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VOLUME 26.

THE TEACHING OF MATHEMATICS IN
THE UNITED KINGDOM. PART I.

Being a series of Papers prepared for the
INTERNATIONAL COMMISSION ON THE TEACHING OF
MATHEMATICS.



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PREFATORY NOTE TO VOLUME 26 OF SPECIAL REPORTS ON
EDUCATIONAL SUBJECTS.

The present volume is the result of the action taken by the International Congress of Mathematicians at their meeting in Rome in 1908, when, at the proposal of Dr. David Eugene Smith, Professor of Mathematics in Columbia University, New York, it was resolved to appoint an International Commission on the Teaching of Mathematics. This Commission was entrusted with the task of securing a series of Reports on the state and progress of mathematical instruction in various countries of the world.

There were certain difficulties in arranging for the collection of reports in this country, and the Board of Education were approached and asked to give assistance in the matter. The Board were glad to avail themselves of the opportunity which thus presented itself of adding to their series of Special Reports on Educational Subjects a collection of papers dealing with mathematical teaching in this country. In order to assist the Office of Special Inquiries and Reports in the collection of suitable material, the President appointed an Advisory Committee consisting of:—

Mr. C. E. Ashford.
 Sir George H. Darwin, K.C.B., F.R.S.
 Mr. C. Godfrey, M.V.O.
 Sir George Greenhill, F.R.S.
 Mr. G. H. Hardy, F.R.S.
 Professor E. W. Hobson, F.R.S.
 Mr. C. S. Jackson.
 Sir Joseph Larmor, F.R.S.
 Professor A. E. H. Love, F.R.S.
 Professor George A. Gibson.

This Committee acted at the same time as the British Sub-Commission of the International Commission on the Teaching of Mathematics. The Reports which have been secured through their assistance are therefore intended to serve a double purpose; on the one hand they enrich the series of Special Reports, while on the other they serve as the Report of the British Sub-Commission, which will be submitted to the International Congress of Mathematicians meeting in Cambridge this year.

The Board desire to take this opportunity of expressing their cordial thanks to the members of the Committee for the assistance they have rendered, and to the various authors who have contributed papers to this and the accompanying volume. In particular they wish to express their acknowledgment of the

invaluable service which has been rendered by Mr. C. S. Jackson Secretary to the British Sub-Commission, on whom has fallen the main work in connexion with the collection and revision of the papers included in this and the accompanying volumes. Mr. Jackson has placed his time and knowledge most freely at the disposal of the Board.

The Board's thanks are also due to the authorities of those Colleges at Oxford and Cambridge who have courteously given permission for the reprinting of their Scholarship Examination papers. The Board are also indebted to the Delegates of Local Examinations, University of Oxford; the University of Cambridge Local Examinations and Lectures Syndicate; the Oxford and Cambridge Schools Examination Board; the Senate of the University of London; the Joint Matriculation Board of the Universities of Manchester, Liverpool, Leeds, and Sheffield; the Delegates of the Clarendon Press, Oxford, and the Syndicate of the Cambridge University Press, for a similar service.

It may be said that the immediate cause which led to the formation of the International Commission was a divergence between mathematical and pedagogic requirements in the schools and Universities. This divergence is well illustrated by the papers which appear in the present volumes. It is hoped that their publication may contribute towards the diminution of this divergence, a task in which all who are interested in mathematical instruction are eager to co-operate.

Though the papers are published by the Board on the recommendation of the Committee, it must be clearly understood that neither the Committee nor the Board accept any responsibility for the statements and opinions contained in them.

Office of Special Inquiries and Reports,
August 1912.

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THE TEACHING OF MATHEMATICS IN
ELEMENTARY SCHOOLS.

THE TEACHING OF MATHEMATICS IN LONDON PUBLIC ELEMENTARY SCHOOLS.

Ten years have passed since the Board of Education ceased to prescribe a fixed syllabus of instruction in Arithmetic for elementary schools. During these ten years each school has had freedom to devise its own scheme, conduct its own examinations, and adopt its own methods of instruction. One would therefore naturally expect to find as the years advanced a gradual divergence from the Board's syllabus as a common point of departure, so that at the present day the syllabuses of the various schools would differ widely from one another and from the original common syllabus. As a matter of fact the divergence, with a few noteworthy exceptions, has been very slight. In the bulk of the schools the Board's Scheme B (*see* Appendix I.), which was the better of the two prescribed courses, still forms the framework of the mathematical curriculum. It has been enlarged by the inclusion of mensuration, and slightly rearranged so as to admit of the earlier introduction of fractions, but rarely has it been drastically altered. This conservatism is partly due to the fact that the Board continued, up till 1905, to publish Scheme B as a standard of proficiency for the labour certificate, and partly to the fact that most of the available text-books for elementary schools are even to this day based upon that scheme. It required therefore a head teacher of some force of character to overcome the inertia of the older system and strike out a line of his own. Some encouragement was given to initiative and experiment by the publication of the Suggestions to Teachers in 1905, for since that date departure from the normal course has been a little more frequent.

TEACHING OF ARITHMETIC IN INFANTS' SCHOOLS.

Instruction in the rudiments of number is given in the infants' school, on the organisation of which a few explanatory words are necessary. On account of exigencies of accommodation children who by age and attainments would normally pass into the lowest class of the Senior Department are in the majority of the London Schools retained in the Infants' Department as Standard I. These children are as a rule seven years of age at the middle of the Educational Year. The children of six, five, and under five years of age are known respectively as Grade III, Grade II, and Grade I or babies. The Standard I. children have for many years past been expected to deal with the four simple rules (*see* Appendix I.). At the end of the

4. *Teaching of Mathematics in London Elementary Schools.*

Educational Year a joint examination based upon this syllabus is held by the head teachers of the Senior and Infants' Departments with a view to the allocation of the examinees to suitable classes in the Senior School in the following year. This examination has been a very important factor in determining the nature of the number instruction in the infants' school. It has served as the objective towards which the work in the grades has been directed. The examination has too often taken the form of a mechanical test, as in Appendix II., but there is an increasing tendency to set simple problems, as in Appendix III. Although this examination serves a useful purpose, its influence on the whole has been more baneful than beneficial; for it has been the custom in past years to work up to this examination right through the grades by the most direct route. "Sums" dominated the number lesson. Children began even in the babies' class to set down little sums, and gradually proceeded towards harder sums. In Standard I. practice in setting out was essential. The four simple rules were always taken in the same order, and each rule assigned a particular place on the paper, so that at the final test the elements of novelty should be reduced to a minimum. But these shallow devices are happily obsolescent, if not entirely obsolete. In the great majority of the infants' schools serious attempts are being made now to impart a sound and intelligent knowledge of the fundamentals of number. And the clue to the instruction is being sought in the child's interests and aptitudes rather than in any logical arrangement of subject matter. No longer are lessons on number given to the babies, and paper sums have disappeared from Grade II. and all but disappeared from Grade III. Rarely is the chanting of the multiplication table heard in the school. The aim of the work is gradually changing. The teachers now try to foster an intelligent grasp of the relations between the first few natural numbers and a facile application of them to simple and familiar things. And this they attempt to do by the analysis of number by the Grube method. The teacher first fixes the notion of unity, and then deals with 2, 3, 4, &c., exhausting the possibilities of one number before proceeding to the next. Everything that can be done with 4 is done with it before 5 is taken up. Each child is supplied with objects—sometimes sticks, sometimes beans, sometimes tablets, &c.—which he manipulates and arranges at the command of the teacher. The results discovered then become the basis of questions in mental arithmetic asked by the teacher and nearly always put in concrete form; for example:—Five little birds sitting on a tree; two fly away, how many are left? The teacher aims at getting the children by frequent repetition of the concrete process to memorise the result so as to avoid counting by units. In some schools numbers up to 5 are dealt with in Grade II., and numbers up to 10 in Grade III. In other schools the two grades go up to 10 and 20 respectively.

These seem to represent the maxima and minima. The attempts made in the bulk of the schools range between these limits. This system is the very keel and backbone of the instruction in number in the London infants' school—the decomposition of small numbers, the manipulation of objects, and oral questions in the grades; supplemented by written sums in Standard I. Sometimes the teaching is limited to this, with no counting and no learning of tables beyond the number dealt with concretely.

It may be claimed that this is a great improvement upon the older ciphering method, for it succeeds in giving a child a fairly intelligent notion of the relationships of the simple numbers which form the basis of our denary system, and it enables him to deal with common-sense problems. Still it is open to serious criticism:—

1. Unity cannot be dealt with alone. The notion only becomes intelligible in contrast with more-than-one.
2. The lessons tend to become very tedious. It is only a very strong teacher who can keep a class of six-year-olds interested in the number 7 for a whole lesson—especially when that lesson has been given several times before.
3. Counting should not be reined in so as to keep pace with analysis, and something should be known about 20 before everything is known about 8.
4. The ratio or measuring aspect of number is inadequately represented.
5. The various operations do not in general arise from any need felt by the children—a very serious objection.
6. The avowed end is rarely achieved. Seldom can a child of five deal adequately with more than 5, or a child of six with more than 10. Even when he does, the process is egregiously slow. The memorisation aimed at rarely takes place, and counting by units is the rule rather than the exception.
7. Finally, I have very grave doubts whether number should be taught at all to children below seven years of age. I have many reasons indeed for thinking that such instruction involves a wastage of teaching power, and a stunting of the growth of child intelligence.

These objections are not unfelt in our schools, and the bare Grube system is accordingly being modified and supplemented. In one instance at least it has been entirely abandoned.

It is a well-known fact that children will learn to count of their own accord. They do it at home, in the playground, and at their games. It is impossible to stop them. It may be that the counting is of that type which Mr. Raymont designates "spurious," a counting which is a mere rythmical sequence of sounds like "ena dena dina dust," from which no real inference is drawn as to the size of a group; still, it forms the

basis of real counting. This spontaneous nature of children's counting is used by some head teachers as an argument for teaching the children in the grades to count up to a hundred; and by other head teachers, curiously enough, as an argument for not teaching it. I tested a school where no counting is specifically taught and found that about 60 per cent. of the children could count up to 20 before they were five years old, and about 70 per cent. could count up to 30 before they were six. On inquiry of the children themselves I found that they had, as a rule, learnt at home from their mothers, brothers, or sisters. I may point out that children incidentally learn to count when finding the page in the reading book.

The question whether initial instruction in number should be based on counting or measuring is one which is agitating the minds of some of the teachers in infants' schools. A child naturally learns to count before he learns to measure. But he should learn to measure as well. In consequence of this belief Tillick's bricks are used in a few schools. They consist of rectangular prisms of equal cross sections, but standing at various heights to represent the first ten numbers. They are compared by guessing, and the guesses are verified by measuring one brick against another. The purpose of this training, however, appears to be defeated in one or two instances by marking off the prisms into distinct units. A certain amount of simple measuring in inches by means of rulers is occasionally to be found in the grades, but generally speaking this work is confined to the standards.

There is to be observed a commendable tendency to relieve the tedium of the number lesson by the use of toys, such as soldiers, wooden animals and trees of the Noah's ark type, as substitutes for counters.

Still more significant is the effort that is made in many of our schools to provide a motive for the various numerical exercises. A competition in skipping or in tossing a shuttlecock takes place before the class, the children counting and recording the score. A ring and hook game is engaged in, the class adding the numbers and comparing the results. Shopping with artificial coins is a device that is rapidly spreading. In at least two schools, rooms not needed for ordinary classwork are roughly fitted up as shops, and rice, sugar, tea and other commodities are weighed or measured in bulk and sold behind a counter by a child who acts as shopman to his companions who act as customers.

But the most revolutionary of these experiments in providing a motive for calculation is to be found in School A.* Here there is no formal instruction in number below Standard I. Number is learnt incidentally through the construction of objects. Articles or designs of various kinds are drawn, modelled, cut

* The schools described as School A, School B, &c. are Public Elementary schools in the London area.

out, or constructed; and the children are encouraged to converse about the things in such a way as to use numerical and quantitative terms. Even in Standard I. there is no written work except that notation is learnt and the answers to problems put down. No "sums" are taught. The intelligence of these children when they pass into the Senior School is beyond question. The mathematics in the boys' department is exceptionally good all through.

This leads up to the final bit of criticism, which strikes at the very root of number teaching in infants' schools. It raises the serious question: Is it worth while? Is there any ultimate gain to be derived from specific lessons in number given to children under seven years of age? However ambitious the class teacher may be, it is very rarely found that Grade II. children can deal with any degree of readiness with numbers beyond 5. These children, as a rule, receive one lesson of 20 minutes per day for five days in the week and forty-four weeks in the year, and yet they only master the numbers 2, 3, 4 and 5. Think of the progress that a child of the same age during the same period can make in motor co-ordination as shown in games, dancing, drawing, modelling, constructing and other activities; think of the number of words that he can thus add to his usable vocabulary; think of the number of songs that he can learn and the number of nursery rhymes that he can memorise; think of the possible increase in his store of nature knowledge; and then compare any of these with his meagre achievements in Arithmetic. Does it not give rise to the suspicion that the child's mind is at that period not quite ripe for that particular kind of training? This suspicion is somewhat strengthened by the results of an experiment that has been going on for some years in School B. Here the number teaching begins in Grade III., and it has been found that at the end of two years (that is when Standard I. course is completed) the children are quite as advanced as they were when the number lessons began in Grade II. and extended over three years. The amount of time per day devoted to the subject remained the same under both systems. Last July 176 of the children in the school were of Standard I. age, and were tested on the Standard I. syllabus. The examinees included all the laggards, the children retarded through ill-health, the migratory children from other schools—in fine, all who were qualified by age for transfer to the Senior School. One of the actual tests set is given in Appendix III. Alternative tests of equal difficulty were given to prevent copying. Out of the 176 children 84 got all four sums right, and altogether 77 per cent. of the sums were worked correctly. In the best of the three classes 90 per cent. of the sums were right. This result is well above the average, and when considered in conjunction with the result of the experiment in School A., it points to the necessity for further experiment in the direction of postponing

the age for beginning to learn Arithmetic in distinct and specific lessons.

As the instruction in number has been gradually changing for the better, so has the instruction in form. While diminishing in quantity it has improved in quality. At one time children were required to name from a card various plane figures, such as triangle, square, oblong, parallelogram, pentagon, hexagon, &c. They folded paper and learnt something about angles, lines and surfaces. By means of the Kindergarten gifts they were even initiated into three-dimensional geometry. In drawing lessons the terms vertical, horizontal and oblique were learnt and illustrated. Nearly all this pedantry has now been abandoned, for it is being recognised that better ideas of form are gained incidentally through handwork and drawing.

There seems to be no department of elementary work where more uncertainty reigns than in number teaching in the infants' school. Many teachers of experience share my scepticism about the value of any formal kind of mathematical instruction given to children under seven years of age. Nearly all are agreed as to the necessity for experiment.

Mention may be made of a few methods that are being adopted in some schools and are not, in my opinion, wholly successful.

A triadic system is used extensively in School C, and to some extent in a few other schools. All numbers up to ten are to be arranged and visualised in threes,

$$\begin{aligned} \text{e.g. } 4 &= \cdot \cdot \cdot \cdot \\ 5 &= \cdot \cdot \cdot \cdot \cdot \\ 8 &= \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{aligned}$$

&c.

In other schools it is insisted on that the children should always arrange their numbers according to some fixed design. They are graphically represented in the form, say, of pips on a playing card or dots on a domino. Thus five must be always pictured

as $\cdot \cdot \cdot \cdot \cdot$ never as $\cdot \cdot \cdot \cdot \cdot$ or as $\cdot \cdot \cdot \cdot \cdot$. This attempt to force

the imagery of children into one mould is in my opinion ill advised, and in any case destined to failure. The investigations of Galton tend to show that each individual has his own characteristic mode of imagining numbers.

Another prevalent practice of a more advanced stage consists in decomposing numbers so as to form a series of tens—

$$\begin{aligned} \text{e.g., } 7 + 8 &= (7 + 3) + 5 = 15 \\ 28 + 9 &= (28 + 2) + 7 = 37 \end{aligned}$$

&c.

In actual practice the method is safe but slow. As a practical method it is more rapid than counting by ones and less rapid than reliance upon memory for the unit figure. For the logical justification of the statement $7 + 8 = 15$, we must go

back to the natural number series and count. The best practical expedient is to memorise the result $7 + 8 = 15$. The step $(7 + 3) + 5$ is a temporary device neither logically nor practically ultimate.

This completes my account of the Infants' Schools.

MATHEMATICS IN THE SENIOR DEPARTMENTS.

Before dealing with the senior departments, it may be well that I should explain the range of experience upon which my report is based. First, comes my knowledge of the schools in my own district; secondly, visits made to typical schools in other parts of the Metropolis; thirdly, conferences with my colleagues; and fourthly, an examination of the syllabuses and methods in use in 70 departments taken entirely at random from the various inspectorial districts. Of these 70 schools, 30 were boys', 30 girls', and 10 mixed departments. The list comprised both Council and Non-provided Schools; in fact, every variety of school except the Higher Grade and Central.

Some of the results of my investigations are herewith tabulated:—

Subject or Method.	Number of Cases in which it is taught or used in the 70 Schools.
Scheme B. Syllabus (modified)	60
Practice	51
Recurring Decimals	37
Compound Interest	33
True Discount	17
Stocks and Shares	36
Square Root	33
Cube Root	11
“Rule of Three”—taught by method of unit exclusively	24
“Rule of Three”—taught by proportion as well	46
Tots	36
Simple Graphs	47
Practical Arithmetic taken systematically	44
Multiplication in which the pupil starts with the highest denomination in the multiplier	3
Both methods of multiplication (starting with the unit figure, and starting with the highest denomination)	14
L.C.M. and H.C.F. taught by repeated divisions only	22
L.C.M. and H.C.F. taught by factors only	24
” ” ” both methods	24
Factors used for extraction of square and cube roots	17

Geometry is taken as a separate subject in 30 cases, and an average of 35 minutes per week is devoted to it. Algebra, on the other hand, is only taken in 16 cases, with an average time of 60 minutes per week.

These investigations confirm me in the view that my own district is fairly representative of the rest of London, and that any generalisations that I may make can be regarded as holding good of at least a considerable majority of the schools.

It will be seen that Arithmetic still remains the one branch of Mathematics common to all the schools, and that Scheme B. is the root from which it springs. It is not, however, possible by a mere examination of the syllabus to form an estimate of the kind of work done in any given school. A far safer clue is to be found in the kind of test set by the Head Teacher. Every class teacher in the London service is supplied with a Report and Progress Book, which contains, among other things, the syllabus in Mathematics. The Head Teacher has to examine the class on the syllabus at least twice a year, and to record the results in the Report and Progress Book. It is this examination that mainly determines the teaching. It indicates the specific interpretation given by the Head Master to his syllabus. It serves as a goal towards which the teachers work, for they feel that their professional reputation is largely dependent on successful records. It is not, therefore, unusual to find two schools with almost identical syllabuses, but widely different methods of instruction. In the one case, the terminal test may consist of a small number of sums of fixed type; in the other case it may consist of a variety of problems each making some fresh demand upon the ingenuity of the pupil. The first kind of examination encourages example-grinding; the second kind encourages intelligent teaching.

It will thus be seen that the influence of the Head Teachers in determining the method as well as the matter of instruction is paramount; and it is pleasant to record that they are rising to their responsibility. Indeed, far more intelligent enterprise is being shown in the devising of tests than in the drawing up of syllabuses.

In Appendices IV. and V. will be found sets of examination questions, which will illustrate better than any other data that I can supply the progress that has taken place in the mathematical instruction of some of the schools during the last six years. The first instance is a girls' school; the second, a boys'. Two sets of tests are supplied in each case, one of each set being given in 1904 and the other in 1910. The other conditions of the examinations—the standard and the time of the year—were almost identical. A comparison of these tests will reveal the following tendencies:—

1. Arithmetic is passing into Mathematics. The claims of Geometry, Mensuration and Algebra obtain at least some recognition.

2. Fixed conventional types of sums are being largely abandoned.
3. The examples are brought into close relation with the children's own everyday life.
4. An abundance of practical work is being introduced.

It may be pointed out that School E. (Appendix V.) is a boys' school where Mathematics is exceptionally well taught. The percentage of marks in 1910 was 66 in the practical paper and 70 in the theoretical, which is remarkably good for an ordinary school attended by an average class of London boys. It must not be supposed that even five per cent. of the schools came anywhere near this mark. To show what can be done in a school of an entirely different type, I give in Appendix VI. a test set in School F in 1910. This is a girls' school, where the children are abnormally poor, and where the special strength is literary rather than mathematical. The average percentage of marks was 60. I do not quote these as model tests, but as indications of tendencies. There is much in them that is open to criticism. The wording of some of the questions could be improved, and the boys' test in particular strikes me as being inordinately difficult.

These are extreme cases; but they represent changes that are going on to some extent in all the schools.

First let us consider the tendency to extend the limits of Arithmetic and to break down the barriers between the various branches of Mathematics. Some five or six years ago Algebra appeared as a separate subject on the time tables of a large number of schools, but it consisted of little more than an unintelligent manipulation of symbols. It was in no way connected with the Arithmetic; the need for a more general symbolisation than the arithmetical digits was never made clear; and the usefulness of the algebraic processes was left in obscurity. But Algebra as a separate subject has now largely disappeared, except in the Central Schools, and attempts are being made to graft a little of it on the Arithmetic. The attempts are laudable, but not very successful. I should not, for example, care to defend the algebraic questions in the representative tests I have appended. It is, perhaps, a debatable question whether Algebra should be taught at all in an ordinary elementary school. I am personally inclined to think that it should, and that it should have its starting point at problems leading to simple equations, or at mensuration formulæ such as: area of triangle = $\frac{1}{2}$ b.h. In any case, the teacher should provide situations where a symbol like x would be a more potent instrument of thought than a symbol like 5, where, in fact, Algebra is more useful than Arithmetic. The teacher should see that the need for the new kind of symbol is felt by the pupil himself. I have rarely, as a matter of fact, seen this done, and the Algebra that is brought into the arithmetic lesson is generally of an adventitious kind which might just as well be left outside.

Geometry.

Geometry, however, seems to me to be upon another footing. It has distinct claims of its own, apart from the use made of it in mensuration. Much practical geometry is absorbed in the practical arithmetic of most schools; but when it is taken as a separate subject (*see* page 10) it generally assumes the form of working problems with compasses and ruler. It is in fact sometimes classed with the art subjects and sometimes with the mathematical. The procedure in each case follows on the lines which were common in the days when the old South Kensington Art syllabus was taught in the schools. The teacher sets a problem and works it on the board, and the children copy it into their Drawing Books. This is the almost universal method, and it is unquestionably a bad method. If heuristic training is justifiable in any subject it is justifiable in this subject. Treated as a progressive series of puzzles to be solved by the child himself, the subject may be made of immense educative value. Unfortunately a fixed syllabus is generally given to the class teacher, and he is required to get through a certain number of problems in a given time. This is a mistake. It is not the master who should set the pace, but the class.

Theoretical geometry is taken only in the Central Schools.

The Teaching of Practical Arithmetic.

Far and away the most significant change that has taken place during the past few years has been the introduction of what is known in the schools as practical arithmetic. The *differentia* of practical arithmetic seems to be the presence of the material objects dealt with. If they are not actually present, they must be represented by natural and not mere conventional symbols. If, for instance, a boy were asked to find the diagonal distance across a rectangular garden, 30 yards long and 25 yards wide, and proceeded to draw the garden to scale and to find the approximate length of the diagonal by measurement, that would be considered an exercise in practical arithmetic; for although neither the actual garden, nor the actual unit of measurement was used; yet each was adequately represented by things which made the same kind of perceptual appeal.

Practical arithmetic aims, in fact, at presenting in full the evidence on which the conclusions are based, and testing the conclusions by fresh application to concrete experience. The advantage of it is that the pupil knows what he is doing; he is going through a bit of real experience. It is well known that a child may be able to multiply 6489 by 76 with ease and accuracy, and yet be unable to show its connection with ordinary life, or, indeed, to attach any meaning whatever to the process. The aim of the practical arithmetic is to remedy

this defect—to give the fullest and clearest meaning to the various mathematical operations carried out. The rationalisation of the arithmetic so as to prevent it from ever degenerating into unintelligent mechanical work is the very soul and secret of the practical arithmetic movement. And if this end is to be achieved, the practical arithmetic must come first and the mechanical drill come afterwards. With these principles in mind, we will proceed to estimate the value of the practical arithmetic instruction given in the schools. In every school *some* practical arithmetic is done; in some schools nearly all the arithmetic is practical. In a few schools it is well done; in a few others it is badly done; in the bulk of the schools it is indifferently done, but is improving. The gravest defect is the relegation of the practical arithmetic to the end of the school year. The mechanical and theoretical part is done first, and then—if there is time—the practical work is attacked. This reversal of the true order of things is partly due to the fact that certain arithmetic books which are almost universally used in the schools provide a large number of the older style of examples, and append merely a few pages of practical work as a sort of grudging concession to newer views; and partly due to the opinion still tenaciously held by a few teachers that mechanical drill should take precedence over every other form of mathematical training. It should come first in order of time, and have the biggest share of practice. They call it dividing the difficulties and presenting them one at a time.

The second fault I have to find is that the practical work is often done by the wrong person. Let us consider an extreme case. The teacher is about to give a class of 60 boys in an ordinary class-room a lesson in weighing. He has on the table in front of him a balance and a set of weights. It is obvious that only one at a time can use the balance; and he elects that he shall be that one, or he lets about half-a-dozen boys do the weighing one after the other, while the rest of the class look on. A boy who watches another boy weighing finds no more pleasure or profit in the business than he does in watching another boy eat an apple. The difficulty of the case under consideration will readily be conceded, but it should not be met by so wasteful an expenditure of the time of the class. A better plan would be to provide each boy with a written problem or set of problems to be solved by weighing, and allow each boy in turn to get out his own results, while the rest of the class is engaged in drawing, silent reading, or ordinary arithmetic. A better plan still may be suggested. It would probably be found possible to furnish each desk with a cheap balance quite delicate enough for practical purposes. The upright could be permanently attached to the desk and rendered collapsible, the crossbeam and scales being detachable. An arrangement of this kind would enable all the class to do the practical work at the same time.

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Another undesirable aspect of much of the practical work is its purposelessness. Weighing and measuring are essential features of practical arithmetic, but they are not the only features. Simple exercises in pure weighing or measuring are both allowable and desirable at the beginning of a course when it is necessary to make the children familiar with the instruments and the units ; but these exercises soon become extremely tedious. Make the weighing or the measuring, however, subservient to some end in which the child is interested, such as the solution of a problem or the construction of an object, and the boredom immediately disappears. The simple processes should in fact form part of a larger purpose within the grasp of the pupil. This subservience of processes to purpose seems to me to be vitally essential to all sound education.

The part of the arithmetic course that seems at present to stand in greatest need of diagrammatic illustration is the theory of fractions. Children of 12 years of age frequently fail to show by the division of a line or rectangle the validity of such simple relations as $\frac{2}{3} = \frac{4}{6}$ and $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Successful attempts to deal diagrammatically with the division of fractions are extremely rare.

The desirability of reference to ultimate sensible data is still more apparent in dealing with decimal fractions. Whether it is because decimals are introduced rather late in the course, or whether it is that the subject presents inherent difficulties to the learner, the fact remains that the pupils seem to feel much more at home with vulgar fractions than with decimal fractions. They cannot think in decimals: they have to convert them into vulgar fractions before they can grasp their full significance. I have frequently tested the children in the top class orally with the following question: Which is the larger, $\cdot 7$ or $\cdot 72$? The answer I almost invariably get is $\cdot 7$; and the reason I get is that the first is tenths, and the second is hundredths, and tenths are larger than hundredths. The notion that $\cdot 72$ is seven tenths, plus a little bit more, does not seem to occur to them. There are, however, exceptions. In a few schools I have had questions of this kind answered promptly and correctly; and in all these cases it turns out that the pupils have had much practice in measuring with decimal scales. In the written exercises children frequently convert decimal fractions into vulgar, even when it would be much more convenient to deal with them in their original form; but they never seem to reverse the process. Even if the school is well provided with rulers divided decimally, the children should themselves construct such rulers of paper or cardboard and use them frequently.

A movement which has for its aim a closer correlation between mathematics and manual training is rapidly spreading. Conferences take place between the instructor at the woodwork centre and the head teachers who send boys to that centre.

They draw up a scheme which will benefit both parties. The woodwork instructor prepares models for use in the practical arithmetic lesson in the school, and the mathematical course is so devised as to facilitate and render more intelligible the instruction given at the woodwork centre.

The tendency towards practical measurement has promoted the development of the mathematical aspect of art, science, and geography. The official science syllabus issued by the London County Council as a suggested scheme of practical work consists mainly of exercises in measuring and weighing. It is in fact a course of practical and applied mathematics.

In a few schools the map studied during the geography lesson is made the basis of calculations respecting distances, areas, and the time and cost of travelling from one town to another.

In a large number of schools graphs representing the daily changes in temperature and barometric pressure are kept up to date by monitors and hung upon the walls of nearly every class-room. Everywhere, in fact, we find traces of the invasion of practical arithmetic.

Theoretical side of the Instruction in Arithmetic.

Some weaknesses in the more purely theoretical side of the instruction call for comment. My experience leads me to the conviction that the theory of proportion is very inefficiently taught. The symbols between the terms remain cryptic, the abstract nature of a ratio is not grasped, and the notion of equality is not utilised. A large number of children tested could not build up a complete proportion. There would, I think, be a considerable gain in clearness if the older symbols were discarded, and the proportion set forth in the form $a \div b = c \div d$; or, better still, $\frac{a}{b} = \frac{c}{d}$. To find a missing term would then be to solve a simple equation. Fractions and the simple equation known, the principles of proportion become an easy inference. Attempts should be made to connect the notion of ratio and proportion with similar geometrical figures.

It has been customary to postpone the study of averages till the last year of the school course (*see* Appendix I.). The nature of the modern mathematical programme renders this postponement undesirable. There is nothing intrinsically difficult in the notion of average; children in the lower part of the school can understand it; and it is extremely useful in lending greater variety and accuracy to the problems involving measurement. Where exact measurement is desirable, it is always better to take several measurements and strike an average. In finding the length of one's stride or the thickness of a piece of paper, the advantages of measuring several at once and finding the average size of one will be apparent even to pupils in the lower classes.

There are two distinct ideals which make a claim upon the teacher of Arithmetic. The one is speed and accuracy in

calculation, the other is intelligence in the solution of problems. The ideals are not by any means incompatible. It is possible to secure both, but they compete for possession of the limited time at the disposal of the teacher. To adjust the claims of these ideals has always been a difficult problem in the school; and I think we have come nearer than ever before to striking a reasonable balance between them. There is, and I presume always will be, a considerable difference between school and school in the relative importance attached to these two aims. The general tendency during the last two years has been to increase the intelligent work at the expense, if need be, of the mechanical accuracy. The children must understand what they are doing, even if they fumble at it. There are a few teachers who still think that "rules" must be learnt first and applied afterwards; that it is no use, for example, to set problems involving long division until the children can work quite difficult examples with facility; that, in fact, the greater part of the year should be spent in working abstract examples and the tail end devoted to problems. The one thing to be said about excessive drill is that it deadens interest; and to say this, is to condemn it root and branch. Even when the purport of the exercises is understood, they are to most children very tedious. When this purport is not understood they become the veriest drudgery. Facility should first be fostered by practice in the solution of simple problems. When the meaning and application of the various operations are thoroughly understood, there is no harm done by giving practice in these operations to secure rapidity and accuracy. But there are limits. It is possible to make too great a sacrifice in securing this result.

There is a prevailing opinion that the London elementary school children of to-day are slower and less accurate in computation than they were ten years ago.* I have searched for evidence in support of this contention, but have failed to find it. I am, therefore, inclined to relegate the belief to that group of opinions which have reference to the annual deterioration of Academy pictures, the increasing degeneracy of each new generation of men, and other palpable fictions. But even if there has been a slight loss of accuracy, there has been a great gain in intelligence; and intelligence is an equipment incomparably more valuable than facility in calculation.

Progress has taken place, too, in the ability to explain the various operations and rules in the understanding of the theory of pure number. It has been contended that mechanical processes should not be used until the reason for the processes has been thoroughly understood. This is a counsel of perfection. Although such a standard has probably never been completely attained in any school, it will readily be conceded that no process should be used until *some* meaning is attached

* There is, however, abundant evidence that the arithmetic is not so accurate as it was 20 years ago, when much more time was devoted to the subject.

to it. A child should, for instance, not be set to work a subtraction sum on paper unless he can state it in the form of a concrete problem and give a reasonable interpretation to the result. He must have some idea of what he is aiming at; but if the teacher waits until every child in the class understands the *rationale* of what used to be called "borrowing" in subtraction before the device is practised, progress will be indefinitely blocked. As a matter of fact some attempt is made in the schools to render the process intelligible; and the reasoning seems to be understood by the brighter children. The question of method in teaching subtraction is sometimes a source of contention between the head teachers of the Senior Departments and the head teacher of the Infants' Department; for it is manifestly desirable that the same method should be adopted throughout the school. The head teacher of the Infants' Department finds the method of decomposition the easier to explain; and the head teachers of the Senior Departments find the method of equal addition the easier for practical calculation.

While there is much excuse for neglecting to make clear the reason for all mechanical operations in the lowest classes, there is no excuse for it when dealing with the older children. As I have already pointed out, fractions and proportions are not, as a rule, well understood; neither are L.C.M. and H.C.F. rationally taught. The difficulty with L.C.M. and H.C.F. is that the more convenient methods are the most difficult to understand. The same is true of square and cube root. In many cases the teachers get over the difficulty by using factors as an intelligible process, and teaching the ordinary methods as frankly rule-of-thumb. Square root is usually required in the application of Euclid I. 47 to mensuration, and here an approximate result may be obtained by drawing to scale and measuring. This method was found largely used in one school. If lines or bits of string were used to illustrate the method of repeated division in finding the H.C.F. the reasoning would not, I think, be too difficult for a bright lad of 12. Much may be said, however, for entirely omitting this method from the elementary syllabus. Its usefulness is virtually limited to operations with vulgar fractions of large numerators and denominators. But these fractions have in actual life been superseded by decimals. To find the H.C.F. of big numbers is, in fact, waste of time.

It is lamentable to find, as one occasionally does, the top class in the school spending a whole year on a syllabus confined almost exclusively to percentages, averages, discount, and stocks. From the point of view neither of general nor of vocational training can such a programme be defended. The pupils waste their time (and very precious time, too) in applying the principle of proportion, which they do not understand, to commercial transactions which they will never carry out. Fortunately cases of this kind are rapidly disappearing, and there is

observable, particularly in some districts, a steadily growing tendency to make the last year a recapitulatory year. The whole arithmetic course is reviewed, and a large variety of problems are set bearing on the experiences of every-day life. This practice is wholly commendable. I would, however, suggest that another step be taken. The pupils should, during this year, be led to make a careful and critical examination of our system of notation and of all the mechanical processes that they have already used. Reasons but dimly apprehended before will be clearly apprehended now. It will, I contend, provide the same kind of training as a good course of algebra, and a far better kind of training than that afforded by the algebra that is usually taught.

Year after year has the Chief Examiner for the London County Council Junior County Scholarships reminded us that the arithmetic papers worked by the girls are considerably inferior to those worked by the boys. Is their inferiority due to a lower average of mathematical ability among girls, or is it due to defective teaching in girls' schools? With a view to finding an answer to this question, one of my colleagues carried out a few years ago some careful investigations. Two different kinds of tests were given in mixed schools where both sexes went through precisely the same training. One set of tests consisted entirely of mechanical examples, the other entirely of problems. The results went to show that the girls did the mechanical work better than the boys; but the boys did the problems very much better than the girls. On the whole, the boys were considerably ahead of the girls. Investigations I have made at the mixed schools in my own district have confirmed these results. There can be no doubt whatever that, generally speaking, girls have not the same natural aptitude for mathematical reasoning as boys. There can be little doubt, too, that this natural difference is accentuated by somewhat inferior mathematical instruction in girls' schools. I have come across brilliant exceptions, but it must be confessed that women teachers do not, as a rule, teach Mathematics well. The question arises, Should the mathematical training in a girls' school differ from that in a boys' school? I think we must agree that it should. In determining the ideal course for any given pupil, three factors at least have to be considered: natural aptitude, present environment, and future environment. In each of these conditions does a girl differ to some extent from a boy. This fact is obtaining recognition in the schools, for there is noticeable a tendency for the Mathematics in the two kinds of schools to fall wider and wider apart. As a concession to predilection and natural ability, less time is being spent in girls' schools over Mathematics and more over literary studies. Certain parts of the mathematical course which are of theoretical rather than of practical interest are generally omitted. On the whole, the terminal tests set to girls are distinctly easier than those set to boys, and they cover less

ground. This is, it seems to me, as it should be. To be encouraged, too, is the practice of giving more household arithmetic to girls. An interesting course of domestic arithmetic is adopted in the top class of the N girls' school. Each pupil keeps a weekly account-book for an imaginary family chosen by herself and recorded on the first page. This family has a regular weekly income, also chosen by the pupil, but not allowed to differ widely from 30s. Accounts are kept week by week, and looked over and criticised by the teacher. General accounts, detailed accounts, and banking accounts are separately treated. Children are encouraged whenever they have a fair balance to put a little money in the bank. Occasionally they have to deal with sickness and unemployment in the family, and have to adjust their accounts accordingly. The girls seem to take much interest in this kind of work, and it must be admitted that they are more profitably employed in contemplating the putting by of a little money in the savings bank for a rainy day than in contemplating the transfer of a lordly sum of money from consols to railway debentures.

SUITABILITY OF THE MATHEMATICAL SYLLABUS.

This brings me to the question of technical as distinct from liberal training. To what extent is the Mathematics syllabus modified to suit the future occupations of the pupils? There is of course a sense in which all education is special rather than general. The pupil is trained for a particular environment. The London child is taught English and is not taught Japanese because he will probably live in England and not in Japan.

It is tolerably certain that the majority of the girls in the school will at some time or other have something to do with housekeeping. They are all accordingly given training in domestic subjects. They attend cooking, laundry, and housewifery centres. Their arithmetic has not in the past been correlated with the work done at these centres; and what little correlation there is at present should be greatly extended.

With the boys the case is not so simple. There is no general occupation which the bulk of the boys will follow when they leave school. Is any deliberate attempt made in the schools to prepare the pupils for specific callings? As far as the ordinary school is concerned, the answer is no.

Confining our attention to the Mathematics we can find nothing in the syllabus that is intended to be technical. The arithmetic done is not better suited for a future clerk than it is for a future engineer—at least it is not designedly so. The scale drawing done at the woodwork centre is not primarily intended to serve as a training for craftsmanship, but as a means of general culture. All the boys, whatever they are destined to be when they leave school, go through precisely the same course.

In the Central Schools, however, the case is somewhat different. These schools demand a word of explanation. The Council is carrying out a policy of rendering more effective the old system of Higher Grade Schools, inaugurated by the late School Board for London. The full scheme comprises the establishment of 55 Central Elementary Schools, distributed equitably over the whole of the London area. These schools are to be fed entirely by children of 11 years of age, transferred from neighbouring elementary schools, and a four years' course of instruction is to be provided. By April 1911 the number of schools already organised as Central Schools was 39.

Although not yet officially recognised as Central Schools, many of the old Higher Grade Schools temporarily serve the same purpose. All the most intelligent children who have failed to secure County Scholarships to Secondary Schools are, if their parents consent, drafted into these Central Schools. To these two characteristics (that they absorb the best brains of the ordinary schools, and that the children are supposed to remain till they are 15) must be added a third, viz., that each school is definitely given a technical bias.

This bias is of an extremely broad nature, and may be of two kinds, somewhat vaguely termed industrial and commercial. Any school consisting entirely of girls is given in addition a domestic bias.

Of the 55 Central Schools 12 are to have a commercial, 16 an industrial, and 27 both a commercial and an industrial bias. Nearly half the schools, in fact, will have two more or less distinct courses of instruction.

Let us now consider the question—how far will these Central Schools provide mathematical instruction suitable for special trades or occupations?

At present it is only possible to report what is actually being done in the schools already working on the lines prescribed, and to venture on a few criticisms and comments. It may be said at once that no preparation is made for any one specific trade. The instruction is kept on lines so broad that nothing is really lost, even though at the end of the time the pupil changes his intentions. For the course has been devised so as to possess cultural as well as utilitarian value. The precise nature of the technical bias is gradually being defined by experience. At present the schools are cautiously feeling their way. The syllabuses are tentative and differ from school to school.

I have been trying to discover what parts are regarded in these schools as common to the commercial and industrial courses in Mathematics, and what parts are regarded as peculiar to each course. In the face of many differences of opinion I can merely give the more commonly accepted view. The common programme, which is considerably heavier than that of the ordinary school, comprises arithmetic, algebra, drawing

to scale, mensuration, and experimental geometry. The arithmetic includes the use of graphs, logarithms, and contracted methods of multiplication and division of decimal quantities; and the algebra includes quadratics, square and cube roots, surds, variations, and progressions. Peculiar to the commercial syllabus are long and cross tots, foreign exchange, bankruptcy, brokerage, commission, and other commercial transactions; and peculiar to the industrial syllabus are simple trigonometry, graphic algebra, kinematics, and the use of such instruments as the slide rule, vernier, micrometer, and theodolite. Deductive geometry occupies an anomalous position. It has been proposed for both sides, for neither side, for the industrial side only, and for the commercial side only.

The above sifting of material is, however, somewhat misleading, for the common factor is not so great as the above classification would suggest. Many of the items of the common syllabus receive different treatment from the different sides. Graphic methods, for instance, are used by the commercial pupils to present at a glance the rise and fall of prices and quantities, and to indicate various tendencies in trade. The work, in fact, resolves itself into the plotting of statistics where the fluctuations do not conform to any discoverable law; the industrial pupils, on the other hand, have to go through a much more severe training in graphs, for they have to discover the laws between observed quantities, and to carry co-ordinate geometry much further than the commercial pupils. The theory of geometrical progression is applied on the one side to machinery, and on the other to compound interest. Where the slide rule is used by both sides it is used for far different purposes.

In the "M" Central School a special feature is made in the industrial classes of carrying working drawings through a complete series of processes. The boy begins with the object. He first draws in freehand the elevation, plan and (if necessary) the section; measuring the object and recording the dimensions. This enables him to make an exact scale-drawing, colouring the various parts in accordance with the builders' scheme. A tracing is then made in Indian ink, and finally a blue print is taken from the tracing. Very complicated work of this kind is done with great neatness and accuracy.

It would seem at first sight as though a very strong claim could be made out for book-keeping as an essential part of the commercial syllabus; but the Council at first disallowed it, probably actuated by the consideration that there is no one form of bookkeeping adopted by all the London business firms, not even by a large number of them. For although the double entry system is almost universal, each firm has its own special adaptation of the system. After a lad has entered an office, a few years, as a rule, elapse before he is allowed to have anything to do with accountancy, and in the meantime he can

get at an evening school all the special instruction he requires. These circumstances combine to render the teaching of book-keeping in the elementary schools a matter of doubtful expediency. The Council are now, however, reconsidering their decision.

As distinct from what is consciously aimed at as a preparation for livelihood, it may be pointed out that much of what is as a matter of fact done even in the ordinary schools is manifestly of vocational value. When the head of a business firm is asked what quality he most prizes in a beginner, he invariably answers "Accuracy." The beginner should, he thinks, be able rapidly and accurately to make out invoices and add up a column of figures. Beyond this, an acquaintance with foreign weights, measures and currency should also, he thinks, be demanded of him. These modest requirements are satisfied to some extent at least by the ordinary elementary school; but a meagre equipment of this kind will not carry a boy far. He is likely to remain among the lower ranks of clerks unless he goes through some such training as is provided at a Central School.

The practice in measurements and in solid geometry given in all schools in connection with the woodwork centres will indisputably prove of great service to a boy who enters any branch of the building or engineering trade. Of greater service still will prove the more advanced practical mathematics done at the Central Schools. These schools will probably do much to promote intelligent craftsmanship, and to remedy some of the evils incident to the decay of the apprenticeship system.

The specialisation in domestic science at the Central Schools for Girls carries a little further the training given in the ordinary schools. Arithmetic is brought to bear on home-making and scale drawing is applied to dressmaking. It must not be understood that the girls at the Central Schools have anything like so full a syllabus of mathematics as the boys. They do the same kind of work, but of a less advanced type.

The problem of providing technical education in the London Elementary Schools is complicated by the fact that there is no one skilled craft characteristic of London as a whole, or, if we may except Woolwich, of any London district. The citizens of London are, in fact, more concerned with the distribution than with the production of wealth. Out of the 50 boys who, during a certain period, left the "M" Central School, which has a dual bias, 41 easily found places in City offices, while the remaining 9 had much difficulty in getting into workshops. I have reason to think that this case is far from representing the relative demand for the two kinds of training, but it illustrates the fact that London is preponderantly commercial.

There is the further difficulty of discovering in reasonable time the proposed future occupation of the pupil. In the ordinary school the boy rarely knows, even on the day when he

leaves school, what he is about to do for a livelihood. Nor do his parents. They generally wait for something to turn up. A large proportion of the boys join the ranks of unskilled labour, and a large proportion of the girls enter factories or become domestic servants. In the Central Schools, however, the difficulty is not so great. The first year's course is strictly general, so that an opportunity may be afforded for ascertaining the natural aptitudes of the children. Then the parents are consulted and a decision arrived at as to the special course to be adopted.

Under the circumstances it seems clear that technical training in the elementary school is being carried as far as is at present expedient. It is very possible that in the future further developments may take place. The Central Schools may, perhaps, specialise a little more; the residual schools may be re-organised in groups so as to permit of massing the upper classes into one of the schools of the group and giving that school some sort of technical bias. But it is difficult to see how in any case the Mathematics will be vitally affected. Indeed, I have myself serious doubts of the educational soundness of devoting much time to tots and mechanical processes useful in an office. One thing may safely be said of the industrial mathematical course: it affords an excellent general training. Indeed, the headmaster of the "M" Central School is of opinion that the boy who has been through the industrial course will make more headway in an office than the commercial boy. He has gone through a more varied and more concrete mathematical training. He has more intellectual alertness, initiative, and resource. Personally, I agree with this headmaster as to the distinct superiority of the industrial course.

The business employer seems to me to attach far too much importance to a lad's actual skill—and that of a very limited kind—at the time of his entering the office, and far too little importance to his possibilities of development. Rapidity in calculation may be vastly improved by a few months' practice; a trained intelligence is the product of an education extending over many years. Unguided mechanical practice will secure the one; much intelligent guidance is necessary for the other. The employer, in fact, often shows a preference for the cheaper and more showy article.

The question of specialisation on the part of teachers is one which has received insufficient attention in the elementary school. Indeed, I know of no instance where Mathematics is taken by a teacher outside his own class. In the boys' schools this rigid adherence to the class system is, as a rule, less open to objection, but in the girls' schools, where it often happens that a few teachers are weak in Mathematics and others exceptionally strong, it is manifest that a certain amount of specialisation is particularly desirable. The obstacle to this plan is that the displaced teacher is inclined to regard the

arrangement as a reflection on his efficiency as a teacher. A teacher is rarely ashamed of being unable to teach music or art or physical exercises—and there is no reason why he should be—but he will never acknowledge that he cannot teach Mathematics. It is, however, not a question of efficiency or inefficiency, but of relative aptitude.

Occasionally there is a slight shifting of the scholars from class to class for the arithmetic lesson. A re-organisation of this kind is quite feasible if, as frequently happens, arithmetic is taken at the same time by all the classes in the school. A special classification for Mathematics is justified on the ground of the great variation in natural mathematical ability among the pupils, quite distinct from their ability in other directions. In School G, an easy-working cross classification is successfully achieved.

The amount of time devoted to the subject seems to average about $3\frac{1}{2}$ to 4 hours per week, rather more perhaps in boys' schools and less in girls'. The usual place in the time-table is immediately after the closing of the morning register, *i.e.*, from about 10 to 10.30 or 10.45.

METHODS OF TEACHING.

The methods of teaching are unquestionably in every direction improving. The most signal reform has been the adoption of more practical methods to which reference has already been made. This in itself has facilitated the apprehension of the *rationale* of mechanical processes and diminished the amount of explanation on the blackboard. The practice of allowing the children to make up problems of their own is one which has proved very fruitful in some schools and may well be extended to all schools. If a class is periodically tested, I see no reason why this test should not be devised occasionally by the children themselves. They might be told to prepare, either as homework or during school time, a test consisting of, say, six problems. The teacher would look over them, select the best test and set it to the class. Mary Jones's examination would possess for the class some points of interest which the teacher's test would lack. Moreover, the clear formulation of a question goes a long way towards answering it. It at least guarantees an intelligent answer, if not an accurate one.

Experimental Systems.

It remains now to describe a few systems of a more or less experimental nature. In School G, an interesting concentric scheme is adopted. Even the lowest class solve problems as far as possible of the same nature as those solved by the higher class, but differing in difficulty and complexity. For the first three years the work is almost entirely mental—or more properly "oral" since all arithmetic is mental. At the end of the third

year the problems involve such numbers as necessitate the use of definite methods, and then, for the first time, the conventional "rules" are introduced. The boys are encouraged to attack the problems by their own methods and no method is rejected so long as it is clear and logical. The time allotted to the subject in the lower part of the school is much below the normal. The lessons, in fact, only last 20 minutes. The time thus saved is devoted to reading. The scheme has excellent points. It unquestionably develops intelligence in dealing with numbers. And this is saying a great deal.

The mathematical occupations found in Standards I. and II. of School H are similar in character to the number lessons in Infants' School A (*see* p. 6). The objects constructed form the basis of simple observation in form and colour. The kind of work is gradually displaced in the higher classes by the more usual type of practical arithmetic. The scheme is too young yet to enable me to pass an opinion on its practical efficacy. It however promises well.

Paper folding and cutting is extensively used in all the classes of School K to illustrate fractions and geometrical principles. This work is good.

In School L a special number board invented by the headmaster is used by the children. This board, which is divided into squares, ten in each vertical column, is used to represent numbers up to 150* so that their relative magnitude may be immediately apparent to the eye. The common numerical operations are easily and clearly represented on the board, and the system has the merit of always ensuring that the symbols used have real meanings behind them.

Summarising briefly, I may state that there is every evidence of steady improvement in the teaching of Mathematics in London Elementary Schools. In the infants' schools the method of rigid and systematic analysis of small numbers is almost universally adopted. But this method is open to somewhat grave objections and is gradually giving way to methods which are more in accordance with the natural interests and activities of child life. In the senior schools the most significant changes are the introduction of practical arithmetic and the increased importance attached to ability to deal with common-sense problems. There is everywhere to be found less example-grinding and more thinking.

P. B. BALLARD.

* There is no special reason for stopping at this number beyond considerations of space. It is desirable that there should be more than 100 squares on the board in order that the hundreds column may be exemplified.

APPENDIX I.

BOARD OF EDUCATION.

Standards of Examination in Arithmetic.

SCHEME B.*

Standard I.

The four simple rules. Divisors and multipliers not exceeding 6. No number higher than 99 to be employed in the questions or required in the answers.

Standard II.

Compound rules (money). Divisors and multipliers not exceeding 12. Sums of money in the questions and answers not to exceed 10*l*.

Standard III.

Simple rules and compound rules (money). Divisors and multipliers not exceeding 99. No number higher than 99,999 to be employed in the question or required in the answer. Sums of money in the questions and answers not to exceed 99*l*.

Standard IV.

Compound rules applied to the following weights and measures (length, weight, capacity, time). In length, yards, feet, and inches; in weight, tons, cwts., qrs., lbs., ozs.; in capacity, gallons, quarts, pints; in time, days, hours, minutes, seconds—are the only terms that will be required in this, and in the fifth standard. Divisors and multipliers not to exceed 99.

Standard V.

Vulgar fractions (simple fractions only). Practice. Bills of parcels. Common weights and measures.

Standard VI.

Decimal fractions (excluding recurring decimals). Simple proportion or single rule of three by the method of unity. Calculation of simple interest upon a given principal. Common weights and measures. Mensuration of rectangles and rectangular solids; the extraction of square and cube roots is not required. (Boys only.)

Standard VII.

Vulgar and decimal fractions. Averages and percentages. Investments of savings. Consols.

* This scheme, which appeared annually in the Code from 1894 to 1906, is no longer used in the examinations for Certificates of Proficiency, but has been replaced by the Syllabus of Examination in Arithmetic, which is to be found in Schedule VI. of the present Code.

APPENDIX II.

TEST FOR STANDARD I., INFANTS' DEPARTMENT.

43	94	28	5) 75
9	36	4	—
14	—	—	—
38	—	—	—
—	—	—	—

APPENDIX III.

TEST FOR STANDARD I., INFANTS' DEPARTMENT.

1. How many times can 6 be taken from 96?
2. A man catches 70 fishes, and puts them in a basket, 17 get out. How many are left?
3. We come to school 5 times a week. How many days do we come in 19 weeks?
4. In a class there are 38 girls and 39 boys present, and 16 are absent. How many altogether in the class?

APPENDIX IV.

SCHOOL D (GIRLS'). TERMINAL TEST SET TO STANDARD IV., 1904.

1. 9 lbs. 9 oz. 14 dwts. 21 grains \times 23.
2. Reduce 95,040 inches to furlongs.
3. How many acres in 11,495 square yards?
4. Divide 313 mls. 5 fur. 8 pls. 4 yds. 1 ft. 11 ins. by 19.

TERMINAL TESTS SET TO THE SAME STANDARD IN 1910.

Theoretical.

1. A milkman started his round with 15 gallons of milk. He called at 92 houses and at each left 1 pint. How much did he have left, and how much money did he receive if each pint cost 2d.?
2. I spent $\frac{1}{2}$ of my money and placed $\frac{1}{3}$ in the bank. What fraction did I have left?
3. I owe 2s. 6d. to a number of people. My total debt is 13l. 17s. 6d. How many people were there?
4. What do I pay for 37 bags of flour marked 16s. 8d. each, if I am allowed 10 $\frac{1}{2}$ d. off the marked price?
5. Arrange three classes of 45, 63, and 81 children respectively in one large room. Make the lines of children of the same length, but as long as possible without mixing the classes. Of what number of children will each line consist?

Practical.

1. Provide a dinner on paper for 8 people. You may spend any amount up to 3s. 8d.
2. Measure your paper and tell me how many times it will go into the length of the board.
3. What is the height of the girl sitting next to you? Who is the taller, your partner or yourself? By how much?
4. Weigh any suitable and available article as accurately as the weights provided will allow. Write name of article and weight upon your paper.
5. How many times does the serviette ring turn in rolling across the table? Find out in any way you please.
6. Count the girls in the class, and tell me how many are away, and who those girls are. How many times could each girl have attended school this week? If we were cent. per cent., how many attendances would have been made altogether?

APPENDIX V.

SCHOOL E (BOYS'). TERMINAL TEST SET TO THE HIGHEST CLASS IN 1904.

1. How much money was realised by selling out 1,500l. of Railway 4 per cent. stock at 87?
2. Express in £ s. d. the value of .2989375 of 100l.
3. Find the amount of 87l. 16s. 3d. for 2 years at $3\frac{1}{2}$ per cent. (simple interest)
4. A boy wrote .61 instead of .61; express in decimals the amount of the error.
5. Find the value of $\frac{1}{2}$ of $1/9\frac{1}{2}$ of 1l. 4s. 0d. + .0375 of 15s. and express the result as the decimal of 5l.

TERMINAL TEST SET TO THE HIGHEST CLASS IN 1910.

Practical.

1. Describe a circle which shall have as nearly as possible 12 square inches of area.
2. Find as nearly as you can the area of the irregular field on a scale of 1 cm. = 20 yards.
[This was a figure of irregular outline, drawn on paper ruled finely in millimetre squares.]
3. A cistern can be emptied by a tap in 3 hours and filled by another in 5 hours. If the cistern be empty and the filling tap be allowed to run and after 2 hours the emptying tap be set going, how much of the cistern will be filled at the end of 4 hours from the time of setting the filling tap running?
4. If 3 horses and 5 cows cost 190l., and 2 horses and 1 cow cost 80l., find the cost of a horse and also of a cow.
5. Find a line which shall be $\sqrt{125}$ cm. long.
6. Draw a graph to show the path of a body moving as follows:—

Time taken in minutes	-	-	2	4	5	8	9	10
Distance travelled in yards	-	-	150	250	325	525	650	700
7. A and B run a race of 500 yards. A starts from scratch and finishes the race in 4 minutes. B has a start of 50 yards but commences $\frac{1}{2}$ a minute later at the rate of 120 yards a minute. Which shall win?
8. Two steamers lie E. and W. of a certain point 5 miles apart. The westerly one sails in a N.E. direction at 10 miles an hour, the easterly one

sails due N. at 8 miles an hour. Draw a graph showing where they meet and at what time.

