_i + \delta q_i)} &= 1 + \frac{\Sigma q_i \delta p_i}{\Sigma p_i q_i \left(1 + \frac{\Sigma p_i \delta q_i}{\Sigma p_i q_i}\right)} \\ &= 1 + \frac{\Sigma q_i \delta p_i}{\Sigma p_i q_i} \left(1 - \frac{\Sigma p_i \delta q_i}{\Sigma p_i q_i}\right) \\ &= 1 + \frac{\Sigma q_i \delta p_i}{\Sigma p_i q_i} \end{aligned}$$

Hence the limits coincide and

$$P_{i..i+\delta} = 1 + \frac{\Sigma q_i \delta p_i}{\Sigma p_i q_i}$$

$$\text{Similarly } Q_{i..i+\delta} = 1 + \frac{\Sigma p_i \delta q_i}{\Sigma p_i q_i}$$

These equations also give

$$\log P_{i..i+\delta} = \frac{\Sigma q_i \delta p_i}{\Sigma p_i q_i}$$

$$\text{and } \log Q_{i..i+\delta} = \frac{\Sigma p_i \delta q_i}{\Sigma p_i q_i}$$

This property can be used to supply an additional test of the correctness of any formulæ that may be



proposed for the price index and the quantity index.

5.—From the differential equation

$$\delta \log P = \frac{\Sigma q \delta p}{\Sigma p q}$$

and the above formula for  $\log P_{t..t+\delta}$  another property of the price index can be deduced.

$$\begin{aligned} \text{Since } \log P_{t..t+\delta} &= \frac{\Sigma q_i \delta p_i}{\Sigma p_i q_i} \\ \log P_{t..t+\delta} &= \delta \log P_{0t} \\ &= \log P_{0..t+\delta} - \log P_{0t}. \end{aligned}$$

$$\text{Therefore } P_{0..t+\delta} = P_{0t} \cdot P_{t..t+\delta}$$

Hence if the differences between  $p_2$  and  $p_1$  and between  $q_2$  and  $q_1$  are infinitesimal  $P_{02} = P_{01} P_{12}$ .

6.—We turn now to properties of  $P_{01}$  and  $Q_{01}$  as we have determined them.

The writer's formulæ comply with Professor Fisher's "time reversal test,"<sup>1</sup> though that test was not used in determining them. We have

$$\begin{aligned} \log P_{01} &= \left( \log \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \right) \frac{\Sigma (p_1 q_1 - p_0 q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1 q_1}{p_0 q_0}}}{\Sigma (p_1 q_1 - p_0 q_0)} \\ \log P_{10} &= \left( \log \frac{\Sigma p_0 q_0}{\Sigma p_1 q_1} \right) \frac{\Sigma (p_0 q_0 - p_1 q_1) \frac{\log \frac{p_0}{p_1}}{\log \frac{p_0 q_0}{p_1 q_1}}}{\Sigma (p_0 q_0 - p_1 q_1)} \end{aligned}$$

<sup>1</sup> Irving FISHER, *op. cit.*, page 64.

and it is easy to see that

$$\log P_{10} = -\log P_{01}$$

whence 
$$P_{10} = \frac{1}{P_{01}}$$

Similarly 
$$Q_{10} = \frac{1}{Q_{01}}$$

7.—The formulæ determined are such that they can readily be extended to other factors, if such factors exist. This results from the manner in which they were calculated, as the general propositions on which they are based are applicable to any number of factors. But we may prove this property in another way.

Let us suppose that, besides price and quantity, there is a third factor, represented by  $r$ . Treating  $qr$  as a single factor we have :

$$\log P_{01} = \left( \log \frac{\sum p_1 q_1 r_1}{\sum p_0 q_0 r_0} \right) \frac{\log \frac{p_1}{p_0} \frac{\log \frac{p_1 q_1 r_1}{p_0 q_0 r_0}}{\log \frac{p_1 q_1 r_1}{p_0 q_0 r_0}}}{\Sigma(p_1 q_1 r_1 - p_0 q_0 r_0)}$$

$$\log (QR)_{01} = \left( \log \frac{\sum p_1 q_1 r_1}{\sum p_0 q_0 r_0} \right) \frac{\log \frac{q_1 r_1}{q_0 r_0} \frac{\log \frac{p_1 q_1 r_1}{p_0 q_0 r_0}}{\log \frac{p_1 q_1 r_1}{p_0 q_0 r_0}}}{\Sigma(p_1 q_1 r_1 - p_0 q_0 r_0)}$$