9. Plot the points (6, 8) (6, -8) (-2, 2) (-2, -1). Join the 4 points and find the number of units of area in the quadrilateral thus formed.

Theoretical.

1. A walks at 4 miles an hour and starts at 10 a.m. B starting from same place at 10.45 and going in same direction, walks at 5 miles an hour. How far will B have walked when he catches A?

2. Find the number of degrees between the hands of a clock at 20 minutes past 6. How long after this will the minute hand cross the hour hand?

3. Two trains start from the same point and run on parallel rails in the same direction. One travels at 35 miles an hour and the other at 30 miles an hour. After what time will the faster train be $4\frac{1}{2}$ miles in front of the other and what distance will the faster train have travelled?

4. One tap lets 50 gallons per hour into a cistern while 10 gallons leak out every hour. If the cistern holds 1,000 gallons when full and the tap is set running when the cistern is empty, how much water will have run into the cistern by the time it is full?

5. Explain what is meant by stock at par, stock at a discount, 5% stock at 120. How much shall I have to pay for 2,500l. stock at par and for 1,200l. stock at $\frac{1}{2}$ discount?

6. A cylinder 3" radius and 10" deep is filled with water. A ball of iron diameter 3" is gently dropped in. How much water will overflow and how much will be left in the vessel?

7. Find the cost of gravel at 9s. a cubic yard required to cover to a depth of 3 inches a path 5 feet broad round a circular pond of diameter 50 feet.

8. A map is made on a scale of $\frac{1}{4}$ inch to a mile. How many acres on a map will be covered by a penny piece; diameter $1\frac{1}{2}$ inches?

9. There are 3 casks of wine. One contains 5% water, the second $7\frac{1}{2}$ % water and the third $12\frac{1}{2}$ % water. One gallon of the first is mixed with 2 gallons of the second and 4 gallons of the third. Find the percentage of water in the mixture.

10. Find the value of the following if x stands for 5s. and y for 2s. 6d.

$$\left(\frac{3x + 2s. 6d.}{7} + \frac{5y - 2x + s.}{30 + y} \right)$$

APPENDIX VI.

SCHOOL F (GIRLS'). TERMINAL TEST SET TO THE HIGHEST CLASS IN 1910.

NOTE:—Ten sums are to be done out of 12, but those marked with a * must be done.

1. Find the value of $xy + ys$ when $x = 3\frac{1}{2}$, $y = 4\frac{1}{2}$, $s = 5\frac{1}{2}$.

2. Express 5l. 7s. 3 $\frac{1}{2}$ d. as decimal of 1l. Divide it by 12 and verify by division in £ s. d.

3. The area of a triangle is 50 square cm. and the base is 85 mm. Find the height.

4. A boy obtains 165 marks out of a maximum of 185. Bring this to a percentage.

* 5. Draw an equilateral triangle ABC. Side 3 inches. Bisect each side in P, Q, and R respectively. Join P, Q, R and find area of triangle P, Q, R.

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6. Express the difference between 1*l.* 2*s.* 6*d.* and 1*l.* 6*s.* 8*d.* as a percentage of 1*l.* 2*s.* 6*d.*

7. Find the contents in gallons of a tank 3 feet long, 2 feet 8 inches wide and 1 foot 9 inches high.

8. Add $18\frac{1}{2}\%$ to 24*l.* 12*s.* 6*d.*

9. A man buys goods for 5*l.* 10*s.* 6*d.* and sells at a profit of $27\frac{1}{2}\%$. Find the selling price.

10. Draw 3 straight lines, length 5 cm., 7 cm. and 4 cm. respectively. Draw a fourth line whose length is to 4 cm. as 7 cm. is to 5 cm.

11. If 10 eggs cost 1*s.*, how many can be bought for 3*s.* 6*d.*? Work (a) graphically on squared paper and (b) arithmetically.

* 12. Plot the points A (10.10), B (20.20), C (30.10), D (20.0). Join the points. This figure represents the plan of a cube. Find the volume.

Illustrate wherever possible.

THE TEACHING OF ELEMENTARY MATHEMATICS IN ENGLISH PUBLIC ELEMENTARY SCHOOLS.*

The object of this memorandum is to give a brief historical sketch of recent changes in the teaching of Elementary Mathematics in an English public elementary school and to indicate (a) what, in the opinion of an elementary school teacher in daily contact with the realities of the schoolroom, are the main difficulties and the most serious defects in the present-day teaching of this subject, (b) some principles and methods by which Elementary Mathematics may be made of increased value even under present conditions, with specimen syllabuses for Infants', Senior, and Higher Elementary or Central Schools, and (c) to make some suggestions of a practical and definite character for applying the principles advocated to the mathematical work of the Higher Elementary or Central Schools.

The school period of the English worker's child may be said to extend from 4 to 14, or possibly 15 years of age. The first four years of this period are spent in the preparatory or infant school, from which the child passes at the age of $7\frac{1}{2}$ to 8 years to the upper or senior department, where he remains until the leaving age of 14 to 15 is reached. In a fair number of cases, however, especially in town schools, it will happen that the scholar, on account of more than average ability, finds his way, at the age of 11 or 12, to an upper standard or to a secondary school. In this review the secondary school has no place and will not receive further notice, but the Higher Elementary or Central Schools are now so numerous, and play so important a part in our elementary school system that it is impossible to omit them from any survey of English elementary education.

Great changes amounting almost to revolution, have taken place in the aims and methods of our schools during the last decade, and particularly in mathematical teaching. The old practice of labouring mechanically through a number of complicated arithmetical processes, imperfectly understood, is gradually giving place to objective teaching, visualisation, and the actual individual handling of objects; to problems within the range of the child's intellect and experience; and to progressive and systematic training on the purely intellectual side. But this development is by no means complete. So rapidly have changes come that the older teacher, the man of 20, 30, 40 years' experience, has had difficulty in adapting himself to new demands, and still feels the hampering effects of the old conditions and methods.

* The acknowledgments of the writer are due to Mr. John Scott, of the Ashburnham School, Fulham, and to the staff of the Bloomfield Road School for help in the preparation of this paper.

Space does not permit of any extensive survey of these conditions obtaining, say, prior to 1900, but some short review is necessary for our purpose. Suffice it to say that schedules defining in the baldest manner the work to be done in Arithmetic were issued yearly by the Education Department as instructions to teachers, that formal examinations were held annually covering these schedules, and that money grants were made conditional upon the results obtained—the famous system of “payment by results.” Each year’s work was compartmentally distinct. Previous work was rarely revised, the pupil spending his year upon the exact portion of work allotted to that year.

The character of the old schedules will be made clear by one or two extracts from them, *e.g.*—

Standard III. (age 9–10).—Notation and numeration to 1,000,000; long division; compound addition and subtraction.

Standard IV. (age 10–11).—Compound rules and reduction of money; common weights and measures.

Two criticisms of the time are quoted from Government Blue Books* to illuminate the results achieved:—

“The progress made in Arithmetic is out of proportion small in comparison with the extraordinary labour bestowed upon it.”

And—

“I come across children [of 11, 12, 13] who can add and subtract intricate vulgar fractions, but are incapable of writing down from dictation a sum in simple addition, which a class of well-taught infants would do easily. In a class of 58 children in Standard V. only 11 could encounter with success the difficulties of a sum in simple addition.”

A change was obviously inevitable, and a schedule known as “Scheme B.” was issued in 1894. This, though in some respects defective, was a notable advance upon the older schedules involving the manipulation of large numbers, and, as it still forms with various modifications the basis of work in many schools, it is given in detail in Appendix I. to Mr. Ballard’s paper in this series (*see* p. 26).

The change begun in 1894 was completed a few years later by the withdrawal of all schedules of work, and the abolition of annual examinations for payment of money grant—a step which almost immediately began to give a new complexion to our mathematical work and methods, in that each school had to make its own syllabus. The change has, up to the present, perhaps, been more one of aim than of practice. Schemes of work in different schools are found to present wide variations both in matter and treatment, and many faults incidental to an experimental stage have naturally arisen, so that, notwithstanding

* Reports of the Committee of Council on Education, 1878–79 (p. 522), and 1879–80 (p. 396).

great improvement during the last ten years, arithmetic in most of our schools is still laboriously rather than well taught.

There is ample proof of this, but, if no other evidence were available, the reports of the Examiners for Junior Scholarships in London, would suffice to show that in the metropolis at any rate the teaching of arithmetic is still far from satisfactory. In the Junior Scholarship Examination over 20,000 boys and girls are examined each year, so that the data for criticism are fairly ample. Of these, reckoning the boys only, some 54 per cent. in 1906, 54 per cent. in 1907, 79 per cent. in 1908, and 57 per cent. in 1909 obtained less than half marks. The girls were not so successful, and making every allowance for the fact that the examination is competitive, and designed to discover the best candidates, these results cannot be accepted as in any real degree satisfactory. After a fair consideration of the questions set and the Examiners' reports, the unavoidable conclusion is that in a large number of London schools either arithmetic is taught under very adverse conditions or is not taught on sound lines. Probably both these alternatives are true to some extent, and thus we shall be led later to consider some difficulties of school administration, and some serious defects in the treatment of mathematical subjects in the classroom.

The further discussion of these two questions is postponed until the writer has indicated what in his opinion is possible even under present-day conditions, if only better syllabuses and methods were adopted.

In considering present possibilities we may take the Mathematics of the primary school to include :—

Arithmetic as ordinarily understood, practical arithmetic, simple geometry treated experimentally with some constructive work, and in the higher schools perhaps a little deductive work.

Simple mensuration.

Algebra as generalised arithmetic and in its simplest forms, leading up to the use of the simple equation in the solution of arithmetical problems.

In setting out to consider what should be the purpose or aim of Mathematics as a primary school subject, one is confronted with a very real danger, that of dwelling upon theoretical or paper possibilities. The practical teacher must by every means in his power endeavour to avoid this and to seek out and fix the actual living possibilities of his own classroom. Perhaps the predominant feature of modern educational development is the entry into school life of the expert adviser, and his assumption that his own subject has peculiar and special claims to consideration. It becomes particularly important therefore that the teacher should preserve a true sense of proportion, of essentials and non-essentials, an appreciation of what is possible

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in the average school to the average teacher and to the average child. He must remember that Mathematics is one subject out of probably ten or twelve, and under ordinary conditions will command no more than perhaps from three to six hours out of a working week of some 25 hours.

With these actualities in mind he has to determine what value and aim Mathematics has in the primary school. It may be laid down :—

- (i) That young children shall obtain, hold, and use intelligently the fundamental ideas and processes of number. This is the essential work of the Infants' Department.
- (ii) That the child during his school period shall acquire sufficient rapidity and accuracy of calculation to serve the ordinary purposes of daily life.
- (iii) That he shall be able to apply the principles of his mathematical work to these purposes.
- (iv) That he shall obtain such an intelligent understanding and use of number manipulations and the quantitative aspect of things as shall fit him to learn and understand any special business or industrial methods requiring number that may fall to his lot. (It does not from this by any means follow that the special methods of business or industry must be taught in the school.)
- (v) That the nature of the teaching in Elementary Mathematics should provide special intellectual training of a kind not so well obtained from other subjects. And here lies the real value of the primary school mathematics as mathematics, unique of its kind, and in its own field, demanding of the pupil thought, investigation, analysis, synthesis, comparison, reasoning, deduction, induction.

These five claims may be regarded as possible in all schools, and may therefore be considered the minimum. For schools fortunately circumstanced, well equipped and staffed, there may be others tending towards special preparation for various departments of life, commercial and industrial.

The realisation of these minimum values begins in the Infants' School, and it is in the highest degree important that the child shall get his conceptions of number and of the fundamental processes intelligently and in the true way—that is, through the concrete. The teacher must find out what kind, what variety, and how much concrete experience is necessary to enable the child to reach these elementary number concepts and operations. He must see that the child gets this experience, that he passes from the concrete to the abstract of his own accord as it were, and unconsciously, and that he is able to re-translate his symbols into concretes when required.

Coming now to the work of the senior department, it may be assumed that the child has obtained the fundamental ideas of simple numbers and of the operations of addition and subtraction, multiplication and division with numbers up to say 10 or 20. Our other four values already noted may be made to fall into two classes. We have :—

- (1) The utilitarian aspect and the practice of everyday life. This side is increasingly regarded by many authorities as important. It touches so many sides of life ; it appeals to the boy and to the man ; it means the power to "do" something ; it aims at rapidity, accuracy, skill, insight, and common sense.
- (2) Mathematics as a mental training, dealing with the principles underlying the numerical processes, an exercise in investigation and deduction, a method of thinking, an attitude of mind.

Authorities differ as to which of these two aspects is primary and which only secondary. The writer holds that each is essential to the other, and one object of this memorandum is to show that a really valuable mental training may be obtained from an intelligent treatment of work of an essentially utilitarian kind, and that the practical teacher will be wise to adapt and limit the mathematical work of his school to the probable demands of ordinary work-a-day life, *i.e.*, to exclude everything in the nature of useless, absurd, or obsolete computation and to endeavour to get by suitable treatment of his subject the maximum amount of mental training from the work laid down.

Under the first, he will note that the demands made upon Mathematics in the ordinary man's life are after all very modest, and that by confining the arithmetic syllabus strictly to work bearing upon the ordinary matters of everyday life, much time may be gained for securing rapidity and accuracy of calculation and for making the mathematical work of his school the medium of a really valuable mental training. To this end, therefore, he will throw overboard all mathematical lumber in the shape of long and involved calculations, long reductions, useless or unfamiliar weights and measures, complicated vulgar and decimal fractions, formal and involved greatest common measures, and least common multiples, square roots, stocks, shares, brokerages, complex and little understood formulæ, mathematical tricks and dodges, clock sums, train sums and the like, and will concentrate upon the strictly utilitarian side of arithmetic.

We now come to consider utilitarian mathematics in its aspect as mental training of a special and valuable kind.

Regarding this mental training it is exceedingly important to have clear conceptions of its kind, value, and possibilities in ordinary schools under ordinary teachers. It is here that the

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We now come to consider utilitarian mathematics in its aspect as mental training of a special and valuable kind.

Regarding this mental training it is exceedingly important to have clear conceptions of its kind, value, and possibilities in ordinary schools under ordinary teachers. It is here that the

theorists and experts usually fail us by giving abstract and general schemes instead of concrete instances which we can use in the world of school experience.

We may regard Mathematics taught for mental training as an Elementary Science, and consider it as :—

A.—Classified knowledge.

B.—A method of thinking.

A. is science in the sense that it represents systematised knowledge, organised information to be utilised for work-day purposes. This classified knowledge may be held in two ways. The child may have it as acquired and remembered information or knowledge, or instead of burdening his memory with a multitude of details he may know where to find it when required for use. The readiest means of obtaining it is from books, and one of the functions of the primary school is to train children to use books as tools, to know where to find what is wanted and how to use it.

Mathematics dealt with as suggested in this paper ought to make the pupil in his last years able to use a mathematical text-book as a tool, to find there material useful and necessary, tables, definitions, formulæ, proofs, special rules and devices, sets of examples, &c.

We may, however, regard Mathematics in the higher sense B as a training in the art of thinking, of arriving at satisfactory and logical conclusions, of passing from mere opinion, impression, guess-work, to true knowledge.

This art can only be acquired by intelligent and conscious practice. The pupil himself must take part in the making of his real knowledge, the essential point being that the child shall do as much as possible, the teacher helping as little as possible. The teacher finds something in the child's experience to start with, prevents useless but not useful labour, makes the child decide between alternatives for himself, and obviates aimless wandering and floundering. The child must in short make the journey for himself, the teacher seeing that he makes it in the most effective way.

The method will thus be somewhat as follows :—

1. Place the problem before each child concretely as far as possible, using, say, cardboard coins, a given circle, a cylinder.
2. Insist on a clear understanding of (a) what we know or have, (b) what we are asked to find.
3. Let the child reflect on these and see if his knowledge or past experience will suggest anything.
4. Require the children to try their own suggestions on their concrete material.
5. Test, examine, or discuss the suggested solutions, select the most satisfactory, and verify.
6. Apply the method to other cases.

This is the method of science:—a problem—analysis of data—reflection thereon—formation of hypotheses—reflection on these and reference to previous knowledge or experience—test or verification of the likeliest—decision between alternatives—and so on, to wider circles and use in the world of experience.

In experimental geometry we constantly use such methods as:—

- (a) Something noted after experiment. Will it always be so? Must it always be so? Why?
- (b) Suppose I do so and so, what will be the result, and why?
- (c) A problem is given. Children suggest possible solutions. Try them and select the best.
- (d) State results of work already done. Reflect on these to see which way they point. Make simple deduction or general statement and verify.

The reasoning in such a problem as, "If 9 lbs. cost 27s., what will 4 lbs. cost?" is something as follows:—

"I can find the cost of 4 lbs. if I know the cost of 1 lb. I can find the cost of 1 lb. if I know the cost of 9 lbs. But I know the cost of 9 lbs. Hence I see my way to finding the cost of 4 lbs." Or generally, "I can prove or find this if I can prove or find that. I can prove or find that if I can prove a third thing. But I can prove this third thing, so I can prove the first."

This is a simple method of reasoning, exceedingly valuable in all departments of life. In these and other similar ways very much of the arithmetic and geometry of the elementary school may be so taught as to give valuable training in simple analysis and synthesis, in seizing upon essentials and excluding non-essentials, in simple investigation upon scientific lines, and in arriving at well founded conclusions.

Illustrative syllabuses for Infant and Senior Departments are here submitted.

ILLUSTRATIVE SYLLABUS IN ARITHMETIC.

INFANT DEPARTMENT.

Time given per day, 30 minutes.

Time given per week, 2½ hours.

GRADE I. (3–4½ years).

No number teaching except incidentally in connection with games, toys, &c.

GRADE II. (4½–5½ years).

Elementary idea of number up to 6. Taught by games, concretes, drawing, &c. Number pictures to 6 arranged in bricks, cubes, counters, shells, beans, beads, &c. Number pictures drawn in dominoes, squares, circles.

GRADE III (5½-6½ years).

Comparison of lines, surfaces, solids. Drawing of lines equal to, longer or shorter than line drawn by teacher.

Measurement using inch and foot. Exercises to teach inch and half-inch marks on ruler. To estimate these lengths. 6 in. rulers marked by children (paper) (1 in. square tablets used). Draw, mark, and cut from 1 in. squared paper, squares and oblongs of 1, 2, 3, 4, 5, 6 in. side.

Foot, half-foot.—Exercises to teach, 1 foot = 12 inches; half-foot = 6 inches; pound and half-pound; sticks of 1, 2, 3, 4, and 6 inches used to construct foot lengths. Tested by 12 in. ruler.

Pint and half-pint.—Exercises with actual measures and water.

Half and quarter.—Taught by paper folding and cutting.

Half.—Square folded into oblongs and also into triangles.

Quarter.—Square folded into squares and also into triangles; paper pies, apples, &c., cut into halves and quarters.

Number.—To count to and from 10 by ones, twos, &c. Analysis of numbers 1 to 10 into parts and into factors. Keeping shop. No formal addition or subtraction. Very simple exercises. Concrete examples only. To make number dominoes and pictures.

Apparatus.—Strips of cardboard cut into lengths of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 inches, one set kept by each child. Bone counters, square tablets, beads, sticks, Gift III., Gift IV., beans, shells, cardboard coins, measures, squared paper (1 in. squares). Four-inch paper squares.

STANDARD I.—*First Half-Year.*

1. Analysis of numbers 1 to 9 into parts (addition and subtraction).
2. Analysis of numbers into factors (multiplication and division).
3. Simple lessons on halves and thirds.
4. Introductory Geometry lessons dealing with points and lines to indicate distance. Measurement of length, estimation of length, drawing on squared paper of squares and oblongs.
5. The number 10, its parts, factors and multiples.
6. Rapid addition and subtraction.
7. Buying and selling, weighing and measuring to illustrate number and fraction work.

Apparatus for illustration.—Cubes, bricks, cardboard slips, and square counters, sticks of different lengths, cardboard coins, scales and weights, set of fluid measures, yard stick graduated in feet. Rulers, 12 in., marked in inches, halves, and quarters. Paper squares, unruled. Paper squares, 3 in. ruled on one side into nine 1 in. squares. Four-inch squares ruled into 16 squares. Drawing paper, ruled, 1 in. squares. Ruled drawing paper, ½ in. squares, 5 in. × 5 in., containing 100 squares.

Lessons on following general plan.—Mainly oral, the answers obtained mentally, afterwards written down under the teacher's guidance.

Step I.—Exercises with concrete objects as units.

Step II.—Drawing exercises to illustrate the same "number fact" as in Step I., and in which the 1-in. line or one of the squares on the squared paper is taken as the unit.

Step III.—Simple exercises without the actual objects, but in which the numbers are associated with the names of objects pictured mentally.

Step IV.—Exercises of similar type in which abstract numbers are used.

Second Half-Year.

1. Extension of number work to 100. Notation, parts of numbers, factors of numbers, odd and even numbers. Experimental work, drawing exercises, exercises and problems as before. Two numbers only taken in addition. Formal addition and subtraction, but no formal multiplication and division. Tables gradually built up.

2. Further lessons on halves, thirds, quarters, including the addition and subtraction of "like" fractions.

3. Further use of money, weights, and measures.

4. Geometry, the straight line as indicating direction.

5. Rapid addition and subtraction. Apparatus and method as in first term. Addition and subtraction taught on "the completion of the ten" system.

SENIOR DEPARTMENT.

STANDARD II.—First Half-Year.

The 100 group to 500.

Revision of multiplication tables and use in four rules.

Manipulation, analysis, and synthesis of numbers.

Addition of hundreds.

Multiplication of tens by units.

Subtraction of tens and easy hundreds.

Division of tens by units.

Money to 2*l.* limit.

Length, feet, inches, and half inches, to 5 ft. limit. } The four rules and reduction.

Use of half and quarter.

Measure and draw common objects.

Deal concretely with money and length.

Practical Geometry.—Measure and draw from common objects twice and half size.

Test straightness of lines and edges.

Place straight edges parallel.

Fold square and rectangle for equalities.

Fold to make right angle and to halve angles, guess perpendiculars.

Estimations of length, breadth, depth by the eye.

STANDARD II.—Second Half-Year.

The 10 and 100 groups to 900.

Multiplication table and use in four rules.

Manipulation of numbers.

Addition to 900.

Multiplication of hundreds by easy tens.

Subtraction of hundreds.

Division of hundreds by units.

Money and length. The four rules and reduction.

Use of half, quarter, three-quarters, third, and very easy oral addition, subtraction, multiplication and division.

Use of 2*s.* 6*d.*, 10*s.*, 2*s.*, 5*l.* note.

Use of straight-edge in foot and inches, half and quarter inch.

Use of tape measure and yard.

Folding for half, quarter, third, three-quarters.

Use of lbs. and half lbs

Practical Geometry.—Fold to make right angles and to divide angles. Measure with large compass angles as of opening door.

Fold square, rectangle, equilateral triangle for equalities.

Copy objects and figures, same, half and double size.

Fold paper to show varying angles.

Draw figures on squared paper.

Draw square and rectangle of given dimensions.

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STANDARD III.—*First Half-Year.*

The 10, 100, and 1,000 groups.
Manipulation of numbers. Factors.
Addition to thousands.
Subtraction of hundreds.
Multiplication of hundreds by units and tens by tens to 1,000 limit.
Division of hundreds by units. Easy thousands by units.
Money and length, yards, feet, inches.
Reductions, pounds to pence, yards to inches, and inversely.
Use of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{2}{3}$ and four rules orally.
Use of lbs., half lbs., quarter lbs., three-quarter lbs., ounce, 2s. 6d., 10s., 2s., 5l. note.
Use of folded paper and straight-edge in addition, subtraction, multiplication and division of above fractions—oral work only.
Measuring objects to $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ inch.
Draw figures to show above fractions.
Use of yard and tape measure.
Measure curves by string, paper, rolling.
Measure diameters of circular objects.
Practical Geometry.—As in previous class but more advanced.
Bisection, &c. by folding.
Rulers and set squares for right angles. Squares. Rectangles.
Measure opposite sides, diagonals.
Simple geometrical patterns and familiar objects on squared paper.

STANDARD III.—*Second Half-Year.*

The 10, 100, 1,000 and 10,000 groups.
Manipulations and factors.
Addition to four figures carrying to 10,000.
Subtraction, thousands.
Multiplication by easy hundreds.
Division by easy factors. }
Money. } Four rules and reduction. Multipli-
Length: yards, feet, inches. } cation and division by units and easy
Liquid: gallons, quarts, pints. } factors until principle is understood.
Weight: lbs., ounces. }
Easy questions on method of unity by first finding the amount for one mentally. Written examples later.
Use of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{5}$ to $\frac{4}{5}$. $\frac{1}{10}$ to $\frac{9}{10}$. Addition, subtraction, multiplication and division of these from rulers, with coins and by folding rectangle, oral only, and multiplication and division by integers only.
Practical Geometry.—Bisection, &c. by folding.
Folding square, rectangle, equilateral and isosceles triangles for equalities.
Geometrical patterns and familiar objects made up of straight lines with rulers and set-squares.
Simple scales and figures to scale 1 inch to foot.

STANDARD IV.—*First Half-Year.*

Notation and numeration to tens of thousands.
Factors. The four rules to four figures carrying to five.
Division by factors and easy long division.
Money.
Linear: yards, feet, inches, easy miles and yards. Distances from school of well-known buildings. Height of school and tall chimney. Height, length, breadth of classroom and hall.
Weight: Tons, cwt.; and lbs., oz., ton of coal, 10 sacks, stone as 14 and 8 lbs.

Liquid: gallons, quarts, pints. 9 and 18 gallon casks.

Four rules and reduction with small numbers. Division by factors and easy long multiplication.

Method of unity by first finding for one. Easy direct and indirect.

Use of tenths from rulers, squares, half-sovereigns and shillings. Four rules orally.

Decimal expression. Four rules with easy integers and tenths.

Common fractions with denominators 2, 3, 4, 8, 10, 12. Four rules. Multiplication and division, whole numbers only.

Practical Geometry.—As in previous class, more advanced.

Fold rhombus.

Bisect by folding sides of any triangle at right angles.

Bisect by folding angles of any triangle, and note results.

STANDARD IV.—*Second Half-Year.*

Four rules and reduction. Money. Linear. Weight. Liquid as in previous class occasionally, as revision only. Add chains, yards. Time: Months, weeks, days, and hours; minutes, seconds; years, days; years, weeks; calendar months.

Short methods of working multiplication of money.

Addition and subtraction of common fractions with easy and small denominators.

Multiplication and division by integers.

Use of tenths, hundredths. Decimal expression and four rules, working answer to two places.

Method of unity, direct and indirect, and using very easy money and weights and measures.

Areas of squares and rectangles. Distances round rectangles.

Idea of standard yard and lb.

Four rules with common and decimal fractions from divided square and rectangle and from rulers.

Diagrams to show $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16}$, &c.

Use of metre rulers. Centimetre.

1 metre = 39½ inches nearly. 1 inch = 2½ cm. nearly.

Estimate parts of $\frac{1}{6}$.

Use of graduated pint glass.

Gallon of water weighs 10 lbs.

Use of callipers and wedge for internal and external diameters.

Practical Geometry.—Superposition of triangles. Two sides and included angle. Three sides equal.

Use of compasses to carry distances, to make circles, and geometrical figures involving circles.

Folding circle to get 180°, 90°, 45°, 22½°.

Parallels with set-square and by eye. Non-parallels meeting. Parallelograms.

Scale drawing, 1 inch to foot and 6 inches to inch from actual measurement of objects.

STANDARD V.—*First Half-Year.*

Four rules with common fractions. Small numbers and very easy complexes to include easy fractions of concretes (money, linear, weight, liquid, time) and easy reductions.

Four rules with decimals; reduction to two decimal places in answer—to include as above.

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Short methods of working multiplications of money.

Method of unity. Direct and indirect to include the above weights and measures.

Mensuration. Squares, rectangles. Simple carpet problems by strips. Acre = 4,840 square yards = 10 square chains. Very easy volumes of rectangular solids.

L.C.M., G.C.M. (or H.C.F.) by inspection and mentally (not by rule).

Use of callipers and wedge—internal and external diameters.

Use of metre rule. cm., mm.

Use of burette.

Estimation of parts of tenths .05, .03 nearly.

Square feet in square inches. Square yard in square feet. square cm. in square mm.

Cubic feet in cubic inches.

Area of neighbouring park, &c., in acres.

Cubic foot of water weighs 1,000 ounces.

Simple areas on squared paper.

Difference between three square inches and three inches square.

Gram wts., 0.5, 0.2, 0.1, 0.05, 0.02, 0.01.

1 metre = 39½ in. 1 in. = 2½ cm., all nearly.

5 miles = 8 kilometres. 1 litre = 1¼ pts., 1 kilogram = 2.2 lbs. and use in problems.

Practical Geometry.—Plane surfaces (e.g. still water). Test with straight-edge. Intersection of planes shown by flat card and water.

Examination of surfaces of cone, cylinder, hemisphere, prism, pyramid.

Plumb line. Spirit level.

Bisection of lines and angles with compasses. Locus of equidistance from two points. Position of point equidistant from three points. Question when in straight line. Circle through three points.

Construction of equilateral and isosceles triangles and proofs by folding. Fold for bisectors at right angles of side, for bisectors of angles, for perpendiculars from angles on opposite sides.

In connection with the above, some simple Euclidean reasoning.

Angles of circle by folding.

For boys only.—Very easy plans and elevations from models.

STANDARD V.—*Second Half-Year.*

Common fractions. Reductions with small numbers, and to include fractions of concretes as money, linear, weight, liquid, time.

Decimals. Reduction to three places of decimals in answer. To include as above.

Method of unity. Direct and indirect with cancellation to include the above money, weights, and measures and simple very easy fractions.

Shortening processes, especially of multiplication of money.

Practice only as alternative method and when truly shorter. Simple and compound, but not heavy types.

Invoices and bills. Items and prices only those likely to be met with in actual shopping and commercial life. Examples worked mentally.

Receipting bills.

Show from ruler, $5 \div \frac{1}{4}$, $5 \div \frac{1}{2}$, $\frac{3}{4} \div \frac{1}{2}$.

Show graphically by rectangles:—

$$\begin{aligned} \frac{2}{3} &= \frac{1}{3} \text{ of } 3. \\ \frac{1}{2} \text{ of } \frac{1}{2} &= \frac{1}{4}. \\ \frac{2}{3} &= \frac{1}{3} \\ \frac{1}{2} &+ \frac{1}{4}. \\ 3 &\times \frac{1}{4}. \\ \cdot 1 \text{ of } \cdot 7 &= \cdot 07. \\ \frac{3}{4} &= \cdot 75. \end{aligned}$$

Rulers in twelfths to show :—

$$\frac{1}{2} \text{ of } \frac{1}{2} = \frac{1}{4} \text{ of } \frac{1}{2}. \quad \frac{1}{3} \text{ of } \frac{1}{2} = \frac{1}{6} \text{ of } \frac{1}{2} \text{ \&c.}$$

Use of graduated vessel in c.cm.

Gram weights, 0.5, 0.2, 0.1, 0.05, 0.02, 0.01.

Practical Geometry.—Angles. Bisection by folding and with compasses.

Bisectors of adjacent angles. Vertically opposite angles by rotation.

Circle to show 180° , 90° , 45° , $22\frac{1}{2}^\circ$, $67\frac{1}{2}^\circ$, $157\frac{1}{2}^\circ$, 270° , 315° , 330° . Make and use simple protractor 0° to 180° .

Circles inscribed in and circumscribed round triangles.

Perpendiculars from point in and outside line.

Parallels with set-squares and by construction—as rotating lines—as crossed by line and testing angles.

Parallelograms by construction.

Simple Euclidean reasoning where suitable.

For boys only.—Easy plans and elevations from models.

STANDARD VI.—*First Half-Year.*

Common and decimal fractions as in previous class. Common fractions as decimals, and idea and very simple use in addition and subtraction of recurring decimals and multiplication and division by units.

Method of unity, direct and indirect, to include easy common and decimal fractions.

Invoices and bills occasionally for revision.

Interest, meaning and use of terms. Easy examples.

Easy generalised arithmetic. Rules worked with letters.

Algebraical equation. Preliminary exercises to understanding the use of the equation in arithmetic. Very simple substitutions. Meaning and use of negatives. Principle of the equation—

$$\text{If } a = b. \quad a \pm c = b \pm c; \quad a \times c = b \times c.$$

Solution of equations.

Use of ruler in cm. and mm.

Volumes by use of cubic inch blocks, by use of graduated vessel by emptying or displacement.

Practical Geometry.—Locus of points equidistant from a given point.

Locus of points at given distance from line—at equal distance from two points, and find position of a point equidistant from three points.

Circles passing through two and three points.

Various proofs that three angles of a triangle equal 180° .

Possibility of triangles from any three angles or sides.

Relation of exterior to interior angles of a triangle.

Lines crossing equidistant parallels. Equality of intercepted portions.

Hence to divide a line into equal portions, and proportionally. Hence also simple graph uses.

Simple Euclidean reasoning where suitable.

For boys only.—Plans and elevations with simple sections.

STANDARD VI.—*Second Half-Year.*

Interest, involving simple common and decimal fractions, formulae and use.

Percentages. Easy problems.

Meaning of list price, cash price, net price, discount, wholesale, retail.

Method of unity. More difficult problems to include fractions and easy rates, taxes, dividends.

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Mensuration. Triangle, rhombus, rhomboid. Simple volumes.

Algebraical equation. Problems requiring use.

Generalised arithmetic.

Areas of triangles and irregular figures with squared paper.

Relation of circumference to diameter.

Graphs of class attendances. Inside and outside daily temperatures. Hourly temperature. Prices of things.

Graphical illustration of rule for area of triangle, rhombus, &c.

Plan of school and playground from actual measurements.

Practical Geometry.—Circles intersecting and touching, internally and externally, to find centre. Bisection of chords at right angles. Squares in and about. Angles of semicircle and of segment. Locus of apex of right angle whose sides always pass through two given points—ditto of 60° . Touching converging lines.

Triangles, question of equality, test by superposition. Three sides equal. Two sides and angle equal. Three angles equal. Two angles and side equal. Simple Euclidian reasoning where suitable.

For boys only.—Plans and elevations to simple scales.

STANDARD VII.—*First Half-Year.*

Method of unity to include rates and taxes, insurances, bankruptcies, dividends, easy work and time and cistern sums. Very easy "compound" questions.

Interest formulae and use.

Percentages—more difficult, and including common and decimal fractions.

Averages.

Proportional parts (easy).

Mensuration—carpet and paper problems; areas for painting, &c.; circle; volumes and surfaces of rectangular solids from actual measurement.

Use of algebraical equation. Generalised arithmetic.

Areas of fields from rectangles and triangles.

Volumes by cubic inch blocks and use of graduated vessels.

Graphs to show easy proportion, interest, average area of circle on squared paper from area of square on radius, from constituent triangles.

Practical Geometry.—Equality of triangles on same base and between same parallels.

Construction of triangles.

The ellipse with string.

Square on hypotenuse of right-angled triangle and squares on sides.

Ratio and proportion.

Consideration of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ &c., adding very nearly to unity—the more nearly, the more terms taken.

Graph of a curve always approaching a straight line, but never actually reaching it or coinciding with it.

Construction of geometrical figures and designs, and colour washing.

STANDARD VII.—*Second Half-Year.*

General revision.

Investment of savings :—Post Office, banks, purchase of land and houses, very simple idea of mortgage, annuities, insurance, cheques, accounts, receipts and expenditure, discount, bankers' and shop only, meaning of commission, commercial traveller, agent.

Very simple illustration of company formation—shareholders, capital, stock, shares, dividends, investments.

Very easy questions on stock bought, income, interest or investment, better investments.

Simple general measuring, including volumes of cylinder, cone, sphere.

Practical Geometry.—General revision, construction of geometrical figures and designs and colour washing.

SOME DIFFICULTIES OF SCHOOL ADMINISTRATION.

The difficulties and defects already outlined on page 33 may now be considered in greater detail.

(a) The large classes to be taught. This from the teacher's point of view is the most difficult problem in school management with which this generation has to deal. Improvement is slowly being effected, but there are still large classes of 50 and 60 pupils which make the teaching of arithmetic upon any intelligent and practical basis exceedingly difficult, and the actual handling of concrete material almost impossible. Where these adverse conditions exist any great improvement of method is hardly to be expected. Some subjects of the school curriculum may perhaps be passably taught to large classes of children, but Mathematics more than any other requires personal attention to minute detail and intimate intercourse between teacher and individual pupil. With classes of even 30 or 40 this ideal is difficult to attain, with larger classes it is out of the question and thus arithmetic in many schools remains little more than a mechanical class drill on paper.

(b) Another primary difficulty is the increasingly heavy curriculum. The work of an average elementary school in London covers not only arithmetic with its cognate subjects geometry and some algebra, but English (comprising reading, writing, composition, spelling, grammar and literature), history, geography, drawing, science, needlework for girls, manual training or domestic subjects, music and physical exercises. Arithmetic will command some 30 to 60 minutes per day, possibly from 2½ to 5 hours per week.

(c) Further the head teacher of the large town school has in these days too frequently ceased to be in any real degree a teacher. He has become an organising clerk dealing with multifarious duties formerly unknown in school life. The classroom sees him only at intervals. He organises chiefly on paper. Nor is his staff altogether exempt from the same burden, for the pressure of clerical work is being also felt in the classroom to a degree hitherto unknown. There is thus a grave danger of the larger school experiencing a lack of uniformity of method

and treatment, and of that progressive continuity so necessary in mathematical work. Some teachers have sought to avoid this and to give a general direction to their staff by adopting one of the numerous "Teachers' Books" in arithmetic published during the last ten years, books which present at length the method of approach and the treatment of the rules and examples given in the pupils' books. In other schools conferences of the staff are held at frequent intervals. In still other cases the head teacher issues what may be called "covering instructions" supplementary to his syllabus, amplifying and explaining it by his own detailed examples of treatment. Some able teachers are of opinion that in these days the teacher should not be regarded as an expert in all the subjects of his school curriculum, and that detailed schemes of work should be officially prepared for him. Detailed schemes and methods by teacher-mathematicians who have successfully worked them in average schools are undoubtedly of great value, but the writer of this memorandum holds it as eminently desirable that each head teacher should compile his own scheme of work, giving special consideration to his own local conditions, the class and ability of his pupils, their probable future, their school period, and the strength of the teaching staff. It is a matter of some surprise that a very large number of teachers, although encouraged to compile their own syllabuses of work, still utilise "Scheme B" of the Code previously referred to. That this scheme has its merits will readily be admitted, but it has also serious defects. It is self-contained, it co-ordinates with no other subjects, it indicates no practical work, and it seems distinctly weak at the top. It introduces vulgar fractions at the age of 11-12, and decimal fractions at 12-13, both too late. Its chief merit is that it deals with small numbers and quantities.

SOME DEFECTS IN CLASSROOM TEACHING.

(a) Neglect of concrete aids in teaching in arithmetic and geometry. There is an undoubtedly increasing effort among teachers to develop this side of school method, and in many schools sticks, cards, bricks, paper squares, strips, rulers, coins, scales, weights and measures, are in daily use. Unfortunately it cannot be affirmed that this effort is by any means general, and in too many cases the pupil is still found labouring among symbols which have no vital meaning for him. Many experienced teachers regard this disinclination to rely on the perceptive side as at present the most serious defect of our class teaching. The fundamental processes of addition, subtraction, multiplication, division and their use in the solution of simple problems are best demonstrated to children objectively and should first be performed by the child himself on concrete objects and afterwards transferred to the blackboard or to paper. Money rules, reduction, weights and measures, place value and decimals, vulgar fractions and ratio, can all be thoroughly mastered by a well constituted scheme of concrete teaching.

But even when the value of practical work and of what is known in the school as "measurements" as a necessary complement of ordinary class teaching is rightly recognised, its true place and limitations do not appear to be always clearly understood. An age of swift transitions like the present naturally lends itself to some unscientific experiment. "Science is measurement," and thus the conclusion has been easily reached, that "measurement is science," with the result that time is often wasted by the repetition of purely mechanical exercises masquerading under the guise of "science."

Measurement is excellent and necessary in that it leads the pupil to use his hands and eyes, and to acquire a certain useful dexterity; to obtain a correct idea of the common units; to be able to make a fair estimate of length, area and volume; to institute comparisons and to get a practical notion of approximations, but it has its limitations and is not "science" unless it produces the scientific attitude of mind or develops the reasoning faculty.

It has been well said that measurement may become "liturgical" only and be in no sense scientific.

The following examples are taken from a modern text-book for Standard VII., boys (age 14), and are given after decimals, vulgar fractions, ratio and proportion, percentages and interest have been studied:—

1. Make a square of 3 in. side. What is the area of the surface?
2. Draw an oblong $4\frac{1}{2}$ in. by $3\frac{1}{2}$ in. Divide it into squares, each with one-half inch side. How many such squares are there?
3. Find as near as you can in square inches the area of the back of your copy book, the surface of your slate, the surface of your desk.

Such exercises for boys of 14 are almost entirely mechanical, tending, if perpetuated, towards waste of time, and it is here suggested that exercises in measurement should not only have a practical and useful purpose, but should, where possible, demand some thought in the application of principles. Generally their value may be increased by linking them up with calculations leading to some definite and useful end—just as no one in actual life measures for the mere sake of measuring.

(b) The neglect of oral in favour of written work, especially in the lower grades of the Senior School. By the term "oral work" should be understood not only the teaching at the blackboard but the actual interchange of ideas between the pupils and teacher, their common effort towards the solution of the problem, and the attempts of the class in what is usually termed "Mental Arithmetic." The object of this work is to teach the process or rule, to familiarise the pupils with it, and to give them abundant practice in applying it, to revise previous

teaching, to clear up old difficulties which reappear, to secure accurate and rapid mechanical computation, to give practice in using common sense, and to keep a class more continuously busy. It is found that calculation on paper throughout the school is laborious and uncertain, and that there is much unnecessary marginal work which would not result if oral work occupied its legitimate place in the Arithmetic lesson. The question is causing some discussion among teachers, and as a consequence written work in certain schools is being deferred until the age of 8, 9, or even 10 is reached. Any experiment in this direction should be made with care, and probably the middle course will eventually be found the best, *i.e.*, let written work be done throughout from the earlier stages but (in the lower standards especially) let oral work, with use of concrete material where necessary and possible, form a part of almost every lesson, the children getting independent practice on paper at intervals.

(c) Another defect found probably in most schools is the inadequate attention given to the few fundamental principles and processes, and in its place a slavish adherence to rules worked mechanically and contributing but little to the intellectual development of the scholar. He is not sufficiently thrown upon his own resources, works too much by types, looking for the key-word suggesting the type, or saying, "I have not done one like this before," instead of dealing with it in the light of 'common sense and fundamental principles. Some of the "pupils' books" already referred to tend in the hands of an unskilled or inexperienced teacher to perpetuate this vicious method of work. Others see the difficulty and endeavour to avoid it by varying the form of example in every possible way and by providing for frequent brief revision.

The following problem was given to children of 11 to 12 at the London County Council Junior Scholarship Examination: "Seven lights each 6 ft. apart are placed along the sides of an electric railway car so as to leave a space of 3 ft. at each end of the car. What is the length of the side of the car in yards? You may draw a diagram to assist you."

The examiner's comment is that a large majority went wrong on the problem, and that very few indeed troubled to draw a diagram.

(d) It is a common charge that arithmetic is not correlated with the other subjects of the school curriculum, each forming a watertight compartment. The question of co-ordination is a difficult one, and the word has latterly fallen into some disrepute. It must be admitted that there is much co-ordination that is so only in name. Algebra, geometry, manual training, science, and geography are the subjects where correlation is of most value.

. It may be (1) parallel development of Mathematics and the other subjects; (2) application to Science, &c. of what has

been learned in Mathematics, or application to Mathematics of what has been learned in Science, &c.

Co-ordination in the elementary school will be chiefly of the latter kind, the essential point, as far as Mathematics is concerned, being that geometry and algebra shall advance along with arithmetic, all three being mutually helpful and interconnected and that benefit shall come to Mathematics from the work done in Science, Manual Training, &c.

Algebra.—It is a practical advantage to the pupil that he should, even at an early age, make some acquaintance with literal symbols and learn to use “ x ” in solving his problems. The work may be limited to : The use of letters as symbols and their simple manipulation, a little work on symbolic expression, the use of a bracket, negative quantities which may be illustrated graphically, and the simple equation.

The Geometry Course.—The geometry of the old school was of the constructional type and almost purely mechanical. Each problem was a self-contained exercise with ruler and compass, largely a matter of memory, requiring neatness and accuracy, but not otherwise educative. The boy drew, he did not investigate. On the other hand, the study of geometry as a purely deductive science is out of the question for the ordinary elementary school. A line of work has accordingly been evolved which, whatever its limitations, tends to make the pupil think. The old constructional exercise is made to develop work of an investigative nature, leading to some knowledge of very simple inductive and deductive reasoning, as well as of geometrical properties discovered experimentally. By means of ruler, compass, protractor, folding for symmetry, superposing, the pupil discovers for himself many of the results contained in the first and third books of Euclid. He may be led to see that reasoning from a single instance is insufficient and dangerous, but that if all trials in the class produce the same result there is good ground for coming to a general conclusion.

Manual Training.—The preparation of working drawings, involving as it does measuring, geometrical and scale drawing, is a valuable training in elementary mathematics, the more so that the use, and the need for accuracy, are so obvious to the boy. Equally valuable is the making of the simple plane figures and the simple solids, in wood or metal from drawings made in the classroom—the square, rectangle, triangle, parallelogram, circle, hexagon, the cube, right prism, cylinder and cone—one set being produced by each pupil for use in the classroom.

It is advisable that simple scale drawing, the notion of a plan and elevation, and of a simple section, the bisection of angles, drawing of angles, properties of the circle, chord and tangents, be introduced into classroom work at an early stage.

Many objects produced in the manual training school may be profitably retained for arithmetical measurements and calculations.

The paper and cardboard modelling introduced of late years into the lower standards of the Senior Department furnishes useful exercises in simple measurements, introductory drawing to scale and geometrical drawing.

Science, Domestic Subjects and Geography may be correlated with Arithmetic in so far as they may be made to provide problems in mathematical work. In the average elementary school the possibilities of correlation with these subjects will be but slight.

(e) The syllabus is often overloaded with work of a too ambitious or of an unnecessary or useless type, with operations of a kind and size not met with in daily life, with heavy types of questions and wearisome overpractice upon rules.

The uselessness of some of the answers produced in class-work may be illustrated by reference to practical results and the demands of ordinary life, *e.g.*, 14.2345 inches and 5 cwt. 3 qrs. 12 lbs. 7 ozs.

(f) The introduction of vulgar and decimal fractions is often too long delayed. Place value and the real significance of a decimal receive unintelligent treatment or are neglected with the result that decimals are converted into vulgar fractions in the working of problems.

A friendly familiarity with the decimal system should be ensured by teaching decimals from the ruler, say at the age of nine, and by their frequent use throughout the course.

(g) Estimation and forecasting and the application of rough checks should be encouraged. The average pupil still works his sum on paper and proceeds to the next question without a thought as to how far his result is likely to satisfy the problem given. The pupil should be taught to obtain a rough forecast of the result—to examine his answer after formally working the problem—to apply any commonsense check available; and, just as no commercial man would leave his result unchecked, the child should learn to prove his answer by some second method—a very valuable training. He should further learn to distinguish between the “business method” of solution, which has for its object conciseness, rapidity and accuracy, and the “analytical method” which should show a clear comprehension of the various stages involved in the solution.

(h) Graphical work in some elementary schools has been somewhat overdone during the last few years. Experience shows that a class can cover a fair amount of such work without gaining any clear idea of the principles involved. The pupil rarely develops the ability to “read” a graph, *i.e.*, to grasp its significance by inspection, or to draw the graph from data given in a slightly unusual form.

In the ordinary elementary school, graphs, as such, should be introduced with caution, work in the shape of graphic representation being substituted, and regarded as an aid in the teaching of mathematics, science, nature study, practical geography, &c.

THE HIGHER ELEMENTARY OR CENTRAL SCHOOL.

The Higher Elementary or Central School demands a treatment of mathematical subjects wider and fuller than that possible in the ordinary elementary school, but the principles, methods, criticisms and suggestions contained in the earlier portion of this memorandum may, with some modifications which will be noted, be taken to apply equally to higher grade work. The Higher Elementary or Central School will endeavour to develop the utilitarian side of Mathematics, and to get from this as much as possible of the mental training already indicated. But there will be a more definite preparation for industrial or commercial life, and the more thorough cultivation of self-reliance, initiative, adaptability, common sense and reasoning power.

As the Higher Elementary or Central Schools vary considerably in different localities with respect to character, aim and possibilities, it is difficult to describe them briefly as a type, or to make general suggestions for their conduct. In London they tend to range themselves in three classes, developing (1) a commercial, (2) an industrial bias, or (3) towards the end of the course combining both these tendencies, but not in any case giving technical training as usually understood.

For the purposes of this paper it is proposed to consider a London Central School favourably situated in an industrial district, and accordingly a working syllabus is here given, with comments on its interpretation and use.

ILLUSTRATIVE SYLLABUS IN MATHEMATICS.

CENTRAL SCHOOL.

First Year.

Arithmetic.—Units of length. Use of ruler, foot rule, metric rule, inch and mm. Decimals to hundredths. A percentage. Measurement to half mm. Decimals and the metric system. Easy ratio and proportion. Easy vulgar fractions, prime factors, easy H.O.F. and L.C.M. Averages and percentages, easy cases. Relationship between long and square measure. Areas:—Square, rectangle, parallelogram, triangle, quadrilateral, simple properties of these figures.

Volume:—Cubic measure, cube, rectangular prism. Volumes by displacement. Very simple scales and readings to scale, selection of suitable scales.

Simple graphic representation. Exercises on measurement of concrete objects and simple exercises in land measurement.

Algebra.—Letters as symbols, definitions, meaning of a , $2a$, ab , abc , a^2 , $\frac{a}{b}$, $\frac{ab}{c}$, &c.

Simple substitution in symbolical expression.

Simple symbolical exercises.

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Simple addition, subtraction, multiplication and division.
Very easy simple equations.

Geometry.—Definition of a line, surface, solid, circle, angle, parallels, and the common figures.

Facile use of pencil, compass, ruler, set-squares, and protractor.

The right angle as unit.

Bisecting lines, angles, construction of circles, squares, rectangles, triangles. Drawing very simple scales, copying figures to scale. Drawings from boy's own measurements of hall, playground, &c. Simple pattern drawing.

Experimental geometry (paper folding, &c.) should be introduced when possible. Discuss properties of the square, rectangle, equilateral and isosceles triangles, parallels and transversals, *e.g.*, square :—sides and angles equal, diagonals bisect at right angles.

Equilateral and isosceles triangles :—Angles at base equal. Straight line from apex to middle point of base is perpendicular to base, &c.

Any triangle :—Three angles together equal two right angles.

Rectangle :—Diagonals bisect each other and are equal.

Rhombus :—Diagonals bisect each other at right angles.

Second Year.

Arithmetic.—More difficult examples of vulgar and decimal fractions. Averages. Square root. Ratio and proportion. Percentages. Contracted methods of multiplication and division. Estimations. Approximations. Percentage error. Simple interest (easy cases).

Graphic representation. Simple problems on mass and other work from laboratories.

General mensuration of length, area and volume in easy cases. Further exercises on scales and scale drawing.

Easy exercises in mensuration of land, areas, &c. and construction of map or plan.

Further exercises in measurement of length, area and volume.

Algebra.—Removal of brackets, substitution in formulæ. Exercises on the first four rules.

Simple factoring.

Simple equations and problems more difficult than in the first year.

Geometry—Continue work on simple scales and drawing to scale, choosing suitable scales, drawing and figuring.

Freehand sketching of simple rectilinear figures. Insert dimensions and draw to scale.

Squares, rectangles, triangles, circles, quadrilaterals, pentagon, hexagon, and octagon on given line or inscribed in given circle.

Simple plans and elevations of parallelipeds met with in laboratories, manual training centres, &c.

Pattern drawing.

Show experimentally the properties of figures drawn, *e.g.*, the angle in a semicircle is a right angle.

Angles in same segment of a circle are equal.

Equal chords of a circle cut off equal segments, &c.

Fold triangle to show area.

Measure and add angles of triangle, quadrilateral, and polygons.

Measure exterior angle and the two interior and opposite angles of a triangle, and show that the exterior angle is equal to the two interior and opposite angles.

Show that the greater angle of a triangle is opposite to the greater side, &c.

Third Year.

Arithmetic.—Still further treatment of vulgar and decimal fractions. Ratio, proportion, involution, square root, averages, percentages.

Simple interest. Find interest, principal, rate, time.
Measurement and problems from the laboratories.
Contracted methods and approximations for workshop purposes.
Estimation, approximations and their value.
Percentage error.
Rough checks, intelligent anticipation of result.
Further work on scales and drawing to scale.
Diagonal scale.
Graphic representation and introduction to graphs.
The straight line graph.
Mass, force, density, pressure, &c., from laboratory work.
General mensuration. Length, area, volume in more difficult cases than in first and second years.
Trigonometrical ratios, sine, cosine, tangent.
Use of logarithms in multiplication, division, involution, evolution in simple cases, e.g., x^2 , x^3 , x^4 , x^5 .
More difficult work in outdoor measurements, sketches and plans to scale.

Geometry.—Work of the first and second years and in addition:—

Circles in connection with lines and rectilineal figures (inscribed and described).

Tangents to a circle.

Proportionals.

Plans, elevations and side elevations of cubes, prisms, cylinders, cones, and spheres in easy positions (main axis parallel to one plane).

Areas and distances found from maps, &c., drawn to known scale.

Graphic solution of simple problems—straight line graphs only.

Very easy vector addition and subtraction, e.g., sum of distances in varying directions.

Pattern drawing.

Prove by deductive geometry the chief theorems connected with triangles and parallels.

Congruence of triangles.

Properties of the isosceles triangle.

Greater side opposite the greater angle.

Exterior angle equal to two interior and opposite angles.

Properties of parallels and transversals.

Algebra.—G.C.F. and L.C.M. by factoring. Fractions.

Decimal substitutions in formulae.

Evaluating formulae.

The four rules, using decimal coefficients.

Simultaneous equations.

Quadratic equations (factors).

Problems.

Fourth Year.

Arithmetic.—The boys of this year should be able to solve any ordinary arithmetical problem, especially those of a practical nature, or arising out of the laboratory course.

Text-books on workshop methods should be consulted wherever practicable. Contracted work, approximations, relative error, and workshop methods of dealing with workshop calculations should have especial consideration.

Exercises on volumes, weights, densities (solids and liquids), forces, stress, &c., will be given.

A boy should be able to work out a gradient, survey a few acres of ground and present a finished plan.

Solutions of problems by graphical methods.

Quadratic graphs.

Trigonometrical exercises in heights and distances.

Use of logarithms in solving the formulae used in mensuration and laboratory work.

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Algebra.—Fractional coefficients and indices.

Fractions, formulæ, quadratics, graphs, problems, surds.

Meaning and use of logarithms.

Progressions (arithmetical and geometrical).

Geometry.—More difficult examples of work done in previous years.

Common tangents.

Circles inscribed in circles, &c.

Ellipse, loci. Very simple problems on constrained motion.

Chief exercises on areas, and reduction of polygons to equivalent triangles.

Easy vectors (velocity, force, &c.).

Plans and elevations of regular solids in easy positions.

Easy sections.

Pattern drawing.

Trigonometrical ratios found by geometry.

Constructions of angles with given sines, cosines or tangents.

Graphic representation and proof of:—

$$a(a + b) = a^2 + ab,$$

$$(a + b)^2 = a^2 + 2ab + b^2,$$

$$(a + b)(a - b) = a^2 - b^2,$$

and conversely.

Algebraic proofs of the main theorems and exercises on Euclid, Book III.

Graphs to solution of ordinary quadratic.

Prove by deductive geometry or experimentally the theorems used in connection with lines, angles, triangles, parallelograms, chords and angles of circles, &c.

This syllabus is, chiefly on account of correlation with science, handicraft, mechanical drawing and practical geography, constructed on a concentric plan, year by year extending and introducing new and greater difficulties. It must be interpreted in the light of the principles and methods already advocated. Its aim is to find a preparatory covering ground common to the chief artisan trades of the school district. It should be adequate to the intellectual quality of the boys admitted, to their school period and to their probable future requirements. It develops on the practical rather than on the bookish side.

The boy is judged not only by what he knows but by what he can do and his method of doing it. He must approach his allotted task in a spirit of investigation and quiet confidence, working by head and not by rule. He requires in connection with mathematical work bold and rapid freehand drawing, scale and mechanical drawing, handicraft and practical computations of all useful and educative sorts.

The times allotted to mathematics and concurrent subjects are roughly:—

	Hours.
Geometrical drawing	1
Other drawing	2
Arithmetic, algebra, theoretical geometry and practical geography	5½
Science	2½
Handicraft	2½

In such a syllabus it is almost impossible, and indeed in practice is unnecessary, to indicate all points of detail. Latitude of treatment within reasonable limits must be allowed to the

class-master as new points of view and of attachment arise. Further, the syllabus should be interpreted modestly. The ground covered is extensive and would be regarded as excessive for some London Central Schools unfavourably placed, but the work is intended to be simple, practical and in no way pretentious. Throughout the course, complicated calculations and those of a purely commercial type are kept as few as possible.

Such a school as that instanced will provide a four years' course for scholars of 11 years selected from the elementary schools of the neighbourhood. These will begin in the "first year" and work progressively through the course to the time of leaving. Unfortunately a varying but large proportion of the pupils of such a school do not at present complete the course.

The classes are of manageable size, averaging from 30 to 40 pupils. A physical and a chemical laboratory, an art room, and wood and metal handicraft rooms will probably be provided, each in charge of a specialist teacher.

Some 60 to 70 per cent. of the pupils will leave to take up skilled industrial work, chiefly mechanical. A few pass into commercial life and the rest fall into the ranks of unskilled labour.

A strong industrial bias will be given to the curriculum, increasing towards the end of the course. Mathematics will include arithmetic, mensuration, some algebra, and practical and theoretical geometry, all with an eye to "workshop" requirements.

The course will aim at giving the power of accurate and rapid calculation with manageable numbers; abundant exercise on well-chosen concrete material, including both indoor and outdoor measurements and calculations based thereon; ability to deal in an independent and commonsense way with any simple problem that may present itself; ability to read and use any ordinary workshop formulæ; good scale drawing, including a simple rough sketch from machine details, or from observation or description, and finished drawings to scale; as much construction work in plane geometry as is necessary for handicraft, mechanical drawing, and general workshop demands; some experimental geometry leading to a habit of investigation and a knowledge of the simple properties of the common geometrical figures and solids and an introduction to the Euclidean method of reasoning; some knowledge of literal expression and the application of algebra to the solution of problems; a knowledge of the trigonometrical ratios and ability to calculate heights and distances; daily practice in the use of logarithmic tables in calculations; the ability to construct or read a simple graph; the ability to use profitably a good mathematical class text-book.

In seeking to attain these results the syllabus aims at covering much ground by dealing with easy examples illustrative of principles and methods. As advocated for the elementary school, so here, formal greatest common measures,

least common multiples, and practice, elaborated fractions, discount, stocks, shares, and problems of a complicated, far-fetched or not really useful type are avoided. All mathematical subjects are to be broadly treated and the whole made naturally related parts of one course of study. Mathematical processes are to be practically applied, and measurements, framed to encourage independence of method, utilised in every available direction; much of this work being done out of doors. Stress is laid on estimating by judgment, checking by calculation, approximating short and alternative methods, familiarity with decimals and the metric system, ratio, variation, averages and percentages.

Arithmetic.—The first year's work is specially preparatory. Most of the boys newly admitted will be of more than average ability, but will work slowly and by rule and have little acquaintance with measurements and the application of algebra to arithmetic. Much oral practice will be given for revising work already done, to familiarise them with principles and rules and to secure rapid accurate calculation. Easy problems involving small quantities and illustrated by concrete material, exercises in easy formal mensuration, easy measurements in and out of doors and calculations arising from them will be prominent. Certain portions of the syllabus required in other concurrent subjects must be treated at an early stage: the decimal notation and its use in measurements, the metric system, manipulation of easy decimal and vulgar fractions, the simple form of equation, averages, percentages, and ratios. All work must be of the simplest character carefully graduated, and home lessons should be strictly limited to, say, 45 or 50 minutes per evening. The grave danger in this year is that the boy in beginning his new work may feel discouraged by the accumulation of difficulties.

The second year's work is intended to be a careful development of the first. This development is not easily shown in a written syllabus, but can be indicated in practice by detailed reference to class text-books. The work is still confined to easy types, gradually increasing in difficulty so as to prepare for the manipulations of the third and fourth years.

Stress is still laid on principles, but practice in contracted methods of multiplication and division, independent estimation of length, area, volume, weight, and cost of concrete material, the value of approximations and checking and the rejection of useless parts of the answer begin to receive special attention. The work is also made closely auxiliary to that of the laboratories and handicraft rooms that these may not spend time in repairing mathematical deficiencies. Thus laboratory problems on mass, density, &c. form a substantial part of the class work in arithmetic.

The third year is the year of expansion. The pupil by this time should be able to depend largely upon his own intellectual resources, especially in dealing with practical work. The use of mathematical text-books for work and reference becomes more

prominent. Formal mensuration is developed and in geometry some knowledge of the Euclidean method is obtained.

Simple trigonometry is introduced and outdoor measurements with the aid of the plane table and sighting ruler receive much attention. Small areas are surveyed and measured, finished plans made and the results checked by reference to an Ordnance map. The graphic work, too, is of a more advanced type. The straight line graph is used for illustration and the solution of suitable problems and such relationships as temperature and pressure, extension of spring, &c. are illustrated graphically in the laboratories.

The fourth year in arithmetic is largely devoted to revision and practice upon more difficult examples. The pupil should now be familiar with the use of text-books and the logarithmic tables; be able to apply his knowledge and intelligence to the solution of any ordinary arithmetical problem not of a commercially technical type; solve any such useful problem possible by the aid of algebra; estimate, measure, and calculate, using data obtained from any given simple concrete object, *e.g.*, a mass of brickwork (volume, weight, cost), a cupboard, an iron spouting, the woodwork of a laboratory bench (quantity and cost), &c.; survey and plan two or three connected roads, six or seven acres of common land or the gradients of a hill road; make a finished working drawing of a simple mechanical detail (*e.g.*, a fork end, a wall box), a piece of laboratory apparatus or handicraft work, and demonstrate or illustrate any important proposition from a modernised equivalent of the first three books of Euclid's Elements.

The practical side of the course will require close consideration. Its limitations and tendency to abuse have already been noted. Care must be taken to prevent its degeneration into a mere mechanical routine. All measurements are to be useful in themselves, to furnish useful calculations, to develop a habit of estimating, forecasting, and checking, and to give the pupil a clear first hand knowledge of the working units and of the uselessness of fine drawn answers. Records of results should, when possible, be kept for future reference.

In the syllabus submitted some of the most valued material for measurements is obtained from the local geography syllabus. The work is done on a co-operative plan, two, three, or four boys working in combination as required. The simple materials necessary are plane table and tripod, sighting ruler, spirit level, rough theodolite, and clinometer, all furnished profitably as part of the handicraft course.

The formal mensuration of the syllabus is developed throughout the whole course and is kept in close relationship with the geometry. It finally includes the measurement of regular figures and simple solids. Models of the plane figures are obtained in metal, and those of the solids in wood, again as part of the manual training course of each pupil.

Algebra.—The place and value of this subject in the curriculum constitute a difficulty in many of our Higher Elementary

or Central Schools at the present time. As a separate, formal, and academic subject it is slowly disappearing, being replaced by generalised arithmetic and the use of the equation in the solution of problems. Some examining bodies still demand it as a separate subject and hence its retention in the syllabus submitted, being required for "Trade" Entrance Examinations. The old stodgy type of sum is, however, now seldom met with, and it is increasingly felt that time spent on complicated highest common factors, least common multiples, algebraic fractions and the like may be more profitably employed by the boy who leaves school at 15 years of age, or earlier, to become an artisan. Thus although algebra appears somewhat prominent in the representative working syllabus attached, the writer suggests that the subject be reduced as may from time to time be found practicable until finally it embraces only such exercises as are merely extensions of Arithmetic, together with symbolical expression, the solution of equations and problems solved by them, the application of a formula, e.g., $V = \frac{\pi r^2 h}{3}$,

some elementary graphic work, the use of logarithmic tables and the little trigonometry necessary for the determination of heights and distances.

The graphic work may well be limited in the first and second years to graphic representation of the usual kind:— the growth of a bean, class attendance, temperature, barometric pressure, rainfall, bright sunshine, market prices, and the results of work in the laboratories. In the third year, the straight line graph may be used for illustration and the solution of suitable problems, and in the fourth year the graph applied to the solution of simple quadratics. Abundant practice should be given in graphic work for the illustration of laboratory results so that the pupil may acquire facility both in the construction and reading of his graph.

Geometry.—Geometry in such a syllabus as that submitted must be regarded as a subject of primary importance, since it is related so closely to other mathematical work and to handicraft, mechanical drawing, and practical geography. No subject of the higher grade curriculum presents greater difficulty or needs more careful treatment. There are conflicting elements and points of view which make the necessity for compromise quickly apparent. The text-book, too, exactly suitable for our purpose is still wanting.

The general aim is determined by (a) the industrial character of the course, and (b) some considerations of that mental training value already discussed. It is therefore twofold. In a course including handicraft in wood and metal, mechanical drawing and the development of workshop methods, some constructional plane geometry of the old type is obviously necessary and to this must be added a really useful course of solid geometry. On the other hand it is desirable that the pupil should obtain some knowledge of reasoning processes as applied to geometrical figures.

Practical Work.—Plane Geometry.—The mechanical character of the old type of constructional work has already been indicated, and it is suggested that it be here limited to constructions in common use such as are required in concurrent subjects, *e.g.*, the drawing of lines and angles, bisections, perpendiculars, parallels, proportionals, circles, tangents, the simple geometrical figures and some work on patterns.

This part of the course aims at giving (a) facility in handling pencil, ruler, compass and protractor; (b) accuracy and finish in the work; (c) a working stock-in-trade of the geometrical constructions required in science, machine drawing and handicraft; (d) opportunities for exercises in measurement and experimental geometry.

Solid Geometry must be introduced at the earliest stage. The work should not be abstract or ambitious, but thoroughly practical in scope and aim, one object being the quick and clean production of a working drawing. The fourth-year pupil should be able to make a plan, elevation and simple section of any of the common geometrical solids, or if working from the actual object, of any simple machine detail.

Geometry as Mental Training.—The question here is between the experimental and Euclidean methods of reasoning. To what extent shall the experimental method of illustration, investigation and so-called proof be carried? Shall the Euclidean method supersede it, and, if so, at what stage?

In an industrial course, although the maximum amount of mental training is sought, an extended course of abstract reasoning is scarcely possible. In the illustrative syllabus, therefore, the experimental method is employed throughout the first and second-year course, much of the constructional work lending itself to this treatment. Simple Euclidean reasoning is introduced towards the end of the second year, developing simply and naturally from the experimental work.

The extent to which the pure deductive method shall be carried will furnish further matter for controversy. It is here intended that the more important propositions of Book I. be proved by the modernised Euclidean method, that Book II. be illustrated algebraically, and that Book III., wherever possible, be worked experimentally by measurement, superposition, &c. The pupil will thus get an acquaintance with the geometry of the first three books, and some insight into the method of pure deductive reasoning, the time saved being given to more practical work. In the poorer type of Higher Elementary or Central School the study of pure geometrical reasoning after the manner of Euclid will probably be entirely omitted.

The mechanical drawing of the course embraces the production of working drawings from models or actual specimens of some 30 or 40 machine details, such as nuts, bolts, pedestals, fork end, angle irons, knuckle joint, eccentric, piston, &c.

In conclusion the following suggestions on the method of teaching arithmetic to get from it its full utility value, and

as much as possible of the mental training value, are additions to, or definite and concrete maxims based on, the principles of method advocated in this memorandum.

(i) Arithmetical meanings, relations, processes, &c., to be obtained as far as possible practically and as a result of individual experiment on the part of the child himself.

(ii) Wherever possible, especially when the rule or process comes first, get the answer by actually solving the problem by means of concrete material:—counters, cardboard coins, rulers, measures, squared paper, &c. Use the experimental method, see that the children clearly understand what is to be done, let them try for themselves and evolve the best way.

(iii) Let the children have realities behind their symbols:—actual measurements from concrete objects, diagrams, standards of length, area, volume, paper folding, use of squared paper, graphic illustration and experimental demonstration.

(iv) Find out and use alternative methods of work, using one as a check upon the other.

(v) Let the lessons be concerned with things within the experience of the children, or possible to their experience, or within their imagination.

(vi) Let written work follow and be rather an appendage to oral work.

(vii) Introduce decimals early and tenths earlier and illustrate from the ruler.

(viii) Rely upon the few fundamental principles and processes rather than upon a number of fixed rules. Try to apply commonsense methods to all operations.

(ix) Gradually “generalise” the ordinary arithmetic, make the fullest use of the symbol “ x ” and introduce the algebraic equation at an early stage.

(x) Deal with small numbers, and so cover much more ground in the lesson.

(xi) Cover all the course, and every kind of problem, process and principle by oral work, using small numbers. To ensure that every child does his share of the work let all answers be written down. Let the blackboard be freely used as an aid in oral work, which should be an integral part of most lessons and form part of each “term test.”

(xii) Aim at *real* correlation with other school subjects where the gain to arithmetic is a real gain.

(xiii) Introduce symbols only when needed or their value appreciated. Make the children understand clearly the short-hand nature of symbols, and accustom them to pass at will from concretes to symbols and symbols to concretes.

(xiv) As soon as children of their own accord begin to drop the concrete, encourage them to work mostly with the abstract in each field of experience as it is met.

(xv) It does not necessarily follow that because children reproduce the language of reasoning, they have reasoned.

THE TEACHING OF MATHEMATICS TO YOUNG CHILDREN.

That skill in handling numbers is one of the fundamental bases on which to rear an educational structure is no new idea. Number has formed a part of the most scanty and elementary schemes of education through all historic time, and we may therefore assume that its value is undisputed even by those unable to realise it in exact terms of intellectual training and power.

Those, however, who are deeply interested in the teaching of the science of number realise that, even though it may never be of practical use to the student, yet a true knowledge of this subject will give him such important knowledge as will stand him in good stead in his future dealings with men and affairs. To them no apology is necessary for the exceeding care which we consider must be bestowed on teaching children to really "think mathematically."

Taking as our working definition that "education is an atmosphere, a discipline, a life," it follows that we realise that education must surround and be a part of the child from his infancy; but until he is ready for school at the end of his sixth year it is to be an education by means of his senses, of his unstudied games, by means of his natural and not of an artificially prepared environment.

The conscious teaching then of number, as of other definite lines of thought, is to be begun in the schoolroom with a pupil whose age is not less than six years.

School Mathematics.—When children begin their regular school course, lessons last for two hours, or two-and-a-half hours, every morning, with a long interval; *number* for 20 minutes a day is one of the lessons. We generally find that the children, when they enter school, are able to count, but know nothing of the properties of numbers.

The number *one* is taken during the first lesson; the children point out to the teacher *one* window, *one* fireplace, *one* piano; in fact, everything in the room which exists singly; then the symbol for one is learnt. Whenever we see a stroke 1 we know that it stands for one of something. The children pick out the ones from groups of figures, and finally learn to write one; getting it as straight and perfect as possible.

The children have a small blackboard and piece of chalk each, and on these they first write the numbers; afterwards a book ruled in $\frac{1}{4}$ inch or $\frac{1}{2}$ inch squares and a lead pencil are requisitioned. The next number to one is two, as the child probably knows; he learns then to write '2,' first on his board,

and then in his book ; picks out 2 from a group of figures, and does little sums involving the number 2. Three is taken in the same way ; and then four, which the pupil must realise is made up of two twos, or of 3 and 1, by very simple little problems such as will readily suggest themselves to any teacher. He learns to count up to 4, and backwards from 4 ; thus realising slowly the idea of a series of symbols denoting a series of quantities whose magnitudes continue to grow greater. The idea of an *order* of things, which is conveyed by a number, is perhaps grasped most easily by counting a series of things ; and that of the relative magnitudes represented by numbers by the little sums in addition and subtraction.

In this way all the numbers from one to nine are learnt, the examples becoming more numerous as the numbers grow larger, and involving, besides simple subtraction, simple factors such as two threes make six, and three threes make nine. Each number is begun from a concrete set of things, beads, &c., and several questions are asked and answered with the help of the beads. Then these are put away, and for the next lesson work is done on the number without the aid of the concrete.

When several of the numbers have been learnt the meanings of the signs +, -, and = are explained to the child ; + means "is added to," or "is put together with," - means "is taken away from," and = means "is the same thing as." Now we have the added joy of being able to write sums in our books. This is always considered a privilege, and is only indulged in on mornings when the children are working well, and during the final lesson on some particular number. Writing is still a laborious effort, and is apt to take attention away from the most important matter in hand. The sums are of course always worked orally first, and then written down, *e.g.*, if your little sister is two years old now, how old will she be in two more years ? When the answer 4 has been obtained the children write in their books $2 + 2 = 4$; then they read it ; two years added to two years make four years. This writing of sums, however, is very sparingly used, and all the work is oral.

During this stage too we give occasional examples dealing with pure number ; there are mornings when the little ones are bright and eager, and more than ever anxious to do innumerable sums ; this is an opportunity to be seized by the teacher ; let us leave the boxes of beads and counters alone, let us even leave out sleep and motor cars, and have nothing but numbers. "How much left if you take 3 from 5 ?" "How much to be added to 4 to make 7 ?," and so on, quick question and quick answer, all easy and simple, so that the children may feel at home with the numbers, and feel that they have a real grasp of them, for, though the ability to work with pure number is undoubtedly a function of some minds only, yet it, like an ear for music, can, to a certain extent, be cultivated, to a very limited extent it may be, but even that is worth striving after with our pupils.

The Number Ten.—This number we of course learn just as we did the other numbers, *i.e.*, working out addition and subtraction sums involving its use and analysis; then we learn the meaning of the word *unit* and here a few match-sticks seem to serve as the best vehicle to convey the idea we wish to impart. "When we have 10 things we tie them together and call it a 'ten bundle.'" Count out several sets of 10 sticks each and tie them together, now we have several "ten bundles," and each of the sticks in a bundle is called a *unit*. Now count out, 1 unit, 2 units, 3 units, 4 units 9 units, 10 units, and tie them together. This is done several times; and, as one must be careful not to confuse in the child's mind that which is general with that which is particular, let the counting out be done with beads, buttons, pencils, &c., the beads and buttons being threaded in sets of 10 and the pencils, &c., tied together. The name "ten bundle" is used for each of these.

When this convention has become habitual to the pupils, it will be time to write the number ten; it is written on the squared paper, one square having 1 written in it for *one* ten bundle, and the next square 0 for "no units"; now all the other numbers from 1 to 9 are written down under one another, and then 10 is written so that 0 comes under 9, and the 1 is out by itself (to the left) in the "ten bundle" square.

After 10 is thoroughly mastered the numbers from 11 to 20 are always learnt very quickly as 1 ten bundle and 1 unit; 1 ten bundle and 2 units, &c., &c., use being made of the bundles of matches, threads of beads, &c., though these must be dispensed with as soon as possible. The children practise counting too, both backwards and forwards, and learn to write the numbers in columns so that the idea of the local value of the digits may be impressed upon their minds.

We try at this stage, and indeed at any stage in the analysis of numbers from 10 to 100, to obtain from the children themselves the composition of each new number that occurs, *e.g.*, we know all numbers from 10 to 13; we have had 10 and no units; 10 and 1 unit, called eleven; 10 and 2 units; 10 and 3 units; then the next number will have to be 10 and 4 units, the next 10 and 5 units, and so on. We find that this sort of counting is necessary, as the children are apt to get an idea of a number by itself, and are unable to realise its position with respect to other numbers.

After the number 12 we have to stop for a while, as we are here able to introduce money; for, though it makes a break in the proper sequence of lessons in numbers from 10 to 20, yet it provides such a valuable teaching asset for the future that it is worth the break in the train of thought we have pursued so far.

We are all provided with purses containing shillings, sixpences, and pennies, and after obtaining from the children the information that lots and lots of pennies would be very heavy to carry about and take a long time to count, we show how instead of six pennies we have a small silver coin and we call

it sixpence. Suppose we went into a shop and bought six 1d. pieces of chocolate; then instead of counting out six pennies we should just give the shopman *one* coin—one of our sixpences; if we bought 12 penny pieces we might count out 12 pennies (they are counted out by the children), or else we might have,—how many sixpences? (Each set of six pennies is replaced by a 6d.); we might have two sixpences, or another coin which we call a shilling and which may be used for two sixpences, or 12 pennies.

After this introduction we have practice in changing from pennies to shillings and sixpences, and *vice versa*, e.g. "count out 18 pennies, we don't want to carry so much, so we change 12 of them into a shilling—count how many are left." "Six." "Yes, so that we may have a sixpence instead of those pennies, and we should only take out with us two small coins instead of 18 large ones." Or, "count out 13 pennies, re-place 12 by one shilling, we have one penny over, so that we have two coins instead of 13: count out eight pennies, a sixpence and two pennies," and so on. As the children deal with numbers larger than 12 the work has to be purely experimental.

A great many simple sums on money follow the experimental work, the numbers involved never exceeding 12; e.g., I bought three penny balls, two-pennyworth of toffee, and two two-penny balls of string. What change did the shopman give me out of a shilling? Or; I won a prize of 5s. and gave 2d. out of every shilling to my brother. How much did I give him?

We are now able to use money sums to provide us with examples in dealing with all the other numbers; money sums appeal to children, and at this early stage, as well as a little later on, questions involving money appear to be worked very easily.

The number 20 is easily explained as 2 ten bundles, and one finds that the path from 20 to 100 is a smooth one for the child of ordinary intelligence. From 30 onwards the numbers are taken in sets of ten, and frequent practice in counting and in writing the back figures is given. At the number 20 the children are introduced to the sovereign, and at 24 to the two-shilling piece, and at 30 to the half-crown.

Examples in pure number are also given, such as—add 25 and 41—we add the ten bundles first:—

$$\begin{array}{r} 2 \text{ ten bundles} + 4 \text{ ten bundles} = 6 \text{ ten bundles, and} \\ 5 \text{ units} \quad \quad + 1 \text{ unit} \quad \quad = 6 \text{ units,} \\ \text{Answer} = 66, \end{array}$$

these are the steps followed in working; in the pupil's book the sum is written as $41 + 25 = 66$.

The analysis of numbers from 1 to 100 occupies the first year; with the second year we begin the four rules and tables. The work carried on does not claim to be original, but is simply a modification of existing methods of teaching, and is taken from existing and well-known text-books. I should like, if I may, to mention what appear to us to be some of the principal features of the course of teaching in these books for the

benefit of those who have perhaps not seen them for themselves. These features are :—

- (1.) The very thorough way in which the work is done; nearly the whole of the first year's work consists of the analysis of numbers from one to a thousand, every one of which is approached from all directions, and exhausted of all its possibilities by the pupil and teacher. The synthesis from pairs of smaller numbers, factors, fractional parts, application to sums of money, to weights and measures (at 60 the pupil begins to work with hours and minutes, at 36 with inches and yards), are all taken with each of the numbers to which they apply, and particular terms, such as dozen and score, are introduced. For instance, the number forty is dealt with thus :—

$$\begin{aligned}
 40 &= 39 + 1. \\
 &= 38 + 2. \\
 &= 37 + 3. \\
 &= 36 + 4. \\
 &= 35 + 5. \\
 &= 34 + 6. \\
 &= 33 + 7. \\
 &= 32 + 8. \\
 &= 31 + 9. \\
 &= 30 + 10 = 10 + 10 + 10 + 10 = 4 \times 10.
 \end{aligned}$$

$$\begin{aligned}
 40 &= 4 \times 10 = 10 \times 4. \\
 &= 5 \times 8 = 8 \times 5. \\
 &= 20 \times 2 = 2 \times 20 = \text{two score.}
 \end{aligned}$$

$$\begin{aligned}
 10 \neq^* 40 & \quad 4 \text{ times.} \\
 4 \neq 40 & \quad 10 \text{ times.} \\
 5 \neq 40 & \quad 8 \text{ times.} \\
 8 \neq 40 & \quad 5 \text{ times.} \\
 20 \neq 40 & \quad 2 \text{ times (twice.)} \\
 2 \neq 40 & \quad 20 \text{ times.}
 \end{aligned}$$

The half of 40 is 20. The fifth of 40 is 8.
 The quarter of 40 is 10. The eighth of 40 is 5.
 The tenth of 40 is 4. The twentieth of 40 is 2.

Then examples such as :—

- (a) How many shillings in £2? Cwts. in 2 tons?
- (b) How many horses have 40 legs between them?
- (c) If one book costs 8 shillings, how many can I buy for £2, &c., &c.

- (2.) The four rules are introduced all together from the very beginning by means of little problems, e.g., "Out of a journey of ten miles I have walked seven miles. How much further have I to go?" or, "How many twopenny balls for a shilling?" or, "How many legs

* $\neq =$ is contained in. This symbol was proposed by Messrs. Sonnenschein and Nesbit. It has not been generally accepted, and unfortunately it resembles closely the sign universally employed for "does not equal."

have four sparrows?" The four signs $+$, $-$, \times , \neq , are given to the children almost at the beginning of the course, they are told the meanings of the signs and are given exercises on their interpretation. Terms such as subtrahend, multiplicand, &c., are given as soon as the children arrive at the sections particularly devoted to subtraction, multiplication, &c.

- (3.) A special apparatus particularly devised is largely used throughout the course.

Even this very scanty indication of the lines of the work will perhaps suffice to show how tempting such a course is to the teacher of young children; it takes the subject in a fascinating and exhaustive way and appears at a first reading to be the ideal course of work laid down for the beginner; but it has from experience been proved to possess certain disadvantages for which reason it is used with these alterations—

- (1.) The whole use of the special apparatus is omitted. The use of complicated apparatus specially designed to teach the child certain facts about numbers tends to form in his mind an iron connection between the facts and the apparatus; he is unable to separate them or to realise that the facts are general. We therefore think it advisable, where apparatus is necessary, to use match sticks, buttons, pencils, everyday articles, and a sufficient number of them.
- (2.) The examples given, though of an interesting nature, are often much too hard for the children. Their acquaintance with the science of number is still in its first stages; their work is largely oral, and the examples we find it best to give them have therefore to necessitate for their solution one arithmetical operation performed once only; addition, subtraction, or whatever it may be, *e.g.*, "John had 1s. more than Mary, he gave her 1d. How much had he then more than Mary?" is too difficult a question for a pupil who is just unravelling the mysteries of the number 12.
- (3.) All tables of weights and measures of time or length seem better omitted until later on. To be told that there are 20 cwt. in a ton cannot convey much to a child of six; and the divisions of time are an abstraction that will convey still less. We keep therefore simply to sums involving money and familiar everyday objects.
- (4.) The signs \times and \neq are not given until the pupil arrives at the multiplication table, and terms such as subtrahend, addenda, are omitted.
- (5.) The later numbers are not dealt with so exhaustively; about a week (or 10 days at most) would be spent in learning about the numbers 40–49 for instance. The children do no homework at all, and their arithmetic is therefore very largely oral work.

We have found that the children, when taught entirely on this system, become very proficient in the analysis of numbers,

but that they have, as it were, specialised in that branch of the science, and are unable to work in any other way; problems, if presented as the analysis of a given number, are easily solved, but not when presented in other ways. Books, giving such courses of work, however, are a very valuable guide and assistance in our work.

Addition and Subtraction.—Here the first sums given are those dealing with money. Addition of pence, e.g., $7d. + 4d.$ and then $7d. + 6d. + 2d.$ is followed by addition of shillings and pence, e.g., $4s. 3d. + 2s. 6d.$ and then $4s. 9d. + 17s. 4d.$, then of pounds, shillings, and pence, the numbers of £ being small :—

$$\begin{array}{r} \text{£} \quad \text{s.} \quad \text{d.} \\ \text{e.g.,} \quad 9 \quad 10 \quad 5 \\ \quad \quad 3 \quad 17 \quad 8 \\ \hline \end{array}$$

Answer 13 8 1, 8 pennies and 4 pennies make 1 shilling, and leave 1 penny over; 1 shilling and 17 shillings make 18 shillings, which, with 2 shillings, make 1£., and leave 8 shillings over; £1 + £3 + £9 = £13. Our answer is, therefore, £13 8s. 1d.;

and, finally, farthings are introduced. After the children have worked some little time at the addition of money, sums on pure number are given them; e.g., $674 + 215$; this is to be written in a new way :—

$$\begin{array}{r} 674 \\ 215 \\ \hline \end{array}$$

the hundreds, tens, and units in their own places. 4 units + 5 units = 9 units, to be written in the units' place. 1 ten + 7 tens = 8 tens, to be written in the tens' place. 2 hundreds + 6 hundreds = 8 hundreds, to be written in the hundreds' place. The finished sum is then :—

$$\begin{array}{r} 674 \\ 215 \\ \hline 889 \text{ Answer.} \\ \hline \end{array}$$

The next step is a sum like :—

$$\begin{array}{r} 519 \\ 392 \\ \hline \end{array}$$

In a case like this the children are taught to add in tens, e.g., 2 units + 8 units = 10 units and 1 = 11 units, i.e., 1 unit

for the units' place and a ten to be added on to the others. 1 ten + 9 tens = 10 tens, or one hundred, and there is still one ten. We therefore have one hundred and one ten; and the one hundred + three hundreds + five hundreds give us nine hundreds:—

519

392

—

911 Answer.

After this longer sums may be given, with three, or four, or more sets of figures, the children always adding in tens and thus attaining a mechanical accuracy. The children are always able after a little judicious questioning to tell us what to do with the 11 units and 11 tens in the sum given above; *i.e.*, to write the 11 units as one in the units' place and put the ten in with the other tens; and with amounts like 13 pennies, which is one shilling and one penny; or 27 shillings, which is a pound and seven shillings, if it were a money question. The analogy between the changing of pennies into shillings, shillings into pounds, and the changing of units into tens, and tens into hundreds, makes the latter seem quite easy and natural to the children. Interesting problems to be worked by the addition of money, or of pure numbers come next, *e.g.*, if a bicycle cost £5 0s. 0d., a tool-case 2s. 6d., a bell 1s. 9d., and a lamp 5s. 6d., what was the total cost?

Subtraction is introduced as addition was, by little money sums practically presented at first, *e.g.*, (a) if I have 6d. in my purse and give a porter 2d. for carrying a parcel, how much have I left; or (b) If I have six nuts and I want 9, how many more must I get? Examples of the nature of (b) are advisable, and should be the more numerous because the children can add more easily than they can subtract; and, also, such examples give them the idea of subtraction as the complement of addition; and this view of subtraction is now generally accepted as the right one to be given to beginners, and is used in later lessons on this part of the work. After one or two such examples we begin sums involving shillings, and begin to use the term "take away." Take away 1s. 2d. from 5s. 8d., what is left?

s. d.

5 8

1 2

—

4 6. Two pennies and 6 pennies make 8 pennies, and
1 shilling + 4 shillings = 5 shillings.

Next come sums like this: I had 10 shillings and I paid a man 3s. 9d., what had I left? Just at first this should be done experimentally; the child attempts to pay 3s. 9d. out of a purse containing 10 shillings and finds the 9d. a difficulty for a little while, though it will soon occur to him to change it into pennies. He then works several sums in his book, mentally changing a

shilling into pennies, and presently pounds into shillings. Even an exceptionally dull child appears to grasp subtraction without any difficulty when it is presented in this way.

When we arrive at subtraction sums involving pure number it is a comparatively simple thing to change tens into units after having already become familiar with the changing of pounds into shillings and shillings into pence; e.g., subtract (we have now learnt that this is the same as "take away") 27 from 41:—

$$\begin{array}{r} 41 \\ 27 \\ \hline \end{array}$$

We cannot take 7 units from 1 unit—what shall we do? Someone in the class will probably suggest "Turn a 10 into units." We now have 11 units, and $7 + 4$ units = 11 units. Then we take 2 tens from 3 tens as 1 has been turned into units; and our sum is:—

$$\begin{array}{r} 41 \\ 27 \\ \hline 14 \end{array} \text{ Answer.}$$

If this should not be clear we go back to our bundle of match sticks and break one up into its component units. Though it is generally advisable to give a class only one method of subtraction and the "Decomposition" method just mentioned is the one most easily explained, yet, we may just glance at the method of "Equal Additions" which some teachers prefer.

We begin with our bundle of match sticks. To take 28 from 47, we have before us 4 ten bundles and 7 units—take 8 from 7, we cannot, therefore, instead of untying one of the ten bundles we have, we get a whole new bundle and untie it; we now have 4 tens and 17 units and taking away 8 we have 9 units left. Now we are to take 2 tens away, take them and remember to remove the ten bundle which we put in, i.e., take 3 ten bundles away. This last little "Rule," if we may call it so, is arrived at by the pupil after he has worked several examples. The method of equal additions is eventually a more rapid way of working subtraction and is therefore worth attention. Problems follow involving subtraction of money and pure number and embodying all that has been learnt so far.

Multiplication and Division.—Multiplication is at first presented as an extension of addition, e.g., "If 4 children had 6d. each, how much had they altogether?" would be worked $6d. + 6d. + 6d. + 6d. = 24d. = 2s.$ Several examples like this are given before we suggest that it may be written down more shortly, thus $6d. \times 4$, where " $\times 4$ " means multiplied by 4, i.e., each of the quantities mentioned is to be taken 4 times, so that $6d. \times 4$ means 4 sixpences, $2s. \times 10$ would mean 10 2s. pieces,

and so on. We work a few simple questions, getting the children to write them on their blackboards with the multiplication sign and using easy numbers for which a knowledge of the multiplication table is not necessary. These elementary examples give to the children an idea of what "times" indicates and we can then begin Tables.

To help the children to see the rationale of the multiplication table, they at first construct each one for themselves with the teacher's assistance, *e.g.*, suppose it were 4 times, the teacher begins "I write down one 4 on the board with a "small 1 above it, to show how many fours I have. Then I "write down another four, how many have I?" "Two." "How much have I now, two fours that is?" "Eight." Put eight down underneath the second 4. Now write down another four, we have three fours or 12, similarly four fours or 16, five fours or 20, and so on to the end of the table, 12 fours or 48, until the whole table stands:—

1	2	3	4	5	6	7	8	9	10	11	12
4	4	4	4	4	4	4	4	4	4	4	4
8	12	16	20	24	28	32	36	40	44	48	

The children look at this for some time, visualising it as an aid to committing it to memory, and then say it through several times. The teacher then rubs out several figures here and there in the table and lets the children fill in the gaps thus left. Then the whole table is written out again with several gaps to be filled in by the pupils. The whole table is then said through again by each one.

There is no royal road to the multiplication table; it *must* be learnt by heart. This is a fact which faces every teacher of elementary arithmetic, and which each must prepare for in the best way possible. They must be learnt by each child individually and not in a chorus. The tables are learnt both forwards and backwards as it were, *i.e.*:—

6 times 1 = 6.

6 times 2 = 12, &c., and also

One 6 is 6.

Two 6's are 12.

Three 6's are 18, &c., &c., and are said not in consecutive order, but in a variety of ways, *e.g.*:—

Four 6's are 24.

Three 6's are 18.

Six 6's are 36.

Seven 6's are 42.

Ten 6's are 60, &c.,

and then again in another order.

As each table is mastered examples involving its use and that of previous ones are given, always in the nature of problems beginning with money questions as in addition and subtraction, and proceeding to the manipulation of pure

number; e.g.:—A boy was collecting shells, he got 23 every day for 72 days, how many had he then? The sum is written:—

$$\begin{array}{r} 72 \ 23 \\ 23 \\ \hline \end{array}$$

and the multiplication is worked by 20 first and 3 afterwards. The information as to how it is to be done is obtained from the child, he is able to tell you that you can do it by multiplying by 20 and 3 and adding the products, and that multiplying by 20 is easy because it is just multiplying by 2 and adding a cipher. The finished sum is:—

$$\begin{array}{r} 72 \times 23 \\ 23 \\ \hline 1,440 \\ 216 \\ \hline 1,656 \text{ Answer.} \end{array}$$

Great care is bestowed in seeing that the units, tens, and hundreds are kept in their proper columns all through the sum.

To multiply by hundreds and thousands is also found quite simple when worked in the same way.

Division.—We have now two different meanings to convey: the one an idea of continuous subtraction, the other an idea of fractional parts. The first one can impart by exercises in “sharing” worked with real nuts or pennies just at first; share 12 nuts out into heaps of three, how many boys must there be if each is to have three nuts, how many if each were to have two? How many sticks of chocolate could you buy if each cost 2d. and you had 6d. to spend? This sharing into heaps of 2 or 3, &c., is called division by 2 or 3, &c. Suppose I asked you to divide 15 into threes, what would you do? Divide it into heaps of 3 and we should have 5 heaps because there are 5 threes in 15. If I divide 8 by 2, how many heaps have I? How many if I divide 10 by 5? 18 by 6? &c., &c. The sign \div for division is explained, and a few simple money sums worked next, e.g., £6 12s. 4d. \div 2 = £3 6s. 2d. After this they learn how to divide when the numbers involve tens, and to write the division as short division, e.g., 2)48

24 e.g., two tens and a four;

this is demonstrated with “ten bundles” if necessary. We have by now discarded our idea of “heaps” of 2, and realise that 48 \div 2 means the number of two’s there are in 48. 4 tens, \div 2 = 2 tens, and 8 \div 4 = 4 units; the answer to 48 \div 2 is therefore 24.

Long Division.—Some children may not be able to do this, but it is worth trying. We begin with money sums, and a small number as divisor, and say that this is a new way of putting down division sums, so that we can *see* the Remainder after dividing into the pounds, shillings, and pence; or hundreds, tens and units, *e.g.*, 4)924(2

$$\begin{array}{r} 8 \\ \hline 1 \end{array}$$

4 into 9 hundreds gives 2 hundreds and we have 100 over, *i.e.*, we shall have two hundreds in our answer. What shall we do with this 100? Analogy with subtraction will perhaps lead the children to suggest that it should be turned into tens, though we are almost certain to have the question "Why can't you divide 100 by 4?" That would be dividing 100 units, and we must find out how many tens there are in the answer before doing the units. We then have ten tens and two others to add in, *i.e.*, 12 tens into which 4 will divide 3 tens times, we shall have 3 *tens* in the answer, and into the 4 units left 4 will divide once, so that the sum stands—

$$4)924(\underline{231}. \text{ Answer.}$$

$$\begin{array}{r} 8 \\ \hline 1 \text{ hundred} \\ 10 \\ \hline 10 \text{ tens} \\ +2 \\ \hline 12 \text{ tens} \\ 12 \\ \hline 4 \text{ units} \\ 4 \\ \hline \end{array}$$

The other aspect of division, *e.g.*, that suggesting fractions, is now taken. If I had 8 oranges and gave away half of them, how many would that be? How many if I gave away one-quarter? If I had 9 and gave away one-third? This last probably gets no answer at first. What did giving away one-half mean? Dividing into 2 parts and giving away 1. What then do you think giving away one *third* would mean? Dividing into 3 equal parts and giving one away? Yes, so we divide 9 into 3 equal parts and give 1 away. The third, one-half, one-quarter are written $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{4}$, and mean 1 of 3 parts, 1 of 2 parts, 1 of 4 parts, $\frac{1}{5}$ would mean one of 5 parts. Now we give many problems, all involving the use of simple apparatus, pennies, pencils, nuts; asking the children for simple fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ of numbers and letting them obtain the answer by dividing into heaps. Then one obtains from them the information that $\frac{1}{5}$ of 20 equals 4 and $20 \div 5 = 4$, so that $\frac{1}{4}$ of 20 means $20 \div 4$

and one gives them a number of simple sums to be worked orally, and introduces them to the notation for farthings in English money sums. The subject of fractions is left here; it is only touched upon at all in order to give the children a complete notion of division; the subject is not taken up again until the children have mastered decimals.

Weights and Measures.—This part of the subject is reached at the beginning of the average child's ninth year, and is the concluding part of the course in elementary arithmetic. The child weighs and measures for himself and makes his own tables; we let him, dressed in a pinafore, measure out pints and quarts of water, very careful not to spill anything, and weigh pounds and ounces of sand or anything clean and easily handled. Our own weights and measures, of which they have probably heard, are taken first. There are, of course, limitations to schoolroom weighing and measuring: tons and cwt., they are told, are the big weights used when men want to weigh very heavy things like beams of iron; and these weights are added to the table already made from drams to pounds.

After the tables have been made the children read through them once or twice and then have a number of rapid oral questions. How many ounces in two pounds? In 1 lb. 10 ozs.? How many cwts. in 1 ton 6 cwt.? &c., &c., and then a large number of problems to be worked in their books. The metric tables are given as those used in other countries, and the children take to them as ducks to water; every British teacher of arithmetic must bemoan the fact of British weights and measures. With our adoption of a decimal system of coinage and measurement a great deal of what seems such unnecessary labour on his or her part will be saved; the rational and logical aspect of this working in tens seems to appeal to the children as well as to the teacher, and they are always delighted when metric measures are involved in their sums.

The children work practical problems as much as possible at this stage, measuring the girth of trees, the furniture of the schoolroom, &c., and making thus a table of comparative weights and measures; metric expressed in British units and *vice versa*.

Measures of Area.—How should we be able to tell how big the wall is, or the floor? We can tell how long the picture rail is, how *long* and how *wide* the door, but we must get a new way of telling how big it is altogether. Look at a page of your arithmetic book; it is divided up into squares, and you can tell how big it is by counting the squares. Draw a picture of a book 6 squares long and 5 squares wide. How big is it? 30 squares big, because there are 5 rows each containing 6 squares. Draw the top of a pencil box 2 squares wide and 10 squares long; there are 2 rows of 10 squares each; it is 20 squares big. Now suppose we have each square 1 inch long and 1 inch wide, it is then called a square inch: if we had it

1 foot long and 1 foot wide we should call it a square foot, and we measure walls and doors by dividing them up into square feet or square inches. This kind of bigness is called *area*. We draw many rectangular figures on paper divided into square inches, and find the area of each by counting the number of rows and number of squares in each, *e.g.*, a rectangle 5 inches long and 2 inches wide.

$$\begin{aligned} \text{area} &= 5 \text{ rows of } 2 \text{ squares each.} \\ &= 5 \times 2 \text{ squares.} \\ &= 10 \text{ squares or } 10 \text{ square inches.} \end{aligned}$$

Thus the children finally arrive at the rule for finding areas, and then make themselves a table of square yards, feet, and inches. They now measure and find the area of hearthstones, windows, steps, blackboards, &c., and work numerous problems in their books. Metric square measure is taken in the same way, *i.e.*, by measuring rectangles divided into centimetre squares, and by measuring the furniture of their schoolroom in metres and centimetres.

Cubic Measure has to be taken in much the same way as square measure, dice or cubes of some sort serving for the preliminary stage, which is shorter than with square measure.

This concludes the fourth year of school life with the ordinary child, and though he has not perhaps progressed very rapidly or "done" many rules and varieties of sums, he has worked thoroughly and discovered much for himself.

During the last year he has begun *Geometry*, taking the subject quite experimentally and practically, learning how to handle the instruments, to use his eye as a judge of lengths and areas, to know the names and some properties of certain geometrical forms, *e.g.*, he discovers by drawing and measurement facts about the intersection of the diagonals of a square and a rectangle, and of parallelograms in general, and the meanings of some geometrical terms, *e.g.*, bisect, perpendiculars, &c., so that when the study of the subject is properly begun he starts with a certain equipment, the foundational ideas are mastered to some extent, and he is able to concentrate his attention on the reasoning out of propositions.

During his sixth and seventh years spent in Class I, he has learnt how to work in carton; to model cubes and chairs and many other things; each model involving the drawing of straight lines of given lengths, the keeping of square corners, and very neat and accurate cutting out. All this serves him in good stead now, his fingers can hold a ruler steady and straight, he draws neat and tidy lines, and is accurate.

To use Ruskin's words, anent *Geometry*: "You have now learned, young ladies and gentlemen, to read, to speak, to think, to sing, and to see Here is your carpenter's square for you, and you may safely and wisely contemplate the ground a little, and the measures and laws relating to that,

“seeing you have got to abide upon it:—and have properly looked at the stars; not before then, lest, had you studied the ground first, you might perchance never have raised your heads from it. Geometry is here considered as the arbitress of all laws of practical labour, issuing in beauty.”*

The Geometry lessons begin with points, and with lines, straight and curved, the children's own definitions being taken if possible; problems are given them which will elicit certain facts about lines, *e.g.*, “draw as many straight lines as you can through a certain point,” or “take 3 points and join each one to every other, how many lines have you?” We also give a lot of practice in judging of the lengths of lines both in centimetres and inches, the amount of error always being found; it is all done of course by work on the concrete, the schoolroom and things out of doors provide an endless supply of straight lines. We learn the meaning of “bisect” and “trisect,” and try to perform these operations without measurement. The next step is to draw plans to scale. I want to draw a picture of the window on a piece of paper not nearly large enough. How shall I do it? Why, draw it smaller, of course. How much smaller? This invites suggestions of which the best should be taken. Eight times as small? Very well, but how shall I be sure that it is eight times as small? Measure it. How? Measure each side and make each eight times as small. This is done and the window drawn and divided into its component panes, the scale of course is very carefully put in. After this doors, books, &c., are drawn to scale, a plan of the schoolroom made with windows, doors, and fireplace in. Then we get out our atlases and see that all maps are really plans drawn to scale, and we can find out by measuring and using the scale how far places are from each other, though here we must go carefully, as the scale given on a map is not by any means accurate. The scale on all maps must necessarily differ in different directions, and the results obtained can therefore be only approximate, though sufficiently so to be of use and interest to the children. After practice in map-measuring, we have a few problems, such as:—“If I look west from my house, I see a lighthouse 4 miles away, if I look west I see a church spire 3 miles away. Draw a plan (scale 1 mile to 1 inch), and find out how far the lighthouse is from the tower.”

The circle comes next, the children practise drawing them, and learn the meaning of radius, centre, circumference, arc, chord, diameter; then they draw circles, concentric, intersecting, &c., &c., and have problems connecting circles and their former work, *e.g.*, a lighthouse whose light has a radius of $4\frac{1}{2}$ miles, is 32 miles from another light, with a radius of 3 miles. Draw a plan to find what space of darkness a ship would have to cross in going from one to the other.

* John Ruskin. *Mornings in Florence, being simple studies of Christian Art for English Travellers*, p. 134. (London: George Allen, 1894.)

Angles are taken next, and are generally defined as "corners" by the children, they see the 2 ways in which a "corner" may be formed; by 2 lines meeting or by 1 line revolving about another; they also learn that blunt and sharp angles are called obtuse and acute, and what a right angle is like. They also find out how the magnitude of an angle can be tested, and that it is not affected by the length of its arms, for two equal angles can be formed by the opening out of a large and a small pair of compasses, for instance. The use of the protractor comes next, and when the children are proficient in this, they can find out the properties of vertically opposite angles, angles in a circle, &c., by drawing and measuring for themselves. The clock-face provides useful practice in the magnitudes of angles and in drawing circles. Plans are given involving the measuring of angles and distances. The questions must all have some purpose in them; a question like "Draw XY 4 inches long, at X draw a line making an angle 58° with XY, and at Y a line making an angle an angle 48° with XY," leaves an unfinished idea in the child's mind and makes his geometry lesson a play at purposeless lines and angles; or makes his work machine-like; he stops like a machine at the drawing of that third line because he sees no reason for stopping there; and he is ready like a machine for the next such pointless demand.

After angles, we are able to take the 16 points of the compass, with many exercises from maps or from plans which the children draw. Work on parallels follows very easily from this; as parallel lines are simply those in the same direction, *e.g.*, railway lines, or roads in the neighbourhood; it can be ocularly demonstrated that two people walking always in the same direction and apart, never draw any nearer each other. The names of the angles "alternate," "exterior," and "interior," are learnt: this is really advisable as an aid to the future when the proposition dealing with parallel lines is taken. If the pupils are not already familiar with the names of the angles formed by a pair of parallel lines with a transversal, we find that the proposition involving the properties of these angles invariably proves a source of confusion and difficulty. The children are next introduced to set squares, and shown how to use them, and are taught how to draw parallel lines in this way. Throughout this early part of the work we try to avoid as far as possible any approach to a proposition in the style of Euclid; all the work is a preparation for the logical proving of propositions that is to come; and any approach to it now is apt to result in some of the parrot-work so much affected by former students of Euclid. All we teach them, therefore, at this stage is to bisect a straight line (this being a very simple proposition can be taken as one of the ways in which circles may be used), and perhaps to bisect an angle and make one angle equal to another, though, as these latter mean necessarily

a mechanical effort of memory, it is advisable to omit them with a slow class. Throughout the work *accuracy* in all measurements and drawings is insisted upon, as this is absolutely necessary when so much is done visually.

All the work in Mathematics is greatly assisted by the "geography" walks taken by the children, *i.e.*, walks during which exercises in pacing, in compass reading, in judging of heights and distances by eye are given; *e.g.*, "make a plan of the road from our gate to the market square putting in telegraph poles, houses on the route, &c., &c.;" this is done by pacing the distances and taking directions with a pocket compass; or, "find the amount of water per minute brought down by the stream above and below the water-works; hence find the amount of water taken by the works"; this would come in with the arithmetic lessons on cubic capacity.

The rate at which the stream flows is determined by timing a floating stick down a measured length of the bank; and the width and depth are measured at what seem to be the best points for giving an approximately average estimate.

After a year of geometry the children have obtained a good idea of direction and distance, some familiarity with ordinary mathematical instruments, habits of neatness and accuracy which are of value in one's dealings with later Mathematics, and some small power of logical reasoning acquired by the tackling of the problems presented.

One's aim throughout in the teaching of Mathematics in these years is twofold:—

- (1.) To prepare the soil from which in after years it may be possible for a good mathematician to spring, *i.e.*, to teach a child the significance of $2 + 2$, keeping before one's vision the day when he might be seeking the significance of $\frac{dy}{dx}$, and
- (2.) To make sure that, even if the child's mathematical training should end with his ninth year, yet the intellectual development it has occasioned and the powers it has called forth are such as will be a valuable asset to him in all his future work; for, to quote again from Ruskin, we reveal to him "the Science of Number. Infinite in solemnity of use . . . including, of course, . . . the higher abstract mathematics and mysteries of numbers, but revered especially in its vital necessity to the prosperity of families and kingdoms."* And, again, "And how often in greater affairs of life the arithmetical part of the business must become the dominant one! How many and how much have we? How many and how much do we want? How constantly does noble Arithmetic of the finite lose itself in

* *loc. cit.*, pp. 135-137.

“base Avarice of the Infinite, and in blind imagination
 “of it? In counting of minutes, is our arithmetic ever
 “solicitous enough? In counting our days, is she ever
 “severe enough”?”*

Here, then, we obtain authority for our hope that our training is to help in the training of a good and capable citizen of his country. Stress is laid then upon one or two points in the general teaching:—

- (1.) The children are taught to formulate rules for themselves by working out several examples from first principles, and when the rule is formulated to use it immediately to shorten their work, *e.g.*, a child works several sums such as, “Find the cost of 12 things at 3*d.* each, 4*d.* each, &c.,” and hence formulates for himself the rule that “the number of shillings per dozen is the same as the number of pence apiece,” this leads to the habit of investigation so essential to the higher mathematician.
- (2.) We insist also upon concentration of thought throughout the lessons which range in duration from 20 minutes at first to 25 minutes in the last year; during that time attention and concentrated thinking are required; the children generally have an easy lesson, such as handicrafts or writing, to follow so that their brains are rested after the effort expended.

To every teacher of this subject it is now clear that the historical presentation of the subject is the easiest and most natural, *i.e.*, that it is to be presented to the child as it presented itself to the race; beginning with the concrete and working back to the abstract generalisation; and having as far as possible a practical bearing on matters of everyday life. Thus much is clear, but what is not always kept so distinctly before the teacher is that the concrete in our case is simply the means and not the end: the function of the concrete is to be simply a preparation for the abstract, or a means of symbolic illustration; it must therefore be discontinued as soon as possible. We find that the children get worried and bored by counters and beads, and work drags wearily on, always the same counters and beads, until the children's attention and interest have both vanished, and the lesson is actively productive of harm. A variety of objects is necessary in these lessons with the concrete, a child may, for instance, learn all about the number 10 from certain cubes of wood, but when the cubes being absent or something else in their place, he is asked about the number 10, all knowledge of it has vanished; 10 pertains specially to cubes and does not exist without them. Once the concrete has been passed, it is better not to go back to it for assistance; it thus becomes just a stage on the way to something fuller, and not a prop to be constantly leaned upon.

* *loc. cit.*, pp. 135–137.

We often hear that sums dealing with interesting things like oranges or tops, or dolls, should not be given to young children, as they are apt to fix their attention on the tops or dolls and not on the numbers; this we are inclined to think is entirely the teacher's fault; though the question is always approached by means of a problem, a beginner can soon be taught that for the working out the *numbers* are the primary things; this is made easier by writing in arithmetic books only the figures involved, until the answer is obtained, *e.g.*, "I went out with 6*d.* and spent 4*d.* on biscuits; I then met a friend, who gave me another 6*d.*, how much did I come home with?" The sum would be stated—

$$\begin{array}{r} 6 \quad 2 \\ 4 \quad 6 \\ \hline 2 \quad 8 \text{ pennies. Answer.} \\ \hline \end{array}$$

The answer must be noted as pennies and hailed as an important and interesting conclusion by some remark such as "So I came home richer than I went out."

There are of course several excellent text-books providing intelligent examples and lucid explanations; but at these very early stages of the subject a text-book can be almost entirely dispensed with; rather than consult text-books the teacher must use her ingenuity, must utilize other lessons and walks, &c., to point her definitions and amplify her instruction.

IRENE STEPHENS.

THE CORRELATION OF ELEMENTARY PRACTICAL GEOMETRY AND GEOGRAPHY.

Now that Practical Geometry has become so important a part of the Elementary Mathematics in the Junior and Middle School, leading on as it does to the more logical deductions, it is well to find what useful correlation can be made with other subjects in the curriculum. Many Schools connect it with Elementary Physics, though this merely requires the accurate measurement and drawing of lines and areas and brings in little else. But the meaning of the word Geometry, *i.e.*, earth measurement, suggests a subject more nearly cognate, *viz.*, Geography, and it is the aim of this paper to show in some ways how the correlation may be made, using Elementary Geometry as a basis for Scientific Geography which can be introduced to pupils about twelve and a half years of age.

MEASUREMENT OF LINES.

As soon as the units of length have been explained in the mathematical lesson, lines can be measured, either as between two points A and B, or between two towns on a map, *e.g.*, London and Birmingham, Manchester and Southampton. Incidentally this accustoms the class to finding places in a map. Following on this as quickly as possible comes the idea of *scale*, the representation of large lines by small, and *vice versa*. The results obtained previously from the map in inches or centimetres can now be translated into actual distances by means of the given map scales. Interest will be aroused if the results are checked as to reasonable accuracy by means of railway timetables, though if this method is adopted care must be taken to select towns connected by a direct railway line. Problems on the converse can also be worked: the class take their maps and, with a given town as base point, answer such questions as, What is the port 150 miles away? Find a lake 50 miles away, &c. This introduces the use of the compasses in determining the position of the free end of a given finite straight line, the other extremity of which is fixed. Arithmetical problems may be correlated with these lessons as the parliamentary fare gives a rough clue to the distance between two towns, and much useful information will thus be absorbed incidentally.

At this stage it is important to practise making rough estimates of linear dimensions. Boys probably have a rough unit of length in the cricket pitch, but many girls have not this advantage. Each member of the class should know

the length of his or her pace, should measure out in the playground a yard, a hundred yards, &c. The principle of linear measurement can be applied to the sphere. Two places on the surface of the sphere are connected by a piece of thin string drawn as tightly as possible, and the distance between the two places can then be measured. In the Geography lessons it can be explained why ships plying between ports do not always take the shortest route as found on the model sphere. Again, if a tracing wheel (such as is used by dressmakers) be employed, the principles of linear measurement can be applied to curved and irregular lines, *e.g.*, the length of a river from source to mouth, the extent of coast line of a country, &c., can be measured by counting the number of revolutions that the wheel makes as it traces out the line, and then multiplying the distance traced out in one revolution by the number observed.