Analogous formulæ can be obtained for  $\log Q_{01}$ ,  $\log (PR)_{01}$ ,  $\log R_{01}$  and  $\log (PQ)_{01}$  and it will readily be seen that

$$P_{01} Q_{01} R_{01} = \frac{\sum p_1 q_1 r_1}{\sum p_0 q_0 r_0}$$

and that  $Q_{01}R_{01} = (QR)_{01}$ ,  $P_{01}R_{01} = (PR)_{01}$  and  $P_{01}Q_{01} = (PQ)_{01}$ .

Treating the "ideal" formulæ in the same way we get

$$P_{01} = \sqrt{\frac{\sum p_1 q_0 r_0}{\sum p_0 q_0 r_0} \cdot \frac{\sum p_1 q_1 r_1}{\sum p_0 q_1 r_1}}$$

and  $(QR)_{01} = \sqrt{\frac{\sum p_0 q_1 r_1}{\sum p_0 q_0 r_0} \cdot \frac{\sum p_1 q_1 r_1}{\sum p_1 q_0 r_0}}$

with analogous formulæ for  $Q_{01}$ ,  $(PR)_{01}$ ,  $R_{01}$  and  $(PQ)_{01}$ .

These formulæ fail to comply with the conditions

$$P_{01}Q_{01}R_{01} = \frac{\sum p_1 q_1 r_1}{\sum p_0 q_0 r_0}$$

and  $Q_{01}R_{01} = (QR)_{01}$ , etc.

8.—It is only a special case of the same property that the formulæ are applicable to theoretical cases in which there is a different fundamental relation between  $P_{01}$  and  $Q_{01}$ . Let us suppose, for example, that

$$P_{01}^2 Q_{01} = \frac{\sum p_1^2 q_1}{\sum p_0^2 q_0}$$

However the calculation may be made, the writer's formula gives

$$\log P_{01} = \left( \log \frac{\sum p_1^2 q_1}{\sum p_0^2 q_0} \right) \frac{\frac{\log \frac{p_1}{p_0}}{\log \frac{p_1^2 q_1}{p_0^2 q_0}}}{\frac{\sum (p_1^2 q_1 - p_0^2 q_0)}{\sum p_0^2 q_0}}$$

The "ideal" formula, on the other hand, gives

$$(P_{01})^2 = \sqrt{\frac{\sum p_1^2 q_1}{\sum p_0^2 q_0} \cdot \frac{\sum p_1^2 q_1}{\sum p_0^2 q_1}}$$

$$\text{or } P_{01} = \sqrt{\frac{\sum \phi_1 \phi_0 q_0 \cdot \sum \phi_1^2 q_1}{\sum \phi_0^2 q_0 \cdot \sum \phi_0 \phi_1 q_1}}$$

according as we treat  $\phi^2$  as one factor and  $q$  as the other, or  $\phi$  as one factor and  $\phi q$  as the other.