MEASUREMENT OF ANGLES.

The idea of an angle and its measurement must enter at an early stage in order to lead on to more practical problems. As soon as the pupils have made a protractor in cardboard, with the chief angles marked upon it, they are ready to study the points of the compass. Their early lessons in Geography have given them ideas, probably vague, of the four cardinal points, and they can crystallise these by constructing a compass card accurately with the help of their protractors, and from the division of the right angles, either by folding or by geometrical construction, they obtain the new points, N.E., N.N.E., &c. The use of the compass needle can be shown practically by letting the pupils make a map of their route to and from school. In the first instance the plan should only be asked as regards *direction*, but later on it may be made more accurate by taking the pace as a unit of length and drawing all lines to scale.

MAP-MAKING.

When the measurement of lines and angles is thoroughly understood, the making of a map follows. In the Geometry lessons, dictated maps can be drawn, beginning with simple problems such as:—A man walks five miles east, then two miles south, and then returns home. Draw a map of his walk. In what direction does his last road run? These lead on to more difficult maps. The pupils should be taught to draw their first line in all directions so that a horizontal line in their note books does not always represent ore from west to east. At the same time, in the Geography lessons, practical map-making can be taken. First the pupils learn how to determine the south, either by finding the position

of the sun at noon, or by the help of the mariners' compass.* When this has been determined, the class should go to a level piece of ground which is fairly high, (a flat roof is excellent), and make a map, or rather panorama, of the surrounding district, purely as regards direction. A piece of paper is placed on the ground with a fixed line on it pointing to the south, and lines are drawn on the paper pointing to prominent places in the neighbourhood, *e.g.*, the church, a school, a factory, a pond, &c. The angles made by these lines with the base line are then measured, and on the return to the class-room the pupils draw a map of their observations.

Next linear and angular measurement can be combined. A ground plan of the class-room should be made first, the actual measurement of the walls and corner angles being made, and then reduced to a convenient scale for the note books. The class should be made to understand thoroughly that a reduction of scale only affects the lines and not the angles. As a rule, class-rooms are rectangular, so it is advisable to proceed to the playground, if that be more irregular. Here, two other methods of drawing a map may be introduced, that of triangulation, which brings in much elementary Geometry, and that of the field book used in surveying, the base line and offsets. The class will be interested in seeing that any one of the three methods produces the same results as the other two, and later on they find that the third method enables them to calculate the area of the map most conveniently. The data for making various maps can be dictated, either method of measurement being adopted. If these maps are of real places, and most guide books contain ground plans of churches, old castles, &c., the plans become more living to the pupils, and there is a certain stimulus of fascination in wondering what the design is going to be. The class can now solve quite easily problems such as:—London is 40 miles

* Some teachers to determine the south use the following rough-and-ready rule: hold a watch horizontally, with the hour hand pointing to the sun, then the line bisecting the angle between the hour hand and the figure 12, measured counter clockwise, will point to the south. But this method is inaccurate and should only be used for such practical purposes as determining the aspect of a room, or the general direction of a road, where the variation of a few degrees is immaterial. It may, however, be of interest for the pupils to find out how this rough rule arises. They know that the sun appears to take 24 hours to go from south on one day back to south again, a revolution, while the hour hand of a watch revolves completely in 12 hours, *i.e.*, the hour hand moves twice as quickly as the sun. Next let them observe the shadow of a stick during six hours, say 9 a.m. to 3 p.m., it describes a quadrant of a circle while the hour hand describes a semi-circle. Thus again the watch hand moves twice as quickly as the sun. Hence the rule appears to hold good. But by closer observation they find that the shadow does not describe the same angle in each hour, which shows that the angular motion is not uniform. They see then why they cannot assert that at any given time the watch hand has gone twice as far as the sun in its angular motion, and that the rule only gives the approximate position of the south.

east of Reading, Cambridge is 52 miles north of London, and Norwich is 55 miles north-east of Cambridge. Draw a map and find what would be the fare from Reading to Norwich (1) by rail *viâ* London and Cambridge, (2) by rail if a direct line existed. The objection may be raised why should not the class find the answer direct from the map, but the answer to this is a practical one. The railways run direct between the places given, and thus the data are obtained, but there is no railway direct from Reading to Norwich. When the class have obtained their results by geometrical construction, they should compare them with those obtained from reading the map and the problems thus become more concrete.

Another use for angular measurement is to study the direction of the wind. If a vane is near the school, a record can be kept of the wind's direction at a certain time each day, and this, combined with a record of the weather prevailing at the same time, leads to interesting deductions. If no vane is near, a simple one can be made with pieces of cardboard radiating from a stick run through a fixed compass card in such a way that the arms can rotate when they are blown by the wind.

Visualisation of heights might now be introduced in connection with the more accurate finding of a height by means of a base line and an angle of elevation. Many persons are able to roughly estimate the length of a horizontal, better than that of a vertical line.

Gradient.

The idea of gradient can also be explained, and the pupils can find out the gradients of roads in the neighbourhood in the following simple way. In the Geometry lessons they should discover that the acute angle between two perpendiculars to two given lines is the same as that between the original lines. This principle is involved in the construction of a simple clinometer. Take a rectangular piece of cardboard and to the middle of one side fasten a length of fine thread weighted at the other end. If the cardboard is held in the air so that this side is at the top, and horizontal, the string will hang vertically; fasten the string in such a way that the vertical line is at right angles to the edge of the cardboard. To determine the gradient of any slope, place the lower horizontal edge of the board along the slope, the lead line will still fall vertically, but will now make an angle with the original line, which will be at right angles to the slope. The angle between the two lines gives the slope, and if a protractor be fastened to the board, this angle can be read at once.

Contour Lines.

If the pupils have measured the gradients of known slopes in their neighbourhood, and have drawn angles in their note

books to represent slopes of 1 in 200, 1 in 50, &c., they will be able to appreciate better the meaning of contour lines on a map, and the use that may be made of them in determining the physical features of a country. After making a cross-section of a piece of land, taking as is usual different scales for vertical and horizontal distances, it might be useful to draw the actual slope on paper for a portion of the cross-section. (When the rise in height is only a very small fraction of the horizontal distance, this will not be advantageous.) The steepness of slopes of the sides of valleys and the consequent effect on the rivers flowing through them can thus be made more intelligible to the pupils.

MAP-MAKING FROM SHADOWS.

Hitherto the positions of places on the earth have been considered in relation to other places there. Now they can be regarded as in space with reference to the sun. The simplest beginning can be made with shadows. Practical observation with a lighted candle soon elicits the fact that the shadow of any solid body is thrown opposite to the light. If then a person is walking in an easterly direction at noon, his path will be at right angles to his shadow, which will be on his left hand; if he is walking south-west, his path is at an angle of 135° to his shadow, which is on his right hand to the back. (If the sun is not shining, a practical demonstration can be made with a lighted candle and a pencil.) When this principle has been mastered, a map of a walk should be made, from observation of the apparent alteration of the direction of the shadows, the unit of distance being the pace. Too long a time should not be spent in walking, as in that case the position of the earth with regard to the sun alters and affects the direction of the shadow. Country schools can do more in this way than those in towns, though most of the latter are near to a park or open space where tall buildings do not completely overshadow the street.

So far only the direction of the shadow has been observed. The next step is to show how the length of the shadow varies with the length of the object and also with the position of the sun, *i.e.*, the time of day. Two experiments with a lighted candle should be performed: first take a stick of fixed length, move it along a straight line perpendicular to the axis of the candle and record (*a*) the length of its shadow, (*b*) the angle of elevation of the top of candle from top of stick. The pupils find that as the angle of elevation increases, the length of the shadow decreases, but not in exact inverse proportion. Secondly, each pupil should take a stick of different length and keeping the angle of elevation constant, they find that the length of the shadow always bears a constant ratio to the height of the stick when the angle is fixed. When this is realised, they can proceed to calculate the heights of inaccessible

objects by observations of the lengths of their shadows compared with that of a known height, *e.g.*, if the shadow of a tree is 24 times as long as the shadow of a foot rule, then the tree is 24 feet high.

The class should make a systematic record of (1) the length of the shadow of a metre stick at half-hourly periods in one day, (2) the length of the shadow of the stick at noon during the year. Graphs of the results can be drawn, and as the stick remains of constant height, the graph shows the class that the altitude of the sun above the horizon alters. The altitude can be found on any particular day by constructing the right-angled triangle whose two sides are proportional to the length of the stick and its shadow. Elementary lessons on the seasons and on day and night fit in quite well at this point, but they do not require Elementary Geometry.

AREAS.

After the definition of a unit of area has been taken in the mathematical lessons, its use may be found in the geography lessons. Tracing paper divided into 1-inch squares can be used to find the area of a country, or its divisions. This method gives such very rough results when the map is on a small scale that it is advisable only to use it for *comparing* areas drawn on the same scale, rather than for determining the actual area of a country. Indeed but few maps show *areas* in their true relative proportion. The area of the Thames basin may be compared with that of the Severn or of England, but it is better not to attempt to find the actual area.

It is important that the class should realise that if the linear unit is reduced, say to one-tenth of its original size, then the surface unit is reduced to one-hundredth. Once again visualisation of areas should be taken, and if possible a piece of ground, *e.g.*, one square yard, or a tenth of an acre, marked out in the playground.

SPHERICAL GEOMETRY.

Up to this point only plane Geometry has been involved with respect to lines, angles, and triangles. The consideration of the properties of circles can be approached from observation of a sphere. A plasticine model or a ball is useful, but the cardboard or paper model made from the following description seems to make the ideas of the pupils even clearer than the solid does, probably because they can actually see the planes of the great circles.

"Cut out an even number of equal circles, the meridians, and slit them all up a radius; slit half of them on further, half-way to the circumference, and slit the other half of the circles from the circumference inwards, half-way to the centre

along the same radius. Now cut out another circle, the equator circle, of the same size as before, and draw on it as many diameters as there are equal circles. (Unless the equator circle is very large, eight is about as many as can be easily managed). Cut out of this circle a smaller concentric circle; also slit along the marked radii a very little from the smaller circle towards the circumference to hold the meridians firm; and cut the meridian circles correspondingly, a little from the circumference inwards. Fit the meridian circles together in pairs by slipping each into the other along the slit radius. Then insert the pairs into the equator circle. The fitting of the meridians into their places requires a little care, but owing to the elasticity of the paper can be done very neatly, provided that the cutting has been properly done beforehand. When hung up and rotated it shows the sphere excellently.*

After practice in making this model correctly, the pupils can insert smaller circles to show the circles of the Tropics and the Arctic and Antarctic Circles, though they may require help in the determination of the size of these circles. With the model in front of them they see that circles passing through the poles are all the same size, and that circles parallel to the plane of the equator vary in size decreasing towards the poles.

LATITUDE AND LONGITUDE.

The terms Latitude and Longitude can now be introduced, and the model enables the pupils to see that the number of miles moved in changing longitude by one degree decreases towards the poles, while the number of miles moved in changing latitude by one degree is constant. (Later on it can be explained that the earth is not a true sphere, and that the last statement is not quite correct near the poles.) The approximate calculation of the number of miles in a degree can be obtained by measuring the large globe which should always be used in conjunction with the class models.

The fixing of the position of a place on the earth's surface by determining its latitude and longitude shows the use of this division of the globe. The rotation of the sphere round a lighted candle while rotating round its own axis explains the ideas of day and night and of the seasons. The pupils may be convinced by experimenting with the model that two places at which it is noon at the same instant must be on the same meridian, and they may find by how much noon in Greenwich is later than noon in Paris or Berlin. It is advisable at the beginning, when these problems are taken, that the model should be handled as much as possible, so that the actual rotation is made. The spherical model will prove useful in

* Taken by permission from "The First Book of Geometry," by G. C. Young and W. H. Young.

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later stages when Mathematical and Astronomical Geography are being attacked, but, as these branches involve more than Elementary Practical Geometry, they are outside the range of this paper. If these notes have stimulated any interest in the correlation of the two subjects, and lead to the suggestion of new ideas and methods, they will have served their main purpose.

HELEN BARTRAM.

THE TEACHING OF MATHEMATICS IN
SECONDARY SCHOOLS.

THE POSITION OF MATHEMATICS IN SECONDARY SCHOOLS IN ENGLAND.*

There is in England no conception of school Mathematics as a limited whole which should be studied in its entirety by all who complete a full school course. Quite apart from the facts that many pupils leave early before getting into the higher forms and that some schools have a longer course than others, there are even between schools of the same general type marked differences in the amount of Mathematics included in the course offered, to say nothing of the still greater differences often to be found between individual pupils or forms in the same school according to the special courses they are following.

Ignoring, however, the case of those who leave prematurely or who through slowness fail to reach the higher forms, it may be said that the minimum course given in grant-earning Secondary Schools is that required to pass the Oxford Senior Local Examination in Mathematics; that is, Arithmetic, Algebra roughly to "progressions," and Geometry equivalent to Books I.-IV. of Euclid. Schools could easily be found in which neither the Algebra nor the Geometry is quite so extensive even as this. In other schools permutations and the binomial theorem for a positive integral index and the equivalent of Euclid Book VI. (proportion and similarity) are added; but the last subject, interesting and important as it is, is too often wholly omitted because it is not included in the syllabus for the London Matriculation Examination, and therefore is actually discouraged in those schools (still too numerous) where it is regarded as dangerous to go a hair's breadth beyond the examination syllabus. It is similarly discouraged in those schools which use the examinations of the Oxford and Cambridge Joint Board, and in the diminishing number of schools which use the Preliminary Examination of the Board of Education for the Elementary School Teachers' Certificate. It seems highly desirable that proportion and similarity should be included in the scheme of elementary Geometry, and that this should be adopted as a common policy by all examining bodies.

Taken at its fullest as including Algebra to the binomial theorem and the substance of Euclid I.-IV. and VI., this course can only be regarded as providing the "beggary elements" of a mathematical training, and this is all that is offered by many schools and much more than all that is taken by a majority of their pupils. Much stress has been laid recently by those interested in reform on the need for including Solid Geometry in the ordinary course, but little progress has been made as yet in the matter. It

* An amplification of a statement which has already appeared in the Annual Report of the Board of Education for 1910-1911.

is true that some slight reference is made to solids and with great advantage in the earliest stages of the course in many schools, but more than this is needed. Euclid's treatment, dull and barren as it is felt to be, is generally followed where the subject is taken at all.

Mensuration is not infrequently taken as an additional subject, and use is generally made of the opportunity for practice in the use of logarithms. Occasionally it is dealt with, as it ought to be, as a valuable training in Geometry, but too often formulæ are learnt and applied mechanically.

Of higher subjects the first and most frequently taken is Trigonometry. Sometimes this is read by a few individuals or by a small group, but in a fair number of schools it may be regarded as part of the normal course of work at least for boys. Until the last few years the order generally followed was to take the analytical development from the addition theorem before numerical solution of triangles; now it is common to develop the treatment of the single angle as far as possible and to get to the solution of triangles early, deferring (not infrequently never reaching) the analytical work. This, of course, has been done in sympathy with the more practical tendency given to Geometry, and the subject is occasionally, but far too rarely, illumined and enriched by actual practical work in the open air.

It is not uncommon to find that, owing to examination requirements, Trigonometry after being started, say, in the fifth form, is dropped in the sixth, or taken for a term or two and then dropped because it is not wanted for an examination. In the Senior Locals, it is true, Trigonometry may be added as a single additional subject to the minimum Mathematics, but in the London Matriculation Examination Trigonometry forms a part with other branches of "higher Mathematics" and cannot in effect be taken without these other branches, and is therefore actively discouraged in the numerous schools that limit their course by the standard of this examination.

The position of Mechanics is by no means satisfactory. In comparatively few schools is it taken as a class subject at all—much more rarely than Trigonometry. Quite commonly some practical work in Statics, and much more rarely in Dynamics, is included in the Science course—the Statics (parallelogram of forces, moments, pulleys, and inclined plane) often at a very early stage of that course; but it rarely happens that the experimental results are adequately discussed and built up into a sound body of knowledge in the Science course. Too frequently, when the subject is taken up again as part of Mathematics, little benefit is derived from the previous practical work, either because it has been too superficial, or because the mathematical teacher adheres to the old-fashioned abstract mathematical treatment and is not capable of making use of the practical work which has preceded. Here and there happily

there are exceptions to this, and in a few schools sound work is being done, from both the mathematical and the experimental point of view.

Hydrostatics is in an even more unsatisfactory position. Very rarely, indeed, is it now included with Mathematics, and though some parts of it nearly always come in the early part of the Science course, it is seldom dealt with in such a way that boys master its principles.

It is not common, even in boys' schools, and very rare in girls' schools, to find regular class work of a higher type than that already described, but individual pupils, and in the larger schools small groups, often proceed much further. Analytical Geometry is too often confined to the straight line and circle under the influence of the London University Intermediate Examinations, but the examinations of the Joint Board, the Locals and the Northern Universities Matriculation Examination encourage work of a wider type. Higher Trigonometry (de Moivre, &c.) and Algebra and more rarely the elements of Calculus are included: but probably in no case are these higher subjects taken by all boys who reach the top of a school, but only by those who are to some extent specialising in Mathematics.

The work of boys reading for scholarships at Oxford or Cambridge includes a greater amount of Algebra and Plane Trigonometry, modern Geometry, a full treatment of Analytical and Geometrical Conics, a considerable amount of Calculus and Mechanics. But more important than the extension of the range of work is the improvement necessary in its quality and the development of the power of handling problems of some difficulty. This renders the work of these boys altogether different in character from that of boys taking merely the ordinary course. Similarly valuable higher work is done in some cases by boys preparing for competition for Woolwich and for certain branches of the Civil Service.

Several causes have contributed to the limitation of the ordinary range of work in this subject in spite of its importance both as a Science and as a discipline.

(i) In the older schools the mathematical course for many of the boys has been limited by the requirements for entrance to Oxford and Cambridge. It is quite usual therefore to find, at least on the classical side of a great school, that the mathematical course for the bulk of the boys is extremely slight. The tradition thus established has limited the requirements for other examinations, with the result that nowhere does the obligatory course extend at the furthest beyond Algebra to the binomial theorem and the elements of Plane Geometry.

(ii) While the extent of the work has not been enlarged, it is probable that the difficulty of it has. Examinations inevitably lead to the elaboration of detail. An examiner, to test the real power of candidates, wants to set questions that have not been

set before. But every new problem set tends to become part of the necessary routine, and fresh sets of examples are constructed to make it familiar. The result is an ever increasing bulk of text books and sets of examples, built on "problems," which, while useful as problems for the independent application of knowledge and power, become burdensome and ridiculous when made the material for instruction. It is interesting to note that precisely the same trouble affects Mathematics in German schools, though their examination system is so different from ours. In both countries the feeling is growing among the more thoughtful teachers that, if school Mathematics is to attain its proper scope and influence, it must be disburdened of these undesirable accretions. In this country the Mathematical Association published in March 1911 a valuable report of conclusions on the point, but these have not yet found their way into practice. Their report deals only with the reform of Algebra and the necessity of including Numerical Trigonometry; but the lightening of Arithmetic is at least as necessary. Geometry has on the whole been made far more manageable and fruitful by recent changes and, as already urged, its minimum scope should be extended by the inclusion of proportion and similarity and Solid Geometry.

(iii) This reform, however, will be hindered as all other reform has been hindered, by the inadequate qualifications of a large proportion of mathematical teachers. Such teachers are almost inevitably bound by the particular text book they use; change is difficult to them in any case, and it is almost impossible, if indeed it were desirable, to set forth the changes required or proposed in such detail and in such a form that a poorly qualified teacher will escape serious misconceptions and mistakes. Not a few, also, of well qualified teachers have necessarily become accustomed to existing practice—sometimes, perhaps stiffened in it and unable to see the necessity for change. This latter difficulty is inherent in human nature and cannot be got rid of, but the former and greater one—the need for better equipment of the younger generation—calls aloud for attention.

(iv) In the newer schools which grew up under the influence of the Science and Art Department, the great stress laid on Science, though in some ways it has helped to improve the Mathematics, has probably on the whole tended to obscure the need for improvement. Care had to be taken at one time that the demand for Science should not be treated as met to some extent by what was then the more familiar subject, Mathematics. Special care was necessarily exercised as to the competence of teachers of Science, and the grants were based at first almost wholly on the efficiency of the Science equipment and teaching. Science teachers were often paid higher salaries than others, and money was freely spent on laboratories and scientific apparatus. All this, inevitable as it was, tended to

thrust other subjects into the background, and though the disturbing cause has been removed the effects still remain.

(v) In the older schools, where the classical tradition is strong, the mathematical course, as already stated, is limited by tradition. In the newer schools it is often limited by the shortness of school life. On the Continent where a large number of boys stay at school till 19, taking a complete nine years' course in one and the same school, the work naturally goes farther than it does with us in the case of boys who do not specialise. The normal course in our schools is comparable rather with the six year course in the German schools which qualifies for military privileges than with the nine year course which is required for entrance to the Universities and the professions. But it is not likely and probably not desirable that, if the ordinary school life with us is extended, all pupils will, as in the German schools, follow the same curriculum, pursuing Mathematics for instance to the end of their school career. Some degree of concentration and intensification of work in the later years of school life is generally felt among us to be desirable, and probably, if the normal course for boys can be made to include Trigonometry and Mechanics as well as Algebra and Geometry, it is as much as can be expected. Ample opportunity must of course be given for some boys to follow a wider course including Analytical Geometry and the elements of Calculus. For girls the case may be different and this point is dealt with subsequently.

(vi) When all difficulties of tradition and organisation are allowed for, the chief difficulty remains—the poverty of much of the teaching. Until the teaching, of beginners especially, is better informed, more illuminating and more stimulating, we shall never realise what is possible even under present limitations.

It has often been stated in full inspection reports on individual schools that the mathematical qualifications of many of the mathematical teachers were scanty. An attempt has been made to collect statistics that would serve to indicate the general state of affairs in this respect. The following statement is founded on such statistics gathered from over 650 schools. In the case of about 100 they were obtained at the time of inspection, when it was possible to examine more closely into the matter. The other returns were made by the Board's District Inspectors or by the school authorities through the Inspectors. The net results of the two sets of returns are practically identical and may therefore with the more confidence be accepted as substantially accurate.

Of the total of nearly 7,000 teachers in these schools, about 2,500 are engaged in the teaching of Mathematics beyond Arithmetic. Of these just less than 50 per cent. have read Mathematics to some extent at a teaching University and 40 per cent. have an examination qualification which includes Calculus.

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If men and women are taken separately, it appears that 44 per cent. of the men and only 27 per cent. of the women teach Mathematics; that the proportion of these teachers who have read some Mathematics at a University are 47 per cent. and 52 per cent. respectively; but that the proportion who have included Calculus are 44 per cent. and 33 per cent. As might have been expected, therefore, it appears that a much smaller proportion of women than of men teach Mathematics and that, in the case of those who do, the qualifications of the women are on the whole lower than those of the men. This is strikingly borne out in the case of "mixed" schools containing both boys and girls. In 149 of these included in the return there are 824 men and 633 women teachers. Mathematics is taught by 439 men and 138 women; 44 per cent. of the former and only 15 per cent. of the latter include Calculus in their qualification.

It is not of course suggested that the efficiency of individual teachers or even of a considerable body of teachers can be measured by their academic qualifications. This is only one, and not perhaps the most important, element determining efficiency. It is a common experience that teachers with poor qualifications may from their personal power be highly effective and that real scholars may be quite useless as teachers. None the less when the question is not of an individual or of a small group but of a large number, it remains true that the lack of good qualification must seriously limit the efficiency of teaching and further perhaps that the lack of such qualification is to some extent an index of the lack of that personal power which in individual cases may compensate for the lack of special knowledge.

It is then a matter for grave concern that half of our mathematical teachers have had no instruction in Mathematics beyond what they have had at school. It may be said that this is an over-statement, that many who have not studied Mathematics at a University have nevertheless studied it elsewhere. This is of course true, for study for the external London University degree has not been included under this head. But much of such study has been pursued under very unfavourable circumstances, and often not under the influence of living teachers. Further, allowing the utmost weight to this consideration, it is probably more than balanced by the fact that of those who have studied Mathematics at a University a considerable proportion have not carried the study beyond the stage of the intermediate examination, i.e., beyond one year. The statement may therefore be regarded as sufficiently near the truth.

As to the Calculus qualification, it may be asked what is the importance of this? Few will need to teach Calculus at school, and there are plenty of well qualified teachers to deal with the few boys and girls who need to go as far as this at school. The answer is twofold. First, it is clear that no one who has not read Calculus has made any very serious study of

Mathematics, and, taking cases in bulk, it is probable that his acquaintance even with the lower work that he has to teach will not be very thorough. Secondly, it is a matter of outlook. The Calculus probably does more than any branch of the subject to open the eyes to the meaning and possibilities of Mathematics, throws much light on the earlier work, and gives a sense of power in dealing therewith. It should be added that recent developments of even elementary school work both in Geometry and Algebra make a training in Calculus of immediate value to all teachers.

If the University qualifications are looked into more closely the result is still more unsatisfactory. Out of about 700 men whose qualifications are not obviously inadequate, 82 have obtained a first class in Mathematics in one or other part of the Oxford or Cambridge course, 87 similarly a second, 56 a third or lower class; 38 have obtained honours in the degree examinations of one of the other Universities. That is 263, or about a third, give evidence of a prolonged and more or less successful study of Mathematics. Most of the rest hold the pass B.Sc. degree of London or one of the newer Universities, including either Pure or Applied Mathematics, or, in some cases, both.

Of some 900 women teaching Mathematics, it has already been said that half have studied at a University, but a large number only to the intermediate stage, and only a third have included Calculus. Of these 300, 10 have taken first classes in one or other stage of the Oxford or Cambridge course, 42 second classes, 53 lower classes, and 25 have obtained honours elsewhere, making 130 in all. The rest hold only pass degrees, mostly in Science. Thus, of the 2,500 teachers of Mathematics, barely 400 can be said to have a high qualification.

For the sake of comparison it may be stated that the minimum special qualification for teachers of Mathematics in Prussian schools implies a course of University work in the subject at least as wide as that required for the pass B.Sc. degree of London, and that this only qualifies for teaching the subject in the lower classes. For the full qualification the requirement is more akin to an honours course at an English University.

In quite an appreciable number of schools—about 20 per cent. of the whole—it appears that no member of the staff has read Mathematics as far as Calculus, which may fairly be taken in general as an indication that there is no one in these schools really competent to advise and help the less qualified teachers, or even to do properly the most advanced teaching that ought to be given. The position in these schools is perhaps most obviously unsatisfactory. But even in schools where there are one or more well qualified teachers, it is almost always the case that a great deal of the work is in the hands of teachers whose mathematical qualifications are exceedingly slight. It may of course be said that for teaching beginners

this does not matter. But in the main this is not true: and it is just the present, conscious or unconscious, acceptance of this view which is responsible for much, perhaps most, of the poor results of mathematical teaching. The beginnings and early stages of most subjects and certainly of Mathematics are the most critical and call for the most skilled and the best informed teaching. Personal qualities, it is admitted, may make good great defects of knowledge; but exceptional personal qualities are not always available and it is certainly the case that much of the teaching of beginners and middle forms is dull and ineffective and that the one most obvious step to meet this defect is to insist on better qualifications, and therefore a wider outlook, in the teachers. It seems to be too readily assumed that because Mathematics has been part of every teacher's own school work every one is competent to teach the subject, at least in its lower stages. No such doctrine has ever been held, say, in regard to Science, or is now held in regard to Modern Languages or Art. It is indeed probable that the insistence that has been laid in recent years on Science teachers being properly qualified has re-acted somewhat detrimentally on the older and for the time being less noticed subject, and it is obvious that, in the case of many of the moderately qualified teachers of Mathematics, Science is their primary subject, Mathematics a very secondary one.

At the same time it is not suggested that the present position is worse than that of 10 or 20 years ago—quite probably it is far better. But no such materials as are now available are available for the past, so that no reliable comparison is possible. It is of course true that the number of schools and the number of teachers required is much larger than it was; probably the average qualification of teachers has also considerably improved, and not improbably the mathematical qualifications of teachers have improved on the whole; but the popularity of Science and the demand for special qualifications in Science have drawn many teachers away from the study of Mathematics, so that quite probably the improvement in mathematical qualifications has not kept pace with that in other subjects. However this may be, it seems quite plain that there is great need for the improved equipment of mathematical teachers, and obviously the first step towards this is the realisation on the part of school authorities that such improvement is necessary.

There is more divergence in practice, even in schools closely under the influence of the Board of Education, as to the treatment of Mathematics in girls' schools than in those for boys, and much more difficulty in deciding what is the right course to take.

In the main the pioneers in the education of women took as their ideals those already established for the education of men; and the Board, legislating on the whole in the light of existing practice, did not differentiate between boys and girls in relation

to Mathematics. More recently, however, distinctions have been drawn, and the Board have accepted proposals from girls' schools as to Mathematics which they would not have accepted from boys' schools, and in some cases have even suggested differentiation. Thus the Board's Regulations specifically provide that Housecraft may be substituted for Science or Mathematics in classes where the average age exceeds 15. Exemption of individual pupils from Mathematics has also been much more freely allowed with girls than with boys.

There are two fundamental differences of opinion on the matter. First, the difference between those who hold that there is an actual difference of capacity for the subject between boys and girls and those who hold that there is not. Second, among those who hold the former opinion, some argue that girls should not be worried with a subject that is essentially uncongenial to them; others on the contrary argue that they need the severe logical discipline all the more, and that they should not be allowed to escape it. There are of course minor disagreements arising from the limitations of school time and the claims of other subjects, but these are less fundamental.

In these circumstances the Board have adopted a waiting observant attitude, and while they have given considerable liberty of experiment to those schools that desire it, they have not in general encouraged differentiation between boys and girls or allowed it to go very far. In the comparatively few girls' private schools with which the Board has come into connexion through inspection it is common to find Mathematics neglected or taken by only a few girls; but on the whole the reasons for the neglect have been insufficient and the results have not been very encouraging.

The opinion of those who hold that Mathematics is essentially less congenial to girls than to boys, whatever truth may ultimately be found in it, must for the present be discounted by the fact that hitherto on the whole girls have not had as good a chance as boys. In private schools where the wishes of parents carry more weight than in public schools, the belief that Mathematics is unsuitable for girls is freely acted on, and this certainly brings with it the risk that the determining factor is really the wish (or even the whim) of the girl herself, often not unwilling to escape what she regards (from hearsay or from scanty experience) as a difficult subject. The tendency is increased and any deduction from it rendered quite unsafe by the fact that so much of the teaching of the early stages of Mathematics has been so dull and ineffective. Boys have got to live through this stage whether they like it or not, but if girls can escape naturally they will.

To a considerable extent girls are at a disadvantage as compared with boys in the character of the mathematical teaching they get. It has been shown above that the qualifications of the women teaching Mathematics are, in the mass,

somewhat more limited than those of the men. The natural result is (though there are of course many individual exceptions) that their outlook is more limited and their teaching less interesting and stimulating; and the very fact that they tend on the whole to be more minutely conscientious than the men makes them more likely to be pedantic and wearisome and to draw less personal effort and thought out of their pupils. Until there is really effective introductory teaching of a subject in a school, the experience and opinion of that school is clearly valueless in regard to the suitability of that subject for the pupils.

In "mixed" schools where boys and girls are taught actually together or under closely similar conditions, and in which exemptions are rarely if ever granted, it does not appear that in the first two or three years of mathematical work any clear difference can be established between the capacity of boys and of girls for Mathematics or even with regard to their interest in the subject. In a considerable proportion indeed of such schools the staff consider that up to a certain stage the girls hold their own or even more than hold their own with the boys. But there is considerable evidence that the average girl comes to the limit of what she can do in Mathematics with advantage sooner than the average boy. It is common to find, in fact, that when boys go on to Trigonometry or Mechanics girls do not. In part undoubtedly this is due to considerations of relative expediency and the varying claims of other subjects, but there is at least a possibility that when girls attempt these branches of work (still more when they attempt higher ones) they show on the whole less capacity for them than boys and profit less from them.

The provisional conclusion on which the Board, and for the most part the schools, may be said to be acting is that every girl should be given a fair chance of showing what capacity she has for Mathematics and that to this end she should have at least two years' instruction under a good teacher in Geometry and Algebra. Most girls should probably take a longer course, reading what has been described above as the ordinary minimum, viz., Algebra to progressions or the binomial theorem and the equivalent of Euclid I.-IV. and VI. Higher work should only be taken by those who show capacity and liking for the subject, though with better elementary teaching the number of these will no doubt be far greater than it is at present.

The two subjects, Physics and Mathematics, are so closely related that it is evident that the general introduction of Physics into school curricula ought to have an important influence on the Mathematics, possibly in regard both to the scope of the work and to the method of treatment.

Physics may be introduced in either of two ways—late on in the school course when a fair knowledge of elementary

Mathematics may reasonably be expected, or quite early when Mathematics is only being begun. The former was at one time the practice in England so far as the subject was introduced at all, and is to-day and by deliberate choice the practice in Germany. The latter is now the usual course in this country. It follows that Physics is being learnt concurrently with the most elementary Mathematics, and before any development either of mathematical knowledge or of mathematical power has been obtained. If the one end aimed at were the rapid acquisition of a sound knowledge of Physics, this course would be clearly unjustifiable, for much of the work to be done necessitates some maturity of mathematical thought. The modern English position of Physics can only be justified if two conditions are secured. First, no part of the subject must be introduced until the pupil is fit to deal with the Mathematics required for its reasonable treatment. Second, the Physics must be so treated as to illustrate and illumine the concurrent work in Mathematics. If the first condition is not observed the work in Physics cannot be sound. If the first condition only is secured, the system has no advantage over the German system, and as it is more difficult and more expensive to work, it would stand condemned in the comparison. When it is said that the pupil must be fit to deal with the Mathematics required, it is not meant that he must already have a knowledge of this Mathematics, still less of its developments, but only that he must be capable of understanding what is essential for the purpose in hand. For instance, a boy may properly begin Optics before he has learned Trigonometry, but when he comes to the study of refraction he ought to be capable of understanding what is meant by the sine of an angle and of learning to use a table of sines; it is not necessary that at that stage he should proceed to the full development of Trigonometry.

In any case the proper treatment of Physics demands some change in the traditional mathematical course. Simple Trigonometry, graphs and the fundamental ideas of the Calculus are essential. A wider knowledge of the facts of Geometry is wanted; less stress may be laid on the methods by which they are established. On the other hand many of the traditional elaborations of Algebra can be dispensed with. So in Arithmetic, while much of the traditional work is useless, a more thorough grasp of fundamental methods and facility and accuracy in applying them are required, while acquaintance with logarithms is obviously desirable. Whichever course then is adopted—the German or the English—the Physics is likely to react sooner or later on the mathematical course, altering possibly the selection of branches included in the course, certainly the relative emphasis laid upon them. The tendency to change thus set up is undoubtedly being felt both in this country and in Germany.

To come now to the two conditions laid down as necessary for the justification of the English system. Little need here

be said about the first; as it concerns Physics rather than Mathematics, but an example or two may be given of cases in which it is not observed. Certain experiments in Hydrostatics come into the Science course, but the fundamental principles—the laws of fluid pressure—are slurred over. Boys use balancing columns to determine densities, but often do not know, even as matter of fact, whether uniformity of bore is necessary, still less often understand why it is not necessary. Much time is spent over experiments to verify Archimedes' principle, but no attempt is made to explain its true inwardness, and it remains in general a mere rule of thumb. It may be that at the point when this work is commonly taken boys are not mature enough to deal properly with it—if so, the conclusion would seem to be that it should not be taken till later; but it is at least possible that boys could understand if the teachers would face the difficulty. The notion is too common that in a laboratory experiments are the one and only legitimate form of activity, and that thought and reasoning are out of place.

It is quite common again to find boys doing Magnetism without the knowledge of moments and couples which is essential to the proper understanding of their experiments. Their study of the refraction of light and of the tangent galvanometer is very commonly hindered by want of acquaintance with the trigonometrical functions. The investigation of the horizontal component of the earth's magnetism loses much of its value if, as is generally the case, the formulæ required have to be accepted without investigation.

Other examples might be given, but these are perhaps sufficient to show that the course in Physics is often drawn up with too little regard to the stage of development reached in Mathematics. It is equally true that the traditional order in which the various branches of Mathematics are introduced, and the manner of their development, are not those best adapted to assist the teacher of Physics.

Turning now to the second condition necessary if advantage is to be gained from the English system of the early introduction of Physics, it is clear that some of the facts just mentioned show not only breaches of the first condition, *i.e.*, mean unsound work in Physics, but also the loss of opportunities to benefit the Mathematics.

Acquaintance with the sine-function is needed in Light. If, as often happens, a clumsy substitute is adopted, because boys have not got this acquaintance, not only is the Physics damaged, but a most valuable opportunity is lost of illustrating the value of what, if studied in the mathematical classroom only, may well appear a mere uninteresting abstraction. Or if Statics is taken before Light and the triangle of forces studied experimentally, this affords an equally useful opportunity of introducing boys to the same function at a time when they actually need it—an opportunity generally ignored.

It may be said at once that in the most elementary work the influence of laboratory work in Physics on the Mathematics has been most beneficial. Decimals and fractions are now understood from experience of their actual use in the laboratory in a fashion which was comparatively rare a few years ago. The distinction between area and volume or between square and cubic measures is now commonly grasped for the same reason, and generally "units" have become realities. The whole subject of Mensuration has thus become more real and intelligible. In some schools much more than this has been realised, and in most schools some traces of mutual influence of the two subjects can be observed, but it would probably not be true generally to say that outside this preliminary subject of Mensuration much has been effected. Mechanics, Statics at least, is often included in the early Science work, but is treated too much as a collection of isolated experiments, leading to no coherent knowledge, and it is not used as it should be to enrich and illumine Mathematics, either by the incidental use of simple Trigonometry, or as leading to the full mathematical study of the subject itself.

Very few schools as yet have ventured on the experimental study of Dynamics, though simple methods for doing so are now available; and generally the mathematical treatment is still very abstract and out of touch with realities. Light is essentially a geometrical subject and can be used to give very valuable geometrical training both in the way of reasoning and in the way of drawing, but this possibility is very rarely made use of.

This shows that there is not yet that identity or closeness of touch between teachers of Mathematics and teachers of Physics which is necessary if either subject is to rise to its full value. It is certainly desirable that at least to some extent the same person should teach both subjects; if the teachers are different, it is necessary at least that each should be fully acquainted with both subjects and with their treatment in the school, and should have regard to his colleague's work.

An outstanding example of the lack of co-operation is given by the style of elementary Arithmetic in the laboratory. All measurements are of course approximate, and boys ought to be trained to do their calculations in such a way that they do not arrive at results pretending to a degree of accuracy unjustified by their measurements. Approximate or "contracted" methods of multiplication and division are it is true taught as part of Arithmetic, but taught from a point of view quite different from that necessary in the laboratory; taught too in such a way that no boy ever uses them unless he is compelled to do so.

The mathematical master presents them as methods of saving labour in long sums, his typical example being "multiply 537.4685 by 49.36894 correct to 2 places of decimals." He lays

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down rules for curtailing the work according to the degree of accuracy specified. A boy finds these rules troublesome—he would generally prefer to do a tedious straightforward piece of work, rather than take the trouble to think at the outset; the process never becomes habitual and remains therefore valueless. Such numbers never occur in the laboratory, and the problem there is not to save trouble, but to keep within the limits of honesty—to get all that can fairly be got out of the measurements and to pretend to nothing more. Science masters ignore the point altogether and have been known to tell boys for instance that “densities must always be calculated to 3 places of decimals,” irrespective of the fact that their measurements may only justify one. The difficulty can easily be met if both teachers will take the trouble; the method needed is much simpler than the one generally given in Arithmetic books and taught by the Arithmetic teacher, but for want of joint consideration it has generally escaped attention.

It is plain that if substantial progress is to be made, the study of both subjects—Mathematics and Physics—and, if possible, actual experience in the school teaching of both, must become much more general than it is at present. On this point German practice is of interest: the Prussian “*Prüfungsordnung*” of 1898 still in force lays it down that “Pure Mathematics and Physics must always be taken together.”

W. C. FLETCHER.

MATHEMATICS IN THE PREPARATORY SCHOOL.

Our most successful administrators have said that the young public school man, notwithstanding all that has been said and written about him to his disadvantage, is one of the best aids the country has in administering our Colonies. The administration of the Empire is to a great extent in the hands of those who have passed through the public schools. Thirty years ago a large percentage of boys entering public schools were educated at home in their early years, and recognised preparatory schools could be counted on the fingers of two hands. At the present moment there are 500 preparatory schools preparing boys for more than a hundred public schools, to thirty of which might be applied the name "the Great Public Schools." From this it will be seen that the vast majority of boys, who enter the public schools, receive their early training at the preparatory schools. As a result of this, there is a distinct break in the training, and an examination into the effects of this break, in any particular branch of study, is of great importance.

This paper deals with the effects of the break upon mathematical teaching, and may be divided into three heads:—

- (I) Effects due to public school scholarships offered to boys under 14 years of age.
- (II) Effects due to change of schools.
- (III) Recommendations.

As a help to the consideration of these effects, a *questionnaire* has been sent to the great public schools, and a separate one to the headmasters of 40 or 50 well-known preparatory schools, and I have been able to obtain much valuable information.

(I) At the outset, it may be said that it is a matter of regret that public schools find it necessary—and without doubt always will find it necessary—to offer entrance scholarships, for in an ideal system no scholarships would be offered to boys of thirteen years of age. In the keen competition which arises when 500 schools are doing work that might be done with half that number, there must be a certain number of masters who look to the publication of scholarship lists to give them a much desired advertisement. It is, however, a healthy sign that, without exception, teachers in public and preparatory schools are opposed to all forms of cramming.

(a) It might be supposed that many preparatory schoolmasters, in order to obtain scholarships, would cut off mathematical hours in order to win classical or modern language

scholarships, and *vice versa*; this, however, is not the case as a general rule. In a very few cases, boys are given a little extra work; and sometimes, again, in the last month before examination, some, "polishing" is done at the expense of a few hours which, under ordinary conditions, would have been given to other subjects. Most masters rightly let scholarship boys work separately from the rest of the school, in order that they may not be kept back by less proficient pupils. We may say that in nearly all schools boys take their scholarships "in their stride." It should, however, be noted that my information comes from 40 or 50 well-known preparatory schools, and that, in the remaining 450, there may be masters who, in order to win scholarships at the public schools, resort to cramming.

(b) In preparing boys for Mathematical Scholarships there is always a danger of cramming. In former days, when bookwork formed a large proportion of the test in mathematical papers, there was a much greater danger of this. Unfortunately in this subject a boy, even at the age of 13, may show great facility (I might even call it a fatal facility) in mechanical work and in the reproduction of bookwork, and from that point, when the higher ranges of the subject present themselves, he may fail to maintain his early promise. During the last few years the papers set for mathematical scholarships have improved very greatly, and very few are set in which a boy could score well who had only a superficial knowledge and lacked real mathematical ability. Though it may be invidious to mention the papers of any particular school, where so many are admirable, I feel I must say that the papers set for the Winchester College election are, in the opinion of many, very nearly ideal. These papers do not demand a great range of reading—not so much as in the cases of the papers of many other schools—but they test, not mere ability to learn mechanical processes, but originality and power. In Appendix I. of this paper will be found the papers set at Winchester in the year 1909.

(c) It is necessary to examine the evidence derived from the public schools as to the mathematical knowledge expected of classical and modern language scholars. It is very clear that the mathematical pass papers do not count; they are looked at with a very tolerant eye. The questions asked of the Headmaster on this subject were:—(1) "Does the pass paper in Mathematics mean anything (i.e., count anything) in Classical and Modern Language Scholarship papers? (2) What percentage of marks as a rule do classical and modern language scholars get in mathematical papers which are *not* pass papers?" In most schools the pass paper is a very simple one and in several it counts "practically nothing," though in one important school an unusually bad paper in Mathematics caused a boy "to drop from 7th to 12th place in the scholarship list," and in another "the pass paper in Mathematics might differentiate two boys equal in Classics." In (2) the papers referred to are not pass

papers and classical boys as a rule seem to score anything from 0 to 20 per cent.

(d) It is important to consider, further, the ultimate success or failure of mathematical scholars. In former days of the purely classical public school, when modern sides were unknown, the boy whose bent was mathematical was under great disadvantage. To a question as to the success of these scholars in the general life of a public school and afterwards in life, the answer is almost invariably that they are successful; they take a good position in the school life, and many go on to the Universities as mathematical scholars. This is largely due to the fact that they are required to attain to a fair standard in other subjects before being allowed to specialise in Mathematics. Further, public schools do not look out for young mathematical prodigies, but are quite ready to take boys of undoubted ability, who have had the ordinary training obtained in a preparatory school.

(II) "Is there a break in continuity in mathematical teaching owing to the passage from the preparatory to the public school?" Ten years ago the answer to this question would have been "Yes." To-day, though the answer could not be "No," the break is much less pronounced, owing to the more definite agreement as to sequence. It will be worth while to look into the causes of this agreement.

(a) The chief factor has been, without doubt, the institution of the Common Entrance Examination for admission into the public schools. Instituted in 1904, and directed by a board of six, three drawn from public school headmasters and three from preparatory school headmasters, this examination is at the present time adopted by 52 public schools, which in fact include all the great public schools, with one notable exception. Before its institution each public school had its own particular examination, and preparatory school masters endeavoured to arrange that each of his pupils was prepared for the special papers he was likely to have at the school for which he was entered. It is not necessary to point out how much the work at preparatory schools has been unified as a result of this common examination. I find that in answer to a question to both public and private schools as to the utility of the examination, the answer has invariably been that as pass papers the mathematical papers are excellent, though there is a very general feeling that the time devoted to Mathematics is too short, and that alternative papers for boys of different standards would be more effective than alternative questions, and would ensure a more accurate estimate of a boy's capacity and capability.

(b) The co-operation of the Head Masters' Conference and the Association of Preparatory Schools, moreover, has brought about the issue of a schedule of mathematical work suitable

for boys from the ages of 9-16* ; it will be noticed that the limits of age envelop the break from one school to the other. This schedule has been submitted to all the public schools and to all the members of the Association of Preparatory Schools. Unfortunately a large number have either not troubled to look at it, or have not studied it with care. On the other hand, in nearly every case where it has been considered, the scheme is approved of, and in most cases adopted. From one great public school, however, comes the answer : "Not approved of and not adopted" ; and from another important school : "We only agree with a small portion of it—to the bulk of it we are in strong opposition." These are the only strong dissentients, and in the latter case the master in charge of the mathematical department has strong views of his own on mathematical teaching ; but, though he carries them to a very successful issue in his own school, his methods would not be generally possible in other schools.

(c) To the public schools the following question was sent :—
"Do you find boys coming from Preparatory Schools in any way uniform in their methods?" This question was a natural one, as, in a given list of 20 new boys at a public school, it would be observed that they were drawn from a dozen or even 15 different schools. The answers to this question are hardly satisfactory. One school, indeed, replies frankly : "No, and not expected." To the question was added : "This may not be the case in geometry. Are they uniform in arithmetic and algebra?" It is a remarkable fact that, notwithstanding the revolution in the teaching of geometry in the last ten years, there seems to be more uniformity in geometrical teaching than in arithmetic and algebra. There are still teachers who regret the old Euclid with its rigid chain of reasoning, but it has been realised that the earlier propositions in that book were beyond the powers of the young boy, and should be left till later in the course. By assuming a certain number of facts, by aid of which easy riders can be proved, a young boy can be led on to reason according to Euclid's method, when he will find the propositions come to him naturally and easily. The use of hypothetical constructions has done away with many of the old clumsy "problems" of Euclid, and the so-called "geometrical drawing" strengthens his grasp of the subject. There is always a danger, especially in the case of teachers who are not mathematicians, of overdoing this branch of the work. I believe, however, that the majority of teachers would say that the change in the method of teaching geometry has been for good, and it will be found that, if the headmasters' schedule be adopted, the break in passing from school to school will entirely

* This schedule will be found in Appendix II ; it is now sent out with information relative to the common entrance examination, and papers are set in accordance with the scheme.

disappear. For the most part arithmetical teaching shows the greatest lack of uniformity. This is, again, due, as in the case of the schedule, to the fact that variations in text-books are infinite and that headmasters in their busy lives lay aside reports which are sent to them and allow them to grow dusty on their shelves, if indeed they are not at once consigned to the waste-paper basket. The Mathematical Association issued in 1907 a report of their committee on "The Teaching of Mathematics in Preparatory Schools."* If this report were followed, methods would become uniform; the suggestions there both as to arithmetic and algebra are thoroughly sound and, if followed, would in no way hamper the individuality of any particular teacher. Further, the same Association published a report on the teaching of algebra in 1911.† There is a danger in preparatory schools of covering too much ground in algebra and it will be noticed that in the schedule which is given in Appendix II. the range suggested is comparatively small.

(d) Far more satisfactory are the answers to the question: "Do you think boys come from Preparatory Schools better equipped in Mathematics than (say) 10 years ago?" From two schools comes the reply: "Cannot say;" and from one: "Cannot see much difference;" but in every other case the answer comes "Yes," and in one case: "Decidedly so." This is due, without doubt, to the fact that the preparation is more uniform (though, as has been shown above, not completely so), by reason of the all-round preparation for the Common Entrance Examination.

(III.) It may be well to state clearly here the recommendations which, in the opinion of many, should be followed, and which have been brought out in the course of this paper. If they were followed there is no reason why the ill-effects due to the scholarship system should not be minimised, and the break, outside the personal one, between the two types of school, be entirely bridged:—

(a) More scholarships for proficiency in all subjects. As has been said above, a boy of 13 may show a high standard in one subject, in which his early promise may never be fulfilled; the boy, on the other hand, who shows all-round ability, is very likely to develop strongly later on in some special subject.

(b) More weight should be attached to the mathematical attainments of boys competing for classical and modern language scholarships. A definite percentage should be required in the pass paper, and unusual excellence in Mathematics in a would-be classical scholar should receive substantial reward.

* Published by Geo. Bell & Sons. Price 3d.

† Report of the Mathematical Association Committee on the Teaching of Elementary Algebra and Numerical Trigonometry, 1911. (London: G. Bell and Sons. 3d.)

(c) Whilst using their own individuality in teaching, all mathematical teachers in public and preparatory schools should adopt the schedule issued by the Curriculum Committee of the Headmasters' Conference, setting forth the course to be taken by boys aged 9-16.

E. KITCHENER:

LOS ANGELES CITY SCHOOLS.

Table with columns: Name of Occupation, Nature of Occupation, Preparation Necessary, Demand for, Opportunity for Advancement, Money Returns, Other Conditions affecting Desirability, Professional Outlook with College Course. Section: ART WORK. Includes occupations like Commercial Art, Hand-wrought Metal Work, Interior Decorating, Leather Work, Pottery Work.

The GARDENA AGRICULTURAL HIGH SCHOOL, as its name indicates, has for its most distinctive function the training of boys and girls for agricultural pursuits.

Table with columns: Name of Occupation, Nature of Occupation, Preparation Necessary, Demand for, Opportunity for Advancement, Money Returns, Other Conditions affecting Desirability, Professional Outlook with College Course. Section: AGRICULTURAL OCCUPATIONS. Includes occupations like General Farmer, Speciality Farmer, Truck Gardener, Landscape Gardener, Nurseryman, Dairy Farmer, Poultryman, Farm Mechanic.

The following high schools offer a four-year Commercial Course:—MANUAL ARTS, POLYTECHNIC, HOLLYWOOD, and SAN PEDRO. All the high schools offer some commercial work.

Table with columns: Name of Occupation, Nature of Occupation, Preparation Necessary, Demand for, Opportunity for Advancement, Money Returns, Other Conditions affecting Desirability, Professional Outlook with College Course. Section: COMMERCIAL WORK. Includes occupations like Office Service (Multigraph Operator, Filing Clerk, etc.), Accountancy (Accountant, Auditor, etc.), Stenographic (Stenographer, Reporter, etc.), General Work (Shipping Clerk, Business Manager, etc.).

Every high school in the Los Angeles system offers training in Domestic Science and Domestic Art. The schools offering a distinct four year course in these subjects are HOLLYWOOD, with its excellently appointed and equipped Household Arts building, and its strong course of study; MANUAL ARTS, POLYTECHNIC, and the GARDENA AGRICULTURAL HIGH SCHOOLS, which are also well equipped, and offer strong courses in this field.

For the vocations below listed, it is generally necessary for the student to supplement her school training with some practical experience in the desired field.

Table with columns: Name of Occupation, Nature of Occupation, Preparation Necessary, Demand for, Opportunity for Advancement, Money Returns, Other Conditions affecting Desirability, Professional Outlook with College Course. Section: DOMESTIC SCIENCE and DOMESTIC ART. Includes occupations like Caterer's Assistant, Cook, Teacher, Housekeeper, Waitress, Dressmaker, Millinery, Seamstress.

The SAN PEDRO HIGH SCHOOL, because of its location and the interests of the community which it serves, offers courses leading to the vocations here listed.

Table with columns: Name of Occupation, Nature of Occupation, Preparation Necessary, Demand for, Opportunity for Advancement, Money Returns, Other Conditions affecting Desirability, Professional Outlook with College Course. Section: MARINE VOCATIONS. Includes occupations like Boat Builder, Engineer (Marine-Gasoline), Merchant Marine, Naval Architect, Marine Biology (Aquarium Attendant, etc.).

THE POLYTECHNIC HIGH SCHOOL devotes itself particularly to pre-technical instruction. While, therefore, other high schools in the system offer work tending to prepare for some of the vocations listed below, the Polytechnic High School offers courses leading to all of them. All of the high schools offer shop-work, either wood-work, iron-work, or both, and related mechanical drawing.

In almost all the vocations here listed, the student needs practical experience in addition to the training of the school before he can be considered thoroughly efficient.

Table with columns: Name of Occupation, Nature of Occupation, Preparation Necessary, Demand for, Opportunity for Advancement, Money Returns, Other Conditions affecting Desirability, Professional Outlook with College Course. Section: TECHNICAL and SEMI-TECHNICAL VOCATIONS—TRADES. Includes occupations like Assayer, Blacksmith, Cabinet Making, Chemistry, Draftsman (Architectural, Mechanical), Foundry, Electrical Work (Central Station, Telephone, Electrician), Machine Shop Work, Pattern Making, Surveying.

APPENDIX I.

WINCHESTER COLLEGE MATHEMATICAL
SCHOLARSHIP PAPERS.

JULY 6TH, 1909.

MATHEMATICS.—I.

[N.B.—In assigning marks to this paper, consideration will be paid to the ages of the candidates.]

I. Find all the factors which are common to the three numbers, 24, 60, and 72; and the common multiples of these three numbers which lie between 1000 and 3000.

II. Simplify:—

$$(1) (12 \cdot 41 + 2 \cdot 09) \times 1 \cdot 2 - (4 \cdot 5 - \cdot 9) \div 1 \cdot 8.$$

$$(2) 12 \cdot 41 + 2 \cdot 9 \times 1 \cdot 2 - 4 \cdot 5 - \cdot 9 \div 1 \cdot 8.$$

$$(3) \frac{1}{x+2} + \frac{1}{x+3} \div \frac{1}{x+2} + \frac{1}{x+3}.$$

III. A pint of water weighs $1\frac{1}{4}$ lbs.; one kilogram = 2·206 lbs.; one cubic centimetre of water weighs 1 gram. Find to three significant figures the number of pints in a litre.

IV. Solve the equations:—

$$(1) \frac{x-1}{2} + \frac{x-2}{3} = 3.$$

$$(2) y + 2x = 5.$$

$$2x - 3y = 1.$$

$$(3) (x-3)(x^2 - 5x + 5) = x - 3.$$

Is the equation $(x-1)(x-2)(x-3) = 2x^3 - 7x^2 - 15x - 30$ satisfied by any of the values $x = \pm 6$; or ± 4 ?

V.—(1) What is 5 per cent. of £20?

(2) £20 is 5 per cent. of what?

(3) What must goods be sold for if they cost £20, and a profit of 5 per cent. is to be made?

(4) What must goods, have cost, if a profit of 5 per cent. is made by selling them for £20?

(5) A man sells goods, which cost him £900, through an agent. The agent keeps 10 per cent. of the selling price himself, but this still leaves the owner with a profit of 10 per cent. For how much did the agent sell the goods?

VI. On inch paper mark the points A 1, 1. B 3, 2. C 3, 3. D 0, 3.

Find (1) the locus of points equidistant from the lines AB and CD .

(2) A point O , $\frac{1}{2}$ inch from CD and equidistant from A and B .

Give a short explanation of your method, and do not rub out construction lines.

VII. At 10 o'clock a steamer is observed 10 miles due north of a lighthouse. Half an hour later it is 5·18 miles north-east of the lighthouse. By means of an accurate figure, find:—

(1) The course the steamer is steering and its speed in miles per hour.

(2) When it will be at its least distance from the lighthouse, and what the least distance is.

VIII. Defining a parallelogram as a four-sided rectilineal figure with its opposite sides parallel, prove:—

- (1) That its opposite sides are equal.
- (2) That if its diagonals are equal, the figure must be a rectangle.

Draw two parallel lines an inch apart, and another pair, also an inch apart, intersecting the first pair. Prove that the figure formed by the lines is a rhombus.

IX. ABC is a triangle right angled at A . Squares $ABGF$ and $ACKH$ are described outwards on the sides AB and AC . Prove that A is equidistant from BC and FH .

X. Prove that the sum of two opposite angles of a cyclic quadrilateral is 180° .

What does this property become when two adjacent corners of the quadrilateral move together and ultimately coincide?

XI. Find the factors of

- (1) $14 + 5x - x^2$.
- (2) $(ax + b)^2 - (cx + d)^2$.
- (3) $x^3 - a^3 + x + a$.

Simplify:—

$$\frac{x^3 - 7x + 12}{x^2 - 16} \div \frac{x^2 - 27}{x^2 + 4x}$$

JULY 7TH, 1909.

MATHEMATICS.—II.

[N.B.—In assigning marks to this paper, consideration will be paid to the ages of the candidates.]

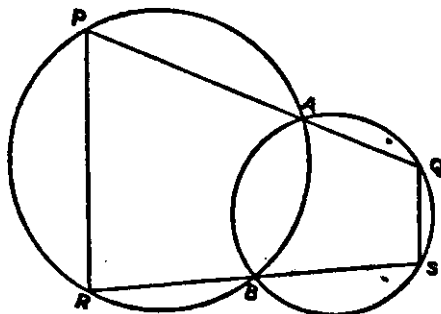
I. Show that in the equation $x^2 - 6x + 4 = 0$ the sum of the roots is 6 and the product 4. [You need not actually find the roots unless you like.]

A cuboid has edges of lengths a, b, c cm., where a, b, c are the roots of $x^3 - 9x^2 + 22x - 12 = 0$. Find the surface and volume of the cuboid.

II. Prove that $x(x + a) + b$ and $x(x + b) + a$ have a common factor for all values of x if and only if $a + b = -1$, supposing that a and b are not equal.

III. You may take it as a known theorem that if two circles cut at A and B and chords PAQ, RBS are drawn to meet the circles in P, Q, R, S then PR is parallel to QS . Do not prove this, but enunciate the particular cases of this theorem when

- (1) Q and S are very close together.
- (2) Q and A are very close together.
- (3) A and B are very close together.



IV. Two circles APB and CPD intersect at P , and the straight lines APD , BPC meet these circles in A , B , D and C . Prove that if the radius of the circle APB is double that of CPD then the chord AB is double CD .

V. Two circles touch externally at A and BC is a common tangent. Prove that the circle on BC as diameter touches the line joining the centres at A .

VI. A train has to run at x miles an hour in order to keep up to time. It has been delayed and the engine-driver finds he is M minutes late. He increases the speed to y miles an hour. What distance will he have to run at this speed before he is up to time again?

VII. If one mile in x seconds is the same pace as y miles in one hour, find the numerical value of xy .

VIII. The side of a hill is a plane, and the steepest path on it (*i.e.*, straight up the hill) makes an angle of 45° with the horizontal. Find (graphically or otherwise) the inclination to the horizontal of a path on the side of the hill which makes an angle of 45° with the steepest path.

IX. Given that x_1, y_1 are the roots of $\left. \begin{array}{l} 2x - y = 12 \\ x - 2y = 7 \end{array} \right\} ;$

x_2, y_2 the roots of $\left. \begin{array}{l} 2x - y = x_1 \\ x - 2y = -y_1 \end{array} \right\} ;$

x_3, y_3 the roots of $\left. \begin{array}{l} 2x - y = x_2 \\ x - 2y = -y_2 \end{array} \right\} ;$

and so on: find the value of $x_n + y_n$.

X. Three pipes A , B , and C , used to fill a swimming bath with horizontal floor, supply respectively a gallons, b gallons, c gallons per minute. A and B together raise the level p inches per hour, while B and C together fill the bath in q hours. Find the depth of the bath.

APPENDIX II.

SYLLABUS OF MATHEMATICAL TEACHING FOR BOYS AGED 9 TO 16, FOR NON-SPECIALISTS.

Issued by the Curriculum Committee of the Headmasters' Conference.

It is assumed that revision of previous work will form part of the work of each stage.

ARITHMETIC.	GEOMETRY.	ALGEBRA.	TRIGONOMETRY.
<p>I. Notation and numeration. The four simple rules. Unitary method (sums not involving fractions). Prime factors. Tables. Reduction. Compound Rules. (£ s. d.; mile, yard, foot, inch; ton, owl., pound. Questions involving English units should be the simplest possible). Areas and volumes of rectangular figures (not involving compound quantities). <i>This section to be illustrated by squared paper and solid models.</i> Square and cube root by factors.</p>	<p>(References are to the Cambridge Schedule reprinted below.)</p> <p>Fundamental concepts. Drawing and measurement of lengths and angles.</p>		

APPENDIX II.—Syllabus of Mathematical Teaching for Boys aged 9 to 16, for Non-Specialists—*continued.*

ARITHMETIC.	GEOMETRY.	ALGEBRA.	TRIGONOMETRY.
<p>II. } Vulgar fractions. Unitary method, leading up to solution by means of ratios. Decimals and metric system (<i>illustrated by actual practice in measuring and weighing</i>); 4 rules. Expression of vulgar fractions as decimals and vice versa (omitting the method of expressing recurring decimals as vulgar fractions). Reduction in square and cubic measure (including metric system). <i>Illustrate as in Stage I.</i> Areas and volumes of rectangular figures.</p> <p>III. } Decimalisation of money, &c. Percentage and ratio; profit and loss. Simple Interest.</p>	<p>The fundamental facts about— (a) Angles at a point. (b) Angles made by parallel lines and a transversal. (c) Sum of the angles of a triangle, and of a polygon. (d) Congruent triangles. (e) Equality of the base angles of an isosceles triangle. Schedule B, 1-11. <i>(Formal proofs of the above should be postponed to Stage V.)</i></p> <p>Constructions. Cambridge Schedule A, 1-6. Parallelograms. Cambridge Schedule B, 14, 15. Loci. Cambridge Schedule B, 21, 22. Area of parallelogram and triangle. Cambridge Schedule B, 16, 17, 18; Cambridge Schedule A, 7. Geometrical illustrations of algebraic identities. Cambridge Schedule B, 19.</p>	<p>Elementary notions; notation; negative numbers. Four simple rules (multiplication and division by monomials only). Simple equations in one unknown, numerical coefficients; problems.</p> <p>Simple equations in two unknowns, illustrated by graphs (numerical coefficients). Problems. Multiplication and division—binomial operator. Factors. Fractions—numerical denominators.</p>	

<p>IV. { Square root by rule. Revision.</p>	<p>Theorem of Pythagoras, and converse (not the extension); part of B, 20.</p> <p>Circle { Chords & arcs; Schedule B, 23, 24, 27. Tangent Properties: Schedule B, 28, 29; Schedule A, 8. Angle Properties: Schedule B, 30-34. Constructions: Schedule A, 9, 11.</p>	<p>Quadratic equations (numerical coefficients); problems. Further work with fractions.</p>	
<p>V. { Mensuration of plane and solid figures. Simple business Arithmetic, such as is needed for ordinary affairs.</p>	<p>Revision of theoretical geometry up to above stage: Schedule A, 1-9; 11. Schedule B, 1-34. Proportion; similar triangles. Schedule A, 10, 12; Schedule B, 35-40.</p>	<p>Fractional and negative indices The use of logarithms. Variation. Equations in two unknowns, one equation being linear.</p>	<p>Numerical trigonometry, up to the solution of triangles by the sine and cosine rules, and the use of tables of log, sin, &c.; problems of surveying, &c.</p>

It should be borne in mind that this Syllabus does not limit the teaching of each stage to one year. No doubt boys at Preparatory Schools may reach the end of the Fifth Stage with success. It is assumed that the normal standard for the non-specialist on leaving a Preparatory School is represented by Stage III.

SCHEDULES REPRINTED FROM THE CAMBRIDGE
UNIVERSITY ORDINANCES.

(In the Cambridge Ordinances the constructions and theorems are not numbered, and no definite sequence is imposed. Numbers are given here for reference.)

SCHEDULE A.

1. Bisection of angles and of straight lines.
2. Construction of perpendiculars to straight lines.
3. Construction of an angle equal to a given angle.
4. Construction of parallels to a given straight line.
5. Simple cases of the construction from sufficient data of triangles and quadrilaterals.
6. Division of straight lines into a given number of equal parts or into parts in any given proportion.
7. Construction of a triangle equal in area to a given polygon.
8. Construction of tangents to a circle and of common tangents to two circles.
9. Simple cases of the construction of circles from sufficient data.
10. Construction of a fourth proportional to three given straight lines and a mean proportional to two given straight lines.
11. Construction of regular figures of 3, 4, 6 or 8 sides in or about a given circle.
12. Construction of a square equal in area to a given polygon.

SCHEDULE B.

Angles at a Point.

1. If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles; and the converse.
2. If two straight lines intersect, the vertically opposite angles are equal.

Parallel Straight Lines.

3. When a straight line cuts two other straight lines, if
 - (i) a pair of alternate angles are equal,
 - or (ii) a pair of corresponding angles are equal,
 - or (iii) a pair of interior angles on the same side of the cutting line are together equal to two right angles,
 then the two straight lines are parallel; and the converse.
4. Straight lines which are parallel to the same straight line are parallel to one another.

Triangles and Rectilinear Figures.

5. The sum of the angles of a triangle is equal to two right angles.
6. If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles.
7. If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.

8. If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.

9. If two sides of a triangle are equal, the angles opposite to these sides are equal; and the converse.

10. If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

11. If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.

12. If two sides of a triangle are unequal, the greater side has the greater angle opposite to it; and the converse.

13. Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.

14. The opposite sides and angles of a parallelogram are equal, each diagonal bisects the parallelogram, and the diagonals bisect one another.

15. If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

Areas.

16. Parallelograms on the same or equal bases and of the same altitude are equal in area.

17. Triangles on the same or equal bases and of the same altitude are equal in area.

18. Equal triangles on the same or equal bases are of the same altitude.

19. Illustrations and explanations of the geometrical theorems corresponding to the following algebraical identities:

$$k(a + b + c + \dots) = ka + kb + kc + \dots,$$

$$(a + b)^2 = a^2 + 2ab + b^2,$$

$$(a - b)^2 = a^2 - 2ab + b^2,$$

$$a^2 - b^2 = (a + b)(a - b).$$

20. The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other two sides, according as the angle contained by those sides is obtuse, right, or acute. The difference in the cases of inequality is twice the rectangle contained by one of the two sides and the projection on it of the other.

Loci.

21. The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

22. The locus of a point which is equidistant from two intersecting straight lines consists of a pair of straight lines which bisect the angles between the two given lines.

The Circle.

23. A straight line, drawn from the centre of a circle to bisect a chord, which is not a diameter, is at right angles to the chord; conversely, the perpendicular to a chord from the centre bisects the chord.

24. There is one circle, and one only, which passes through three given points not in a straight line.

25. In equal circles (or, in the same circle), (i) if two arcs subtend equal angles at the centres, they are equal; (ii) conversely, if two arcs are equal, they subtend equal angles at the centres.

26. In equal circles (or, in the same circle), (i) if two chords are equal, they cut off equal arcs; (ii) conversely, if two arcs are equal, the chords of the arcs are equal.

27. Equal chords of a circle are equidistant from the centre; and the converse.

28. The tangent at any point of a circle and the radius through the point are perpendicular to one another.

29. If two circles touch, the point of contact lies on the straight line through the centres.

30. The angle which an arc of a circle subtends at the centre is double that which it subtends at any point of the remaining part of the circumference.

31. Angles in the same segment of a circle are equal; and, if the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.

32. The angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

33. The opposite angles of any quadrilateral inscribed in a circle are supplementary; and the converse.

34. If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

35. If two chords of a circle intersect either inside or outside the circle the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

Proportion: Similar Triangles.

36. If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally; and the converse.

37. If two triangles are equiangular their corresponding sides are proportional; and the converse.

38. If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the triangles are similar.

39. The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle, and likewise the external bisector externally.

40. The ratio of the areas of similar triangles is equal to the ratio of the squares on corresponding side.

COURSE IN MATHEMATICS FOR MUNICIPAL SECONDARY SCHOOLS.

This paper offers for consideration a suggested course of Mathematics which the writer considers suitable for his own school, but as the type of school is by no means an isolated one and probably is becoming yearly more common, it may be found of interest and possibly useful or suggestive to others. The school is a Municipal Secondary School of about 300 boys, the majority of whom are between 11 and 17 years of age; a few are above the higher limit, none below the lower. Very few boys proceed to the older Universities of Oxford and Cambridge, a fair proportion, probably four or five annually on an average, proceed to the local University; a *good* proportion remain till 16 to 17.

Of those who leave a considerable number enter engineering firms as non-premium apprentices, or smaller manufacturing firms; a fair number enter as junior chemists into chemical manufacturing firms, *e.g.*, coal-tar distilleries, gas-works, alkali and acid works, &c.; some into surveyor's and architect's work, while a number inevitably find their way into the clerical work of banks, insurance offices, &c. Of those who leave it is found that a large number continue their education, chiefly in the technical school, but also in various commercial classes. A fair number while at school take the London Matriculation Examination chiefly in order to qualify themselves for future Degree work (probably 9 or 10 annually on an average), a few—two or three annually—take the Intermediate Science, either of the London or of the local University. This preface may enable the reader to judge of the class of student for whom the course is intended, and to what extent it resembles or differs from those in which he is more particularly interested.

In such a school, therefore, there is no need to consider the possible candidate for a mathematical scholarship at Cambridge and to subordinate the general curriculum to him. It is probable that in many schools this has been done with ill effects; the majority of the pupils who leave the school having mastered as a result an incomplete and comparatively useless and lifeless portion of a course intended to proceed to a higher level. They never reach the point at which they can utilise their knowledge, either for the purpose of obtaining pleasure for themselves from its use, or for the purpose of assisting their progress in other work in which they are interested. As a consequence, on leaving school their Mathematics falls into disuse, is neglected, and soon suffers from the atrophy to which sooner or later all knowledge which has not reached the *useful* stage is liable. To be of any real good it is necessary that a subject studied should be carried to the useful stage—whether the subject is classics, modern

languages, history, science, &c., or mathematics—and it is a painful fact that to many pupils who leave secondary schools Mathematics, with the exception of quite elementary arithmetic which possibly they learned elsewhere, is not useful. By “useful” knowledge is, of course, not meant capable of being materialised into money, of assisting the wage-earning ability of the possessor, but “usable” knowledge, *i.e.*, knowledge which the possessor can apply either for his own enjoyment, or that of other people, or for assisting in obtaining such knowledge. There are, of course, many grades of such usefulness from that of mere simple arithmetical knowledge of numbers and the ability to add and multiply, up to the ability to advance the domains of Science and Mathematics by independent mathematical research, but whatever be the extent to which the study of Mathematics is carried care must be taken that the work done shall result in useful knowledge, and that the student shall be able to utilise his Mathematics, to handle it as a tool, and be able to apply his knowledge, however elementary or advanced it may be, to whatever work or study he is interested or engaged in. He should not be obliged after leaving school to attend classes in Mathematics either under the title of “Practical Mathematics” or “Commercial Arithmetic” in order to understand the applicability of the mathematical knowledge he is already supposed to have attained, although of course he may, and it is hoped he would, desire to attend classes to extend it, and carry it to higher levels. He should be able to do ordinary calculations and computations accurately and with reasonable speed and should be familiar with the various labour-saving devices of general utility, *e.g.*, logarithms, slide rules, &c. Accuracy is an essential; it is useless to know *how* to use logarithms in the computation of $\frac{2.723}{1} \times \frac{751}{760} \times \frac{273}{296.5}$ if the result is obtained incorrectly when they are used. The means by which this accuracy is to be obtained is perhaps a difficult question—some indication will be given later—but it is largely a question of age and the sense of responsibility with regard to the work as well as of interest in it. He should be able to solve accurately any linear equation or simultaneous linear equations, with any reasonable number of unknowns, and to solve any quadratic equation. He should have a clear and ready knowledge of the properties of the chief geometrical figures, triangles, parallelograms, circles and, though less extended, of the conic sections. He should be able to complete the solution of any triangle when sufficient data are given and should understand the methods of obtaining the necessary data in surveying, *i.e.*, the theodolite, sextant, and plane table. He should have a clear knowledge of the conditions necessary for equilibrium of forces, whether parallel or otherwise, and have clear conceptions of elementary dynamics, and of the meaning of energy and its transformations. A knowledge of simple harmonic motion and its

relation to uniform circular motion is desirable, together with the form of periodic curves such as the sine curve, and the effect of superposition of two or more such curves in different phase. He should have also—and this is important and should be attained even if other parts are omitted—a working, if elementary, knowledge of the calculus, sufficient for the differentiation of simple functions, sufficient also to check integrals met with in physical, chemical, or engineering work by differentiation, capable of utilisation in questions of maxima and minima, and for the determination of areas, volumes, centres of gravity and moments of inertia. He should also be able to understand a graph, to read fully its significance, and to form an idea of the probable form of function indicated. The idea of functions and the interdependence of the two variable quantities in a graph is an idea of such fundamental importance and such far-reaching utility that it should be instilled from quite an early period of mathematical work.

It is of course almost inevitable that the objection will be raised, that in the above outlines of the desiderata of the mathematical training no consideration has been taken of the educational aspect of Mathematics, of its efficacy as mental training, as discipline and as "education" in its true sense; *i.e.*, the widening of the mind. This objection is, however, in the writer's opinion, one of no weight. It is only necessary to draw attention to the lack of initiative, the feebleness of resource, and the stunted mental condition of the average pupil fed on the ordinary mathematical pabulum, to show that some change is necessary, and that the orthodox mathematical training has not inspired sufficient confidence in its educative value to be regarded as permanent. Further, the educative value of a subject does not depend so much on the work which is covered and the knowledge which is attained as on the methods adopted for attaining this knowledge, and it is *prima facie* probable that the methods employed to render any acquired knowledge useful will be more truly educative than those which merely aim at the acquisition of knowledge with no thought of its utility. Further, the writer believes that work of the type of interpretation of a graph, determination of a maximum by calculus, or the consideration of the solution of an actual problem by, it may be, the united aid of trigonometry, geometry and calculus, is far more stimulating and educative than the laborious algebra of complex and continued fractions, long G.C.M.'s or artificial and disguised quadratics, which, in spite of the protests of mathematical teachers, still figure in many examinations, and are still a source of irritation to the clever, and despair to the slow boy in our schools. An objection is sometimes raised to the style of mathematical teaching here suggested, that the work is made too easy, that modern methods and modern schemes do not give difficulties to surmount, and that the effect is to make boys mentally flabby.

The objection does not appear to be a real one; there are sufficient real difficulties to be overcome without fictitious ones; these real difficulties need not be shirked and they can supply all necessary stimulus, but it is our duty as teachers to ease and not to create difficulties, to smooth the road and not to lay obstacles upon the track. If a thing can be done easily it is wrong to let students flounder through long and difficult methods; further, it is far harder, as also it is more important, to persuade boys that Mathematics is easy than to let them believe it is hard. Doubtless also some will object that the scheme of work suggested cannot be covered by boys at the age of 16 to 17. These objectors probably overlook the very large amount of time which is spent, it might be said wasted, on work of a highly artificial character, work which seldom or never is related to the actual demands of practical, physical, engineering, or commercial problems. I believe, however, that those who work on the lines suggested will find that the work can be done, and that it will prove more stimulating and interesting, alike to teachers and taught, than the wearisome and stodgy Mathematics through which many of us laboriously trudged.

Arithmetic and Algebra.—The two are grouped together because it appears unnecessary to divide them. Pupils should from the start be allowed to use whatever mathematical tool is required for the solution of the particular problem involved; the same also holds good for geometry, trigonometry, &c. Most of the pupils entering schools of the type considered have probably come from elementary schools. They are for the most part completely ignorant of algebra, but have a fair knowledge of arithmetic. They are capable of doing ordinary questions in multiplication and division of money, or of the ordinary weights and measures, are familiar with the unitary method and simple fractions, their work is fairly accurate but is painfully slow and laborious, and very rarely are they in the habit of checking their answers, either by rough checks at the outset or by testing whether the result obtained is in accord with the statements made in the question. In decimals they are usually distinctly weak. Work can therefore start on the assumption that decimals are not known, and much work, both arithmetical and practical, is requisite in order to familiarise the pupils with their use. Since at the same time measurement work, chiefly in the metric system, is being extensively done as the initial work in physics, a great deal of practice in the use of decimals is obtained, and efforts are made to get pupils to estimate in decimals rather than in fractions, *i.e.*, a length is estimated at 6·4 cms. not as *about* 6½ cms., or as 18·37 cms., not as nearly 18·4 cms., and by this means pupils get to realise that ·37 is less than ·4, a fact which often causes some difficulty to the pupil whose Mathematics is completely divorced from practical work. Here incidentally it may be remarked that the practice of estimation appears to be neglected too much in the science work of schools,

although accuracy of work and self-confidence can seldom be better cultivated than by the habit of the estimation of values as far as is probably accurate. (As an example the writer recollects that, working with thermometers divided into half-degrees and reading by means of a telescope, it was found that the reading by two observers, after some practice, seldom differed by more than .01 degree.) Examples necessitating multiplication and division of decimals are largely obtained from the measurement work; the uselessness of large numbers of decimal figures is hence early evident and the undesirability of stating results to a number of decimal places unwarranted by the data. Practice in giving results to a certain number of decimal places or of significant figures is thus early obtained. At this stage contracted methods are comparatively useless as far as they relate to the actual practical work, because such work seldom introduces sufficient digits to make the contraction really valuable, but inasmuch as examples are frequently set in which the contracted methods are of use, these methods are taught as soon as it is apparent that decimals are completely and thoroughly understood. Throughout this early work accuracy is insisted upon. Accuracy is so largely a matter of habit, that much is gained by early insistence. In all cases also where the question is of such a kind as to render it possible, an approximate value is first estimated, this forming a rough check on the value finally obtained. From the outset also, every care is taken to make pupils refrain from giving answers *obviously* wrong. This is rendered easier by the constant use of actual problems and examples dealing with matters familiar to the student as *e.g.*, the various problems arising out of his practical work in areas, volumes, &c., and in examples in decimalisation of money. Work on percentages and proportion is also done at an early period, *i.e.*, when it is felt that decimals are secure. The teaching of proportion is probably one of the most difficult problems the teacher has to face. On this account it is very largely neglected in schools, and has been supplanted by the unitary method. While the latter is certainly easier to teach, and soon gives the pupil a certain facility in working out examples, yet its use appears to have certain somewhat serious disadvantages. It is very apt to become very mechanical; students are not trained to ask whether or not there is any proportionality, direct or otherwise, between the quantities involved, and they finally leave the subject without any idea of a *ratio*. If this is obtained early, much of the later work is rendered a great deal easier and clearer, various examples in variation can be set without waiting for the chapter on variation in the algebra text-book in use; further, the idea of a *function* is here obtained, though the word need not be used yet (though there appears no objection to doing so). In work on percentages, profit and loss, &c., the process becomes much simplified if students have acquired

an idea of ratio, and realise that actual prices bear the same ratio as the percentage prices; hence such work naturally follows immediately the work on ratio and proportion. The *ratio* idea is also of service in science, and it is advisable for students to realise that 7 is $\frac{7}{9}$ of 9 without having to go through the formality of stating, or of thinking, that 7 is 7 times $\frac{1}{9}$ of 9.

Algebraical work, which need not be distinguished from arithmetic, starts naturally from problems leading to simple equations where the use of the letters appears merely as a form of shorthand to avoid writing down repeatedly "sum of money he had at first" &c. From this study of equations the necessity of addition and subtraction is evident and the convenience of brackets. The latter, however, need not be carried to the extent seen in text-books where excessively cumbersome and lengthy bracket expressions are set to be simplified. The same holds for the work in multiplication and division—lengthy examples are not necessary and time should not be wasted over them. Sufficient examples are required to give complete familiarity with the methods and to set out the terms in proper order, but the work should be limited to the type of expressions liable to be met with *i.e.*, expressions of the first or second order, and certainly no time of the student should be wasted with long division examples in which by use of terms in x , y , and z the average boy can flounder for hours with no result. It is doubtful whether divisors of more than two terms are necessary at this stage.

Simultaneous equations would next follow and should be solved graphically also. Examples of substitution in formulæ should be taken continuously and introduced at various times. It is an important branch of mathematical work which appears to be somewhat overlooked. Examples should include the conversion of the formulæ into other forms according to the particular quantity required. Numerous examples can be forthcoming from the practical measurement work on volumes, densities, &c., which forms part of the science work of the school. The above work would probably be covered during the first year but would not be done with—work of the same type constantly recurs during the following years—new tools may be found but the old ones also are kept in constant use. Throughout the whole course, and especially in the early stages, it is of fundamental importance to ensure habits of accuracy of computation, and of neatness in work and precision of statement. Answers should, as far as possible, be statements and not merely numbers; thus if an area has been found the statement should be "area of field = 6,370 square yards" not Ans. 6,370. Students should not be allowed to associate this precision of statement solely with the science master and be content with slipshod expression of results when working for the mathematical master. At about the end of the first year or the

beginning of the second year logarithms are introduced and students are allowed to use them whenever they desire. The book of logarithms is always present during the mathematical periods. At first four figure logarithms are used but later these are displaced by five figure logarithms, and in the highest forms problems are often set necessitating the use of seven figure tables. For most purposes five figure logarithms are the most useful—the four figure tables are not sufficiently accurate for either commercial or scientific work ; it is obviously absurd to take weighing on a chemical balance to one part in 50,000 if the subsequent calculations are only accurate to 0·1 per cent.

In arithmetic during the second year stocks and foreign money problems, present value, &c., are begun, much practice in computation is given and the mensuration is extended, problems involving geometrical knowledge being introduced. In algebra the work deals chiefly with problems on equations, *i.e.*, ordinary simple equations, simultaneous equations, literal and fractional equations. The latter necessitate some work on fractions, but long examples with complicated fractions are not considered, and in general the denominators consist solely of linear expressions. Quadratics are introduced by problems leading to quadratic equations and the necessity of finding a means of solution. This necessitates the consideration of factors, and when the factors of the quadratic can be seen the equation may be so solved. Thus if the pupil can write $x^2 - 9x + 14 = 0$ in the form $(x - 7)(x - 2) = 0$ he is encouraged to do so and to solve the equation thus. The solution of quadratics by factors is very good training in mental activity. It is, however, very important that the pupil should realise that such examples are of comparatively rare occurrence in practice and that it is more probable that the equation derived from actual practical work will be of a form such as $x^2 - 8\cdot3x + 13\cdot5 = 0$ where exact factors are not obtainable. He should be able to solve such equations to the required degree of accuracy but training him to do so involves much patience on the part of the teacher. The solution of a quadratic equation is always accomplished by the completion of the square ; use of a formula being not allowed. The approximate values should first be obtained ; thus in the above example 7·3 and 1 give a value 7·3 for the third term, while 6·3 and 2 give 12·6, and 5·3 and 3 give 15·9, so that the roots are respectively between 6·3 and 5·3, and between 2 and 3. It should also be insisted on that the pupils should satisfy themselves in all cases that the values obtained do actually satisfy the equation, and certainly that they are possible, so that they should never give values such as 12·3 and 8·6 for the roots of $x^2 - 4\cdot9x + k = 0$ when it is quite obvious that the sum of 12·3 and 8·6 is *not* 4·9. It is essential for this work to be thoroughly grounded, and it requires time and rigid training, but it is time well spent, and it is repaid by the certainty and accuracy of the later work. Beyond the factors for quadratic expressions there is no need

to progress, though the cleverer boys will find no difficulty in the factors for simple cubic expressions—but time need not be spent on them. H.C.F. and G.C.M. may be neglected except as an extension of the factors. When two expressions can be split up into two linear factors their G.C.M. and H.C.F. are obvious, but the actual working out of these by the methods based on the arithmetical solution is unnecessary. During the second year's work elementary trigonometrical ideas are also introduced—at first the tangent, since it is more evidently involved in determination of heights. This is first done graphically, and afterwards the values of the tangents for various angles found and a curve constructed which is used in the solution of simple problems. Tables are next used for the purpose and the sine and cosine introduced. The sine curve is drawn and used, and the relation between sine, cosine and tangent studied, together with their application in the solution of right angled triangles. The examples are, where possible, obtained from practical measurements. If the ground suggested to be covered in the second year appears small it must be remembered that at this stage constant practice is required in the use of the various mathematical tools with which the pupil is now familiar, if he is to have confidence and certainty in their use. For this purpose, too, the form of the example should be constantly altered and actual problems from work employed as far as possible.

The work then naturally passes to simultaneous quadratics—these should be simple and straightforward, and for the majority of the pupils confined to the solution of those consisting of simultaneous quadratic and linear equations. The cleverer boys will probably be able to do harder examples, and may do so. Surds and indices require treatment, but not prolonged, though many examples in indices are necessary to familiarise the pupils with fractional indices. Ratio and proportion involve little work if the idea of ratio has been obtained in the early work and been kept familiar by frequent use. If, however, all earlier work involving ratio has been done by the unitary method the teacher will probably find unexpected difficulty at this stage owing to his assumption that his pupils understand what he means by a ratio when they have no conception of it. Variation is usually treated merely in conjunction with ratio, but the idea of variation should be developed throughout the whole course in areas of circles, volumes of spheres, cones, in the various formulæ for computation, &c.; every chance should be utilised of rendering the idea of a function and of variation clear to the pupil. It is quite common for a boy who is able to calculate without any difficulty the area of a circle of any radius or circumference, to be quite unable to say what would be the ratio of the areas of circles of 9 and 7-inch radius respectively. The ideas of y varying as x , as x^2 , as $\log x$, inversely as x , inversely as x^2 , &c., should have been continuously built up. Arithmetic and geometrical progressions are easily understood and quickly

done. They are both interesting to boys and are frequently required in later work. They can be done, but the work should be plain and straightforward, if is not necessary, neither is it advisable, to try to find hard examples or tricks. Numerous examples on geometrical series are found in questions on compound interest, annuities, and payments by instalments. Elementary work in permutations and combinations—again without trick problems—is found interesting by boys, and a few lessons can be devoted to it, the immense number of permutations of a comparatively small number of things is so unexpected by them that five or six hours spent in its study will usually be found to be stimulating. The binomial expansion and its use for purposes of approximation is of much value, and this can also be done. Examples on the sum of the coefficients, the sum of their squares, &c., the greatest coefficient, the greatest term, &c., are not necessary and should be omitted, and these probably take up nine-tenths of the time required for the examples on the binomial theorem in the usual algebra text-book. As after leaving school the student will not be able to take up a text-book on engineering or on physics without meeting the mystical symbols e^x and $\log x$, it is well for them to learn their meaning at school. The trigonometry after the solution of right angled triangles is naturally extended to the solution of any triangle. For this the relationship

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

is naturally the first. This is readily derived, and its use is the simplest. Afterwards the expression $c^2 = a^2 + b^2 - 2ab \cos C$ is employed. This expression can be used for all cases for which the previous relation is not applicable, and hence all triangles can be determined from sufficient data. This gives all the trigonometry actually required for most practical work, but it is of advantage to extend the work to the various formulæ capable of more convenient logarithmic computation. The trigonometrical functions for the sum and difference of angles and the relationship between the functions of A and $2A$ or $\frac{A}{2}$ can also be utilised as well as the expression for the area of a triangle and the values of the radii of the various connected circles. This is not difficult to cover if the extended and artificial trigonometrical equations are omitted as well as the examples relating to the more or less complicated results which may ensue when $A + B + C = \pi$. It is needless to say that the student should be familiar with the measurement of angles both in degrees and in circular measure, and able to work equally readily in both. Practical examples for trigonometrical computation should be obtained where possible. Even if a theodolite and sextant are not available—though they should be part of the equipment of schools—a great deal of useful practical work can be obtained by means of the plane table.

The survey of the cricket field or of a neighbouring common with chains and theodolite or plane table affords a good combined exercise in practical mathematics, in mechanical drawing, and trigonometrical computation; it is stimulating and arouses interest.

Geometry.—The work at first is chiefly observational or experimental. The average boy—certainly at the age of 12—does not appreciate the logic of Euclid, and sees no gain in an endeavour to prove what appears to be obviously true. But he does appreciate, and enjoy, finding things out for himself, things which he did not know before, and he will get interested in observational and practical geometry. For example, at quite an early age he observes by experiment that the three angles of a triangle are equal to a *straight* angle—the most convenient unit, as being easier to define than the right angle—that is to 180° . This is found from *many* experiments; possibly in the whole class 150. The values obtained for the sum of the three angles seldom lies outside the limits 179° and 181° (paper protractors are used). If a value is found beyond these limits the measurements are repeated. In order to test whether the differences from 180° (the mean) are real or due to inaccurate measurements the *same* triangle is examined by different boys, and it is found that the differences from 180° are of the same order for the same triangle, measured by different observers, as for different triangles. The knowledge of the fact that $A + B + C = 180^\circ$ or thereabouts is thus definitely and firmly obtained. Some advocates of the experimental method of geometry teaching believe in stopping here and doing nothing further. The fact is known; this is the important thing. By knowledge of the fact is meant of course something more than the acquiescence in the statement $A + B + C = 180^\circ$, it means its realisation, its connection with other facts—the placing of it in its proper pigeon hole of the brain. The boy must know at once for example that if A and B are respectively 50° and 75° , then C is 65° , without requiring suggestion from his teacher. It means knowledge so definite and clear that it can be used immediately when required. For this, of course, various examples and exercises based upon it must be done to let the fact establish its importance in the pupil's mind, and to let him be familiar with its use. Nevertheless, although with this argument the writer is so far in accord, yet it has been his experience that with all this, something still is lacking. Many boys, and these often the brightest and more intelligent, are not content with the mere fact. They know the fact is so, and have no doubt concerning it, but feel that surely there must be some reason for it, and are not quite satisfied concerning the fact until this reason has been found. The reason is the same thing as the geometrical proof—it may be called a proof or a reason for the fact. In the course considered the reason or proof is that derived from the rotation of the sides. Examples of the truth of the fact may also, if desired, be adduced from the fact that whenever a triangle is

divided into two, or a new triangle made, by means of a straight line, a straight angle is always added. Thus, if AD divides ABC into two triangles, ABD and ACD, then we add the straight angle consisting of ADB and ADC. When the reason or proof has been given, it is written down with the accuracy of expression that is employed in Euclidean geometry. From this fact others may be deduced immediately, *e.g.*, that the exterior angle is equal to the sum of the two interior and opposite angles. This may be deduced as a necessary consequence of the preceding fact, and if desired, verified by measurement, or, if preferred, it may be discovered experimentally, and then shown to be a consequent. Both methods, *i.e.*, verification of geometrical deductions, and geometrical proof of experimental generalisations, are necessary during the work, and opinions will probably differ regarding the course to be adopted in any particular case. In the case in point it would be preferable to treat it as a geometrical deduction, and test it by measurement, on account of the extreme simplicity of its connection with the established fact. Where this connection is less simple it would be better first to obtain the fact experimentally. As the pupils increase in their powers of geometrical reasoning, the deductions will be extended. In all the reasoning required to give reasons for generalisations obtained by experiment, *i.e.*, in all proofs of theorems, possible constructions may be assumed, if necessary the trisection of an angle, &c. Parallel lines are treated in their geometrical sense as an extension of the triangle to the case where the vertical angle becomes diminishingly small. There will probably be many who would object to this as involving the idea of infinity which young minds cannot grasp. As, however, mature minds cannot adequately do so either, and as the difficulties are more apparent to the mature mind than to the young, the objection is not as strong as at first sight appears. Certainly the writer thought the difficulty would prove a very real one when he first attempted to treat parallels in this way with young boys, but found as a result of experience that his fears were exaggerated, and that reasoning based on the consideration of triangles with a vertical angle zero, did not give the difficulties he anticipated to young boys. By combination of experiment and deduction the greater part of Book I. of Euclid's elements is covered during the first year, although extended practice in deduction based on it will still be needed to make the knowledge an effective and useful tool. Connection of the geometric facts with common experience should be given where possible. Thus the mechanical importance of the fact that three sides fix a triangle whereas four sides do not fix a quadrilateral should be illustrated by the necessity of tie rods in girders and frameworks, &c. During the second year the quantity of deductive reasoning is increased, although experimental work is still employed, especially when new work is started, as *e.g.*, when the circle is considered. The work

covered in the previous year is utilised largely for deductive purposes, easy riders being set. The parts of Euclid, Book I., which have not been investigated are taken.

The work of Book II. is deduced from algebra and utilised for practical constructional work. Figures equal to one figure and similar to another are drawn accurately, and the methods of doing this not treated as purely academic; square roots are determined graphically. Much of Euclid, Book III., is deduced from measurement. In the further years the work becomes more purely deductive, and riders on the first three books of Euclid take up much of the time together with practical applications. The work of Books IV. and VI. can be largely curtailed, many of the propositions being unnecessary. Thus little time is spent over proofs of similarity of triangles when the facts have been accepted as axiomatic during the previous years in all drawing to scale, &c. As the proofs, however, appeal to those boys who have keen mathematical instincts they are not entirely omitted, but are merely indicated for the benefit of such boys. The dividing of lines in given ratio, the finding of mean proportionals, are all taken as practical work, not as mere book methods which are not intended for real use, and many examples, based on the fact that similar areas are proportional to the squares of their linear dimensions, are worked out practically, *i.e.*, by drawing as well as algebraically. Some of the geometrical properties of other figures will also be noted, this being largely the outcome of the graphical work which is being done throughout the whole course, *e.g.*, those of the ellipse, hyperbola and parabola. This graphical work is not actually separated from the other geometrical work, although for convenience it is treated here separately from the work which more resembles the ordinary Euclidean geometry. Students also spend some time in purely mechanical geometrical drawing. The aim of this is only partly to extend geometrical knowledge, but in so far as it does this, it has to be correlated with the work of the mathematical teachers. It is not subordinated to this, however, as it has also the aim of training in the elementary portions of machine drawing and in projectional geometry. The main object indeed is to give boys practice in the accurate use of the pencil and the mathematical instruments so often, unfortunately, entirely divorced from Mathematics. The construction of rectilinear figures from given data is naturally closely connected with their mathematical work, but much work which anticipates later geometrical knowledge is introduced to indicate the necessity of accuracy, *e.g.*, the construction of circles to fulfil various conditions. The drawing of plans and elevations of solid objects and of their sections will naturally lead to the ellipse and the other conic sections, and it is seen that the fact of the area of an ellipse being given by πab (previously found by measurement) is a necessary consequent of its projection from a circle.

The graphical work, independent of this mechanical drawing, is a very strong connecting link between the various branches of Mathematics. Sometimes it seems to the pupils to be algebra, at other times geometry, &c. Graphs have value for two purposes. The first, merely as a convenient representation of known data, the ability to read which is a valuable acquisition. Thus graphs of various statistics, rainfall curves from day to day, &c., can be used for such a purpose even though the quantities represented have no real functional connection—the graph is only a pictorial representation of certain numerical data. The second value of graphs is, however, more important, viz.: their use in indicating functionality, as showing the manner in which one quantity changes with another. Hence although practice in reading the first type of graph, *i.e.*, pictorial statistics, is given, yet far more attention is given to the graph as representing the connection between two variables, that is to the idea of a function.

The linear function is naturally the first to be interpreted and the meaning of the coefficient and of the constant in the equation $y = mx + c$ is considered. The constant slope is emphasised and the fact that the increase of y for a given increase of x is constant. From the outset it must be remembered by the teacher that the knowledge of graphs will later be of the utmost value in the differential calculus, and that familiarity with the underlying idea of the calculus should have been obtained continuously. Various examples from actual science and other data are utilised for graphical representation though not necessarily solely at the stage when linear graphs are introduced. When such relationships are found in later work the graph should be used; it is not advisable that students should think that some work is elementary and has to be dropped when they reach higher forms—one tool is not discarded when a pupil learns to use another, but in practice all must be maintained. Thus the connection between Centigrade and Fahrenheit scales, between the reciprocal of the current and length of wire (in electrical resistance) will in later work form examples of linear functions. The graph for the ratio of the angle of refraction to the angle of incidence in light is useful, for many students are willing to assume that this ratio is constant when the table of experimental results is written, whereas the graph shows the regular departure from the straight line, and illustrates the difference between errors of experiment and a regular trend; the graph for the relation between the sines of the angles shows the straight line together with the errors of experiment. The solution of simultaneous equations by graphs has been mentioned.

The next graph for study would be $y = x^2$ or $y = \frac{1}{x}$, the latter may be more convenient owing to experimental work in science indicating some such relationship. In the parabola the variation of slope is emphasised. As this curve can be utilised to a

covered in the previous year is utilised largely for deductive purposes, easy riders being set. The parts of Euclid, Book I., which have not been investigated are taken.

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considerable extent it is advisable to have the curve accurately drawn and of fair size. It can then be used for the rapid solution of quadratic equations, a piece of tracing cloth or celluloid with a straight line ruled on it being used in conjunction with the graph. The student should also realise that solutions of equations of almost any type can be effected by the use of graphs and he should be given practice in finding first the approximate solution and then a more accurate solution by constructing the portions of the graphs near the solution values on a larger scale. By change of origin it will be seen that the curve $y = a + bx + cx^2$ is part of a parabola and the interpretation of the values of a , b , and c should be known. Students should endeavour to find such expressions which fit experimental numbers, obtaining a and b from the tangent at $x = 0$ and computing c . For example, the expansion of alcohol in a dilatometer would give numbers which could be utilised. Logarithmic curves should also be drawn after $y = x^n$ has been graphed and the students should see that $y = ax^n$ can be written as $\log y = \log a + n \log x$ and hence be able to deduce n and a by plotting the graphs for the logs of the variable. Examples for use can be obtained from the bending of beams considered as a function of (i) length, depth, or (ii) from variation of the tangent of the angle of deflection of a magnetic needle with the distance of the deflection magnet in the Gauss positions and from the variation of time of oscillation of a simple pendulum with the length of the string. The use of interpolation, and the use—as well as the danger and abuse—of extrapolation should be noted.

It must be emphasised, however, that graphical work requires much care and attention on the part of the teacher. It is very apt to become slipshod if the teacher—perhaps not being sympathetic towards graphical work—does not insist on accuracy; also it necessitates a considerable amount of time to ensure the desirable accuracy and neatness. Points which must constantly be impressed are the drawing of the axes and the clear marking of their scales, together with the statements of the variables and of the units employed. The points should be marked with a needle or pin and their positions indicated by small circles or crosses. Thought is necessary in choosing the scales so that the curve can be drawn to the extent desired and be of suitable size. Further, the desirability of choosing a suitable number of squares per unit—*i.e.*, 10, 8, &c., not 7, 11—has to be pointed out.

Calculus.—Some knowledge of the calculus is desirable—it might be said essential. The period at which it is started may vary according to the desire of the teacher; certainly it can be much earlier than is commonly supposed. Indeed there seems little reason why the nomenclature could not be introduced quite early and why, when the constant slope of $y = mx + c$ is observed, the statement could not be written as $\frac{Dy}{Dx} = \text{constant}$.

&c. Probably, however, most teachers will prefer to wait until a fair foundation of algebra and trigonometry has been laid. There seems, however, little doubt that the easiest method of approach is through the graphical work, and that the rate of increase of y compared with that of x is more readily understood by consideration of various curves previously obtained. The method of obtaining this rate by calculation should then be studied. The meaning of a horizontal tangent in application to maxima and minima opens up a large field of interesting work of practical utility and fortunately there is no lack of suitable simple engineering and physical problems to afford practice and arouse interest. At an early stage when a differential coefficient is found the corresponding integral should be written down, so that the student constructs for himself a table of the more common differentials and integrals. Later the idea of the integral as a sum is introduced which leads immediately to the determination of areas, volumes, moments of inertia, centres of gravity, &c., and can be applied to innumerable examples of interest in physics and engineering. Even if the teacher finds himself unable to proceed any further in the study of the calculus, yet even this small amount will be found of very great value to the student and will probably stimulate him to further study, for there appears no branch of Mathematics which so whets the appetite as the elementary portions of the calculus. Those who, while approving of the ideals suggested, yet feel that the time will not allow their attainment, should endeavour accurately to estimate the amount of time which in the ordinary course is devoted to the portions suggested for omission and to complicated algebraical and trigonometrical relationships, &c., and they will probably be surprised to find the large proportion of time a student spends in the *useless* portions of Mathematics. Then, and probably not till then, will they be convinced of how much *useful* Mathematics a boy of 16-17 could have learnt.

LIONEL M. JONES.

MATHEMATICS IN SCOTCH SCHOOLS.

INTRODUCTION.

The principal object that I have set before myself in preparing this report has been to give an account of the present position of Mathematics in the schools of Scotland. I have tried to present the curricula of typical schools in such a way that a reader of the report may obtain a fairly clear idea of the general aim and scope of the mathematical teaching, and I have refrained from criticism of curricula or advocacy of merely personal views. The Scotch system, though by no means stereotyped, has nevertheless assumed a definite character, and an account of the system is likely to be of greater interest and value than the statement of individual opinions in regard to the methods or content of school curricula. Many valuable papers that discuss didactic questions of a more or less controversial character have been prepared for the International Commission on the Teaching of Mathematics, and there is, therefore, less reason for entering into debateable matters in this report.

To understand the position of mathematical instruction in the schools of Scotland it is essential to consider it in relation to the general organisation of the schools, and the following brief statement may, therefore, be useful.

The funds which are provided by Parliament in aid of school education in Scotland are administered by a Department called "The Scotch Education Department." In the State-aided schools education is carried on under the Code of the Department for Day Schools, or, especially as regards the more advanced work, subject to Regulations contained in various circulars issued by the Department. Even when the schools are not in receipt of financial aid from the State (and the number of such schools is relatively small) the organisation and curricula conform, in the main, to the requirements of the Department, and a statement based, like the following report, on the system administered under departmental supervision will embody all the essential features of school education in Scotland.

The classification of schools adopted by the Department—a classification based solely on distinction of curriculum—is as follows :—

Primary School.—A school, or a department of a school, giving an education based entirely upon English, to pupils who are, as a rule, below the age of 14. A Primary School may contain pupils, or small sections of scholars, who are being instructed on the lines of an Intermediate School.

Intermediate School.—A school providing at least a three years' course of instruction in Languages, Mathematics, Science,

and such other subjects as may from time to time be deemed suitable for pupils who, on entering, have reached the stage of attainments in elementary subjects indicated in Article 29 I. of the Code ("the qualifying stage").

Secondary School.—A school providing at least a five years' course of instruction as aforesaid beyond the qualifying stage (Article 29 I. of the Code).

In the Primary School the normal organisation for children up to the age of 12 consists of (a) the Infant Division, for children under 7 years of age; (b) the Junior Division, for children between the ages of 7 and 10; and (c) the Senior Division, for children between the ages of 10 and 12. This is the organisation laid down in Article 19 of the Code for schools receiving grants under the Code; but the education of children up to the age of 12 follows essentially the same lines in all Scotch schools, and the stage of attainments in elementary subjects reached at that age by pupils who have completed the course satisfactorily is called the "qualifying stage."

With the completion of the qualifying stage divergence of curriculum begins, and pupils may take (i) a Supplementary Course, or (ii) an Intermediate Course, or (iii) a Secondary Course. The Supplementary Courses are designed for pupils who leave school at the age of 14 and are the most advanced work of the Primary School; the Intermediate and Secondary Courses are taken in the Intermediate and Secondary Schools.

It may be noted that the nomenclature in common use for the schools in Scotland is not that adopted in the Department's classification, but has gained currency on other than educational grounds. A school that receives grants under the Code for Day Schools is generally called a Public School; it is elementary if its curriculum does not go beyond the Supplementary Course, but higher grade if it provides an Intermediate or a Secondary Course. Such names as Grammar School, High School, Higher Class School usually indicate schools of the secondary type which receive grants under a Code of Regulations specially applicable to them, and, in almost every case, send forward their pupils to the Leaving Certificate Examinations conducted by the Department. As a rule, intermediate and secondary schools have primary departments and the work of the first three years after the qualifying stage in the secondary school is generally equivalent to the work of the same years in an intermediate school.*

PLAN OF THE REPORT.

The method adopted in preparing this report has been, except in the case of the primary school up to the qualifying stage, to select typical schools and to state as fully as possible

* In *Nelson's Annotated Code* (Edinburgh, Nelson and Sons), will be found the Official Codes, Regulations, Circulars, &c., bearing on Scotch schools.

the various courses of instruction. Teachers in these schools have kindly furnished the necessary details.

Copies of Leaving Certificate and of University, Preliminary and Bursary Examination Papers are given in the Appendices at the end of the report. In conjunction with the description of the courses these will, it is hoped, give a fair view of the principal features of mathematical work in schools.

The account of school work in science, as distinguished from that in mathematics, is not so full as I could have wished. I have, indeed, tried to give due prominence to the fact that the teaching of science is a very important part of every curriculum in Scotch schools, but I have not been able to go into the details of science teaching. A report that would deal adequately with this side of school work demands a much greater familiarity with the methods and scope of elementary science than I possess and would, if it were to be thoroughly satisfactory, require special discussion. It is my hope, however, that readers of this account may obtain a fair idea of the attitude adopted by school authorities towards this branch of education.

The schools are taken in the following order:—

- I.—The Primary School up to the Qualifying Stage.
- II.—The Primary School: Supplementary Courses, with Note on Continuation Classes.
- III.—The Secondary School, represented by three types: School A, a Public Higher Grade School with a five years' curriculum; School B, a Secondary School with a seven years' curriculum; and School C, a Science School.

The most important examinations in connection with the schools are those for the Leaving Certificate granted by the Scotch Education Department; a statement of the regulations relative to these examinations precedes the account of the work undertaken in the secondary schools.

I.—THE PRIMARY SCHOOL UP TO THE QUALIFYING STAGE.

Pupils who have satisfactorily completed the course up to the qualifying stage are expected, besides attaining a fair proficiency in English subjects—

- (1) To know the four rules of arithmetic as applied to whole numbers, easy vulgar fractions, and decimals to three places, and to be expert in applying this knowledge to the calculation, both mentally and on paper, of simple sums in money and in the common weights and measures.
- (2) To be reasonably proficient in the other subjects included in the approved scheme of work of the class.

The work specified under (1) and (2) is that which comes within the range of this report; under (2) are included drawing and nature study.

In recent years the Department has adopted the policy of asking teachers and managers to formulate courses of instruction suitable to the particular circumstances of their schools, and does not prescribe in detail the syllabus of any school subject. As a guide, however, to the manner in which a subject may be treated it has issued a series of Memoranda on the teaching of various primary school subjects.* The Memoranda, it is expressly stated, "are not put forward as final or authoritative documents," but they embody, it is thought, "the ideals and aspirations that animate the best schools; and few, if any, specific recommendations are made which are not supported by the actual experience of teachers of repute."

(1)—*Arithmetic.*

A detailed syllabus of the arithmetical work of a selected primary school would probably be less satisfactory than a statement of its general aim; the following summary of the "Memorandum on the Teaching of Arithmetic" is, therefore, given as representing a very widespread ideal.

Arithmetic in the primary school is regarded as consisting, not of so many isolated rules, but simply of a few fundamental operations which may be applied in a great variety of circumstances. The work is divided into two periods. The first period is concerned mainly with whole numbers and occupies approximately four years of the pupil's school life. During this period the aim of the teaching is (1) the development of the pupil's interest and intelligence in solving simple concrete problems with a clear understanding of the processes employed, and (2) the gradual perfecting of the machinery for the addition and multiplication of abstract whole numbers (within certain limits) and, as a consequence, for the inverse operations of subtraction and division.

In the first period great importance is attached to oral (or mental) arithmetic, written work being considered as belonging specially to the second period. Concrete problems and the gradual introduction of the common units of everyday life are essential. The pupil must handle the coins and know their value and he must be practised in the use of measures till he can give a fair estimate of a length or a weight. He should also learn to state concrete problems by means of the four symbols $+$, $-$, \times , \div and, conversely, to give concrete interpretations of such shorthand expressions.

The attainment of mechanical precision with abstract numbers is indispensable and the Memorandum gives various

* The Memoranda may be obtained from Wyman and Sons, London; they are all contained in Nelson's Code.

suggestions, on lines that are now well known and often followed, towards this end. Of course, work with concrete problems and practice with abstract numbers are not to be mutually exclusive; the main object of the concrete problem, as the course develops, is to maintain the pupil's interest, while facility in the manipulation of abstract numbers will contribute to the rapid solution of the problems.

For the second period, which represents the remainder of the pupil's school life, the work is summarised under four heads:—(1) the completion of the pupil's working equipment by adding to his knowledge of whole numbers a thorough practical acquaintance with fractions and their applications; (2) the concrete treatment of the units of length, area and volume, by practical drawing and measurement; (3) the gradual introduction of symbols for numbers as the pupil learns to differentiate more clearly in his mind between the processes of arithmetic and the mere numbers employed in special cases; (4) the development of speed and accuracy in calculation (including facility in approximate calculation).

Though these four heads are closely connected, there is no definite line of demarcation between them in the teaching. Simple fractions are introduced even in the first period, but it is in the second period that they are systematically studied. Stress is laid on the careful treatment of simple cases by means of concrete examples, so that the pupil may be able to understand thoroughly the meaning of the fundamental operations and not be dependent on a mere rule. The treatment of decimals occupies a considerable space in the Memorandum, and it may be noted that the use of repeating decimals is discouraged, while emphasis is laid on the approximate character of the results in practical applications. Ratio and percentage, as aspects of fractions, are to be explained and applied. Drawing and measurement form elements of the course, but these elements have also to be considered in another connection, and will be referred to immediately. As possible developments of the use of symbols mention is made of the solution of problems by means of simple equations and the employment of formulæ, found by the pupil, for purposes of calculation. Finally, the attainment of speed *with* accuracy is insisted upon, and the opinion is expressed that the pupil has not been properly trained if he has only acquired one of these without the other.

An appendix to the Memorandum treats of Geometry and Mensuration. The main object of the course there outlined is to give the pupil a practical acquaintance with the units of length, area, and volume by exercises in accurate drawing and measurement, with arithmetical calculations based on them. For this work the graduated ruler, protractor, set square and compasses are required, and paper-folding and cardboard modelling are also suggested as suitable auxiliaries.

A great part of the developments under Geometry and Mensuration is usually postponed to the period of the Supplementary Courses, and the statement given for a typical Supplementary Course will furnish a fair idea of the character of the work done before the qualifying stage.

(2)—*Drawing and Nature Study.*

The "other subjects included in the approved scheme of work of the class" are, so far as concerns this report, Drawing and Nature Study. These two subjects have an essential place in the curriculum of every school under the Code, and are dealt with in two Memoranda, on "The Teaching of Drawing" and "Nature Study and the Teaching of Science," respectively. Throughout the course an average of from $1\frac{1}{2}$ to 2 hours per week is given to Drawing. In many schools the boys, at the age of 10 or thereby, take up manual work, which includes cardboard modelling and woodwork, and in several schools there are excellent buildings specially designed for manual instruction. In these courses the use of drawing instruments, drawing to scale in plan and elevation, and mensuration and practical geometry receive considerable attention.

A detailed statement based on the Memoranda would lead us somewhat beyond the limits of this report; but it may be said that both subjects are presented in as practical a form as possible with the view of developing the habit of accurate observation, of fostering the pupils' interest in all that surrounds them, and of opening to them many sources of relaxation and pleasure.

The general lines of arithmetical teaching are the same in all primary schools or primary departments. The trend of opinion and practice is also steadily in the direction of incorporating Drawing and Nature Study as essential elements of instruction in every type of primary school. It is not easy to say how far the aims of the Memoranda referred to above are realised in practice. Undoubtedly many pupils fail to reach the qualifying stage at the age of 12, and complaints are often heard of the weakness of the arithmetic (especially as regards accuracy); at the same time there can be little question that the faults complained of are not due to the teaching, while it is certain that much excellent and fruitful work is being done.