9.—The formulæ possess another property which it is not easy to express simply. Let us suppose that our group of commodities is divided into sub-groups. Let the prices in the first sub-group be  $\phi_{m_0}'$ ,  $\phi_{m_0}''$ ,  $\phi_{m_0}'''$ , etc., in the base year and  $\phi_{m_1}'$ ,  $\phi_{m_1}''$ ,  $\phi_{m_1}'''$ , etc., in the given year. Let the prices in the second sub-group be  $\phi_{n_0}'$ ,  $\phi_{n_0}''$ ,  $\phi_{n_0}'''$ , etc., in the base year and  $\phi_{n_1}'$ ,  $\phi_{n_1}''$ ,  $\phi_{n_1}'''$ , etc., in the given year. Let the quantities be similarly indicated, substituting  $q$  for  $\phi$ . Let  $PM_{01}$  and  $QM_{01}$  be the price index and quantity index of the first sub-group, *considered separately*, and  $PN_{01}$  and  $QN_{01}$  the price index and quantity index of the second sub-group, considered separately, while  $P_{01}$  and  $Q_{01}$  are the price index and quantity index of the whole group. Suppose that, in a hypothetical year, the prices in the first sub-group become  $PM_{01}\phi_{m_0}'$ ,  $PM_{01}\phi_{m_0}''$ ,  $PM_{01}\phi_{m_0}'''$ , etc., and the quantities  $QM_{01}q_{m_0}'$ ,  $QM_{01}q_{m_0}''$ ,  $QM_{01}q_{m_0}'''$ , etc., while the prices in the second sub-group become  $PN_{01}\phi_{n_0}'$ ,  $PN_{01}\phi_{n_0}''$ ,  $PN_{01}\phi_{n_0}'''$ , etc., and the quantities  $QN_{01}q_{n_0}'$ ,  $QN_{01}q_{n_0}''$ ,  $QN_{01}q_{n_0}'''$ , etc. Combine the two sub-groups in the hypothetical year and let  $PMN_{01}$  and  $QMN_{01}$  be the price index and the quantity index of the combined group in the hypothetical year in relation to the whole group of commodities in the base year. Then, if the formulæ are correct, we should have  $PMN_{01} = P_{01}$  and  $QMN_{01} = Q_{01}$ .

The writer's formulæ comply with this condition. Noting that the formula for  $P_{01}$  may be written

$$\frac{\left(\frac{\sum p_1 q_1}{\sum p_0 q_0}\right)^{\frac{\sum V p_1}{P_0}}}{\sum (p_1 q_1 - p_0 q_0)}$$

we have

$$PMN_{01} = \frac{\left(\frac{\sum PM_{01} p_{m_0} QM_{01} q_{m_0}}{\sum PN_{01} p_{n_0} QN_{01} q_{n_0}} + \frac{\sum V PM_{01} p_0 + \sum V PN_{01} p_0}{P_0}\right)}{\sum p_{m_0} q_{m_0} + \sum p_{n_0} q_{n_0}}$$

$$= \left(\frac{\sum p_{m_1} q_{m_1} + \sum p_{n_1} q_{n_1}}{\sum p_{m_0} q_{m_0} + \sum p_{n_0} q_{n_0}}\right)^{\frac{\sum V PM_{01} p_0 + \sum V PN_{01} p_0}{\sum (p_{m_1} q_{m_1} + p_{n_1} q_{n_1} - p_{m_0} q_{m_0} - p_{n_0} q_{n_0})}}$$

But  $\frac{\sum V PM_{01} p_0}{P_0} = \sum V \frac{p_{m_1}}{p_{m_0}}$

and  $\frac{\sum V PN_{01} p_0}{P_0} = \sum V \frac{p_{n_1}}{p_{n_0}}$

Hence

$$PMN_{01} = \left(\frac{\sum p_{m_1} q_{m_1} + \sum p_{n_1} q_{n_1}}{\sum p_{m_0} q_{m_0} + \sum p_{n_0} q_{n_0}}\right)^{\frac{\sum V \frac{p_{m_1}}{p_{m_0}} + \sum V \frac{p_{n_1}}{p_{n_0}}}{\sum (p_{m_1} q_{m_1} + p_{n_1} q_{n_1} - p_{m_0} q_{m_0} - p_{n_0} q_{n_0})}}$$

$$= \left(\frac{\sum p_1 q_1}{\sum p_0 q_0}\right)^{\frac{\sum V p_1}{P_0}}$$

$$= P_{01}$$

Similarly  $QMN_{01} = Q_{01}$ .

The same property holds good into however many sub-groups the group of commodities may be divided, and, using the writer's formula, the price index may be calculated in a series of stages without affecting the ultimate result.

The "ideal" formulæ break down under this test also.

## CHAPTER VII

### THE PRACTICAL APPLICATION OF THE FORMULA FOR THE PRICE INDEX

WHILE the formula for the price index, as determined in the preceding chapters, is more correct than any of the formulæ in use, the calculations involved are so laborious that its use can be recommended only when close accuracy is required or when it is desired to give a complete picture, either in the form of indices or in that of percentages, of the changes in aggregate value due to the changes in price and in quantity of each of the separate commodities. Where fluctuations are not wide the results obtained by using Professor Fisher's "ideal" formulæ and by using the writer's formulæ differ very slightly. But neither the "ideal" formulæ nor any other formulæ at present in use will enable the complete picture to be given.

The following method is suggested for calculating the general price index and the general quantity index, but practice would no doubt enable simplifications to be made.

Arrange the names of the different commodities and the figures relating to them in columns, as follows:—

Col. 1 : Numbers of the commodities.

Col. 2 : Names of the commodities and units  
of measure.

Col. 3 :  $p_1$

Col. 4 :	$p_0$
Col. 5 :	$q_1$
Col. 6 :	$q_0$
Col. 7 :	$p_1q_1$
Col. 8 :	$p_0q_0$
Col. 9 :	$p_1q_1 - p_0q_0$
Col. 10 :	$\log p_1$
Col. 11 :	$\log p_0$
Col. 12 :	$\log q_1$
Col. 13 :	$\log q_0$
Col. 14 :	$\log \frac{p_1}{p_0}$
Col. 15 :	$\log \frac{q_1}{q_0}$
Col. 16 :	$\log \frac{p_1q_1}{p_0q_0}$
Col. 17 :	$\frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}$
Col. 18 :	$(p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}$

In columns 14 to 16, as the items enter into operations of multiplication and division, the logarithms, when negative, should be expressed as completely negative numbers and not in the usual way, in which the mantissa is positive and the characteristic is negative.

Dividing  $\Sigma(p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}$ , the total of column

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 18, by  $\Sigma(p_1q_1 - p_0q_0)$ , the total of column 9, and  
 multiplying by  $\log \frac{\Sigma p_1q_1}{\Sigma p_0q_0}$ , calculated from the totals  
 of columns 7 and 8, we obtain

$$\left(\log \frac{\Sigma p_1q_1}{\Sigma p_0q_0}\right) \frac{\Sigma(p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma(p_1q_1 - p_0q_0)}$$

This is equal to  $\log P_{01}$ , whence  $P_{01}$  can be obtained.

Again,  $\log Q_{01} = \log \frac{\Sigma p_1q_1}{\Sigma p_0q_0} - \log P_{01}$ , so that  $Q_{01}$  can  
 be easily calculated.

If we desire to obtain the price indices, quantity  
 indices and value indices of the separate commodities,  
 additional columns are necessary.

$$\text{Col. 19: } \frac{p_1q_1 - p_0q_0}{\Sigma(p_1q_1 - p_0q_0)}$$

$$\text{Col. 20: } \frac{(p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma(p_1q_1 - p_0q_0)}$$

$$\text{Col. 21: } \frac{(p_1q_1 - p_0q_0) \frac{\log \frac{q_1}{q_0}}{\log \frac{p_1q_1}{p_0q_0}}}{\Sigma(p_1q_1 - p_0q_0)}$$

$$\text{Col. 22: } \left(\log \frac{\Sigma p_1q_1}{\Sigma p_0q_0}\right) \frac{p_1q_1 - p_0q_0}{\Sigma(p_1q_1 - p_0q_0)}$$



$$\text{Col. 23: } \left( \log \frac{\sum p_1 q_1}{\sum p_0 q_0} \right) \frac{\log \frac{p_1}{p_0} (p_1 q_1 - p_0 q_0)}{\log \frac{p_1 q_1}{p_0 q_0}}$$

$$\text{Col. 24: } \left( \log \frac{\sum p_1 q_1}{\sum p_0 q_0} \right) \frac{\log \frac{q_1}{q_0} (p_1 q_1 - p_0 q_0)}{\log \frac{p_1 q_1}{p_0 q_0}}$$

Column 21 may be calculated by subtracting from each item of column 19 the corresponding item of column 20, and column 24 may be calculated by subtracting from each item of column 22 the corresponding item of column 23.

Column 22 gives us, for each commodity,  $\log V_{v_n}$ , whence  $V_{v_n}$ , the value index, can be obtained.

Similarly  $P_{p_n}$ , the price index, can be obtained, for each commodity, from column 23, and  $Q_{q_n}$ , the quantity index, from column 24.

With one additional column we shall be in a position to give the increase in value due to the increase in the price and in the quantity of each commodity, both in absolute figures and as a percentage of the aggregate increase in value.