The method of testing pupils for the qualifying stage, in schools under the Code, may be noted. The class teacher and the headmaster of the school decide who are to be presented for the qualifying examination, and one of His Majesty's Inspectors of Schools, after consideration of the pupil's record as a whole, of the teacher's estimate and of such additional examination as he thinks proper to make, certifies to the satisfactory conclusion of the course, or, it may be, puts back the pupil for further work in the primary department.

II.—THE PRIMARY SCHOOL: SUPPLEMENTARY COURSES.

These courses are designed for "pupils who, having reached a certain well-defined stage of general education, will not enter upon the study of the specifically secondary subjects." The stage of general education referred to is the qualifying stage, and pupils who take a Supplementary Course usually leave school at or about the age of 14. The courses vary according to the circumstances of the school, but are generally classified under the heads (1) Commercial Course, (2) Industrial Course, (3) Course for Rural Schools, (4) Household Management Course (for girls).

THE INDUSTRIAL COURSE.

Mathematics bulks most largely in the Industrial Course which is intended for boys who will enter the engineering, building, mining, or similar trades. The distinguishing feature of this, and also of the other courses, is that "at every stage the pupils shall deal with concrete problems and not with developments of mathematical theory." As regards method, "the teacher may adopt devices drawn from any province whatever of Mathematics, so long as they are found to be within the comprehension of the pupil."

The following details of an Industrial Course in a school situated in a populous district north of the Forth have been supplied by the teacher in charge of the class and may be considered as representing what such a course may profitably contain, though the work as a whole is perhaps above the standard of the average Supplementary Course.

Geometry.

Geometrical figures.—Cube, prism, cylinder, cone, sphere, &c. Their occurrence in nature and in structures. Sections, applications to the drawing of flasks and other pieces of apparatus in the laboratory.

Solid, line, point: their representation to the eye. Lines—straight and curved. Tests of straightness (workmen's tests, &c.). Drawing, naming and measuring straight lines decimally. Determination of a straight line by two points, by one point and a direction. Direction of lines (plumb-line, spirit-level, &c.). Lie of lines with respect to one another: (1) in same direction (parallel), (2) in different directions (meeting at an angle). Angles: suggestions as to methods of comparing, measuring and copying angles. Use of circle, protractor. Problems in position and direction.

Curved lines. Circle: occurrence, parts. Circle as locus. Locus of centre of circle through two points; circle through three points. Examples of intersection of loci to solve problems. Chord and diameter. Chord properties. Relation of circumference and diameter. Data required to fix circle in size and in position. Further consideration of angles: sum, difference, &c.

Parallelism : construction of parallel lines. Angle properties of transversal-diagram shown by protractor, tracing paper, &c. Use of rulers and set-squares for constructing parallel and perpendicular lines. Parallel planes in solids.

Rectilinear Figures. Triangle. Sum property of angles—by folding, cutting out, rotation, use of parallels. Exercises. Conditions of congruency. Data for construction of a triangle (many exercises and comparison of pupils' results). Applications to structures and to various problems (triangulation, &c.) : effect of small errors, &c.

Symmetry : axial, central. Isosceles triangle and kite, and constructions dependent on these (bisection of angle, &c.).

Quadrilaterals. Jointed figures; rigidity. Illustrations from structures. Data required to determine a quadrilateral. Drawing by intersection of arcs and by ruler and protractor. Properties of special quadrilaterals: parallelogram, rectangle, square, rhombus and trapezium. Figures drawn by pupils, properties noted and general conclusions drawn.

Property of transversal cutting a series of parallels : division of a line into any number of equal parts. Construction of scales. Representative fraction of a scale.

Areas. Parallelogram, trapezium, triangle. Dissections. Measure of area. Theorem of Pythagoras. Many numerical examples. Loci. Simple examples in one plane and in space. Intersection of loci and construction of figures.

Circle as locus. Symmetry. Chord properties. Angle properties (locus of apex of cardboard angle whose sides touch two pins). Cyclic quadrilateral. Tangent and relative constructions. Properties of two circles: common tangents, &c. Rectangle properties and calculations.

Drawing paths of points such as occur in mechanisms (wooden models illustrate these exercises).

Similar figures. Triangles: conditions for similarity, with numerous examples. Any rectilineal figure. Areas of similar figures. Volumes of similar solids.

Trigonometry. The numerical treatment of the simpler cases, with examples.

Arithmetic.

Revision and enlargement of Qualifying Stage work. Approximation and contracted methods. Index notation and "standard form" of decimals. Decimalisation of British money. Practice. Invoices (specimens of cancelled school invoices and others). Ratio and proportion, with notions of dependence, variable, constant, function. Illustrations in ordinary life and in science. Graphs of the functions treated. Applications of proportion to percentages, interest, &c. Measurement of length. Averages. Area problems (Theorem of Pythagoras, &c.). Square root. Area of circle and of surfaces of solids. Relation of volume of cone and cylinder, of prism and pyramid of same

base and height shown by weighing brass models and by filling hollow models with water.

All classes in the Supplementary Courses take the first year's course in Physics. The problems dealt with, as material for the lessons in Arithmetic, include examples on density, specific gravity, Archimedes' Principle, flotation, laws of liquid pressure, the barometer, Boyle's law, &c., and are coordinated with the laboratory work.

Algebra.

This course is stated to have proved very valuable, the pupil being required to develop the subject with the minimum of help from the teacher. The chief points noted for the first year are: Early introduction of equations and problems by examples so simple that the pupil can solve them and test his answers long before he realises explicitly any definite plan of solution (transposition, adding equals, &c.). Practice in translation into algebraic language and *vice versa*. Long multiplication and heavy manipulations given a very subordinate place. Graphical work found to be useful and to be readily followed without much help from teacher. (It is found that the idea of a graph as representing the variation of a function is much more difficult for the pupils to grasp than the idea of it as representing the path of a point which moves under specified conditions.) In connection with the graphical work many examples are obtained from the lessons in Practical Geography which form part of the Course.

In the second year the earlier chapters of the textbook are used for revision of the first year's work. The course includes: Literal equations. Easy transformations. Determination of any one of several quantities in a formula when the rest are given. Determination of simple formulæ to satisfy the results of experiments (in conjunction with laboratory work). Logarithms. Slide rule. Further examples of graphical work.

Mechanics.

In the second year a course is given in Experimental Mechanics. Subjects treated include:—Principle of the lever. Parallel forces. Geometric addition introductory to experiments leading up to the parallelogram of forces. Resultants by drawing to scale and by calculation. Centre of gravity. Inclined plane. Friction. Simple machines. Pulleys, wheel and axle, screw jack.

Time Table.

The number of periods per week (a period being three-quarters of an hour) allotted to the different parts of the course is as follows:—Mathematics, 7 periods; Mechanical Drawing,

2 periods; Physics, 3 periods in the first year and 4 in the second; Woodwork, 4 periods; Geography (including Practical Geography), 3 periods.

The teacher remarks that in Supplementary Course work there is frequently a weakness, due to the fact that pupils leave as soon as possible after they become 14, and that, owing to difficulties of organisation in primary schools, only a small percentage of pupils are able to pass the qualifying examination at the date of their 12th birthday, many pupils being 13 or over before they enter the Supplementary Classes. The course sketched above is being reorganised for the sake of pupils who can spend three years in the Supplementary Classes and it is hoped that the fuller study thus made possible may improve the character of the results and strengthen the hands of the teachers in dealing with the pupils. While it would be a mistake to suppose that the results of Supplementary Class work are generally unsatisfactory, there is no doubt that considerable improvement is possible and the extension of the course will probably prove a real boon to the more thoughtful pupils.

NOTE ON CONTINUATION CLASSES.

By the Education (Scotland) Act, 1908, the duty is laid on School Boards of providing continuation classes for the further instruction of young persons above the age of 14. These classes usually meet in the evening and on Saturday afternoon from September to April, though in some cases they are continued during May and June. Pupils attend for two or three evenings per week, the meetings lasting usually for two hours. So far as instruction in Mathematics and Science is concerned the teaching follows very closely the lines of the Supplementary Courses; indeed, the true continuation of the supplementary course is the more advanced work of the "Continuation" Classes. While the general nature of the instruction in Mathematics is that of the supplementary course, considerable variation in detail is to be met with; as far as possible, an effort is made to meet the special needs of the trade or avocation which the pupil is pursuing, such as engineering, the building trades, mining, woollen manufacture, &c. Of course, great attention is given to drawing and graphical work; in many schools the drawing work, judged from the trade standpoint, is exceedingly good and the old practice of making copies has been almost entirely supplanted by methods which require the pupil to make sketches of pieces of machinery, models, &c., take his own measurements and then produce a drawing that is comparable with a working trade drawing. The geometry of the subject is doubtless taught, but chiefly as an aid to the production of good drawings.

A continuation class course extends over two or three years and, if successfully pursued, admits the pupils to the evening

classes of the Central Institutions and Technical Colleges. In the great centres of industry the number of evening students who attend classes in these institutions is very large and the instruction in Mathematics and Science is usually "practical" rather than theoretical—ability to apply mathematical rules and formulæ correctly to the more important subjects of the science courses (mechanics, physics, electrical engineering, mining, &c.), rather than the development of mathematical principles being the chief aim. Theoretical Geometry is seldom taught, but algebra, trigonometry and the more important parts of the infinitesimal calculus receive very great attention, and a good working knowledge of "easy calculus" is quite frequently found among the artisans.

III.—THE SECONDARY SCHOOL.

The curriculum of every Secondary School in Scotland is so closely associated with the Intermediate and Leaving Certificates issued by the Scotch Education Department that the following extracts from Circular 340 (revised 1911) seem necessary in order to show the setting of Mathematics and Science in the educational system :—

The Intermediate Certificate.

The essential purpose of the Intermediate Certificate is to testify to the conclusion of a well-balanced course of general education, suitable for pupils who leave school at 15 or 16 years of age, or, alternatively, to the fitness for entry on more specialised courses of post-intermediate study of such pupils as remain at school till 17 or 18.

The intermediate course must extend over not less than three years, and all candidates for the Intermediate Certificate must have completed the approved course in its entirety.

As a rule, any intermediate curriculum to be approved should provide for instruction in at least five subjects, embracing English (including History and Geography), a language other than English, Mathematics, Experimental Science, and Drawing.

The presentation at the Leaving Certificate Examination (of candidates for the Intermediate Certificate) should in all cases be on the Lower Grade,* and in all the subjects of the approved curriculum of the same year; candidates must also present themselves for oral examination as may be arranged with the inspectors appointed by the Department.

No Intermediate Certificate will be granted or withheld without fair consideration of (a) the opinion of the teachers as to the proficiency of the candidates as shown by their work in the various subjects, and (b) the deliberate judgment of the head master as to the candidate's claim for a certificate on the whole range of his work in school.

* The distinction between "Lower Grade" and "Higher Grade" is explained in the "Regulations" reproduced below (page 147).

The Leaving Certificate.

The Leaving Certificate will be awarded on the satisfactory completion of a course of post-intermediate study approved by the Department.

Pupils enrolled in post-intermediate courses, with a view to the award of the Leaving Certificate, must, as a rule, have qualified for the award of the Intermediate Certificate in a previous session.

Post-intermediate courses shall, for the present, be of not less than two years' duration. (After due notice it may be made a requirement that the course shall normally extend to three years.)

The curriculum must in all cases provide for the study of English on the higher grade level, together with a subsidiary study of History (not reckoned as a separate subject). As a rule, one language other than English, as well as either Mathematics or Experimental Science, should also be studied on the higher grade level, the remaining subject or subjects being selected from a specified list. (The curriculum must provide for the continuous study of not less than four subjects, the number of additional subjects being left to the discretion of the school authorities.)

In considering the candidate's claim for a certificate, comparative excellence in one subject will be allowed to compensate for slight weakness in another. To assist the Department in coming to a decision, the headmaster will be asked to record his deliberate judgment on the merits of each pupil's work as a whole, that judgment to be based on a careful collation of the opinions of the various teachers.

From these extracts it will be seen that during the first three years of every Secondary Course Mathematics, Experimental Science and Drawing are taught as parts of a well-balanced course of *general* education. During the next two years either Mathematics or Experimental Science must—unless special exemption be given—be one of the subjects studied on the higher grade level, and it is quite possible and usual to have both subjects in a pupil's curriculum. The aim of the post-intermediate course is to provide for a certain amount of specialised study, while securing a broad general culture, and in several schools Mathematics and Experimental Science are studied for one or more years beyond the normal two years required for the Leaving Certificate.

It should be noticed, however, that specialisation is restricted by the necessity for the concurrent study of four or more subjects. The pupils are not divided into groups, such as "the mathematical group," or "the classical group," or "the modern language group." They form "classes" that are taught a group of subjects, and every member of any particular class studies, as a rule, all the subjects of the group and

no other subjects. Exceptions to this rule are not common; such as do occur are more apparent than real, the class being subdivided in most of the exceptional cases into two or more sub-classes.

The examination in Mathematics is by written papers; the examination in Science is chiefly oral and practical, and is based in each school on the approved science scheme. Normally, a minimum of three hours per week throughout the Intermediate Course must be provided for in any science scheme submitted to the Department for approval. In regard to the science of the post-intermediate years, no definite minimum is prescribed, but it is suggested that science should not receive "a less liberal" allowance than would under similar circumstances be deemed "requisite for, say, Mathematics or a language other than English."

REGULATIONS FOR THE MATHEMATICAL PAPERS OF THE LEAVING CERTIFICATE EXAMINATIONS.

The Scotch Education Department issues each year a "Note as to the Mathematical Papers" and a statement of the Regulations affecting them, along with a syllabus of the subjects included in the examinations. There has been very little change in the tenor of this document for some years, and it is inserted here as being the most satisfactory method of expressing the general purpose of the examinations, while saving tedious recapitulations in the account of individual schools.

NOTE AS TO THE MATHEMATICAL PAPERS.

The following Regulations have been drawn up after very full and careful deliberation. It is hoped that they will be found to provide a satisfactory basis for mathematical instruction in secondary schools. It is felt that in the study of Arithmetic more attention should be given to the explanation of the ordinary rules, so as to secure their intelligent use, and to the employment of contracted methods, with the object of obtaining results which are accurate to a required degree of approximation, without involving unnecessary labour. Systematic practice in the use of logarithms is also an element in mathematical training that deserves more consideration than it sometimes receives. Again, it is desirable to maintain the feeling of unity between the different branches of mathematical study. Pupils should be made to realise that the fundamental laws of Algebra and of Arithmetic are really the same, and they should be encouraged to employ algebraical formulæ in arithmetical calculation. Similarly the explanation and illustration of algebraical expressions by graphical methods might with advantage be introduced at a comparatively early stage, always provided that due care is taken to secure accurate drawing. With regard to Geometry, it is advisable that certain fundamental geometrical results should be established, as far as possible, in the first place by trial and experiment, involving accurate drawing and calculation, before an advance is made to a deductive proof, and that exercises involving definite numerical data should accompany and follow upon the proof of the principal propositions. The syllabus given below ought to lend itself readily to the carrying out of these principles.

Regulations.

Examinations in Mathematics are held in two Grades, Lower and Higher. Candidates may be presented for examination in either grade, but those who are presented in the Higher Grade and who fail to pass in that Grade will not under any circumstances be credited with a pass in the Lower Grade. In writing out the answers to the questions in the mathematical papers it is essential that the full detailed work should always be given in its proper sequence as part of the answer. The work should be written out with such care and neatness in the first instance that a second copy may not be required. But if from any cause a second copy of any answer is made, this copy must include all the detailed working, and the first copy must be struck out with the pen. *The use of scroll paper* is absolutely prohibited.* In all the papers the diagrams should be careful and accurate. For this purpose candidates must be provided with a fairly hard pencil, a flat wooden ruler graduated on one edge in inches and tenths of inches and on another in centimetres and millimetres, two set squares, a protractor graduated to degrees, and compasses furnished with a pencil point. In all the mathematical subjects marks are given for neatness, arrangement, good style, and well-drawn figures. Tables of Four Place Logarithms of Numbers and Trigonometrical Functions will be provided for the use of candidates in the Higher Grade.

Lower Grade.

The Examination in Lower Grade Mathematics will consist of three papers (Mathematics I., II. and III.), for each of two of which two hours will be allowed, while one hour will be allowed for the third. It will embrace the following subjects:—

Arithmetic.—The elementary rules; prime factors of numbers; weights and measures in common use; the metric system; vulgar and decimal fractions; elementary methods of approximate calculation by decimals; square root; practical problems. The intelligent use of algebraical symbols is permitted, and no question will be set on recurring decimals.

Algebra.—Numerical interpretation of formulæ; simple algebraical transformations; the graphical representation of simple functions; equations of the first degree in one and two variables; easy quadratic equations; problems leading to the above equations.

Geometry.—The main propositions given in Euclid, Books I. and III., with deductions and constructions arising from them; simple loci; application of arithmetic and algebra to geometrical theorems and problems. Elementary drawing to scale. Proofs will be accepted which clearly appear to form part of a logical treatment of the subject.

Candidates who take the Lower Grade Examination in Mathematics may not be presented in any of the Additional Subjects.

Higher Grade.

The examination will consist of three papers (Mathematics I., II. and III.), for each of which two hours will be allowed, and will embrace the following subjects:—

Algebra.—The subjects of the Lower Grade; more difficult transformations, equations and problems; application of graphical methods; elementary theory of indices, including logarithms; surds;

* "Scroll paper" means "loose sheets." A book is given to each candidate, and all the answers, with the relative work, must be written in the book.

the remainder theorem; ratio; proportion; progressions. Arithmetical questions will also be set, including questions on theory, and exercises involving the practical use of logarithms.

Geometry.—The main propositions in Euclid I.-VI. and XI. 1-21, with deductions and constructions arising from them, but excluding the theory of incommensurable quantities; the elementary properties of simple plane-faced solids; mensuration of plane and solid figures; approximate solutions by drawing to scale. Proofs will be accepted which clearly appear to form part of a logical treatment of the subject.

Trigonometry.—Elementary Trigonometry, including the solution of triangles, with the aid of four place logarithms. Graphical solutions of problems.

Candidates who take the Higher Grade Examination in Mathematics may be presented in one (but not in more than one) of the Additional Subjects. Those who have already passed in the Higher Grade may take two Additional Subjects.

Additional Subjects.

1. *Elements of Dynamics* (1½ hours).—Velocity; acceleration; the rectilinear motion of a uniformly accelerated particle; the laws of motion; the composition of forces; equilibrium; the centre of mass (gravity); common machines; work and energy; elementary hydrostatics.

2. *Geometrical Conics* (1½ hours).—The properties of the parabola; the elementary properties of central conics; the sections of the cone.

3. *Analytical Geometry* (1½ hours).—The straight line; transformation of co-ordinates; the circle.

4. *Higher Dynamics* (1½ hours).—The elements of Dynamics, as above, treated with the help of Trigonometry; Atwood's Machine; parabolic motion of a projectile; composition of forces acting on a rigid body; uniform circular motion; impact; kinetic energy; centre of pressure; metacentre.

No candidate may take any of the Additional Subjects unless he is at the same time a candidate in Higher Grade Mathematics, or unless he has already passed in Higher Grade Mathematics. If presented in the Higher Grade, he may take one (but not more than one) such subject. If he has already passed in the Higher Grade he may take two. If a candidate has not already passed in Higher Grade Mathematics no credit will be given for success in the additional subjects unless he attains a certain standard of proficiency in Mathematics.

Bookkeeping and Commercial Arithmetic.

The examination in Bookkeeping and Commercial Arithmetic will consist of the following papers. Candidates must pass in both. *No pupil may be presented for this branch of the examination unless he has either (a) already obtained the Intermediate Certificate, or (b) passed in Lower Grade Mathematics in a previous year.*

1. *Bookkeeping* (3 hours).—Candidates will be required to answer easy questions on the principles and practice of Bookkeeping, to frame the books necessary to record a number of transactions of moderate difficulty, to adjust ledger accounts, and to frame balance-sheets and profit and loss accounts. Special marks will be given for neatness and style.

In the books provided for Bookkeeping candidates may be required to do their own ruling. When the paper is taken, the School Authorities must provide each candidate with a *round ruler*.

2. *Commercial Arithmetic* (2 hours).—A thorough knowledge of Arithmetic will be expected, with special proficiency in discount, interest, approximations, exchange, foreign weights and measures, and such questions as are likely to arise in commercial transactions. An exercise in calculation, involving speed and accuracy, will be given.

TYPES OF SCHOOLS GIVING SECONDARY EDUCATION.

Three schools have been chosen as representing types of secondary education, and details of their curricula have been supplied by teachers in charge of classes in the schools.

School A is a Public Higher Grade School under the Code, with a five years' curriculum.

School B is a Secondary School which receives Grants under Special Regulations and sends forward pupils for the Leaving Certificate examinations; its curriculum extends over seven years.

School C may be designated a Science School, as it has developed a more complete curriculum in scientific subjects than the other two; it was indeed established with the object of providing a secondary education in which Science would have a leading place.

Of the schools in which secondary education is given, the Higher Grade Schools are the most numerous; their curriculum need not extend over more than three years, but in many cases it extends to five years. Schools of type C are not numerous and are only found in large centres.

School A.

The following statement shows the number of hours per week assigned to Mathematics and Science during the five years of the course.

First three years.—Arithmetic, $1\frac{1}{2}$; Algebra, $2\frac{1}{2}$; Geometry, $2\frac{1}{2}$; Science, 3.

Fourth and fifth years.—Algebra (including Arithmetic), $2\frac{1}{2}$; Geometry, $2\frac{1}{2}$; Trigonometry, $1\frac{1}{2}$; Science, 3; Dynamics (in fifth year), $1\frac{1}{2}$.

In this school, as in most Higher Grade Schools, boys and girls are taught together in the same classroom.

The first year's course in science has a specially close relation to the work in the mathematical subjects. In the laboratory the pupils undertake measurements of length, area and volume, determine specific gravities and make simple experiments on the pressure of liquids and gases. In connection with the measurements on solids the lessons in Geometry are developed; the records of the various measurements and experiments furnish much material for the lessons in Arithmetic and Algebra.

In the second and third year's course Geometry becomes more deductive, though drawing and calculation still figure largely. The science of the second year is a course in Heat and

of the third year a course in Chemistry; both courses are mainly experimental and are designed to elucidate the leading facts.

In the fourth year the laboratory work is devoted partly to Light and partly to Mechanics of a simple kind; in association with the experimental treatment lessons are given in Theoretical Mechanics or Dynamics. The fifth year of laboratory work is spent on Electricity and Magnetism.

The syllabus of the work in Arithmetic, Algebra, Geometry and Trigonometry agrees very closely with that prescribed for the Leaving Certificate Examinations. The influence of the laboratory on the classroom, especially in the first year, is considered to be decidedly beneficial; the least satisfactory results seem to be in the field of theoretical solid Geometry and may be partly due to the imperfect equipment of the school in models of solids. In connection with the lessons on Trigonometry a simple theodolite is in use, and by its aid the pupils make observations which are utilised in the classroom. In accordance with recent practice great use is made of graphical work; experience in this school suggests the conclusion that certain forms of graphical work—such as the plotting of statistics—have been overdone, and that much more time than is usually given is required in order to pass from the graph of an equation to the conception of the function represented by the graph.

School B.

In this school the courses of study subsequent to the qualifying stage extend over seven years. The following time table shows the number of hours per week assigned to the work in Mathematics and Science.

First three years.—Mathematics (including Arithmetic), $5\frac{1}{2}$; Science, 3; Drawing, $1\frac{1}{2}$.

Fourth year.—Mathematics, 6; Science, 3 (for some pupils, $4\frac{1}{2}$).

Fifth year.—Mathematics, 6; Science, 4 (for some pupils, 6).

Sixth and seventh years.—Mathematics, $7\frac{1}{2}$; Applied Mathematics, $1\frac{1}{2}$; Science (for pupils who give special attention to Science), $7\frac{1}{2}$.

The work of the first five years follows the lines laid down for the Leaving Certificate Examinations, and the official syllabus is fully overtaken. It does not seem necessary to state the scheme of work sketched in the school programme, since it agrees in all important matters with that suggested for the Leaving Certificate.

During the sixth and seventh years of the secondary curriculum, a fair amount of time is spent on revision of the Leaving Certificate work, with the object of strengthening the pupil's grasp of the logical development of Mathematics, but the chief aim of the course is to extend the pupil's range

of knowledge. Many of the pupils continue their studies during these two years with a view to entering the University and to competing for University Bursaries which demand a wider range of mathematical attainments than are represented by the Leaving Certificate.

The course comprises the additional subjects of the Leaving Certificate Examination, namely, Geometrical Conics, Analytical Geometry and Dynamics, but the range of study is distinctly wider than that given in the syllabus. A brief sketch of the subjects treated is subjoined.

Algebra.—In connection with the law of distribution the subjects of degree, symmetry, homogeneity and identities are treated in some detail. The division transformation is fully discussed and leads up to the consideration of factorisation and partial fractions. A short introduction to determinants is given in treating simultaneous equations of the first degree.

The theory of integral and fractional functions, and the solution of quadratic equations in one and two unknowns are illustrated by careful graphical representation, and the methods of successive approximation to the shape of a graph in the neighbourhood of a specified point lead to the introduction of the derivative of algebraic functions. Towards the end of the course the explicit use of limit theorems and applications of differentiation and integration to simple cases give a fair introduction to the fundamental ideas of the calculus. Irrational functions and the arithmetical theory of surds, complex numbers, permutations and combinations, the binomial theorem and the more important elementary series are included in the course.

Trigonometry.—The elements of Analytical Trigonometry, including De Moivre's theorem and its simpler developments.

Geometry.—Harmonic ranges and pencils. Harmonic properties of a circle. Coaxal circles. Inversion. Modern Geometry of the triangle. Projective Geometry with special reference to conic sections.

Geometrical Conics.—As given in the syllabus of the Additional Subjects, but conic sections are also discussed in connection with the general theory of projections.

Analytical Geometry.—The straight line. The circle. Conic sections and the simpler types of higher plane curves.

Dynamics.—The subjects specified in the syllabus of Additional Subjects under the headings "Elements of Dynamics" and "Higher Dynamics" are fully taken up.

School C.

This school is a School of Science which is intended to provide a suitable training for commercial, industrial and professional pursuits. Though there is a preliminary course

to meet the needs of pupils under the age of twelve, the great majority of the boys have reached the qualifying stage before entering the school and the curriculum is arranged for boys whose ages run from 12 to 18. Throughout the course English, including History and Geography, and one language other than English, are taken by all pupils, the time given to these subjects amounting to 13 hours per week in the earlier years and decreasing to 10 hours per week in the last year.

During the first four years workshop practice forms an important element in the curriculum, the object aimed at being a training in habits of accurate and steady effort, in the coordination of hand and eye, and in the full interpretation of drawings. The construction of objects from drawings prepared by the pupil from models or from instructions is systematically carried out, and in the later years considerable experience is obtained in pattern-making and in metal-work. The workshop is in close relation to the Mechanics and art drawing rooms and is a valuable auxiliary to the mathematical instruction as a whole.

The curriculum is the same for all pupils up to the end of the fourth year, and the number of hours per week allotted during this period to Science subjects is as follows:—Mathematics, 7 hours; Physics, $2\frac{1}{2}$ to 4 hours; Chemistry, $2\frac{1}{2}$ to 3 hours; Workshop, 2 hours; Drawing, 2 hours. In the fourth year Mathematics includes Elementary Dynamics.

In the last two years the pupils may take one or other of two courses: (a) the course in Mechanical and Electrical Engineering, (b) the course in Chemistry; and the distribution of hours per week during these two years is as follows:—

Course (a).—*First year*: Mathematics (including Dynamics), 7 hours; Physics, 4 hours; Applied Mechanics and Steam, 3 hours; Engineering Drawing, 2 hours; Workshop, 3 hours. *Second year*: Mathematics (including Dynamics), 8 hours; Physics, 6 hours; Applied Mechanics and Steam, 4 hours; Engineering Drawing, 2 hours.

Course (b).—*First year*: Mathematics (including Dynamics), 7 hours; Physics, 4 hours; Chemistry, 8 hours. *Second year*: Mathematics (including Dynamics), 8 hours; Physics, 6 hours; Chemistry, 6 hours.

The pupils of the school are presented for the Leaving Certificate Examinations, but they pursue their studies to a much more advanced stage than is covered by the Leaving Certificate syllabus. The following details of the later work seem to be of interest:—

Algebra.—Permutations and combinations. Binomial and exponential series. Limits. Simple infinite series. Complex numbers.

Trigonometry.—Analytical Trigonometry with De Moivre's theorem and its simpler developments.

Geometry.—Maxima and minima. Concurrency and collinearity. Similitude. Coaxal circles. Inversion. Harmonic ranges and pencils. Pole and polar. Circles related to the triangle.

Descriptive Geometry.—This subject is taught in connection with Applied Mechanics, and forms part of a course in Graphical Statics.

Analytical Geometry.—The straight line. The circle. The conic.

Calculus.—A short course in differential and integral calculus.

Dynamics.—In the theoretical course the treatment follows the lines of current text-books; at the later stages free use is made of easy calculus.

Applied Mechanics.—A very full course is given in this subject, and the laboratory and drawing room play an important part in the treatment. The following headings are given in the Syllabus:—Graphical statics. Stresses in structures. Link polygon. Work diagrams. Integral curves. Work and energy, including transmission. Strength of materials. Stress, strain, and moduli of elasticity. Bending moment. Shear. Moments of inertia. Torsion of shafts. Tension in belts.

Steam.—Properties of steam. Fuel. Efficiency of boilers. Wetness of steam. Indicator diagram theory. I.H.P. Brakes and B.H.P. Slide valve. Reuleaux and Zeuner valve diagrams. Behaviour of steam in cylinder. Crank effort diagrams. Boyle's law. Charles' law. Adiabatics and isothermals. Entropy. Carnot's cycle. PV diagram and $\theta\phi$ diagram. Gas engine and petrol motor.

UNIVERSITY PRELIMINARY EXAMINATIONS AND UNIVERSITY BURSARIES.

Before entering on a curriculum that leads to a degree a student must pass a Preliminary Examination, or present a recognised certificate that he has passed an equivalent examination. In Mathematics there are more standards than one; these depend partly on the Faculty in which the student proposes to study and partly on the subjects which he chooses for his degree. At present the regulations for the Preliminary Examination are under revision, and it is perhaps sufficient to state that the highest standard demanded is that of the Higher Grade Leaving Certificate; the University syllabus is not identical with that of the Leaving Certificate, but there is no essential difference, and the possession of the Higher Grade Leaving Certificate exempts the holder from the University Preliminary Examination.

But there are other examinations in Mathematics which demand more advanced work from the entrant to the University;

these are the Examinations for Bursaries. Bursaries of an annual value that varies very greatly, but may be as low as 5*l.* or as high as 100*l.*, are awarded by competition among students who are entering on a course of study at the University. In most cases Mathematics is only one of the subjects of examination, though a few bursaries are awarded after examination in Mathematics alone. The mathematical papers set in the Preliminary Examination may or may not form part of the competition; these are on the standard of the Higher Grade Leaving Certificate. Usually, however, additional papers are set; the scope of these papers is not very strictly defined, but may be described in very general terms as the most advanced work of a secondary school. The Bursary Competitions call forth the most strenuous efforts of both teachers and pupils in the secondary schools. The bursaries are valuable and enable promising youths to obtain a university education who without such aid could in many cases not afford the expense of the university life; the winning of bursaries reflects credit on the school from which the pupils come and thus stimulates the teacher. On the whole, the influence of the University Bursary Competitions has been decidedly beneficial though, of course, the system of competitive bursaries is not without its critics.

CONCLUDING REMARKS.

The most striking feature of the system that has now been described is undoubtedly the emphasis laid on Mathematics as an element in general education. The selection of subjects and the methods of treating them presuppose no special ability and lead up to no favoured profession or avocation; the value of Mathematics as an essential element of general culture is the guiding idea. No doubt the possession of the Leaving Certificate is valuable as exempting the holder from various examinations, such as the Preliminary Examination for entrance to the Scotch Universities; but the training which the Leaving Certificate course in Mathematics provides is quite as suitable for boys who go direct from school to business. It is in the interests of general culture that elementary science has such a definite place in the curriculum; so far as exemption from University Preliminary Examinations is concerned the science teaching of the schools is comparatively unimportant.

Mathematical work of a distinctly advanced kind only appears after the Leaving Certificate stage and chiefly in connection with the preparation of pupils for university bursaries, as noted in connection with School B. The advanced nature of the work shows itself not merely in the wider range of subjects taught, but even more in the spirit of the teaching. Heuristic methods are prominent in the early years of instruction, but, in the better schools, are to a considerable extent supplemented, if not superseded, by more explicitly logical reasoning

before the Leaving Certificate Examinations are reached. In the classes beyond the Leaving Certificate group the logical aspect of the training bulks very largely, although even then pupils keep in touch with concrete realities by means of graphical work, problem solving, applications to mechanics, and work in science generally. In some schools the mathematical attainments of pupils in the last year of their course are more than respectable; such pupils usually do good work in the Honours Courses of the Universities.

The question is often raised whether on the whole the recent changes in teaching, represented specially by the introduction of graphical work and experimental or practical geometry, have been justified, and doubts are not infrequently expressed whether pupils have not lost more than they have gained by the newer methods. It would be strange indeed if all teachers made the transition from the old to the new without loss to their pupils, and it may even be the case that none but the best teachers have succeeded in showing that the newer methods make better mathematicians than the old. Yet one result stands out, namely, pupils can now *do* where formerly they could only *say*. Thus pupils could be trained to repeat the propositions of Euclid, but few could solve even simple deductions; the training such pupils obtained was a training in memory far more than in logic. Now, even if a pupil cannot solve a rider he can usually at least take a step towards the solution by rapidly sketching two or three diagrams that not merely indicate that he knows what the problem is, but that not infrequently, especially if it be a case of loci, lead to suggestions for a solution. The pupil's mind is active and not inert in presence of his problem and that at least is pure gain.

I believe that, in spite of inevitable defects, there is an amount of sound work in mathematics being done that has not been approached in any earlier period, and the improvement is due almost entirely to the relegation of the formal attitude to the later years of study. Under the newer methods pupils can know something about the facts of geometry before they begin to argue about them and are familiar with the use of letters to represent numbers before they are involved in discussions of the laws of algebra or caught in the meshes of the interlocked brackets. While there are weaknesses and failures enough to check any feelings of boastfulness, there is also more than enough of actual achievement to give confidence in the essential merits of the present arrangements.

GEORGE A. GIBSON.

APPENDIX L

LEAVING CERTIFICATE EXAMINATION, 1911.

MATHEMATICS.

LOWER GRADE—(FIRST PAPER).

Wednesday, 5th April—10 A.M. to 12 NOON.

Before attempting to answer any question, candidates should read the whole of it very carefully, since time is often lost through misapprehension as to what is really required.

All the figures should be accurately drawn, and, where geometrical constructions are employed, the construction lines should be shown. Proofs of geometrical constructions need not be given unless specially asked.

The value attached to each question is shown in brackets after the question. In addition 10 marks are allowed for neatness and good style.

SECTION I.

All the questions in this Section should, if possible, be answered.

1. Prove that two triangles which have their three sides respectively equal are equal in all respects.

ABC, DBC are two triangles on the same side of BC such that $AB = CD$, $AC = BD$. If AC, BD intersect in O , prove that the triangles AOB, COD are equal in all respects. (18)

2. Define a parallelogram, giving in your definition no more properties than are necessary for drawing the figure.

If in a parallelogram two adjacent sides are equal, prove that its diagonals are at right angles.

If in a parallelogram two adjacent sides are at right angles, prove that its diagonals are equal. (18)

3. Draw a triangle ABC in which $AB = 5$ cm, $BC = 6$ cm, $CA = 8$ cm. From B draw BD perpendicular to AC and meeting it in D . Through A draw AE parallel to BC meeting BD produced in E .

Measure AD, BD, CD, DE , and calculate the areas of the rectangles contained by (1) AD, BD ; (2) CD, DE .

Prove that if your figures were correctly drawn these two areas would be equal. (18)

SECTION II.

One and only one question out of each of the pairs of alternatives should be answered.

Either

4a. Prove that the three angles of a triangle are together equal to two right angles.

ABC is a triangle in which $AB = AC$, and the angle A is known to be 40° . It is also known that the straight line BD , which bisects the angle B

and meets AC in D , is 2 inches in length. Draw the triangle with the help of your protractor, stating the steps of your construction. (18)

or

4b. Construct a triangle ABC having an area of 2.4 square inches, and the sides AB , BC 2 inches and 3 inches, respectively.

Produce BC to D so that the area of the triangle ABD may be 4.5 square inches.

Explain your method in each case. (18)

Either

5a. If the tangents to a circle at P and Q meet in R , prove that $PR = QR$.

Prove that if two adjacent sides of a quadrilateral circumscribed about a circle are equal, the other two are equal. (18)

or

5b. Prove that if a quadrilateral is inscribed in a circle, its opposite angles are supplementary.

Two circles intersect at C and D , and AB is a chord in the first circle. AC and BC produced meet the second circle in E and F . Prove that the angle $ABD =$ the angle DFE , and that the angle $BAD =$ the angle DEF . (18)

LOWER GRADE—(SECOND PAPER).

Thursday, 6th April—10 A.M. to 12 NOON.

Before attempting to answer any question, candidates should read the whole of it very carefully, since time is often lost through misapprehension as to what is really required.

Square-ruled paper is provided for the graphical work.

The value attached to each question is shown in brackets after the question. In addition 10 marks are allowed for neatness and good style.

SECTION I.

All the questions in this Section should, if possible, be answered.

1. Prove that $2x - 3$, and $3x - 1$ are both factors of $6x^2 + 13x^2 - 41x + 12$, and find its third factor. (15)

2. Solve the equations:

$$(1) a(x - c) - c(x - a) = b(a - c);$$

$$(2) 5x - 3y = 18y - 10x = 10;$$

$$(3) 6x^2 - 13x + 6 = 0. \quad (15)$$

3. Express in words the identity

$$a^4 - y^4 = (a - y)(a + y)(a^2 + y^2).$$

Simplify the expression

$$\left(\frac{a^3 + a^2y + ay^2 + y^3}{a - y} - \frac{x^3 - a^2y + ay^2 - y^3}{a + y} \right) \times \left(\frac{a^2}{a^2y^2 + y^4} - \frac{y^2}{a^4 + a^2y^2} \right). \quad (15)$$

4. Solve the following problems :

- (1) A farmer possessed 1000 sheep, Cheviots and Blackfaced. The number of the Cheviots was 10 more than five-sixths of the number of the Blackfaced. How many were there of each ?
- (2) A person travelled 100 miles in 4 hours, part of the way by train at 40 miles per hour, and the rest by motor at 20 miles per hour. How many miles did he travel each way ? (15)

SECTION II.

One and only one question out of each of the pairs of alternatives should be answered.

Either

5a. What number must be added to $4x^2 - 20x$ to form a complete square ?

If $y = 4x^2 - 20x + 36$, show that y is never less than 11, whatever value be given to x .

Find what value must be given to x , in order that y may be 60. (15)

or

5b. Resolve into their simplest factors

(1) $9x^2 + 6xy - 8y^2$,

(2) $a^2 - b^2 + c^2 - 2ac$,

(3) $x^2 - 7x + 6$, having given that $x - 1$ is one factor. (15)

Either

6a. A starts at 8 a.m. to cycle from Edinburgh to Glasgow and travels at a uniform rate of 7 miles an hour. B starts at 9 a.m. and follows A at the rate of 9 miles an hour. Find by a graphic construction at what time and how far from Edinburgh B will overtake A.

Solve the same problem by algebraical or arithmetical calculation. (15)

or

6b. Solve the equation,

$$x(x + 3) = 16,$$

to 3 decimal places.

Draw a straight line PQ , three inches in length, and on this line as diameter describe a circle. At P erect a perpendicular PR four inches in length. Join R to S , the centre of the circle, and let RS meet the circle in T and U .

Show that the length of RT in inches is the numerical value of one of the roots of the given equation. (15)

LOWER GRADE—(THIRD PAPER).

Thursday, 6th April—1 P.M. to 2 P.M.

Before attempting to answer any question, candidates should read the whole of it very carefully, since time is often lost through misapprehension as to what is really required.

All the working must be shown in its proper position in the answer, and the different steps of the calculation should be shortly indicated in words.

The value attached to each question is shown in brackets after the question. In addition 6 marks are allowed for neatness, arrangement, and style.

1. A workman earned 40s. a week and found that after paying all necessary expenses he could save 5s. a week. He then emigrated to a country where he earns 30 per cent. more, but where his expenses are 35 per cent. higher. Find how much he can now save in a week. (10)

2. A man received a legacy of £2,000. He invested £1,250 at $4\frac{1}{2}$ per cent, and the rest at $3\frac{1}{2}$ per cent. Find his annual income from the legacy after paying income tax on the whole of the interest at 1s. 2d. in the pound, (10)

3. Express the following fractions as decimals correct to two places and arrange them in ascending order of magnitude:—

$$\frac{2}{3}, \frac{5}{8}, \frac{6}{7}, \frac{9}{10}, \frac{7}{11}, \frac{8}{11}.$$

How many decimal places must be calculated in order to decide which of the two fractions, $\frac{1}{3}$ and $\frac{2}{3}$, is the larger? (10)

4. A contractor undertakes to do a certain piece of work in 175 days, and employs 72 men for the purpose. After 130 days one-third of the work has still to be done. How many additional men will he require to employ during the remaining days in order to carry out his contract? (10)

5. A metre is known to be more than 3·280 feet, and less than 3·281 feet. Find the corresponding numbers of square feet between which a square metre must lie.

From your results state the number of square feet in a square metre correct to as many decimal places as possible.

Taking as your scale one inch to the foot, draw on your squared paper, figures representing a square decimetre, a square foot, and a square metre. (14)

HIGHER GRADE—(FIRST PAPER).

Wednesday, 5th April—10 A.M. to 12 NOON.

Before attempting to answer any question, candidates should read the whole of it very carefully, since time is often lost through misapprehension as to what is really required.

All the figures should be accurately drawn.

The value attached to each question is shown in brackets after the question. In addition 10 marks are allowed for neatness, good style and accurate diagrams.

SECTION I.

All the questions in this Section should, if possible, be answered

1. Prove that the straight lines, which join the vertices of a triangle to the mid points of its opposite sides, meet in a point.

ABC is a triangle; *BC* is bisected in *D* and produced its own length to *E*; *AE* is bisected in *F*. Prove that the sides of the triangle *ADF* are equal to the medians of *ABC*. (18)

2. *C* is the mid point of a given straight line *AB*, and *P* is any other point in *AB*; show that the sum of the squares on *AP* and *PB* is twice the sum of the squares on *AC* and *CP*.

If the sum of the squares on *AP* and *PB* is thrice the square on *AC*, and if *AB* is $2a$ inches and *CP* is x inches, calculate the value of x , and give a geometrical construction for finding the position of *P*. (18)

3. Draw a straight line AB , 4 inches in length. With A as a centre, and a radius of 1·8 inches, describe a circle. With B as centre, and a radius of 1·4 inches, describe a circle. Construct the four common tangents to the two circles. State the steps of your construction.

If P be a point in which an external tangent intersects an internal tangent, prove that APB is a right angle. (18)

SECTION II.

One and only one question out of each pair of alternatives should be answered.

Either

4a. ABC is an equilateral triangle inscribed in a circle, and P is a point on the arc BC ; CP and BP produced meet AB and AC produced in D and E . Prove that the triangles BDC , PBC , CBE are similar.

Prove that BC touches the circle that circumscribes the triangle BDP . (18)

or

4b. State and prove a construction for describing on a given straight line PQ a quadrilateral similar to a given quadrilateral $ABCD$, the side PQ corresponding to the side AB .

In a given semicircle inscribe a rectangle similar to a given rectangle. (18)

Either.

5a. AB is a straight line and C, D are points in space. Show how to find a point P in AB equidistant from C and D .

If E and F are the projections on AB of C and D respectively, and if $CE = 10$ inches, $DF = 6$ inches, and $EF = 16$ inches, find the distance of P from F . (18)

or

5b. AB and CD are two straight lines in space which do not intersect and are not parallel. Show that two parallel planes can be found such that one of them contains AB and the other contains CD .

Hence construct a straight line to cut each of the two given lines at right angles. (18)

HIGHER GRADE—(SECOND PAPER).

Thursday, 6th April—10 A.M. to 12 NOON.

Before attempting to answer any question, candidates should read the whole of it very carefully, since time is often lost through misapprehension as to what is really required.

Square-ruled paper and four-place logarithmic tables are provided.

The value attached to each question is shown in brackets after the question. In addition 10 marks are allowed for neatness and good style.

SECTION I.

All the questions in this Section should, if possible, be answered.

1. Prove that

$$\frac{A}{\left(1 + \frac{r}{100}\right)^n}$$

represents the present value of A pounds due n years hence at r per cent. compound interest, payable annually.

Find the present value of £1500·0 due 8 years hence at $4\frac{1}{2}$ per cent. compound interest, using logarithms. (15)

2. Show, either by substitution or division, that $a + b - c$ is a factor of $a^2 + b^2 + c^2 + 2abc - a^2(b + c) - b^2(c + a) - c^2(a + b)$, and find the other two factors. (15)

3. Solve the following equations:—

(1) $x + 5 = 6(x^2 - 5)$;

(2) $x - y = b, x^2 - y^2 = 3a^2b + \frac{1}{2}b^2$. (15)

4. Taking half-an-inch as unit, draw the graphs of (1) $x + 2y = 6$ from $x = -4$ to $x = 6$, (2) $x^2y = 8$, from $x = -4$ to $x = -1$, and from $x = 1$ to $x = 6$.

Write down the coordinates of the points of intersection of the graphs as determined by your figure.

Show that the values of x at the points of intersection are the roots of the equation

$$x^3 + 16 = 6x^2. \quad (15)$$

SECTION II.

One and only one question out of each pair of alternatives should be answered.

Either

5a. A set out at 9 a.m. to walk from P to Q , a distance of 30 miles. At 11 a.m. B sets out to cycle from Q to P and meets A at 1 p.m. Had A walked a quarter of a mile per hour faster, and B cycled half a mile per hour faster, they would have met 10 minutes sooner. Find their rates. (15)

or

5b. The sum of three numbers in geometrical progression is 21, and the sum of their squares is 189: find the numbers. (15)

Either

6a. The radius, OE , of a circle is divided into n equal parts by the points A, B, C, \dots and circles are described with O as centre and OA, OB, OC, \dots as radii. If $OE = r$, calculate the sum of the n circumferences.

Show that if each circumference is multiplied by $\frac{r}{n}$, the sum of the n products is approximately πr^2 , when n is a large number. (15)

or

6b. From a point P , inside a circle of radius a , and at a distance b from the centre, a chord HPK is drawn making a given angle θ with the diameter through P . Find the sum of the squares of the segments HP, PK in terms of a, b and θ .

From your result prove that, if two chords are drawn through P at right angles to each other, the sum of the squares of the four segments is constant. (15)

HIGHER GRADE—(THIRD PAPER).

Thursday, 6th April—1 P.M. to 3 P.M.

Before attempting to answer any question, candidates should read the whole of it very carefully, since time is often lost through misapprehension as to what is really required.

Square-ruled paper and four-place logarithmic tables are provided.

All the figures should be accurately drawn.

The value attached to each question is shown in brackets after the question. In addition 10 marks are allowed for neatness and good style.

SECTION I.

All the questions in this Section should, if possible, be answered.

1. Define the tangent and secant of an acute angle.

Prove, by the help of a figure, that

$$\sec^2 \theta = 1 + \tan^2 \theta.$$

If $\sin \theta = \frac{3}{5}$, prove that

$$\tan \theta + \sec \theta = 2\frac{1}{2}. \quad (18)$$

2. What is a radian?

ABC is an equilateral triangle, the length of each side being $10\cdot5$ cm. With A as centre and AB as radius an arc of a circle is described passing through C . Find the length of the arc BC , taking $3\frac{1}{2}$ as the value of π .

If D be a point in the arc such that the arc $BD = 5$ cm., find the value of the angle BAD (1) in degrees, minutes, and seconds; (2) in radians. (18)

3. With the help of your tables draw in the same figure the graphs of $\cos \theta$ and $\cos 2\theta$ from $\theta = 0$ to $\theta = 180^\circ$, plotting points for $\theta = 0^\circ, 20^\circ, 40^\circ, \&c.$

From your figure obtain two approximate solutions of the equation

$$\cos \theta = \cos 2\theta + \frac{1}{2}. \quad (18)$$

SECTION II.

One and only one question out of each pair of alternatives should be answered.

Either

4a. Draw a triangle ABC in which $AB = 8\cdot5$ cm., $BC = 5$ cm., $CA = 4\cdot5$ cm. Find the cosine of each of the angles A and B by measurement, and check the accuracy of your results by calculation. (18)

or

4b. In the triangle ABC it is given that $AB = 4$ cm., $AC = 6$ cm., and the angle $A = 70^\circ 20'$; find the values of BC and the angle B , and verify your results by means of a diagram. (18)

Either

5a. A tower has an angular elevation of 40° from a point on the same level as its base. The height of the tower is now increased in the ratio of 5 to 3, find the ratio in which its angular elevation is increased.

If the angular elevation of the top had been increased in the ratio of 5 to 3, in what ratio would its height have been increased? (18)

or

5b. A rectangular level space $ABCD$ has a flagstaff AK of height h erected at A . If the angles KBA, KCA are α and β , express BK and CK in terms of h, α, β .

If $BC = a$, use the triangle KBC to express h in terms of a, α, β . (18)

GEOMETRICAL CONICS.

ADDITIONAL MATHEMATICAL SUBJECT.

(Higher Grade).

Wednesday, 5th April—3 P.M. to 4.30 P.M.

Before attempting to answer any question, candidates should read the whole of it very carefully, since time is often lost through misapprehension as to what is really required.

All ordinary symbols and contractions are allowed.

All the steps of the proofs must be given. Preference will be given to proofs which depend on first principles, and in all cases it should be clearly shown on what assumptions the demonstrations are based.

The value attached to each question is shown in brackets after the question. In addition 10 marks are allowed for neatness, good style, and accurately drawn figures.

1. If S is the focus of a parabola, and P is any point on the curve, find the locus of the mid point of SP . (12)

2. Prove that the tangents at the extremities of any chord of a parabola meet on the diameter which bisects the chord.

PQ is a chord of a parabola and V is its mid point. If the tangents at P and Q meet in B and BV meets the parabola in U , prove that $EU = UV$. (16)

3. P, P' are two points on any conic, and the chord PP' , produced if necessary, meets the directrix in Q . Prove that SQ bisects the angle PSP' either externally or internally, and draw figures to illustrate both cases. (12)

4. What property of an ellipse enables us to construct the curve with the help of two pins and a piece of thread?

If A, A' be the vertices, S, S' the foci, C the centre, and PN an ordinate of an ellipse, prove that

$$(1) SP^2 \sim S'P^2 = 4 CS \cdot CN,$$

$$(2) SP \cdot CA = CS \cdot XN,$$

where X is a point on the axis, on the same side of C as S , such that

$$CS \cdot CX = CA^2. \quad (18)$$

5. Prove that the locus of the foot of the perpendicular from either focus of an ellipse on the tangent is a circle, and that the rectangle contained by the two perpendiculars is constant. (14)

6. Define the eccentricity of a conic section.

$APQB$ is any arc of a circle described on a fixed straight line AB and trisected in P and Q . Prove that P lies on one branch of a hyperbola of eccentricity 2; of which A is one focus, and that Q lies on one branch of an equal hyperbola of which B is one focus.

Find the centre and the other focus of each hyperbola. (18)

ANALYTICAL GEOMETRY.

ADDITIONAL MATHEMATICAL SUBJECT.

(Higher Grade).

Friday, 7th April—4 P.M. to 5.30 P.M.

Before attempting to answer any question, candidates should read the whole of it very carefully, since time is often lost through misapprehension as to what is really required.

All the steps of the proofs must be given.

The co-ordinate axes may be assumed to be rectangular.

In all cases where numerical coefficients are given accurate figures should be drawn.

The value attached to each question is shown in brackets after the question.

In addition 10 marks are allowed for neatness, good style, and accurately drawn figures.

1. Find the tangent of the angle which the straight line joining the points (x_1, y_1) , (x_2, y_2) makes with the axis OX .

Find the numerical value of this tangent when both these points satisfy the equation,

$$2x - 4y = 7.$$

Use your result to show that all points whose co-ordinates satisfy this equation lie on a certain straight line. (14)

2. If P and Q are any numbers, prove that the equation,

$$P(y - mx - c) + Q(y - m'x - c') = 0,$$

represents a straight line through the point of intersection of the lines

$$y = mx + c,$$

$$y = m'x + c'.$$

Find the equation of the line drawn through the point of intersection of the lines,

$$2x + 3y - 6 = 0, \quad x - 3y + 2 = 0,$$

parallel to the line,

$$4x - 3y = 5. \quad (15)$$

3. Find the co-ordinates of the point which divides the straight line joining the points (x_1, y_1) , (x_2, y_2) in the ratio of m to n .

Find the ratio in which the straight line, joining the points $(0, -5)$, $(4, 8)$, is divided by the line,

$$3x + 4y = 12. \quad (15)$$

4. Find the equations to the following circles:—

(1) A circle whose centre is the origin and which passes through the point (a, b) ;

(2) a circle whose centre is (a, b) and which passes through the origin;

(3) a circle whose radius is c , and which touches both axes. (15)

5. Prove that the straight line,
 $2x - y = 8$,
 is a tangent to the circle,
 $(x - 2)^2 + (y - 1)^2 = 5$,
 and find the point of contact. (15)

6. Find the length of the tangent from the point (h, k) to the circle,
 $x^2 + y^2 + 2ax + 2by + c = 0$.
 A point P moves so that the tangent drawn from it to the circle,
 $x^2 + y^2 = 3$,
 is twice the distance of P from the point $(3, 0)$: find the equation of the
 locus, and sketch the locus. (16)

ELEMENTS OF DYNAMICS.

ADDITIONAL MATHEMATICAL SUBJECT.

(Higher Grade).

Monday, 10th April—4 P.M. to 5.30 P.M.

Before attempting to answer any question, candidates should read the whole of it very carefully, since time is often lost through misapprehension as to what is really required.

All the steps of the proofs must be given, and the work of the calculations should be shown in full.

The value attached to each question is shown in brackets after the question. In addition 10 marks are allowed for neatness, good style, and accurately drawn figures.

1. Define *change of velocity*, taking direction into account as well as magnitude.

A particle moves along AB , a side of an equilateral triangle ABC , with a uniform velocity of 10 feet per second; when it reaches B it proceeds along BC with a uniform velocity of 20 feet per second. If AB is taken to represent its first velocity, find a straight line which represents the change of velocity in magnitude and direction. (15)

2. Obtain formulae to show the velocity, and position after t seconds, of a body, which has an initial velocity V and a uniform acceleration g opposite to its direction of motion.

A body is thrown directly upwards with a velocity of 150 feet per second; find when and where it will begin to fall. (15)

3. Define *Mass* and *Momentum*. How is a uniform force measured?

A 10 lb. shot is fired from a gun with an initial velocity of 1600 feet per second; what is its momentum? If a constant force, applied for half a second, would produce the same momentum, express its measure in absolute units of force. (15)

4. Two parallel forces, P and Q , act in similar directions at two points, A and B . If the resultant R meets AB in C , find expressions for the magnitudes of R and AC .

If R is a force of 10 pounds, $AB = 10$ inches, and $AC = 3$ inches, find the magnitudes of P and Q . (15)

5. Prove that the difference of the pressure at two points in a liquid depends only on the difference of their depths and the specific weight of the liquid.

If a cubic foot of water weighs 1000 ounces, find what pressure per square inch is caused by a head of 36 feet of water. (15)

6. Explain why a balloon rises in the air.

A balloon contains 16000 cubic feet of hydrogen. If the weight of a cubic foot of air is 1.25 ounces, and the weight of a cubic foot of hydrogen at the same temperature is .09 ounce, and the balloon is just able to rise, find the weight of the balloon. (15)

HIGHER DYNAMICS.

ADDITIONAL MATHEMATICAL SUBJECT.

(Higher Grade).

Monday, 10th April—4 P.M. to 5.30 P.M.

Before attempting to answer any question, candidates should read the whole of it very carefully, since time is often lost through misapprehension as to what is really required.

The value attached to each question is shown in brackets after the question. In addition 10 marks are allowed for neatness, good style, and accurately drawn figures.

SECTION I.

All the questions in this Section should, if possible, be answered.