The column required is :

$$\text{Col. 25: } (p_1 q_1 - p_0 q_0) \frac{\log \frac{q_1}{q_0}}{\log \frac{p_1 q_1}{p_0 q_0}}$$

This may be calculated by subtracting from each item of column 9 the corresponding item of column 18.

Column 9 gives us the increase in the value of each commodity due to the change both in price and in quantity and the total is the increase in aggregate value.

Column 18 gives us the increase in the value of each commodity due to the change in price and the total is the increase in aggregate value due to all the changes in price.

Column 25 gives us the increase in the value of each commodity due to the change in quantity and the total is the increase in aggregate value due to all the changes in quantity.

Columns 19, 20 and 21 (each item being multiplied by 100) give us the same increases in value expressed as percentages of the aggregate increase in value.

To express the same increases of value as percentages of the aggregate value in the base year we need three more columns.

$$\begin{aligned} \text{Col. 26 : } & \frac{p_1q_1 - p_0q_0}{\Sigma p_0q_0} \\ & (p_1q_1 - p_0q_0) \frac{\log \frac{p_1}{p_0}}{\log \frac{p_1q_1}{p_0q_0}} \\ \text{Col. 27 : } & \frac{\quad}{\Sigma p_0q_0} \\ & (p_1q_1 - p_0q_0) \frac{\log \frac{q_1}{q_0}}{\log \frac{p_1q_1}{p_0q_0}} \\ \text{Col. 28 : } & \frac{\quad}{\Sigma p_0q_0} \end{aligned}$$

The percentages are obtained by multiplying each item of columns 26, 27 and 28 by 100.

#### THE PRACTICAL APPLICATION OF THE FORMULA 71

An illustration of the use of the writer's formulæ is given in the following tables, in which they are applied to the group of commodities taken by Professor Fisher himself as the basis of his calculations in "The Making of Index Numbers." The group contains 36 commodities, with prices and quantities for six successive years, 1913 to 1918, and our example relates to the prices and quantities in 1913 and 1917.<sup>1</sup> The difference between the price index as calculated by the "ideal" formula and the price index as calculated by the writer's formula is trifling; the "ideal" formula gives 161.56, while the writer's formula gives 161.74.

<sup>1</sup> For the full list of prices and quantities see pages 489 and 490 of "The Making of Index Numbers." Our  $p_1$  and  $q_1$  correspond to  $p_a$  and  $q_a$  in the full list.

TABLE I.

*Prices and Quantities*<sup>1</sup> *in the Given Year and in the Base Year in the Example Chosen.*

No. of commodity.	Commodity and unit of measure.	$p_1$	$p_0$	$q_1$	$q_0$
1	Bacon, lb. . . . .	0·2382	0·1236	1187·00	1077·00
2	Barley, bu. . . . .	1·3232	0·6263	209·00	178·20
3	Beef, lb. . . . .	0·1672	0·1295	8417·00	6589·00
4	Butter, lb. . . . .	0·4034	0·2969	1842·00	1757·00
5	Cattle, cwt. . . . .	15·6354	12·0396	103·50	69·80
6	Cement, bbl. . . . .	2·0942	1·5800	88·10	85·80
7	Coal, anth., ton . . . .	5·6218	5·0636	7·83	6·90
8	Coal, bit., ton . . . . .	3·5800	1·2700	552·00	477·00
9	Coffee, lb. . . . .	0·0929	0·1113	1320·00	863·00
10	Coke, short ton . . . . .	10·6600	3·0300	56·70	46·30
11	Copper, lb. . . . .	0·2764	0·1533	1316·50	812·30
12	Cotton, lb. . . . .	0·2350	0·1279	3423·00	2785·00
13	Eggs, doz. . . . .	0·4015	0·2468	1882·00	1722·00
14	Hay, ton . . . . .	17·6042	11·2500	94·90	79·20
15	Hides, lb. . . . .	0·2828	0·1727	1113·00	672·00
16	Hogs, cwt. . . . .	15·7047	8·3654	67·80	68·40
17	Iron, bar, cwt. . . . .	4·0600	1·5100	133·00	79·20
18	Iron, pig, ton . . . . .	38·8082	14·9025	38·70	31·00
19	Lead (white), lb. . . . .	0·1121	0·0676	230·00	286·00
20	Lead, lb. . . . .	0·0879	0·0437	1099·80	823·70
21	Lumber, M. bd. ft. . . . .	105·0400	90·3974	21·20	21·80
22	Mutton, lb. . . . .	0·1664	0·1025	474·00	732·00
23	Petroleum, gal. . . . .	0·1242	0·1233	14880·00	10400·00
24	Pork, lb. . . . .	0·2435	0·1486	8427·00	9211·00
25	Rubber, lb. . . . .	0·6477	0·8071	375·90	115·80
26	Silk, lb. . . . .	5·9957	3·9083	29·40	19·10
27	Silver, oz. . . . .	0·8142	0·5980	133·60	146·10
28	Skins, skin . . . . .	5·5208	2·5833	2·70	6·70
29	Steel rails, ton . . . . .	38·0000	28·0000	2·94	3·50
30	Tin, pig, cwt. . . . .	61·6500	44·3200	1·56	1·04
31	Tin, plate, cwt. . . . .	9·1250	3·5583	29·50	15·30
32	Wheat, bu. . . . .	2·3211	0·9131	605·00	555·00
33	Wool, lb. . . . .	1·2841	0·5883	707·00	448·00
34	Lime, bbl., 300 lb. . . . .	1·7604	1·2500	24·00	23·30
35	Lard, lb. . . . .	0·2170	0·1101	927·00	1100·00
36	Oats, bu. . . . .	0·6372	0·3758	1587·00	1122·00