1. Enunciate and prove the *parallelogram of velocities*.

A river, whose sides are parallel straight lines, is 220 yards broad and has a speed of $4\frac{1}{2}$ miles an hour. A motor boat, propelled at the rate of 15 miles an hour, starts from a point on one bank of the river, and is propelled in a direction obliquely up the river making an angle of 30° with the bank. Find how long the boat takes to reach the other bank, and how many yards it travels up stream. (18)

2. Define *work* and *kinetic energy*.

If a particle moves under the action of a force that is constant in magnitude and direction, show that the work done by the force between any two positions, *A* and *B*, of the particle is equal to the difference of the kinetic energies of the particle at *A* and *B*.

A weight of 2 tons has to be raised from the bottom of a shaft 300 feet deep: how many foot-pounds of work will have to be done, and how long will an engine working at 200 horse-power take to do it? (18)

3. Show that, when a body is immersed in a liquid either wholly or partially, it is acted on by an upward force equal to the weight of the liquid displaced.

A piece of iron (specific gravity 7.5) is placed on the surface of mercury (specific gravity 13.5): what fraction of the volume of the iron will be immersed?

If water is poured on the mercury until the iron is covered, what fraction of the iron will then be immersed in the mercury? (18)

SECTION II

One, and only one, question from each of the two pairs of alternatives should be answered.

Either

4a. If a body is projected with velocity V at an inclination α to the horizontal plane, find the range on a horizontal plane through the point of projection.

Prove that the range on a plane, through the point of projection, which makes an angle β (less than α) with the horizontal plane is

$$\frac{2V^2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta}$$

and show that this gives your former result as a particular case. (18)

or

4b. Explain what happens when a body, moving in a straight line, overtakes another body moving in the same straight line, supposing that there is no rebound.

Two equal smooth wires AB, BC , each 4 feet in length, make an angle of 135° at B , and are placed so that AB is vertical, A being above B . A bead slides on the wire, starting from rest at A . Explain how its velocity is affected at B , and find its speed when it reaches C . (18)

Either

5a. Explain what is meant by the *coefficient of friction*.

A mass of 5 pounds slides down a plane which makes an angle of 45° with the horizontal, the coefficient of kinetic friction being $\cdot 6$. Find the acceleration, and the time taken to slide 10 feet down the plane from rest.

Find also the least force acting up the plane necessary to keep the body at rest if the coefficient of static friction is $\cdot 7$. (18)

or

5b. A body of mass m moves in a circle of radius r with uniform speed v ; prove that there is no acceleration along the tangent, and find the acceleration along the radius of the circle.

A mass of 3 pounds, tied by a light string to a point in a smooth horizontal table, is describing a circle with uniform speed and strikes a mass of 2 pounds lying at rest on the table. If the masses cohere, prove that the tension of the string after impact is three-fifths of the tension before impact. (18)

APPENDIX II.

SCOTTISH UNIVERSITIES JOINT BOARD.
PRELIMINARY EXAMINATION, 1911.

ARTS AND SCIENCE.

MATHEMATICS.

(LOWER STANDARD).

Thursday, 7th September, 1911.—3 P.M. to 6 P.M.

A Candidate may obtain FULL marks by doing about two-thirds of this Paper; but the Examiners may reject a Candidate who shows exceptional weakness in any of the sections A, B, C.

Marks will not be awarded to answers where the work is not shown.

The Candidates must write on their books the number of the table at which they sit.

A.

1. Find the value of

$$\frac{1\frac{1}{2} - \frac{2}{3}}{\frac{1}{2} + 1\frac{1}{3}} - \frac{2\frac{7}{11} \text{ of } 2\frac{7}{25}}{7\frac{2}{3} \text{ of } 3\frac{7}{11}}$$

2. Find the difference between
- $5\frac{1}{2}$
- guineas and
- $3\cdot 125\text{l.}$
- , and express the result as the decimal of 50
- l.

3. Find the greatest number that will divide 7651 and 10381 and leave a remainder 7 in each case.

4. A reservoir contains 3,217,428 cubic feet of water; its depth is one-third of its length, and its breadth is half the difference between the length and one-third of the depth. Find the dimensions.

5. What sum put out to simple interest at
- $4\frac{1}{2}$
- per cent. per annum will in 1 year 5 months amount to 301
- l.
- 11s. 5·55
- $d.$
- ?

B.

6. Find the G.C.M. of

$$x^4 + 2x^3 - 7x^2 - 20x - 12$$

and

$$x^4 - 5x^3 + 4.$$

7. Simplify

$$(1) \left(\frac{x}{x+1} - \frac{x-1}{x} \right) \div \left(\frac{x}{x+1} + \frac{x-1}{x} \right).$$

$$(2) a \frac{c-b\frac{r}{q}}{a-b\frac{p}{q}} + b \frac{c-a\frac{r}{p}}{b-a\frac{q}{p}}$$

8. Solve the equations

$$(1) \frac{x+2}{x-2} - \frac{x-1}{x+1} = \frac{9}{2}.$$

$$(2) 4x^2 - 27x = 40.$$

9. Draw the graph of $2x^2 - 4x + 1$, showing its lowest point and where it cuts the axes of x and y .

10. Two trains run without any variation of their speeds over the same 36 miles of rail. One of them travels 15 miles an hour faster than the other, and accomplishes the distance in 12 minutes less; find the speeds of the trains.

O.

11. Prove that in any triangle the greater angle is opposite to the greater side.

ABC is a triangle and D is a point on BC between B and C . If the angle ACB is greater than the angle ABC , prove that AB is greater than AD .

12. At a given point in a given straight line make a rectilinear angle which shall be equal to a given angle.

Show how to trisect an angle which is equal to half a right angle.

13. Produce a given straight line so that the rectangle contained by the whole line so produced and the part produced may equal the square on the given line.

Solve the equation

$$x^2 + ax = a^2$$

by a geometrical construction, a being a positive-quantity.

14. P is a point within a circle whose centre is C . If CP be produced to meet the circumference in Q , prove that PQ is the shortest line that can be drawn from P to meet the circumference.

Find the locus of P if it moves so as to remain at the same shortest distance from the circumference.

15. Prove that the opposite angles of any quadrilateral in a circle are together equal to two right angles.

If the exterior angles of any quadrilateral be bisected by four straight lines, prove that the quadrilateral formed by these lines can be inscribed in a circle.

MATHEMATICS.

INTERMEDIATE AND HIGHER—(FIRST PAPER).

Wednesday, 6th September, 1911.—12 A.M. to 2 P.M.

A Candidate may obtain FULL marks by doing correctly about two-thirds of this Paper; but the Examiners may reject a Candidate who shows exceptional weakness in either of the sections A, B.

The pass mark for Intermediate Candidates is fixed with reference to the total number of marks assigned to questions which fall under the syllabus of the Intermediate Standard; but such Candidates will receive credit for any question they may answer.

Marks will not be awarded to answers where the work is not shown.

Candidates must write on their books the number of the table at which they sit.

A.

(Geometrical Proofs must be used.)

1. Prove that the diagonals of a parallelogram bisect one another.

$ABC, A'B'C'$ are two triangles such that $AB = A'B'$, and $AC = A'C'$, but BC is greater than $B'C'$. If O, O' be the middle points of $BC, B'C'$, respectively, prove that AO is less than $A'O'$.

2. Divide a straight line so that the rectangle contained by the whole and one of the parts shall be equal to the square on the remaining part.

Prove that the difference of the squares on the whole and on the remaining part is equal to the rectangle contained by the whole and that part.

3. If one angle of a triangle be obtuse, the square on the side opposite that angle exceeds the sum of the squares on the remaining sides by twice the rectangle contained by either of these sides and the projection on it of the other.

If the squares on one pair of opposite sides of a quadrilateral be together equal to the squares on the other pair of opposite sides, the diagonals of the quadrilateral will be perpendicular to one another.

4. Prove that the perpendicular drawn from the centre of a circle to any chord which does not pass through the centre bisects the chord.

Find the locus of the middle points of all chords of a circle which pass through a given point other than the centre of the circle.

5. Prove that angles in the same segment of a circle are equal to one another.

Two arcs of different circles have their extremities at two points A, B , and are on the same side of AB . A point P on the inner arc is taken, and AP, BP are drawn and produced to meet the outer arc in Q and R , respectively. Prove that the length of the arc QR is the same for all positions of P .

6. Describe a circle to touch a given side of a triangle internally, and the remaining sides externally.

Show how to describe, with the three vertices of a given triangle as centres, three circles such that each shall touch the other two. How many solutions does the problem admit of?

7. If a vertical angle of a triangle be bisected, the bisecting line divides the base into two segments whose ratio is that of the two adjacent sides.

A, B, C are three points in a straight line. Prove that the locus of a point at which AB, BC subtend equal angles is a circle, except in one special case.

8. Prove that equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

ABC is a triangle, and points P, Q, R are taken in BC, CA, AB , respectively, such that $BP = \frac{1}{3}PC, CQ = \frac{1}{3}QA, AR = \frac{1}{3}RB$. Find the ratio of the area of the triangle PQR to that of ABC .

9. If O be a point in the side BC of a triangle ABC , such that AO bisects the angle BAC , prove that

$$AB \cdot AC = BO \cdot OC + AO^2.$$

Verify this proposition in the case where $A = 60^\circ, B = 90^\circ$.

10. If a straight line be perpendicular to each of two intersecting straight lines, it will be perpendicular to every other straight line in the same plane with these.

OA, OB, OC are three lines each of which is perpendicular to the other two, and P is the orthocentre of the triangle ABC . Prove that OP is normal to the plane ABC .

B.

11. Define the tangent of an angle of any magnitude and sign. Find the values of x between 0° and 360°

(1) when $\tan 2x = 1.0234$,

(2) when $2 \tan^2 x = 1 + \tan x$.

12. If the angle C of a plane triangle ABC be 90° , prove that

$$\sin(A - B) = \frac{a^2 - b^2}{a^2 + b^2} \quad \cos(A - B) = \frac{2ab}{a^2 + b^2}$$

MATHEMATICS.

INTERMEDIATE AND HIGHER—(SECOND PAPER).

Thursday, 7th September, 1911—12 NOON to 2 P.M.

A Candidate may obtain FULL marks by doing correctly about two-thirds of this Paper; but the Examiners may reject a Candidate who shows exceptional weakness in either of the sections B, C.

The pass mark for Intermediate Candidates is fixed with reference to the total number of marks assigned to questions which fall under the syllabus of the Intermediate Standard; but such Candidates will receive credit for any question they may answer.

Marks will not be awarded to answers where the work is not shown.

Candidates must write on their books the number of the table at which they sit.

One at least of the two questions on Arithmetic must be done by all Candidates.

A.

1. A man's income is half of it "earned" and half of it "unearned." On the former he pays an income tax of 9d. in the pound, and on the latter 1s. 2d. in the pound. His net income after payment of the tax is 1,218l. 13s. 4d. What is his gross income?

2. A vessel steams 25 sea-miles per hour. How many metres per second is this equivalent to?

[1 metre = 3·2809 ft., 1 sea-mile = 6077·2 ft.]

B.

3. Find the real factors of

$$x^3 - 4x^2 - 4x + 16 \text{ and } x^4 - x^2y^2 + 16y^4.$$

Prove that the expression

$$(1 - x + x^2)^2 + (1 + x + x^2)^2$$

is divisible by $1 + x^2$, and find the remaining factor.

4. If the sum of two positive quantities be given, prove that their product is greatest when they are equal.

Prove that the cube of the sum of two positive numbers is always less than four times the sum of the cubes of the numbers.

5. Solve the equations

$$(i) \frac{3x - 2y}{4} = \frac{24x + 4y}{7} = \frac{12x + 20}{11};$$

$$(ii) x^3 = 2y + 3, y^3 = 2x + 3.$$

6. Draw to scale, on the same diagram, the graphs of

$$y = x - \frac{1}{2} \text{ and } y = 2x^2 - x.$$

Find the curve of the type

$$y = Ax^2 + Bx + C$$

which shall pass through the points (1, 0), (0, -1), and (2, -3). Give a figure.

7. State and prove the relations between the roots and the co-efficients of a quadratic equation.

If α, β be the roots of

$$x^2 + px + q = 0,$$

prove that

$$(A\alpha + B\beta)(A\beta + B\alpha) = ABp^2 + (A - B)^2q$$

8. A man travels a certain distance in 2 hours, having walked half the time and driven half the time. If he had walked half the way and driven half the way he would have taken 5 minutes more. Compare his rates of walking and driving.

9. Assuming that the volume of a sphere varies as the cube of the diameter, find the external diameter of a hollow spherical shell, having a uniform thickness of one inch, which shall have the same weight as a solid sphere, of the same material, whose diameter is 8 inches.

C

10. The sides of a triangle are 95 ft., 100 ft., 105 ft., respectively: find the angles, to the nearest minute.

Also find the ratio of its area to that of an equilateral triangle of the same perimeter.

11. Prove the formula

$$\cos \frac{1}{2}(B - C) = \frac{b + c}{a} \sin \frac{1}{2}A.$$

Apply this to show that if one side and the opposite angle of a triangle be given, the perimeter will be greatest when the triangle is isosceles.

DYNAMICS.

Friday, 8th September, 1911.—3 P.M. to 6 P.M.

A Candidate may obtain FULL marks by doing about two-thirds of this Paper.

Marks will not be awarded to answers where the work is not shown.

Candidates must write on their books the number of the table at which they sit.

1. Prove that in uniformly accelerated motion the mean velocity in any interval of time is equal to the velocity at the middle instant of that interval.

A falling body passes a certain point with a velocity of 120 feet per second; where was it 2 seconds previously, and where will it be 2 seconds later?

2. Distinguish between absolute and gravitational standards of force.

Find in tons the excess of tractive force over resistance in order that a train of 200 tons may get up a speed of 15 miles an hour uniformly from rest in a space of half a mile.

If, the steam now being shut off, the train comes to rest in one mile, find the original tractive force.

3. A hammer strikes a nail and drives it partially into a fixed block of wood. How does the extent to which the nail is driven in depend (1) on the weight of the hammer-head, and (2) on its velocity just before striking the nail? [Assume the resistance of the wood to be constant, and neglect the recoil of the hammer.]

If the weight of the head be 1 lb., and the velocity 20 ft. per sec., and the nail be driven in half an inch, find the resistance in lbs.

4. State the principles on which problems of direct impact are treated.

A steel ball *A* moving with a velocity of 9 ft. per sec. impinges directly on another ball *B* of twice its mass, which is at rest, and then proceeds with a velocity of 2 ft. per sec. What would have been the velocity of *A* after impact if *B* had been originally moving in the opposite direction to *A* with a velocity of 3 ft. per sec.?

5. Explain a graphical method of finding the relative velocities of two points whose absolute velocities are given.

A vessel is steaming due E. at 10 miles an hour, and the wind is N.W. A vane on the mast-head points N.N.E.: what is the velocity of the wind? In what direction will the vane point if the steamer's course is altered to due N.?

6. A particle is projected under gravity with a given velocity in a given direction; find by geometrical construction its direction of motion after t seconds.

Two particles are projected from the same point and in the same vertical plane with horizontal and vertical velocities (u, v) and (u', v') respectively. Find an expression for the difference in the times of projection so that the particles may collide.

7. State the rules for finding the resultant of two parallel forces. In what case do they fail?

A horizontal beam, loaded in any manner, is supported at two points A, B , 10 ft. apart. If a load of 1 ton is moved along the beam in the direction from A to B , through a space of 2 ft., what changes are thereby produced in the pressures on the supports at A and B , respectively?

8. Show how to find graphically the resultant of any number of forces acting at a point.

AC and BC are two light rods, jointed at C and hinged at A and B to two fixed points in the same vertical, and AC is horizontal. If $AC = 4$ ft., $AB = 2$ ft., find (either by a diagram drawn to scale or by calculation) the tension or thrust in each rod produced by a weight of 20 lbs. suspended from C .

9. Find the centre of gravity of a uniform triangular plate.

If the plate be equilateral, and the triangular portion cut off by a straight line bisecting two sides be cut off, find the centre of gravity of the remainder.

10. State the laws of sliding friction.

A uniform rod, bent into the form of a square, hangs with one of its sides over a rough peg. Having given the coefficient (μ) of friction, find the greatest possible distance of the peg from the centre of the side.

11. A pendulum swings from rest to rest through an arc PP' , the equilibrium position of the bob being A . Prove that the velocity with which the bob passes through the position A is proportional to the chord PA .

What must be the arc of oscillation in order that the tension of the string when vertical may be exactly double the weight of the bob?

12. Obtain a rule for finding the pressure at any depth in a liquid.

Find the pressure in tons per sq. ft. at a depth of 120 ft. in sea-water of sp. gr. 1.026. [The weight of a cubic ft. of fresh water is 62.3 lbs. Neglect the atmospheric pressure.]

13. An open cubical tank, 4 ft. in the edge, is half full of water. If it be tilted about one of the lower edges until the water is on the point of overflowing, find (in lbs.) the pressure on the base in this position.

14. Explain the action of the ordinary bicycle pump.

Supposing that at the first stroke the valve opens when the piston has described one-tenth of its course, and at the twenty-fifth stroke when the piston has described one-half of its course: what is the ratio of the capacity of the tyre to that of the pump?

APPENDIX III.

UNIVERSITY OF GLASGOW.
BURSARY EXAMINATION, 1911.

MATHEMATICS.

ADDITIONAL PAPER.

14th September, 1911.—9 A.M. to 12 NOON.

[Candidates are requested not to attempt more than EIGHT of the following questions.]

1. Prove that the three medians of a triangle ABC are concurrent.

G is the point through which the three medians pass and D, E are the points of trisection of the base BC ; show that GD and GE are proportional to the perpendiculars from G to the sides of the triangle.

2. D is a point in the base BC of a triangle ABC and a straight line drawn parallel to BC meets AB, AD, AC at E, F, G respectively; show that

$$EF : FG = BD : DC.$$

State and prove a construction for inscribing in a triangle ABC a rhombus which shall have one of its angular points at a given point D in the base BC , and one of its sides on the base BC .

3. B is a point outside a circle of centre A ; AB is produced to a point C such that BC is equal to the tangent from B to the circle, and any point P is taken on the line through B perpendicular to AB . If a secant from P cut the circle at Q and R , prove that the triangles CPQ and BPC are similar.

4. Two planes α and β intersect in the line MN ; from a point A in the plane α a perpendicular AB is drawn to the plane α , and a perpendicular AC to the plane β , these perpendiculars meeting the plane β in the points B and C respectively. Prove that BC is perpendicular to the line MN .

If the lengths of AB and AC are b and c respectively, find the distances of B and C from the line MN .

5. The diameter AB of a semicircle is produced beyond B to any point C , the tangent AT is drawn to the semicircle at A , and from C a tangent is drawn which meets the semicircle D and AT at E ; O is the middle point of AB and M is the projection of D on AB , the lengths of OA and OM being a and s respectively. If V is the volume of the cone generated by the revolution of the triangle ACE about AC , prove that

$$V = \frac{\pi a^3}{3} \cdot \frac{(a+s)^2}{as-x^2}$$

Draw a curve to show how V varies as s increases from 0 to a , and show that V cannot be less than $8\pi a^3/3$.

6. Find by inspection the coefficient of x^3 in the product

$$(x^2 + 2x^2 + 3x + 4)(x^2 - x^2 + 2x - 3);$$

and determine the number a so that the coefficient of x^3 in the product

$$(x^2 + ax + 1)(x^2 + x + 1)^2$$

may be zero.

If $x = a + b + c$, $y = bc + ca + ab$, $z = abc$, determine the numbers l, m, n, p so that the equation

$$x^4 + b^4 + c^4 = lx^4 + mx^2y + nxy + py^2$$

may be an identity.

7. Solve the equation

$$a\sqrt{(x^2 + 1)} - \sqrt{c^2(x^2 + 1) - b^2} = bx.$$

If a, b, c are the sides of the triangle ABC , prove that $x = \cot C$ is a solution.

Solve the simultaneous equations

$$x^2 = 3x + 6y, \quad xy = 5x + 4y.$$

8. In the equation $x^3 + 6x^2 + 7x - 18 = 0$, put $y + h$ for x and determine h so that the resulting equation may be of the form $y^2 + ay + b = 0$; then prove that the equation has one real root, and find that root to two decimal places by a graphical method.

9. A and B are the centres of two circles of radii $12\sqrt{3}$ and 3 respectively, and the distance from A to B is 5 ; from a point P in AB produced tangents PC, PD are drawn to the two circles. If $AP = x$, express PC^2 and PD^2 in terms of x , and find the maximum and minimum values of the ratio $PC^2 : PD^2$.

10. Establish the identities:

$$(i) \cos^2 A + \cos^2(A + 120^\circ) + \cos^2(A - 120^\circ) = \frac{3}{2},$$

$$(ii) \sin^2 A + \sin^2(A - B) - 2 \sin A \cos B \sin(A - B) = \sin^2 B.$$

If $2 \tan \alpha + \tan(\alpha - \beta) = \tan(\beta + \alpha)$, show that

$$\tan \alpha = \frac{\sin(\alpha - \beta)}{2 \sin \alpha \sin \beta}.$$

11. On one bank of a straight river stands a tower AT ; on the opposite bank a distance BC is measured, and the angle of elevation ABT of the top T of the tower and the angles ABC, ACB , which BC makes with the lines joining B, C to the foot A of the tower, are observed. If BC and the angles ABT, ABC, ACB are denoted by a and α, B, C respectively, show that the height of the tower is

$$a \sin C \operatorname{cosec} A \tan \alpha.$$

12. Through the centre O of the circumscribing circle of a triangle ABC a straight line is drawn parallel to the side BC to meet the other two sides at D and E respectively; if $DE = a$, show that

$$\frac{a}{a} = \frac{\cos(B - C)}{2 \sin B \sin C},$$

where α, B, C have their usual meaning.

If y and z are the segments intercepted by the other two sides on lines drawn through O parallel to CA and AB respectively, prove that

$$\frac{a}{a} + \frac{y}{b} + \frac{z}{c} = 2.$$

13. Eliminate θ between the equations

$$ax \cos \theta + by \sin \theta = ab = bx \sin \theta - ay \cos \theta.$$

Through the vertex C of a triangle ABC a straight line is drawn which does not pass between A and B ; if the perpendiculars from A and B on the line are x and y respectively, show that

$$a^2x^2 + b^2y^2 - 2abxy \cos C = a^2b^2 \sin^2 C.$$

ELEMENTARY APPLIED MATHEMATICS.

SPECIAL PAPER.

14th September, 1911.—2 P.M. to 5 P.M.

[NOTE.—Candidates are advised to answer a few questions well rather than all indifferently.]

1. Define momentum of a particle. Explain how the momenta of particles moving in different directions are to be compounded.

A straight rod carries two particles, each of mass 40 grammes, at its extremities A, B . The rod turns about a point C , the distances of which CA, CB from C are equal to AB , at the rate of two complete revolutions per second. If AB be 20 centimetres, find the magnitude and direction of the resultant of the momenta of the particles A, B .

2. A particle of mass m moves with speed v in a horizontal circle of centre C and radius a . It is at a point P of the circle at a given instant, and after t seconds is at a point Q . The arc PQ subtends an angle θ at C . Show that when the particle is at Q its momentum parallel to PC is $mv \sin \theta$. Hence, find the mean rate of growth of momentum in the direction PC for the interval t ; and from the result prove that when the particle is at P its acceleration towards C is v^2/a .

If the particle weigh 120 grammes, a be 40 centimetres, and the particle traverse the circle 4 times in a second, find the stretching force in a string connecting the particle to the centre of the circle. [Be careful to specify the units in which the force is given.]

3. A certain road crosses a river by an old-fashioned arched bridge, and the road above the bridge is convex upward in consequence. If the radius of the circle in which the "centre of gravity" of a motor car weighing 2 tons passes over the bridge be 60 feet, find the force exerted by the car on the road when the speed of the car is 20 miles an hour and when the speed is 30 miles an hour.

4. "Force is rate of change of momentum." Illustrate this statement by a solution of the following problem: A fine uniform chain which weighs m grammes per centimetre of length, and has length l centimetres, is held vertically with its lower end in contact with the floor, and is then let go. The part of the chain in the air at any instant has been falling freely under gravity from the commencement of the motion, but each element as it reaches the floor is stopped and added to the pile which has already arrived there. Find at any time t after the commencement of the motion the force applied by the chain to the floor.

How is this force affected by the arrival of the last element of the chain?

5. The resistance to the motion of a train of 200 tons travelling at a speed of 60 miles per hour is a force per ton equal to the gravity of 10 lbs. Find the rate of working of the engine in horse-power.

The engine consumes 10 cwts. of coal per hour, and the burning of a pound of coal produces 11×10^5 foot-pounds of energy. What proportion of this energy is usefully employed by the engine?

6. Assuming that the resistance of similar steamers is proportional to the wetted surface and to the square of the speed, prove that if a model 6 feet long, run at a speed of 2 knots in an experimental tank, experience a resistance of 0.2 pound, a similar steamer 600 feet long having a displacement of 10,000 tons would experience when run at 20 knots a resistance equal to $1/112$ part of its weight, and require over 12,000 effective horse-power.

7. Distinguish between elasticity of shape and elasticity of bulk, by specifying the modulus in each case, or otherwise.

Prove that if the substance be a fluid for which Boyle's law holds, and which is kept at constant temperature, the modulus of elasticity of bulk is simply the pressure.

8. Define pressure at a point in a fluid and prove (specifying clearly the criterion of a fluid on which your proof is based) that the pressure at any point is the same in all directions through the point.

A vessel containing mercury is carried in a lift. Find the pressure at a depth of 10 centimetres (1) when the lift is ascending with acceleration 10 ft./sec²; (2) when it is descending with acceleration 10 ft./sec². [Density of mercury, 13.6.]

9. Give some account of experiments by which Joule showed that work could be completely converted into heat, and that the amount of heat produced was proportional to the work spent. Define a unit of heat and give its work equivalent.

State what you know with regard to the conversion of heat into mechanical work, and the limitations, if any, which prevent complete conversion.

APPENDIX IV.

UNIVERSITY OF EDINBURGH.

EXAMINATION FOR BURSARIES OF THE
FIRST, SECOND, AND THIRD YEAR.

MATHEMATICS.

ADDITIONAL PAPER.

Tuesday, 12th September, 1911.—2 P.M. to 5 P.M.

Examiner:—Professor CHEYRAL.

1. Find the locus of the centre of a circle which cuts orthogonally two given circles; and prove that all the circles which cut two given non-intersecting circles orthogonally pass through two fixed points.

AB is a fixed straight line of given length: AX and BY are two parallel straight lines like directed. D and E are points in AX and BY respectively such that $4AD \cdot BE = AB^2$; prove that DE touches a fixed circle.

2. Establish the pole and polar theorem for a circle.

Deduce a linear construction for the polar of a given point with respect to a given circle traced on the paper.

3. Prove that the inverse of a circle is in general another circle; and state the exceptions. Also prove that the centre of the given circle inverts into a point which is harmonically conjugate to the centre of inversion with respect to the inverse circle.

Invert the following theorem:—"The locus of the centres of all the circles which touch two given straight lines L and M is a pair of straight lines which bisect the angles between L and M ."

4. Establish the formulae for the volume and area of the curved surface of a right circular cone.

A, B, C are the middle points of the edges (each of length a) of a cube which meet at the vertex O ; and E is the vertex of the cube opposite to O . Calculate the volume of the right circular cone whose vertex is E and whose base passes through A, B, C .

5. Show that, if the constants a and b be properly determined, the following symmetric integral function of x, y, z becomes a perfect square; and find its square root:—

$$2x^3 - bzx^2yz + 2zx^2y^2 + ax^2y^2z^2.$$

Express as a single fraction at lowest terms—

$$\frac{b^4 - c^4}{a(x-a)} + \frac{c^4 - a^4}{b(x-b)} + \frac{a^4 - b^4}{c(x-c)} - \frac{b^4 - c^4}{a(x+a)} - \frac{c^4 - a^4}{b(x+b)} - \frac{a^4 - b^4}{c(x+c)}$$

6. Compare the equivalence of the three systems—

$$P = Q, R = S;$$

$$P = Q, PR = QS;$$

$$PR + QS = 0, PR - QS = 0.$$

Solve the system—

$$(x + y - 2)(2x + 2y - 5) = (2x + 3y - 7)(3x - 2y - 2),$$

$$x + y - 2 = 2x + 3y - 7.$$

7. Solve the following systems—

(i) $x^2 - y^2 = 0$, $9x^2y^2 - 27x^2y - 10y^3 + 27y + 9 = 0$;

(ii) $y = x^2 - 4x$, $x = y^2 - 4y$;

and illustrate (ii) graphically by plotting the graphs of the separate equations and marking their intersections.

8. Draw the graph of $y = (x + 1)/(x^2 - 1)$; and find roughly by means of your squared paper the maximum value of y .

9. If 150*l.* amount in 10 years to 245*l.* 10*s.* at compound interest payable half-yearly, find the rate per cent. per annum by means of the tables supplied.

If six successive instalments of an annuity of £*A* amount to the same sum as four successive instalments of an annuity of £*B*, find a formula for determining the rate per cent. of compound interest, payable annually.

10. Find the necessary and sufficient condition that $\sqrt{a + \sqrt{b}}$ may be expressible in the form $\sqrt{p} + \sqrt{q}$, where a, b, p, q are real positive commensurable numbers.

Simplify—

$$\frac{2 + \sqrt{5}}{\sqrt{14 + \sqrt{180}}} + \frac{2 - \sqrt{5}}{\sqrt{14 - \sqrt{180}}}$$

11. Draw the graph of $y = \sin x \tan x$.

Solve the equation—

$$\sin 3x - \sin x - 2 \sin x \sin 2x = 0.$$

12. Express $\sin^4 x$ as a sum series of multiples of x .

Simplify—

$$\frac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x} + \frac{\sin x - \sin 2x + \sin 3x}{\cos x - \cos 2x + \cos 3x}$$

13. In the triangle ABC , AD is perpendicular to BC . Given $B = 35^\circ 16'$, $C = 29^\circ 14'$, $AD = 38.4$, solve the triangle ABC .

Calculate in terms of parts of ABC , the sides and angles of the triangle formed by the centres of the escribed circles of ABC .

Show that in any triangle

$$\sin^2 A \operatorname{cosec}^2 \frac{A}{2} = \sin^2 B \operatorname{cosec}^2 \frac{B}{2} + \sin^2 C \operatorname{cosec}^2 \frac{C}{2} - 2 \sin B \sin C \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2} \cos \frac{B+C}{2}.$$

14. Establish the equation of motion of a body under the action of a constant force, and deduce the equation of energy.

A truck weighing one ton starts from rest and runs down an inclined plane of 1 in 60, passing over 840 feet in 1 minute. Find the final momentum and kinetic energy of the truck, and also the frictional force, assumed constant, which acts against the motion. State clearly in what units the various quantities are expressed.

15. State the conditions of equilibrium of an extended body, illustrating your meaning by reference to one or two simple cases.

Two lads, weighing 7 and 8 stones respectively, occupy the ends of a plank 12 feet long, which is supported, see-saw-wise, at its middle point. The plank is kept in a horizontal position by a third person, weighing 12 stone, who stands with one foot 9 inches to the right and the other 9 inches to the left of the middle point. How will the pressure be distributed on the plank; and how far distant from the middle point will the vertical line through the centre of gravity of the third person be?

16. What is meant by the phrase "the pressure at a point in a fluid"; and what is the meaning of the statement that the pressure at a point in a fluid at rest is the same in all directions?

A cylindrical jar, 1 foot long, is inverted over the surface of mercury partly filling a slightly wider vessel. How much of the cylinder must be immersed in the mercury so that the mercury surface may rise two inches within the cylinder, the height of the barometric column being 30 inches?

17. Prove that the force associated with uniform motion in a circle is measured by the product of the momentum of the body and the angular velocity of the radius vector.

Discuss the motion of a cyclist rounding a corner, and show how the inclination of the cycle and rider to the vertical depends upon the speed and upon the radius of curvature of the track.

APPENDIX V.

JOHN WELSH MATHEMATICAL BURSARIES
AND KELLAND PRIZE.

HIGHER PURE GEOMETRY.

Wednesday, 13th September, 1911.—9 A.M. to 12 NOON.

Examiner:—JOHN URQUHART, M.A., B.A. (Cantab.).

1. PQ is a chord of a circle. If the tangent at T meet PQ in O , prove that $OP \cdot OQ = OT^2$.

O is a fixed point, and LT is a fixed straight line touching a fixed circle at T . On LT any point L is taken, and OL is produced to P so that $OL \cdot OP = \text{square of the tangent from } O \text{ to the fixed circle}$. Prove that the locus of P is a circle passing through O and touching the fixed circle.

2. $A, B, C \dots$ are any number of fixed points in a plane, and P is any other point in the plane. If $lPA^2 + mPB^2 + nPC^2 + \dots$ is constant, where $l, m, n \dots$ are constants; prove that the locus of P is a circle.

A, B, C are three fixed points; and E, F are the projections of a variable point P on AB, BC . If $lBE + mBF$ is constant, l and m being constant, prove that the locus of P is a circle.

3. If P_1, P_2, P_3 be the images of the orthocentre in the sides of a triangle, prove that they all lie on the circumscribing circle of the triangle.

Prove that the orthocentre is the centre of the circle inscribed in the triangle $P_1P_2P_3$.

4. If $ABC, A'B'C'$ be two triangles such that AA', BB', CC' meet in a point (centre of perspective), prove that $AB, A'B'; BC, B'C'; CA, C'A'$ meet in points which lie on a straight line (axis of perspective).

If, in addition, the points A, B, C, A', B', C' lie on a circle; prove that the centre and axis of perspective are pole and polar with respect to the circle.

5. Prove that each of the three diagonals of a complete quadrilateral is cut harmonically by the other two.

Given a straight line and its middle point, show how to draw, with the help of the ruler only, any number of lines parallel to the given line.

6. Prove that if the line p passes through the pole of the line q , then q passes through the pole of p .

The chords AB, CD of a circle meet in X , and AC, BD meet in Y . Prove that XY passes through the point of intersection of the tangents at A and D .

7. Prove that, if two curves cut at a given angle, their inverses cut at the same angle.

D, E, F are the feet of the perpendiculars on the sides of a triangle ABC from O , a point on the circumscribing circle of the triangle. If O be taken as the centre of inversion, and A', B', C', D', E', F' be the inverses of A, B, C, D, E, F respectively, prove that $A'B'C'$ lie in a straight line and are the feet of the perpendiculars from O on the sides of the triangle $D'E'F'$.

8. If two homographic ranges $ABCD \dots$, and $A'B'C'D' \dots$ have in common the point of intersection A of the ranges, prove that the lines joining corresponding points are concurrent.

A, B, C, D are four points on a circle, such that $(ABCD) = -1$. If the tangent at A meets BD in P , and the tangent at B meets AC in Q ; prove that the lines PQ, BC, AD meet in a point.

9. A_1, A_2, A_3, A_4 is a tetrahedron, and O a given point. A_1, O meets A_2, A_3, A_4 in D , and B_1 is the harmonic conjugate of O with respect to A_1 , and D . B_2, B_3, B_4 are similarly constructed. Show that, if A_1, O, B_1 meets B_2, B_3, B_4 in E , then A_1 and O are harmonic conjugates with respect to B_1 and E .

10. A, B are two fixed points, and P is a point on a given curve. Find the position of P in order that the area PAB may have a stationary value.

Hence prove that of all rectilinear figures of a given number of sides inscribed in a circle, that which is regular has a maximum area.

Two circles touch internally at P . Through P the diameter PQ of the inner circle is drawn, and also a chord PAB cutting the circles in A and B . Find the position of the chord, which makes the area QAB a maximum.

11. The normal at P to a conic, whose focus is S , meets the axis in G . Prove that $SG/SP = e$, the eccentricity.

The normals at the ends of a focal chord of a conic intersect in O , and OKL is drawn parallel to the axis meeting the chord in K and the directrix in L . Prove that K is the middle point of the chord, and $OK/KL = e^2$.

12. Two chords PQ, RS of a conic are drawn in fixed directions. Prove that for all positions of the chords the ratio $OP \cdot OQ/OR \cdot OS$ is constant, O being the point of intersection of the chords.

Hence, or otherwise, prove Carnot's Theorem, viz. :—

If a conic cut the sides of a triangle ABC in $P_1, P_2; Q_1, Q_2; R_1, R_2$: then

$$AQ_1 \cdot AQ_2 \cdot CP_1 \cdot CP_2 \cdot BR_1 \cdot BR_2 = AR_1 \cdot AR_2 \cdot BP_1 \cdot BP_2 \cdot CQ_1 \cdot CQ_2.$$

MATHEMATICS AT OSBORNE AND DARTMOUTH.

INTRODUCTORY.

A brief account of some of the conditions and requirements at the Osborne and Dartmouth Royal Naval Colleges may be of service as a preface to Mr. Mercer's more detailed discussion of the scope of the work done there.

It must at the outset be understood that the account is entirely unofficial and, so far as it deals with matters of opinion, it expresses nothing more than the writer's individual views. It should also be borne in mind that the syllabuses and methods pursued in the Naval Colleges are elastic and are liable to modification as experience may suggest or as the changing requirements of the service may render desirable.

The founding of Osborne College was associated with a lowering of the age at which naval cadets enter the service. They now join when they are barely 13. This lowering of the age of entry made it possible to arrange a consistent curriculum from 13 to 17, which should give the emphasis to the humanities essential in the education of such young boys, while by omitting Latin and Greek a greater proportion of time than is usual at public schools could be devoted to Science, and provision could be made for a thorough practical training in engineering workshops. This last could not be obtained in the schools which formed the recruiting area for naval cadets, partly because of the costly nature of the requisite plant, and partly through fashion and tradition.

Every executive naval officer has now, in the performance of his ordinary duties, to make use of complex mechanical and electrical appliances. He cannot rely on calling in an expert to deal with them, but must himself be in some sense an engineer; so his early training should be planned with due recognition of this fact.

At Osborne and Dartmouth roughly one-half of the working time is devoted to Science, Engineering, and Mathematics; of this time from one third to a half is spent in the workshops. The time devoted each week to Mathematics during the two years at Osborne is about $6\frac{1}{2}$ teaching hours and 2 hours of preparation. At Dartmouth during the first year it is about 5 teaching hours and $2\frac{1}{2}$ hours preparation, and during the second year 6 teaching hours and $3\frac{1}{2}$ hours preparation; but this includes also the time for navigation, which occupies about $3\frac{1}{2}$ hours weekly during the second year. After cadets have passed out of Dartmouth they spend 8 months in a special cruiser,

where a still more practical turn is given to the training. The abler boys here carry on their study of pure mathematics, in such time as they can gain through their greater rapidity in mastering other subjects of instruction. The time which can be found in this way cannot be stated exactly.

It will be seen that the aggregate amount of time available for Mathematics is no greater at these Colleges than on the Modern Side of most schools. This indicates clearly a deep-seated change which has been made in the training of naval officers; speaking broadly that training was formerly mathematical, while now it is much less specialised, but has a strong leaning towards mechanics.

A considerable knowledge of the theory and practice of navigation and pilotage is, and has been for many years, an essential for every naval officer; formerly this was the only part of his professional training which could conveniently be begun on shore. In former times, when this shore training ended at 14, it was difficult to provide a sufficient mathematical equipment on which to base it, and educational balance had to be sacrificed for the purpose. As the age of entry rose, since no other claims asserted themselves strongly, the emphasis on Mathematics persisted. In the latter days of the "Britannia," cadets spent at least 14 teaching hours and 5 hours of preparation weekly on Pure Mathematics and Navigation, and an exceptional amount of time had to be given to Mathematics in the education leading up to the competitive examination for entrance. It is now impossible to assign so preponderant a position to Mathematics; and since the demand for efficiency in Navigation is no less imperative now than formerly, it is necessary to devote very great attention to the development of the course in order that it should be possible to cover the necessary ground in the decreased time available.

Some changes in the practice of Navigation facilitate this; increased reliance on chronometers and the consequent abandonment of lunars as a matter of routine, and the introduction of the new Navigation into the British Navy, have considerably reduced the mathematical skill required of the ordinary navigator. Great economy of time results from recent changes in the methods of teaching Mathematics, and still more can be gained by eliminating large portions of the subject which have become traditional, but whose chief justification depends on their utility in defeating an examiner. They have come into existence mainly through the ingenuity of earlier examiners and writers of text-books, not through the demands of the man who uses Mathematics as a tool or who wishes to carry it on to higher levels for its own sake. There seems, too, to be a limit to the amount of time which can be spent with profit on a restricted range of elementary mathematics; after this limit is exceeded the work becomes deadening and mechanical. In the particular subject of navigation this is not a serious drawback, as the ordinary processes

must be practised so often as to become almost automatic. Finally it must be recognised that Mathematics enters very largely into the subjects which have partly replaced it, and that such applications not only provide practice but also enormously vivify a boy's conception of what he learns in a mathematical class-room.

A naval officer requires Mathematics for use in subjects other than Navigation, no less essential to him.

Throughout the four years and a half of training at Osborne and Dartmouth and in the training cruiser, there is a continual call for mathematical knowledge in his work in engineering and science which forms the basis of subsequent study in the separate branches of gunnery, torpedo and engineering. (It may be worth while to explain that the second of these includes electrical engineering.)

A certain proportion, probably less than half of the cadets, will not need to pursue the study of higher mathematics at later stages of their career; they will form the body of "general service officers" from whom there is not required a greater knowledge of navigation, engineering, science, etc., than can be based on the amount of Mathematics acquired before leaving Dartmouth.

But foundations have also to be laid for the future study of more advanced Mathematics in the case of those capable of it. Of the remaining cadets who will be needed for specialist officers, all must attain to a higher standard than is reached on passing out of Dartmouth, and some, selected by a gradual process of elimination, will have to attain to a very considerable standard during a series of courses at Greenwich and elsewhere.

Now the course of Mathematics pursued in the public and grammar schools some half century ago was designed for the production of mathematicians. The average boy was put through as much of the course as he happened to be able to cover before dropping the subject, and the only justification to be found for such treatment in his case was that it furnished a mental gymnastic complementary to his studies in Latin composition, etc. Such Mathematics in its early stages is not of much use to the engineer or scientist, and in the effort to arrive early at the relatively advanced methods which are of use, the technical schools tended to sacrifice the teaching suited to those who would pass on to still more advanced work; a scheme of "practical mathematics" was thus evolved, of great value in economising the time of those whose requirements were strictly limited, and at least as intelligible and convincing to the ordinary student as the traditional course.

The problem to be solved is mainly one of administration. Three groups of cadets, each about 75 in number, enter Osborne every year; each group is divided into a number of classes according to ability in the several subjects studied, and these classes persist without considerable

change throughout the four years. So it was possible to design for the less able, who would not pursue Mathematics beyond the passing-out standard, a course of "practical mathematics" and to supplement it, in the case of cadets who could work more quickly, by the teaching required as a foundation for their subsequent work and desirable as a mental discipline. This supplementary teaching is directed towards the development of the power of abstraction and of deductive reasoning; in courses of practical mathematics every effort is made to keep in touch with the concrete, and to allow intuition and inductive processes, in which the generalisations are often rather bold, to take the place of deductive reasoning.

The supply of teachers competent to carry out such a combined course is at present not very large. A man must have received an essentially mathematical training if he is to realise the importance of sound foundations, and to lay them properly; success in this training demands a concentration of effort incompatible with the practical study of science or engineering which is essential for the teacher of practical mathematics. The plan adopted at Osborne and Dartmouth is to secure trained mathematicians and make all possible opportunities whereby they can develop a scientific habit of mind. It is not sufficient to ensure close contact outside the class-room between mathematician and engineer or scientist, nor to provide for courses, however extended, of "looking on" at science teaching, but a most marked effect is produced by arranging for mathematical masters to give science teaching themselves for a few terms both in lecture room and laboratory. The natural spirit of competition leads to the acquisition of the point of view which they see is effective in the case of science masters taking parallel classes or working with them in the laboratory. It is obviously desirable that these science masters should themselves be in touch with practical engineers to complete the chain. The confidence implied in the adaptability of the mathematical mind has been fully justified in our experience; I am satisfied that even if particular classes have suffered in their science work for a short time, the improvement in the mathematical instruction makes up for it to them, and of course the general efficiency is greatly increased.

There is another peculiarity in the education of naval officers, for which provision must be made in their mathematical teaching. For five years after passing out of the training cruiser they have to serve at sea and concentrate their attention on learning the active duties of their profession. The only breaks in this period consist of short courses of teaching in purely professional subjects, and an examination chiefly on those subjects. The practice of this profession necessarily involves continual use of the theoretical knowledge acquired before going to sea; but organised teaching in Mathematics

is suspended, strong inducements being, however, held out to encourage these junior officers to study it for themselves. At the end of this period those who are fit to be specialists will attend courses involving Mathematics of an advanced nature, and it will be necessary for them to be able to refurbish in a reasonably short time what has grown rusty in the five years spent at sea. It is therefore desirable so to present Mathematics to them in the earlier courses that this fallow period shall be as little detrimental as possible. It is clearly inadvisable to aim at great skill in manipulation of symbols at that stage, for such skill rusts quickly and the whole process would have to be repeated. The aim must rather be to accustom them to rely on their own powers of working with the minimum of material in the way of standard types, processes and results. Although the special circumstances compelled the adoption of this method, there is little doubt that it is preferable in almost all cases, since for the majority of men life subsequent to school and college is a fallow period in this sense.

This general principle of course admits of exceptions. For example, midshipmen get continual practice in the recognised forms for performing manipulations in navigation; and in other professional subjects every one makes use of standard methods for economising thought in operations so frequently repeated that it is less trouble to remember them than to think. But in the general teaching of Mathematics such economy of thought is out of place, except in producing facility in operations which are of frequent use in all walks of life.

The fear has often been expressed that a reduction of the traditional drill in manipulation, by which a knowledge of these standard types was instilled, would lead to a decrease in accuracy, already sadly to seek. It is not easy to bring forward proof or disproof of this view, but experience goes to show that other factors are more important in producing accurate work. First of these is age, or maturity of mind; it is largely a matter of the growth of a feeling of responsibility. Our forefathers relied on the birch to foster it; we have to find a substitute. *A priori*, this feeling of responsibility is not likely to be encouraged by a wearisome course of manipulation which carries no conviction of its value to a boy, while it is likely to grow when he is working at problems which are real to him, producing results whose accuracy he can himself check; and great advantage has been found from developing to the utmost the habit of checking results by independent processes. Again, the methods employed in the various branches of physical science admit of different degrees of accuracy of observation, and attention must therefore be paid to the degree of precision advisable or allowable in the computations based on the observations; this presents the question of accuracy in a light which impresses a boy since he is personally interested in the matter.

He also readily grasps the importance of accuracy in dealing with symbols and reasoning, when he finds by experience that errors in these are of greater effect than are errors of observation, in wasting his time and producing fallacious results.

In this way a demand for accurate work is made on him by a number of teachers in different departments, and he is less likely to think of accuracy as the fad of the mathematician which may be neglected when out of his reach. It will be seen by Mr. Mercer's paper that, in developing the details of the course of Mathematics at Osborne and Dartmouth, full allowance was made for this co-operation of various branches of study in this matter; and I venture to assert that experience has justified the elimination of a very considerable amount of the customary drill in manipulation, in that the boys who have passed through these courses, though as yet naturally lacking the precision of trained computers, are certainly not less accurate than the ordinary schoolboy of the same age; and further that they have some idea of what accuracy and suitable approximation really mean.

In conclusion it may be worth while to point out that the work done in these Colleges has been done on ordinary boys, not, for the most part, on boys naturally endowed with unusual intellectual powers. There is no longer any competitive examination for entry into the Navy; boys are selected for general aptitude for the service and the only written test which they have to pass is "a simple qualifying examination in the usual subjects taught in a good preparatory school." On the other hand the spirit that exists throughout the Navy inspires them with a zeal which would probably cause them to outstrip their former equals, if they could be compared under conditions identical only so far as their class work is concerned. This may cause the results of the methods adopted to give too much support to the ideas on which they were based, but not, it is believed, to such an extent as to render an account of them valueless to those working under somewhat less favourable conditions.

C. E. ASHFORD.

MATHEMATICS AT OSBORNE AND DARTMOUTH.

THE COURSE OF INSTRUCTION.

The colleges at Osborne and Dartmouth have this great advantage over other public schools, from the point of view of curriculum-framing, that all the boys are being trained for the same career, and it is fairly clear what use they are likely to make of their subjects of study in the future. Let it be clearly understood at the outset that Mathematics is taught to naval cadets mainly as a useful tool which can be employed in physics, navigation, engineering, and, later, in special gunnery and torpedo work. This does not mean that the subject is reduced to a number of rules of thumb—far from it. It is true that information must be gained and the memory cultivated, but these are of little use unless at the same time the boy acquires a logical way of thinking, and the power to apply his stock of knowledge to a new problem. We cannot spare the time to teach Mathematics merely as a “mental gymnastic” without regard to its usefulness, but there is no reason why Mathematics which is intended to be of use in the future should not be an effective instrument of mental discipline; properly taught it is probably more powerful in this direction than the old-fashioned type of Mathematics, which, in the earlier stages at any rate, was much too abstract and artificial, and to the ordinary boy seemed to lead nowhere and to have no point of contact with actual experience. We try to steer a middle course between two extremes—one in which Mathematics is treated solely as an instrument for getting results, the other in which it is looked upon merely as a means of mental discipline and no interest is taken in its practical applications.

The statement that all boys are being trained for the same career must be modified to some extent. Cadets will eventually be general service officers, specialists in Engineering, Gunnery, Torpedo, or ultra-specialists in one of these branches. The standard of Mathematics expected from those who take specialist courses will be high. Now it is found at Osborne and Dartmouth, as everywhere else, that boys of the same age differ very widely in their aptitude for Mathematics, and before cadets reach their last year at Dartmouth it is possible to mark out with some degree of certainty fifty per cent. of any group as being those from whom specialists will be drawn. The syllabus provides, first, for a minimum course to be taken by everyone; secondly, for some extensions of this course to be

taken by better boys in the lower half; thirdly, for higher work to be done by the upper half, and fourthly, for still higher work for a very small number of cadets at the top, for it is generally found that there are three or four who stand out from the rest and can with advantage be taken on further. The teaching of Navigation is closely connected with the mathematical teaching and the course in Navigation is the same for all. It is thus possible to arrange that the poorer mathematicians give more time to Navigation, a subject of vital importance to them, and that the abler boys give more time to Mathematics, being able to attain the required standard in Navigation with less effort.

The problem before us is roughly this:—to give the weaker cadets enough of the right kind of Mathematics, sound so far as it goes, to enable them to apply it intelligently to their Navigation, Science, and Engineering, and also to give future specialists more pure Mathematics generally, and to give them some sound notions of the principles and uses of the Calculus, so that they shall be well equipped for continuing their Mathematics after leaving the College. Of the weaker ones particularly it may be said that the intention is that they shall have a few tools, but that those few shall be always bright and ready for use, and that they shall be able to choose the right tool for a particular purpose.

Since it is necessary to introduce many things which have hitherto had no place in the ordinary traditional school course, it follows that much of the traditional course must be cut out to make way for this new matter. At first, one proceeded with some caution, the tendency being to retain things about which there was any doubt, lest, after all, they might turn out to be of use later. Then, as more things claimed admittance, the cutting out had to be done more boldly, the guiding principle being rather that if no reason could be assigned for retaining a certain piece of work, it should go. If it should turn out that something so omitted is required in subsequent work, it can be introduced when the need for it arises.

Many people profess anxiety about this wholesale excision of things which from time immemorial have formed part of the educational food of the English school-boy. They find it hard to believe that what occupied so large a place in their own education can be unimportant and capable of being omitted with advantage. Consider, for instance, the wonderful questions in vulgar fractions that are found in so many text books on arithmetic, totally unlike anything that could possibly occur in any research. It will perhaps be claimed for such questions that they called for patience, orderly arrangement of work, and accuracy. But there is sufficient scope for the exercise of all these qualities in problems which arise in connection with physics, solution of triangles, &c. These problems have the additional advantage of having some obvious

purpose, whereas the others were merely dull and gave point to a saying of Lamartine: "L'enseignement mathématique fait l'homme machine et dégrade la pensée."

It will be convenient to consider the course of instruction under different headings, though in actual practice there is considerable overlapping. We do not want the various parts of Mathematics separated from each other and shut off in water-tight compartments. In solving a problem a cadet is encouraged to make use of any knowledge he may possess. If he finds it advantageous to apply algebra or trigonometry to a problem in geometry, by all means let him do so. His attitude should be "What have I to do? Which of the implements at my disposal shall I use?" Why hamper him by limiting his choice of tools? The abler boy has, of course, a wider choice and will often produce a neater solution of the problem, but better a clumsy solution, if correct and sound, than none at all.

The effect of giving freedom in choice of methods is that the ordinary boy finds more problems within his scope. As an example, suppose that a class meets for the first time the problem: "A circular arc ADB has its chord AB 10 inches and its height CD 2 inches, find the radius of the circle." A clever boy would draw the diameter DCE and use the fact that $DC \cdot CE = CA^2$. Another boy would apply Pythagoras' Theorem to the triangle OCA (where O is the centre) and get an equation $x^2 - (x-2)^2 = 5^2$. But most boys, endowed with no special degree of cunning, would use trigonometry to find the angle ADC (its tangent is 2.5), deduce the angle COA and then get the radius from $OA = CA \operatorname{cosec} COA$, and this last solution is not devoid of merit. Others, again, would perhaps draw a figure to scale and get a result by measurement.

ARITHMETIC.

The old type of arithmetic contained too many financial questions and too many artificial problems involving special tricks. An appalling amount of time used to be spent on stocks and shares, and ingenious puzzles about cows engaging in contests with uniformly growing grass. As a matter of fact, the number of separate rules and principles is surprisingly small, and the application of the correct principle to a particular problem is largely a matter of common sense. Our teaching aims at accuracy and facility in the elementary operations, a sound knowledge of the metric system, the unitary method, and the use of four-figure tables. Boys learn not to be afraid of a long piece of computation. They find that accuracy in this kind of thing is important in Physics and Navigation, and realise that in Navigation, especially, inaccuracy is fatal. It might at first sight appear that arithmetical manipulation had been cut down too much by the omission of so much matter, but it must not be forgotten that in Geometry and Trigonometry, numerical work,

often of a complicated nature, is continually being done, that Navigation requires a considerable power of dealing with masses of figures, and that in Physics and Mechanics a boy is constantly exercised in elementary arithmetical operations.

The question of arithmetical accuracy is an important and difficult one. We can hardly expect the habit of accuracy to be fully developed in boys of 15, but a great deal can be done if every master, whatever his special subject, for whom a boy has occasion to work out numerical results, will refuse to accept careless and inaccurate work. It is not a question of teaching a boy how to add and multiply, but rather of getting him to realise that to know how to do a thing, and yet to do it wrong, is useless. It is, I suppose, inevitable that boys should make such mistakes as these—to say that $9 + 8 = 15$, $\sqrt{0.25} = .05$, $\log 30 = 2.4771$, to take out $\log 78$ instead of $\log 73$, to write 4471 for 4771. What he has to learn, especially in a long piece of calculation, where every step affects all the subsequent work, is to make sure of each step before proceeding to the next; for instance, when he takes out a logarithm, he should look at it again and see that he has taken it out correctly before going on; if he has added three numbers, he should add them again in a different order; if he has multiplied something by 7, he should check his result by dividing it by 7, and so on. This continual criticism of his own work must be insisted on as a necessary part of everything he does until he comes to apply it mechanically. Of course the main responsibility for all this rests on the mathematical master, but science masters can help, and indeed must help, by refusing to accept work which is spoilt by careless arithmetical blunders. Otherwise there is a danger that a boy looks upon the demand for accuracy as a fad of the mathematical master, whereas if he finds that everyone insists upon it, he may be led to believe that it is really important. Mental and oral work are useful in various ways. In introducing a new idea in any branch of Mathematics, the exercises should be easy and often capable of being done mentally, so that the principle is not obscured by a mass of computation. It is important, too, that boys should acquire the habit of estimating the approximate value of an arithmetical expression, and so avoid giving up $.0453$ when the correct answer is 4.53 . Where possible, external probability should be used as a check; e.g. if the H.P. of a locomotive is calculated to be $.213$, it is probably wrong.

To boys who make so many calculations founded on approximate data, the question of degree of approximation is important, and good exercises can be made by making them calculate areas and volumes from measurements made by themselves, and estimating the possible errors in their results, due to the inevitable inaccuracy of their measurements. If the sides of a rectangular sheet of paper are found by measurement to be 9.87 and 7.46 in., and if there is a possible error of $.005$ in. in each measurement, it is dishonest to state the area as 73.6302 sq. in.

without further comment. It will be found that all that can be stated with certainty is that the area is between 73.54 and 73.72 sq. in.; and the result should be stated $73.63 \pm .09$ sq. in.