<sup>1</sup> The prices are expressed in dollars per unit and the quantities in millions of units.

TABLE II.

*Price Indices, Quantity Indices and Value Indices  
of Separate Commodities.*

No. of commodity.	Commodity.	Price indices of separate commodities.	Quantity indices of separate commodities.	Value indices of separate commodities.
		100 $P_{01}$	100 $Q_{01}$	100 $V_{01}$
1	Bacon . . .	100·71	100·10	100·73
2	Barley . . .	100·74	100·26	100·90
3	Beef . . .	101·54	101·48	103·04
4	Butter . . .	101·05	100·16	101·20
5	Cattle . . .	101·69	102·56	104·30
6	Cement . . .	100·24	100·02	100·36
7	Coal, anth.	100·02	100·04	100·05
8	Coal, bit..	106·71	100·92	107·69
9	Coffee . . .	99·89	100·25	100·14
10	Coke . . .	102·19	100·35	102·54
11	Copper . . .	100·71	100·58	101·30
12	Cotton . . .	101·83	100·28	102·45
13	Eggs . . .	101·52	100·38	101·80
14	Hay . . .	103·05	101·22	104·30
15	Hides . . .	100·53	100·54	101·08
16	Hogs . . .	102·74	99·96	102·70
17	Iron, bar . .	101·50	100·78	102·30
18	Iron, pig . .	104·66	101·06	105·78
19	Lead (white) .	100·06	99·97	100·03
20	Lead . . .	100·23	100·10	100·33
21	Lumber . . .	101·72	99·68	101·39
22	Mutton . . .	100·20	99·82	100·02
23	Petroleum . .	100·06	103·04	103·11
24	Pork . . .	104·61	99·19	103·75
25	Rubber . . .	99·81	101·00	100·81
26	Silk . . .	100·27	100·38	100·55
27	Silver . . .	100·16	99·95	100·17
28	Skins . . .	99·93	100·08	100·01
29	Steel rails . .	100·17	99·90	100·07
30	Tin, pig . . .	100·12	100·15	100·27
31	Tin, plate . .	100·45	100·48	101·17
32	Wheat . . .	104·54	100·42	104·97
33	Wool . . .	102·89	100·63	103·55
34	Lime . . .	100·07	100·01	100·07
35	Lard . . .	100·58	99·92	102·77
36	Oats . . .	101·94	101·27	103·24
	General Indices	161·74	118·80	192·23
		100 $P_{01}$	100 $Q_{01}$	$100 \frac{\sum P_{191}}{\sum P_{01}}$

TABLE III.

Changes in Aggregate Value due to the Changes in Price, Quantity and Value of Separate Commodities, in Absolute Figures and as Percentages of the Aggregate Value in the Base Year.

No. of commodity.	Commodity.	Increase in value due to increases in price of separate commodities.	Increase in value due to increases in quantity of separate commodities.	Increases in value of separate commodities.	Increase in value due to increases in price expressed as percentages of the aggregate value in the base year.	Increase in value due to increases in quantity expressed as percentages of the aggregate value in the base year.	Increases in value of separate commodities expressed as percentages of the aggregate value in the base year.
		$\frac{V_2}{P_0}$	$\frac{V_2}{Q_0}$		$\frac{P_1 - P_0}{P_0}$	$\frac{100 V_2}{\Sigma P_0 Q_0}$	
1	Bacon	130-3099	19-3163	149-6262	0-994	0-148	1-142
2	Barley	135-9624	28-9797	164-9421	1-037	0-221	1-259
3	Beef	282-9241	271-1228	554-0469	2-159	2-069	4-228
4	Butter	191-8420	29-5675	221-4095	1-404	0-226	1-630
5	Cattle	310-2420	467-6578	777-8998	2-367	3-569	5-936
6	Cement	44-7348	4-2002	48-9350	0-341	0-033	0-374
7	Coal, anth.	4-1102	4-9697	9-0799	0-031	0-038	0-069
8	Coal, bit.	1201-1197	169-2503	1370-3700	9-165	1-292	10-457
9	Coffee	-19-6607	26-5761	6-9154	-0-150	0-353	0-203
10	Coke	399-7415	64-3915	464-1330	3-050	0-491	3-541
11	Copper	131-5737	107-7813	239-3550	1-004	0-822	1-826
12	Cotton	334-7121	113-4914	448-2035	2-554	0-866	3-420
13	Eggs	279-8866	51-0468	330-9334	2-133	0-390	2-523
14	Hay	555-3436	224-2050	779-5486	4-238	1-711	5-949
15	Hides	98-2186	100-4834	198-7020	0-749	0-767	1-516
16	Hogs	499-8338	-6-9885	492-8453	3-812	-0-053	3-759
17	Iron, bar	275-8271	144-5609	420-3880	2-105	1-103	3-208
18	Iron, pig	844-2146	195-6852	1039-8998	6-442	1-493	7-935
19	Lead (white)	11-3313	-4-8821	6-4494	0-086	-0-037	0-049
20	Lead	42-9222	17-7545	60-6767	0-327	0-136	0-463
21	Lumber	314-6855	-58-5008	256-1847	2-401	-0-446	1-955
22	Mutton	37-2780	-33-4344	3-8436	0-284	-0-255	0-029
23	Petroleum	11-2584	554-5176	565-7760	0-086	4-231	4-317
24	Pork	833-3247	-150-1048	683-2199	6-359	-1-145	5-214
25	Rubber	-34-4719	184-4801	150-0082	-0-263	1-208	1-145
26	Silk	50-6135	51-0116	101-6251	0-386	0-389	0-775
27	Silver	30-1464	-8-7371	21-4093	0-230	-0-067	0-163
28	Skins	12-2097	-14-6116	-2-4019	0-093	-0-101	-0-018
29	Steel rails	31-9766	-18-2566	13-7200	0-244	-0-139	0-105
30	Tin, pig	22-4726	27-6086	50-0812	0-171	0-211	0-382
31	Tin, plate	126-5326	88-2129	214-7455	0-966	0-673	1-639
32	Wheat	821-5364	75-9586	897-4950	6-269	0-480	6-849
33	Wool	406-6373	237-6530	644-3003	3-103	1-813	4-916
34	Lime	12-0803	1-0443	13-1246	0-092	0-008	0-100
35	Lard	107-0443	-16-9933	80-0490	0-817	-0-206	0-611
36	Oats	355-8905	233-6983	589-5888	2-716	1-783	4-499
	Totals	8892-8543	4193-4721	12086-3265	67-857	24-361	92-228
		$\frac{\Sigma V_2}{P_0}$	$\frac{\Sigma V_2}{Q_0}$	$\Sigma (P_1 - P_0)$	$\frac{100 \Sigma V_2}{\Sigma P_0 Q_0}$	$\frac{100 \Sigma V_2}{\Sigma P_0 Q_0}$	$\frac{100 \Sigma (P_1 - P_0)}{\Sigma P_0 Q_0}$