We do not teach the formal contracted methods of multiplication and division. The cases when four-figure tables do not give sufficient accuracy are not numerous enough to make it worth while.

ALGEBRA.

Reasonable skill in straightforward manipulation is expected, but we have no time for dealing with long, tedious examples in simplification of fractions, or with hosts of "elegant devices" for solving problems specially constructed for the purpose. Everyone should have a few simple rules and principles as part of his mental machinery. If these are in constant use, and are backed up by some common sense, quite difficult problems can be solved by very simple means. The process will often be laborious, and a clever boy will see and use many short cuts. The ordinary boy will see the advantage of these short cuts, and should be encouraged to look for possible simplifications of his work, but he should not be discouraged from proceeding because the method he proposes to himself promises to be tedious.

The work on fractions has been much reduced. Questions on addition and subtraction of fractions are useful as giving practice in multiplication, the rule of signs, and the meaning of brackets. Some drudgery is necessary at the outset in order that this kind of manipulation shall come to be done more or less mechanically in the same way as addition and multiplication in arithmetic, but we lose none of the necessary drill by limiting ourselves to fractions with monomial and binomial denominators.

The idea of a formula as a compact general statement of the operations to be performed in all questions of a certain type is the starting-point in algebra, and a boy is practised in deriving formulæ inductively after consideration of a multitude of particular cases. The evaluation of quantities by the substitution of numerical values for the letters occurring in a formula is a matter of arithmetic, and cadets have constant practice in this kind of work throughout the course.

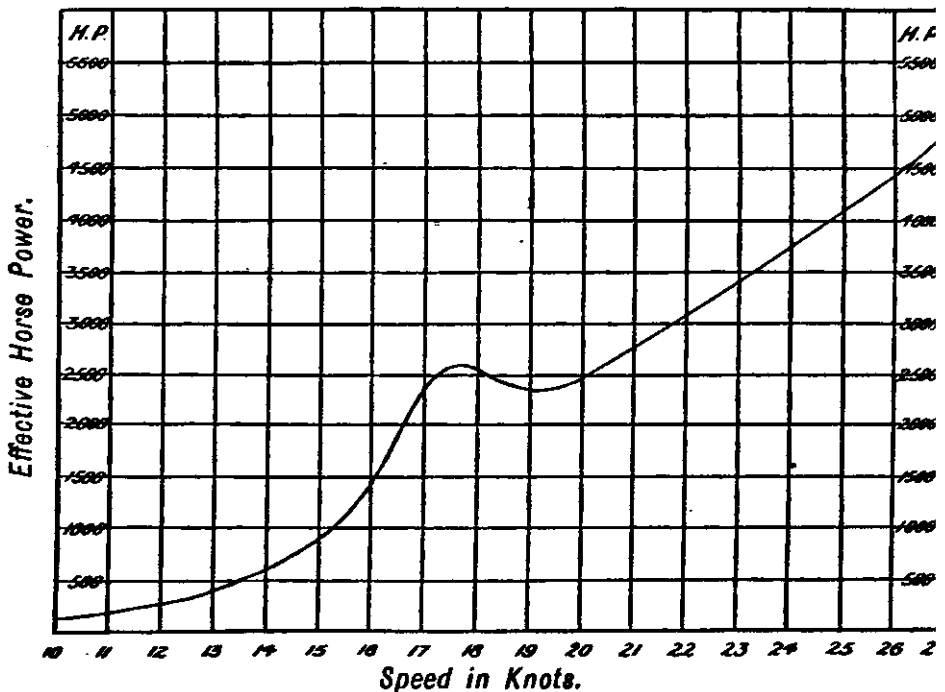
The simple equation, and the problem leading to it, occur at the very outset. Here, as always, problems are introduced where possible which have reference to other parts of their work, such as Geometry and Physics, but the important thing is to be able to express the conditions of the problem in algebraical language, and a few more or less artificial problems may be most useful to this end as involving ideas well within a boy's comprehension. Great pains are taken to inculcate the habit of checking solutions to equations and problems; that is, of verifying that the solution obtained does actually satisfy the conditions of the problem, and no problem is considered completely solved without this check.

As Professor Chrystal says in his "Text Book of Algebra":
 "The ultimate test of every solution is that the values which it
 " assigns to the variables shall satisfy the equations when sub-
 " stituted therein. No matter how elaborate or ingenious the
 " process by which the solution has been obtained, if it do not
 " stand this test, it is no solution; and on the other hand, no
 " matter how simply obtained, provided it do stand this test, it
 " is a solution. A little attention to these self-evident truths
 " would save the beginner from many a needless blunder."

The idea of functionality is constantly brought out and illustrated by graphical representation. Variation provides a multitude of concrete examples, as do most of the ordinary formulæ in use in mensuration, mechanics, &c. Simple numerical questions in geometry can be modified so as to bring out some functional relation, *e.g.*, after calculating the length of the tangent to a given circle from a given point, the change in the length of the tangent as the distance of the point from the centre changes may be studied. The idea of a continuous change in a function of x due to a continuous change in x grows on a boy when he has met it in so many different forms, not only in mathematical class rooms, but in science lecture rooms and laboratories, though it would probably be a mistake to sum it up for him in these words. He learns to look upon a graph not merely as a work of art, but as a picture story which he can read. A few examples of exercises in graph-reading are given.

FIG. 1.

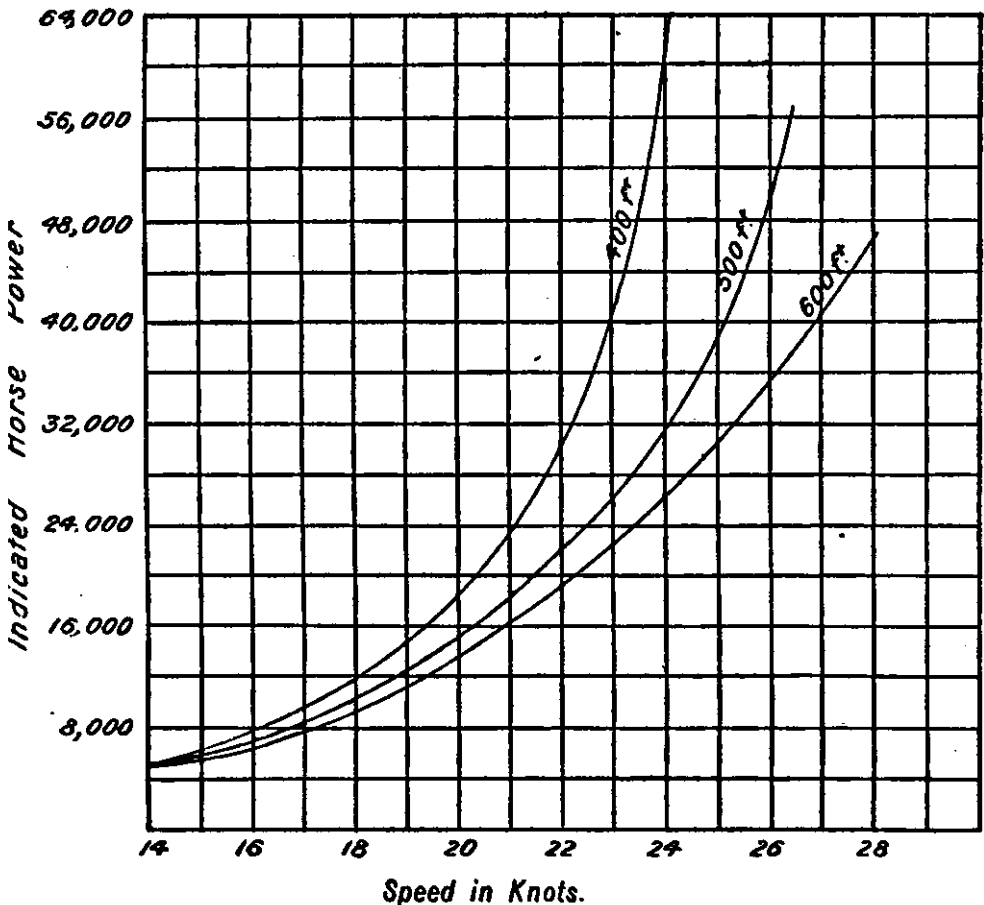
Relation between E.H.P. and Speed of a certain Destroyer in 20 ft. of water.



1. What is the H.P. for speeds 13, 16, 17, 19, 24 knots?
2. What increase in H.P. is required to increase the speed
 - (i) from 14 to 15 knots; (ii) from 15 to 16 knots;
 - (iii) from 16 to 17 knots; (iv) from 21 to 22 knots;
 - (v) from 22 to 23 knots?
3. The H.P. increases with the speed as the speed increases from 11 to 17 knots, also as the speed increases from 21 to 27 knots. What difference do you notice in the two cases?
4. What striking feature is shown by the curve?

FIG. 2.

Relation between I.H.P. and Speed for three Cruisers of lengths 400, 500, 600 ft.



1. What do you gather from these curves concerning the relation between length and H.P. for any given speed?
2. What H.P. would be required to give the same speed to the 400 ft. cruiser that 24,000 H.P. gives to the 600 ft. cruiser.

3. With the same H.P. that gives a speed of 22 knots to the 400 ft. cruiser, what speeds would be given to the others?
4. What do you notice about the increase in H.P. required to produce an increase of 1 knot in the speed?

The application of graphs to the solution of all kinds of equations and the determination of maximum and minimum values of all kinds of functions follow naturally.

A boy is expected to be able to solve simple and quadratic equations in one unknown, linear simultaneous equations in two or more unknowns, and simultaneous quadratics in two unknowns without any special attention to particular artifices for various types, only those being dealt with in which one unknown can be expressed explicitly in terms of the other and the solution effected by substitution.

Quadratics are all solved by factors, that is to say, every quadratic is written in the form $f(x) = 0$, and $f(x)$ is resolved into linear factors. If factors can be written down at sight, so much the better. If not, the method of completing the square is used and the equation written in the form $(x-p)^2 - q = 0$, so that $(x-p+\sqrt{q})(x-p-\sqrt{q}) = 0$. Thus the method of completing the square appears as a device for factorising the left-hand side by writing it as the difference of two squares when the factors are not obvious. This plan of writing equations in the form $f(x) = 0$ will often save boys from omitting roots, *e.g.*, if the equation $x^2 = 3x$ be written $x^2 - 3x = 0$ or $x(x-3) = 0$, a boy will hardly fail to see the two alternatives $x = 0$ and $x - 3 = 0$, whereas from $x^2 = 3x$ he will often deduce $x = 3$ only.

Apart from its use in solving amenable quadratic equations, the resolution of a quadratic function into factors at sight (where possible) is a good exercise in an inverse process. To multiply $3x + 2$ by $4x + 5$ is one thing, to see that $12x^2 + 23x + 10 = (3x + 2)(4x + 5)$ is another. This is one of the first instances a boy comes across of systematic guessing, and at any rate for those who will subsequently do integral calculus, it is important to develop this kind of faculty. Easy questions in partial fractions are useful in the same way, though they should not at this stage be elevated to the dignity of a special chapter. They should be scattered among ordinary questions on addition of fractions. A few equations with "literal" coefficients are solved, the main point being the discrimination between known and unknown. The process is an imitation of that which has previously been employed in many cases in which the coefficients are numerical, and provided that the examples are not made needlessly complicated they provide excellent practice in passing from the particular to the general. A useful exercise of this kind is to rewrite a formula involving several letters so as to show one letter as an explicit function of the rest, *i.e.*, to change the subject of the formula, *e.g.*, $V = \pi r^2 h \therefore r = \sqrt{\frac{V}{\pi h}}$. The

general quadratic $ax^2 + bx + c = 0$ should be set as a "literal" equation to be solved, and the result may be learnt as a formula for solving any quadratic, but it should be noticed that in many cases the method of solution by completing the square is easier.

The teaching of surds is limited to a few fundamental and useful ideas such as $\sqrt{2} \times \sqrt{3} = \sqrt{6}$; $\frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}} = \sqrt{.6667}$ or $= \frac{\sqrt{6}}{3}$; $\sqrt{8} = 2\sqrt{2}$. The more complicated exercises on rationalisation of denominators and the expression (where possible) of $\sqrt{a} + \sqrt{b}$ in the form $x + \sqrt{y}$ do not seem to be of much use. If such expressions ever occur, they can be dealt with quite adequately by straightforward application of tables and present no difficulty.

In indices the graph of 2^x or 3^x derived from the values corresponding to $x = 1, 2, 3, 4, 5$, &c., suggest values for $2^{\frac{1}{2}}$, 2^0 , &c. Interpretations found in the usual way from consideration of the fundamental multiplication law proved for positive integral indices are seen to agree with these. The formal proof of the general truth of the index laws is not given, but a few particular cases are useful as an exercise in going back to the meanings of symbols. Artificial examples in the manipulation of negative and fractional indices are avoided. The graph of $y = 10^x$ constructed by means of a square root table will now enable a boy to appreciate the meaning of his logarithm table.

The actual use of logarithms precedes a thorough understanding of their meaning. A lesson spent on the graph of 2^x will convince a boy of the reasonableness of taking $3 = 2^{1.6}$, $10 = 2^{3.3}$, $30 = 2^{4.9}$, and thus of saying that $2^{1.6} \times 2^{3.3} = 2^{4.9}$, and so on. The table of logarithms can then be presented as a table by means of which any number can be written as a power of 10. Thus $2 = 10^{.3010}$ and $3 = 10^{.4771}$. He will expect to find that 2×3 or 6 is $10^{.3010 + .4771}$ or $10^{.7781}$, or in other words $\log 6 = .7781$, which is the case. A few more examples of this kind and he will soon learn the use of logarithms, and the more detailed study of the meaning of indices and logarithms as outlined above can be deferred.

The chapter on Variation in the ordinary text-book is, as a rule, somewhat scanty, and boys are apt to look upon the word "varies" as a signal for the performance of certain tricks with "k." As cases of variation frequently occur, and provide excellent examples of functionality, the subject deserves more detailed consideration, and a treatment on the following lines seems to lead to more intelligent work.

"y varies as x" means that x and y change together in such a way that whatever factor x is multiplied by, y is multiplied by the same factor. Thus supposing that y is 5 when x is 2, then if x is changed to 3, y will be changed to $5 \times \frac{3}{2}$, for the multiplier that converts 2 into 3 is $\frac{3}{2}$.

Concrete examples should be given, *e.g.*, "The extension of an elastic spring varies as the load" means that if the load is doubled, so is the extension; if the load is multiplied by $\frac{2}{3}$, so is the extension, and so on. Thus if a load of 3 lb. stretches an elastic spring 2 cm., a load of 5 lb. will stretch it $2 \times \frac{5}{3}$ cm.

We might take the value of y when x is 1 as the basis of calculation, working by unitary method. Thus in the last example

A load of 3 lb. stretches the spring 2 cm.
 \therefore " 1 lb. " " $\frac{2}{3}$ " 5 cm.
 \therefore " 5 lb. " " $\frac{2}{3} \times 5$ cm.

" y varies as x^2 " means that whatever factor x^2 is multiplied by, y is multiplied by the same factor. Thus supposing that y is 5 when x is 2, then when x is 3, y is $5 \times \frac{9}{4}$, for in the first case x^2 is 4, and in the second case 9, and the factor which converts 4 into 9 is $\frac{9}{4}$, or we may say:

when x^2 is 4, y is 5
 \therefore when x^2 is 1, y is $\frac{5}{4}$
 \therefore when x^2 is 9, y is $\frac{5}{4} \times 9$.

Since the multiplier, $\frac{9}{4}$, which converts 4 into 9 is the square of $\frac{3}{2}$, the multiplier which converts 2 into 3, we may say that if y varies as x^2 , then whatever factor x is multiplied by, y is multiplied by the square of that factor, and for concrete examples this is better: *e.g.*, Areas of similar figures are said to vary as the squares of corresponding lengths, *i.e.*, if there are two similar figures, and any length in the first = m times the corresponding length in the second, the area of the first = m^2 times the area of the second. Thus, if corresponding sides of two similar triangles are 17 cm. and 31 cm., and the area of the first is 200 cm.², the area of the second is $200 \times (\frac{31}{17})^2$ cm.².

Since the factor by which we multiply y is always the square of that by which we multiply x , it follows that the factor by which we multiply x is the square root of that by which we multiply y , *i.e.*, if $y \propto x^2$, $x \propto \sqrt{y}$; for instance, we may say that corresponding lengths of similar figures vary as the square root of the areas, meaning that if there are two similar figures, and the area of the first is n times the area of the second, any length in the first is \sqrt{n} times the corresponding length in the second.

" y varies inversely as x " may be taken to mean that whatever number x is multiplied by, y is multiplied by the reciprocal of that number; so that if x is doubled, y is halved; if x is multiplied by $\frac{2}{3}$, y is multiplied by $\frac{3}{2}$, and so on.

Thus if y is 5 when x is 2, then when x is 3, y is $5 \times \frac{2}{3}$, or by unitary method,

when x is 2, y is 5,
 \therefore when x is 1, y is $5 \times \frac{1}{2}$,
 \therefore when x is 3, y is $5 \times \frac{2}{3}$.

It should be noticed that the multiplier which converts $\frac{1}{2}$ into $\frac{1}{3}$ is the reciprocal of that which converts 2 into 3, so that the statements, " y varies inversely as x ," and y varies as $\frac{1}{x}$ " are equivalent.

Boyle's law is a familiar concrete case of inverse variation.

In all these cases it is important to see that as x gradually changes, so does y ; e.g., if the shape of a triangle is fixed, and the length of a side gradually changes, the area will gradually change. If one of the triangles has a side 17 cm., and area 200 cm.², we can calculate the area corresponding to a side of any other length, and a graph can be drawn showing how the area depends on the length of the side.

Since the area is 200 cm.² when the side is 17 cm., the area when the side is x cm. will be $200 \times \left(\frac{x}{17}\right)^2$ or $\frac{200}{17^2} \times x^2$ cm.², so that if y cm.² stands for the area of the triangle when the side is x cm., $y = \frac{200}{17^2} \times x^2$, and this will be the equation of the graph.

The significance of this factor $\frac{200}{17^2}$ is, perhaps, made clearer if we work by the unitary method, for

when side is 17 cm., area is 200 cm.².

\therefore when side is 1 cm., area is $\frac{200}{17^2}$ cm.².

\therefore when side is x cm. area is $\frac{200}{17^2} \times x^2$ cm.².

and generally if $y \propto x^2$, $y = kx^2$, where k is the value of y when $x = 1$.

Similarly, if $y \propto x^n$, $y = kx^n$ where k is the value of y when $x = 1$.

The forms of the graphs corresponding to $y \propto x, x^2, x^3, x^{\frac{1}{2}}, x^{\frac{1}{3}}$, $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^{\frac{1}{2}}}, \frac{1}{x^{\frac{1}{3}}}$, &c., should be known.

Special attention should be given to the graphical representation of the relation " y varies directly as x ." It is a straight line through the origin, its equation being $y = kx$, k being the value of y when $x = 1$.

Conversely, if the graph which shows the relation between x and y is a straight line through the origin, y varies directly as x .

It seems a natural step to the case in which the graph is a straight line not through the origin. Returning to the first example given, suppose that the unstretched length of the spring is 20 inches. We might make a table such as the following:

Load	1	2	3	4	5	6	x , lb.
Extension	$\frac{1}{3}$	$1\frac{1}{3}$	2	$2\frac{2}{3}$	$3\frac{1}{3}$	4	y , in.
Length	$20\frac{1}{3}$	$21\frac{1}{3}$	22	$22\frac{2}{3}$	$23\frac{1}{3}$	24	y , in.

If y be plotted against x , we get a straight line through the origin whose equation is $y = \frac{2}{3}x$. If y be plotted against x , we get a straight line, but not through the origin, whose equation is $y = \frac{2}{3}x + 20$.

This may be written $y - 20 = \frac{2}{3}x$; or $y = \frac{2}{3}(x + 30)$, and we may say that $(y - 20) \propto x$, or $y \propto (x + 30)$. Another way of looking at it is to see that the difference between any two values of $y = \frac{2}{3}$ of the difference between the corresponding values of x , or shortly, the change in $y \propto$ the corresponding change in x .

If a table of values of x and y is given, and we find on plotting y against x that we get a straight line, the relation between y and x is of the form $y = kx + l$, and k and l can be found from the co-ordinates of any two convenient points on the line.

If on plotting y against x we get a curve like one of the various types of " $y \propto x^n$ " graphs, the difficulty will be to guess the probable value of n . The fact that if $y \propto x^n$, we get a straight line by plotting y against x^n is not of much use if our guess at the value of n is wrong.

But if $y = kx^n$, $\log y = \log k + n \log x$, or $Y = nX + K$, where X, Y, K , stand for $\log x, \log y, \log k$, therefore if we plot Y against X we shall get a straight line, and n can be found as above.

Cases in which y is a function of several variables are treated similarly, *e.g.*, the statement that for curves on railways the difference in level of the rails varies directly as the gauge, and the square of the highest speed, and inversely as the radius of curvature means that if the highest speed and radius remain the same the difference in level \propto the gauge; if the gauge and radius remain the same the difference in level \propto the square of the highest speed; if the gauge and the highest speed remain the same the difference in level varies inversely as the radius. Thus if the proper difference in level for a radius of 15 chains, a maximum speed of 40 m./hr., and a gauge of 5 ft. 3 in. is 6.8 in., and we want the difference in level for a radius of 25 chains, a maximum speed of 50 m./hr., and a gauge of 4 ft. 8½ in., we say

Difference in level for radius 15 chains, speed 40 m./hr., gauge 63", is 6.8"

∴ Difference in level for radius 15 chains, speed 40 m./hr., gauge 56½", is $6.8 \times \frac{56\frac{1}{2}}{63}$ ins.

∴ Difference in level for radius 25 chains, speed 40 m./hr., gauge 56½", is $6.8 \times \frac{56\frac{1}{2}}{63} \times \frac{15}{25}$ ins.

∴ Difference in level for radius 25 chains, speed 50 m./hr., gauge 56½", is $6.8 \times \frac{56\frac{1}{2}}{63} \times \frac{15}{25} \times \left(\frac{50}{40}\right)^2$ ins.

In practice the last line will be written down straight away, the intermediate steps being done mentally.

We might proceed by unitary method and find first the difference in level for radius 1 chain; speed 1 m. hr.; gauge 1", which would be

$$6.8 \times \frac{1}{63} \times \frac{15}{1} \times \left(\frac{1}{40}\right)^2 = k'', \text{ say.}$$

From this we should get the difference in level for radius r ch., speed v m./hr., gauge g ins. to be $k \times g \times v^2 \times \frac{1}{r}$; in other words, if we are told that y varies directly as g , directly as v^2 , and inversely as r , we have $y = k \cdot \frac{g v^2}{r}$, where k is the value of y when g, v, r are each equal to 1.

GEOMETRY.

The first term at Osborne is devoted largely to practical work. Cadets are supposed to have done some work of this kind at their Preparatory Schools, so that when they come to their theoretical work they have picked up a considerable number of geometrical notions and facts, and are in a position to appreciate what they are going to reason about. The object of teaching Geometry is, I suppose, threefold:—

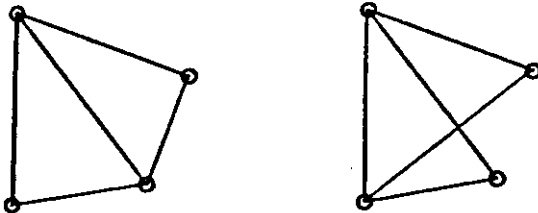
1. That a boy may acquire a certain number of geometrical facts.
2. That he may be able to apply this geometrical knowledge to the solution of problems.
3. That he may get some training in sound reasoning and precision of statement.

There is not much in Euclid's first and third books that a boy who has done some rational work with instruments will not accept as a matter of course. For instance, a boy who has constructed triangles to given data sees that a triangle is uniquely determined if a, b, c are given. The fact that the base angles of an isosceles triangle are equal is sufficiently obvious. To present formal proofs of such propositions on Euclidean lines only fogs a boy's mind, and convinces him at the outset of the unreality of the subject. These fundamental facts must be properly stated and systematically set down, so as to serve as the basis of subsequent work. The number of propositions a cadet is required to write out formally is not very great, and as much as possible he is led on to discover the truths for himself, and build up the steps of the proof. Often a truth to be proved is suggested by experiment. In whatever way the truth and proofs of the standard propositions are arrived at, a cadet is required to be able to write out a proof more or less on Euclidean lines. He must state clearly what is given, and what he is going to prove, and give full reasons for every step in the proof, quoting only facts which have been previously established. He is not tied down to any particular form of words, but must make it per-

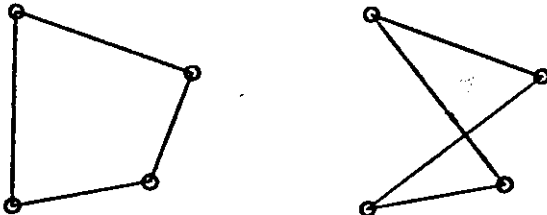
fectly clear that he understands what he is doing. These formal proofs, if properly done, are very valuable in teaching a boy how to marshal his facts, how to select from his stock of knowledge the particular thing wanted for the purpose in hand, and in showing him the nature of a proof. The propositions naturally fall into a few small groups, and where the proof of one depends on another it will of course be necessary to remember the order. As propositions are proved they are added to the stock of quotable material available for solving problems and riders.

When we come to the treatment of similar triangles, much of what used to be proved at great length is more or less a matter of intuition to a boy who has been accustomed to draw to scale, and to deal with maps and models. It is useful to begin by recalling a few facts about congruent figures. A triangle is determined if we know the lengths of the three sides. So that if we wish to copy the scheme of points A, B, C , we can do so if we know the three lengths AB, BC, CA . Another way of putting this is to say that if a framework be made of three strips of wood freely jointed at the corners, it is rigid. To copy four points it will not be sufficient to know four lengths. We want three lengths to copy three of the points, and two more to copy the fourth. A framework of four freely jointed rods is not rigid.

These frameworks are rigid.

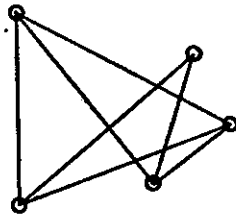
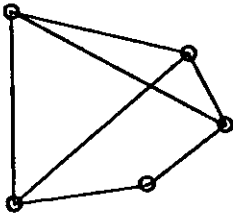


These are not.

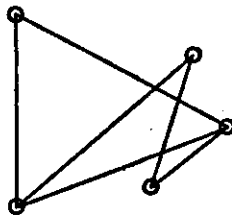
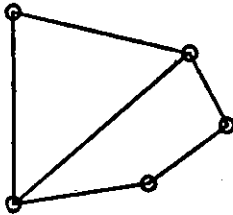


Similarly to copy five points we want $3 + 2 + 2$ or seven lengths, and so on.

These frameworks are rigid.

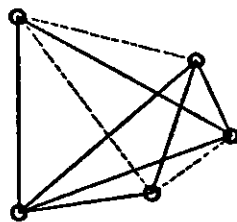
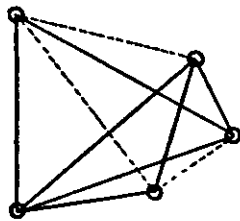


These are not.



It is interesting here to refer to perfect, imperfect, and redundant framed structures which a cadet comes across in his statics work.

In copying a set of points any three may be taken as the first three, and after that the remaining points may be taken in any order one at a time, each being fixed by its distances from any two fixed points already obtained, so that there is considerable freedom in the way in which the construction may be actually carried out. Since the five points in the accompanying illustration are determined if seven of the ten lines joining up pairs of points are known, it follows that if there are two sets of five points, and if of the ten pairs of corresponding lines seven pairs (none of which are redundant) are equal, the remaining three pairs are equal.



The step from congruent to similar figures is easily made, if it be granted that, given one figure, it is possible to have another such that *every* length in the second is k times the corresponding length in the first where k is any positive number. Plan-of fields, cycling maps, models of ships, pictures thrown on a lantern screen, &c., furnish illustrations of this. Consider the problem:—"Given five points, or a figure determined by five points, draw another figure so that *every* length in the new figure is one-third the corresponding length in the old." This

can be done as before with this difference, that instead of using seven lengths from the old figure we use lengths equal to one-third of these. The three remaining lengths of the new figure will then be one-third of the corresponding lengths of the old.

Now consider other methods of fixing a third point C when two (A, B) are given. C is fixed if we know (1) AC and angle BAC , or (2) BC and angle ABC , or (3) angles BAC and ABC , or (4) AC and BC ; so that when two points out of three are given, there are four methods of finding our way to the third, but in each case *two* things must be known. There are thus numerous methods of copying a figure, but all are included in this:—After two points have been obtained, every other is got by completing a triangle, one of whose sides is already known.

Since we get the same triangle whether we copy (1) lengths AB, BC, CA , or (2) lengths AB, AC , and angle A , or (3) length AB and angles A and B , it follows that we shall get a triangle $A'B'C'$ similar to ABC , with linear dimensions k times those of ABC if we make

- (1) $A'B' = k.AB, B'C' = k.BC, C'A' = k.CA$,
 or (2) $A'B' = k.AB, A'C' = k.AC, \angle A' = \angle A$,
 or (3) $A'B' = k.AB, \angle A' = \angle A, \angle B' = \angle B$.

In other words two triangles $ABC, A'B'C'$ are similar if

- (1) $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}$,
 or (2) $\frac{A'B'}{AB} = \frac{AC}{AC}$ and $\angle A' = \angle A$,
 or (3) $\angle A' = \angle A$ and $\angle B' = \angle B$,

and two figures are similar if all pairs of triangles used in their construction are similar.

Exercises on constructions of similar figures are done here to drive these principles home, the process being greatly simplified by the use of the theorem that if XY is parallel to BC a side of the triangle ABC , then the triangle AXY is similar to the triangle ABC . (This is No. 3 above.)

The more formal part consists of exercises based on the fundamental propositions that in two triangles $ABC, A'B'C'$, all the relations

$$\angle A = \angle A', \angle B = \angle B', \angle C = \angle C', \frac{BC}{B'C'} = \frac{CA}{C'A'} = \frac{AB}{A'B'}$$

are true if (1) any two of these, $\angle A = \angle A', \angle B = \angle B', \angle C = \angle C'$, are true,

$$\text{or (2) } \frac{BC}{B'C'} = \frac{CA}{C'A'} = \frac{AB}{A'B'}$$

$$\text{or (3) } \angle A = \angle A' \text{ and } \frac{CA}{C'A'} = \frac{AB}{A'B'} \text{ \&c.}$$

The question of areas of similar figures is approached in the same kind of way. If we have two maps of the same piece of country on different scales, the ratio of two corresponding areas is the same for all pairs. By taking one pair of corresponding areas to be squares, it follows at once that the ratio of any two similar areas is the same as the ratio of the squares on any pair of corresponding lengths. It is well to drive this important truth home by considering a few special cases. If the sides of a rectangle are a'' and b'' those of a similar rectangle will be ka'' and kb'' and the ratio of the second area to the first

$$= \frac{ka \times kb}{ab} = k^2$$

Similarly, if a side and corresponding altitude of a triangle be a'' and p'' , those of a similar triangle will be ka'' and kp'' , and the ratio of the area is

$$\frac{\frac{1}{2}ka \cdot kp}{\frac{1}{2}ap} = k^2$$

The ratio of the areas of two circles

$$= \frac{\pi r^2}{\pi R^2} = \frac{r^2}{R^2} \quad \&c.$$

A similar method of treatment will make it clear that the ratio of the volumes of two similar solids is the same as the ratio of the cubes on any corresponding pairs of lengths. Many numerical problems can be set on these two important principles; in fact, boys will probably have met some before this as exercises in variation.

Constructions depending on areas of similar figures can, at any rate, in the case of weaker boys, be helped out by measurement and arithmetic. For instance, consider the problem of constructing a figure similar to a given figure, but with one-third of its area. If AB is any length in the first figure, and $A'B'$ the corresponding length in the second,

$$A'B' = AB \times \sqrt{\frac{1}{3}} = AB \times \sqrt{.3333} = AB \times .5774$$

and a very good solution can be got by measuring AB and multiplying by .577. This gives $A'B'$, and the construction can be completed. A clever boy will see that $A'B'$ is the side of a square equal to a rectangle whose sides are AB and $\frac{1}{3}AB$, and devise a geometrical construction for finding $A'B'$.

A certain amount of Elementary Solid Geometry, dealing mainly with angles between planes and angles between lines and planes, is done, the subject being approached informally. Most of the matter of Euclid's Eleventh Book is easily grasped with the help of concrete illustrations, the plentiful use of models, and appeals to familiar ideas such as vertical and horizontal. A closed book held with one edge on the table and rotated about this edge, illustrates the fact that if a line OA lies in a plane, any number of perpendiculars can be drawn through

O to OA , whereas an open book held with two edges on the table shows that there is only one line perpendicular to both OA and OB , and that it is perpendicular to every line in the plane OAB . The formal proof of this is a good exercise for the better boys. The main intention of the course in solid geometry is to train boys to "think in space," and it is found that even the weakest boys can become quite good at visualising three-dimensional figures. A little is done in Descriptive Geometry—plan and elevation—enough to give a clear idea of the principles involved. This kind of work is an excellent medium for further developing the "space sense." Practice in the more elaborate technical part of the subject is supplied in the drawing office connected with the engineering shops.

As an introduction to Spherical Trigonometry and Nautical Astronomy, a preliminary course of Spherical Geometry is given in the same term in which Solid Geometry is begun. The cadets draw their own figures and make measurements of sides and angles of spherical triangles on blackboard globes. At the same time wire skeleton models are used so that the connection between arcs of great circles on the surface of a sphere and angles at the centre is constantly in their minds. In other words, they learn to realise figures on the surface of the sphere both from the external and internal points of view. The notions acquired of angles between planes, lines perpendicular to planes, &c., are seen to have their use in Spherical Geometry.

The work is done on blackboard globes 12 inches in diameter,* mounted in such a way that they can be used on the desks in an ordinary class-room. Boys work in pairs, and are supplied with compasses with which great and small circles can be drawn and graduated brass strips with which great circle arcs can be measured in degrees. A great circle is defined as the trace on the surface made by a plane through the centre, and it is assumed that if a string be stretched tightly between two points it will lie along the circumference of the great circle through them. The graduated edge of the brass strip in practice takes the place of the string.

By means of the brass strip several points are marked on the sphere, each of which is 90° from a given point A . They are found to lie on a great circle (tested by the strip). That this must be so is seen by reference to the skeleton globe, for if O be the centre of the sphere and $OP, OQ, OR, \&c.$, are radii perpendicular to OA , then $OP, OQ, OR, \&c.$, all lie in a plane perpendicular to OA . This leads to the definition of pole and polar and to the use of the compasses for the construction of the great circle whose pole is A . The following are a few examples of the kind of thing done.

* These can be bought for 21s. each. Good work can be done with smaller globes about 4 inches in diameter, standing in boxes whose depth is equal to the radius of the globe, with a circular opening of the same diameter as the sphere.

(1) $P, Q, R, S, \&c.$, are points on the same great circle. Construct their polars and verify that they have two points in common. The skeleton globe will help to make it clear that this is merely the theorem that if OA is perpendicular to OP, OQ , it is perpendicular to $OR, OS, \&c.$, which are in the same plane as OP, OQ .

(2) Given a great circle, construct its poles by finding the intersections of the polars of any two points P, Q on it.

(3) The angle between two great circles AP, AQ , is defined as the angle between the tangents at A . Boys should see that this is the same as (i) the angle between the planes OAP, OAQ ; (ii) the angle POQ where P, Q are on the polar of A ; (iii) the angular distance PQ measured with the brass strip along the great circle arc PQ . The last is the one used in globe work. The angle can also be measured directly with fair accuracy by means of a transparent celluloid protractor. (Reference should, of course, be made here to the measurement of longitude.)

(4) Measure the angle between two given great circles.

(5) Through a given point on a given great circle draw another great circle making a given angle with the first.

(6) If the polar of A passes through B , the polar of B passes through A .

(7) The polar of A is at right angles to every great circle through A . (Equator and meridians.)

(8) The angle between two great circles is equal to the angular distance between their poles.

(9) Through a given point not on a given great circle draw another great circle making a given angle with the first.

(10) Construct to given data some spherical triangles and solve them by measurement on the globe, taking note in each case of the sum of the angles.

(11) Sketch a few rhumb lines and notice how they differ from great circles.

TRIGONOMETRY.

This used to be, and still is in many schools, looked upon as a subject for the select few. As a matter of fact, boys find it easier, up to a certain point, than any other part of Mathematics. Many boys who will never make much headway with Deductive Geometry, for instance, soon become quite reasonably good at Numerical Trigonometry. It used to be considered necessary to have done Euclid's Sixth Book before beginning Trigonometry, as otherwise the constancy of the ratio $\frac{a}{b}$ in a triangle with given angles could not be established. As pointed out under Geometry, the truth of this is obvious to the boy who has been in the habit of drawing to scale. The trigonometrical ratios are introduced gradually. The old method of suddenly con-

fronting a boy with six trigonometrical ratios, and then setting him to work on a set of "identities," resulted in many cases in hopeless confusion, and a total inability to grasp what it was all about. The tangent is taken first as being connected with the simplest problems on heights of towers, &c., and it is in connection with the solution of some such problem by scale drawing that the idea of tangent of an angle is introduced. Tables of natural tangents are used straight away, and a boy is interested to find that he has at his command a quicker and more accurate method of solving certain problems than his old one of drawing to scale. The meanings of the tangent, and of the other ratios as they occur, are driven home by easy problems which seem to be much more helpful for this purpose than identities.

When a boy is familiar with the tangent, the sine and cosine are introduced in connection with problems in which the hypotenuse is involved. After a time the convenience of having the reciprocals of these three ratios tabulated is seen, and the cotangent, secant, and cosecant, are defined. At first the natural ratios are used and the arithmetic is made fairly easy, so that a boy's mind is not distracted by a mass of computation. The use of log-sines, &c., will be appreciated when the arithmetic is made more difficult.

In connection with the solution of right-angled triangles, cadets make acquaintance with the Traverse Table, which is very useful for obtaining approximate results in Trigonometry. It is often very convenient to use this table in statical problems depending on resolution of forces. It is, of course, specially important for cadets to be familiar with this table, as they must be able to use it quite freely when they come to Navigation.

The general triangle is then solved by treating it as the sum or difference of two right-angled triangles. Up to this stage nothing but a knowledge of the trigonometrical ratios of acute angles is required, and with no further knowledge a boy has a powerful weapon in his hands.

The solution of many triangles by this method suggests a search for general formulæ, and by following out in the general case the method used in the particular cases the sine and cosine formulæ are easily obtained. At this stage the definitions of sine and cosine are extended so as to be applicable to obtuse angles. The formula $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ is found to be

clumsy in most cases and is replaced by another better adapted to logarithmic computation, viz. : $\text{hav } A = \frac{(s-b)(s-c)}{bc}$ where

$\text{hav } A$ is $\frac{1}{2} (1 - \cos A)$; $\text{hav } A$ is, of course, the same as $\sin^2 \frac{A}{2}$ but the formula for $\text{hav } A$ involves less arithmetic than

$\sin^2 \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, there being no halving of the log, or

doubling of the angle to be done. The haversine is of special importance in spherical trigonometry as applied to nautical astronomy. To avoid the use of the formula $a^2 = b^2 + c^2 - 2bc \cos A$ in the case when two sides and the included angle are given, the formula $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \tan \frac{B+C}{2}$ is established, a geometrical proof being given, as boys are not able at this stage to follow the simpler proof which depends on the factors of $\sin B - \sin C$ and $\sin B + \sin C$. The formulae $\frac{1}{2}bc \sin A$ and $\frac{\sqrt{s(s-a)(s-b)(s-c)}}{2}$ for the area of a triangle come in here. In proving the second of these $\sin A$ in terms of the sides is derived from $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, by means of $\sin^2 A = (1 + \cos A)(1 - \cos A)$.

The problems up to this point are as varied as possible, not merely the ordinary problems on heights and distances, but also questions connected with Navigation, Statics, and Geometry. Even the weakest boys feel that they have something which they can use in all sorts of situations, and they use it quite freely where clever boys would perhaps use algebraical or geometrical methods. Thus, as previously mentioned, they find themselves able to solve many problems, which, under the old system of water-tight compartments and restrictions as to methods, would have been beyond their powers.

The measurement of angles in radians and the simple relation connecting the radius of a circle, the length of an arc, and the circular measure of the angle subtended by the arc at the centre, must be known, as they occur in Mechanics, *e.g.*, in questions of angular velocity. The approximate equality of $\sin \theta$, θ , $\tan \theta$ when θ is small is also an important fact. For those cadets who will later go on to the calculus, it is important to learn to think in radians.

The remainder of the Trigonometry is of a more abstract nature, and therefore more difficult and more peculiarly adapted to the boys with a mathematical bent. It is, in fact, practically a different subject, and this is recognised in continental textbooks by the use of the term "Goniometry." It includes the trigonometrical ratios of $(A + B)$, $(A - B)$, $2A$, the factor-formulae for the sum or difference of two sines or two cosines, a few identities to illustrate and emphasise transformations of frequent occurrence, such as $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$, and the solution of trigonometrical equations, some by graphical methods. This last section includes the extension of the definitions of the trigonometrical ratios to angles of any magnitude. The use of an auxiliary angle to transform expressions of the form $a \cos \theta + b \sin \theta$ into the shape $r \sin(\theta + \alpha)$ is an important piece of work for the better boys. In this part of the subject not much is expected from the weaker cadets. For instance, their acquaintance with the addition and factor formulae will

amount to little more than this: they will have proved the formulae, and used them in establishing such standard results as

$$\frac{a-b}{a+b} = \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}$$

and the formulae which they use in

Spherical Trigonometry.

The better cadets, on the other hand, must have some skill in manipulation of this kind, and a good deal of practice is necessary for gaining a reasonable familiarity with the standard formulae and facility in using them. The forms of the graphs of the ratios, particularly of the sine, should be familiar. These should appear in a restricted form at the outset. The calculation of tangents of various angles from his own figures will make it quite clear to a boy that a gradual change in the angle produces a gradual change in the tangent, and that equal increments in the angle do not produce equal increments in the tangent; and the graph of the tangents of acute angles will show distinctly how the tangent increases more and more rapidly as the angle increases, and emphasis should be laid on the non-straightness of the graph. It is very difficult to prevent boys from making such mis-statements as $\tan 40^\circ + \tan 30^\circ = \tan 70^\circ$, and $\tan 80^\circ = 2 \cdot \tan 40^\circ$, and frequent appeals to the shape of the graph should be helpful. The other graphs will come when the corresponding ratios are being found for different angles. Later, when the definitions are extended to angles of any size, the graphs will be done in their complete form. In connection with the auxiliary angle, it should be noticed that, by suitable choice of scales and origin, the graph $y = \sin x$ can be made to serve as the graph of any of the series $y = a \cos x + b \sin x$.

It should be mentioned that, owing to simplifications made in the teaching of Nautical Astronomy, the number of absolutely essential formulae in Spherical Trigonometry is greatly reduced. A cadet is expected to know:

(1) The fundamental cosine formula,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

(2) $\text{hav } a = \text{hav } (b \sim c) + \sin b \sin c \text{ hav } A$ (used to find a when b, c, A are given, and derived from (1)).

(3) $\text{hav } A = \text{cosec } b \text{ cosec } c \sqrt{\text{hav } \{a + (b \sim c)\} \text{ hav } \{a - (b \sim c)\}}$
 (another form of $\frac{\sin(s-b) \cdot \sin(s-c)}{\sin b \cdot \sin c}$, also derived from (1), and used to find A when a, b, c are given).

(4) $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$ (useful as a check).

CALCULUS.

About half the cadets proceed in their last year at Dartmouth to the study of the Calculus. Much more importance is attached to a proper understanding of the meaning and application of the operations than to skill in manipulation, which can be acquired later, if necessary. Before the notation of the Calculus appears, the notions of "rate of change," "gradient," &c., are discussed at some length, the illustrations used being such as are already familiar to the boy through his acquaintance with easy kinematics, graphs, and geometry.

Every boy who has done any Dynamics has some idea of the meaning of "speed at an instant." He knows that if a body is moving in a straight line with variable speed, the forces acting on it are not balanced, and that if at any instant he could establish a balance among the forces the body would go on with uniform speed, and this speed is what he calls the speed at the instant. Thus the speed at the instant is a real thing in his mind, but for purposes of reasoning another conception is necessary. The first problem he deals with is the calculation of the speed at a given instant from a space-time formula. Suppose, for example, that the formula $s = 2t^3$ is given, and the speed at the end of 4 seconds is required. If the average speed during an interval immediately following the end of the fourth second is calculated, its value will gradually change as the duration of the interval is gradually diminished. Moreover, this average speed will clearly come nearer and nearer to the speed at the instant, and can be made as near to it as we please, if we take the interval small enough. Thus, if a certain speed can be found from which the average speed during an interval immediately following the instant in question can be made to differ by as little as we please by making the interval small enough, that speed is the speed of the instant.

In the particular case which we are considering as an example, the average speeds for intervals of 1 sec., .5, .1, .01, .001 sec., are found to be 122, 108.5, 98.42, 96.2402, 96.024002 ft./sec., and it seems pretty clear that by taking the interval small enough we can get an average speed differing from 96 ft./sec. by as little as we please. To make quite sure, the average speed for an interval of h secs. is calculated. This is $(96 + 24h + 2h^3)$ ft./sec., a result which is seen to include all the numerical results previously obtained, and from which it is quite certain that the speed at the end of 4 seconds is 96 ft./sec. The question of gradient at a point of a curve is treated in a similar way. Everyone has a conception of the meaning of "tangent to a curve." Euclid's definition of a tangent to a circle probably expresses the usual conception of a tangent. For working purposes a modification is necessary. Suppose the gradient of the curve $y = 2x^3$ is required at the point P where $x = 4$. If the gradient of a straight line through P , which cuts

the curve in a neighbouring point, be calculated, its value will gradually change as the line swings round P , so that the neighbouring point comes gradually nearer to P . Moreover, the gradient will come nearer and nearer to that of the tangent at P , and can be made as near to it as we like if we take the neighbouring point near enough to P . Thus, if a line through P can be found whose gradient is such that the gradient of a line through P and a neighbouring point on the curve can be made to differ from it by as little as we please by bringing the neighbouring point close enough to P , that line is the tangent at P , and its gradient is the gradient of the curve at P .

In the particular case under consideration, the gradients of chords joining P (where $x = 4$) to the points on the curve, where $x = 5, 4.5, 4.1, 4.01, 4.001$, are 122, 108.5, 98.42, 96.2402, 96.024002. As before, all these are included in the statement that the gradient of the line joining P to the point on the curve where $x = 4 + h$ is $96 + 24h + 2h^2$, and the gradient of the curve at P is 96.

Other cases of instantaneous rate of change are dealt with on the same plan, e.g., if $pv = 1200$, the rate of change of p per unit increase of v , when $v = 20$, is derived from a consideration of the average rate of increase of p per unit increase of v , as v changes from 20 to $(20 + h)$.

This introductory work is necessarily somewhat laborious and takes time, but it is felt to be important that a boy should have a thorough comprehension of what he is doing in a number of particular cases before proceeding to a study of the calculus on general lines.

(It might be mentioned here that those cadets who are not taken on to the calculus proper do much of this preliminary work leading to the general notion of the representation of instantaneous rate of change by the gradient of a curve at a point, which is of frequent occurrence in their science work; for instance, the speed of a body at a given instant is given by the gradient of the tangent at the corresponding point on the space-time graph; the rate at which p is changing per unit increase of v for a given value of v if $pv = 1200$ is given by the gradient of the tangent at the corresponding point of the hyperbola $pv = 1200$.)

This preliminary course, in which each particular case is dealt with on its own merits, leads a boy to look for some general mathematical process which shall replace this separate detailed consideration of each case. For instance, finding the speed of a body at various instants from the formula $s = 2t^2$ requires that the same set of operations shall be performed on different numbers, and a general formula for the speed at any instant is naturally looked for.

Now, following out in the general case the method adopted in the particular cases, and taking the opportunity of introducing a new notation, we find that in Δt secs. immediately follow-

ing the end of t secs., the distance (Δs feet) travelled is $[6t^2 \cdot \Delta t + 6t \cdot (\Delta t)^2 + 2 \cdot (\Delta t)^3]$ ft., and the average speed during this interval is $\frac{\Delta s}{\Delta t}$, or $[6t^2 + 6t \cdot \Delta t + 2(\Delta t)^2]$ ft./sec. From this we deduce that the speed at the end of t secs. is $6t^2$ ft./sec.

This value ($6t^2$), which $\frac{\Delta s}{\Delta t}$ continually approaches as $\Delta t \rightarrow 0$, and from which we can make it differ by as little as we please by making Δt small enough is denoted by $\frac{ds}{dt}$, so that with the meanings given to s and t in this example, $\frac{\Delta s}{\Delta t}$ gives an average speed during a finite interval Δt , while $\frac{ds}{dt}$ gives the instantaneous speed at the end of t secs.

In the same way a general formula is found for the gradient at any point of the curve $y = 2x^3$. If we take a point P , whose abscissa is x , its ordinate is $2x^3$. If a neighbouring point has abscissa $(x + \Delta x)$, its ordinate, which may be denoted by $(y + \Delta y)$, is $2(x + \Delta x)^3$, and the gradient of the chord joining the two points is $\frac{\Delta y}{\Delta x}$, or $[6x^2 + 6x \cdot \Delta x + 2(\Delta x)^2]$, from which we deduce that the gradient of the curve at P , that is, the gradient of the tangent at P , is $6x^2$, and this is denoted by $\frac{dy}{dx}$. $\frac{\Delta y}{\Delta x}$ gives the gradient of a chord through P , and some neighbouring point distinct from P ; $\frac{dy}{dx}$ gives the gradient of the tangent at P .

Generally if y is a function of x , then in all cases with which we deal a small change, Δx in x , produces in y a small change, which we call Δy , and as $\Delta x \rightarrow 0$, $\frac{\Delta y}{\Delta x}$ tends continually to some limiting value, which we call $\frac{dy}{dx}$. $\frac{\Delta y}{\Delta x}$ gives the average rate of increase of y per unit increase of x , as x increases by a finite amount Δx ; $\frac{dy}{dx}$ gives the *instantaneous* rate of increase of y per unit increase of x .

$\frac{dy}{dx}$ is now found from first principles in a few special cases,

such as $y = x^2$, $y = kx^2$, $y = kx^3 + c$, $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, &c., and the results applied to such questions as the following, so that the significance of the results may be grasped:—

(1) A body is travelling in a straight line, so that its distance (s ft.) from a fixed point in the line at the end of t seconds is

given by $s = 4t^3 + 5$. Find its speed at the end of 3, 4, 5 seconds.

(2) A body is travelling so that its speed (v ft./sec.) at the end of t seconds is given by $v = 7t^2$. Find its acceleration at the end of 3, 4, 5 seconds.

(3) Find the equation of the tangent at the point (5, 4) on the curve $y = \frac{100}{x^3}$.

(4) The radius of a sphere is increasing uniformly at the rate of 3 in./sec. At what rate is the volume increasing when the radius is 3 feet?

(5) The pressure (p lb./sq. in.) and the volume (V c.in.) of a given mass of gas are connected by the relation $pV = 1200$. Find the rate of increase of p per unit increase of V when $V = 40$.

(6) The volume V c.c. of a gram of water at a temperature x° above 4° C. is given by $V = 1 + 8.36 \times 10^{-6} x^3$. Find the coefficient of cubical expansion of water at 20° C.

A general formula is now obtained when $y = x^n$. The particular cases already investigated have probably suggested this formula. A proof independent of the binomial theorem can be obtained if the theorem $\text{Lt}_{z \rightarrow 1} \frac{z^n - 1}{z - 1} = n$ be first established.

The cases $y = kx^n$, $y = kx^n + c$, $y = a + bx^n + cx^m + \&c.$, should be considered, and with no further outfit a great deal can be done.

Problems on maxima and minima, geometrical properties of curves, questions on speed and acceleration, and the rate of increase of one quantity with respect to another in general, approximations (depending on the approximate equality $\Delta y = \frac{dy}{dx} \Delta x$), and relative error, &c., are dealt with, the functional relations involved being always of the type $y = a + bx^n + cx^m + \&c.$, or reducible to this type.

Then the inverse problem—given $\frac{dy}{dx}$ in terms of x , find y in terms of x —is discussed, and this prepares the way for the integral calculus, which is introduced by an attempt to find the area under a curve with a given equation, by a process of summation. The problem is to find the area bounded by the curve $y = f(x)$, the x -axis and the ordinates $x = a$, $x = b$. This is shown to be equivalent to finding $\text{Lt}_{\Delta x \rightarrow 0} \sum_{x=a}^{x=b} f(x) \cdot \Delta x$, and cases are first dealt with in which the summation can actually be effected, *e.g.*, $f(x) = 3x$, $f(x) = x^2$. It is very easy to see that generally the summation would be tedious if not actually impossible, and a different method must be sought. The formula $\frac{dA}{dx} = f(x)$ is established, and the problem of finding A is thus reduced to that of finding a

function of x which when differentiated gives $f(x)$, that is the problem last discussed in the differential calculus.

The notation $\int_a^b f(x) \cdot dx$ is now introduced, the definition being $\int_a^b f(x) \cdot dx = \text{Lt.}_{\Delta x \rightarrow 0} \sum_{x=a}^{x=b} f(x) \cdot \Delta x$, and it is shown that this is the same as $\phi(b) - \phi(a)$ or $[\phi(x)]_a^b$ where $\phi(x)$ is such that $\frac{d \cdot \phi(x)}{dx} = f(x)$.

Other problems are similarly treated. For instance, the distance described between the end of a seconds and the end of b seconds by a body moving according to the speed-time formula $v = f(t)$ is $\text{Lt.}_{\Delta t \rightarrow 0} \sum_{t=a}^{t=b} f(t) \cdot \Delta t$, which is written $\int_a^b f(t) \cdot dt$. Since $\frac{ds}{dt} = v = f(t)$, s can be expressed in terms of t if we can find a function of t whose differential coefficient is $f(t)$. In fact we shall have $\int_a^b f(t) \cdot dt = [\phi(t)]_a^b$ where $\frac{d\phi(t)}{dt} = f(t)$.

Applications of the integral calculus to areas, mean values, volumes of revolution, work, centre of gravity, moments of inertia, &c., follow, these being seen to depend on one general principle if the meaning of the integral calculus is properly understood.

The graphical interpretation of any definite integral $\int_a^b f(x) dx$ as the area bounded by $y = f(x)$, the x -axis and the ordinates $x = a$, $x = b$ is important, as it makes it possible to get a good approximation to the value of the definite integral when it cannot be evaluated by ordinary methods. For example, to evaluate $\int_0^a \sqrt{a^2 - x^2} \cdot dx$ is the same as to find the area of a quadrant of a circle, radius a ; to find the work done by a gas in expanding according to Boyle's law from a volume of a c.ft. to a volume of b c.ft. is $\int_a^b \frac{k}{v} \cdot dv$, and the problem of evaluating this integral is the same as that of finding the area bounded by $y = \frac{k}{x}$, the x -axis and the ordinates $x = a$, $x = b$.

This volume can be approximately found by applying one of the standard rules, such as Simpson's rule or the mid-ordinate rule. (Here again it should be said that cadets who do not take up formal calculus do a certain amount of what may be called graphic integration, without the notation of the subject. They reduce a problem of finding distance from a speed-time formula to that of finding a certain area under the speed-time graph, a problem of finding work done from a force-displacement formula to that of finding a certain area under the force-displacement graph, &c., the approximation to the area being

usually made by application of the mid-ordinate rule. Of course the speed-time formula may in some cases be replaced by a table giving the speeds at certain times from which the graph can be plotted.) Up to this stage there is little scope for skill in the manipulation of the symbols of the calculus. The test that a boy understands his subject is that if he is confronted with a problem he shall be able to make the solution depend on his ability to differentiate or integrate some function of x . The variety of the problems will of course depend on the amount of external knowledge, scientific, geometrical, &c., at the boy's command. The better cadets find time to investigate rules for the differentiation of other functions besides x^n , *e.g.*, $\sin x$, &c., e^x , $\log x$, products, quotients, function of a function, &c., and to do problems, of the same nature as before, requiring for their solution this extended knowledge and some greater skill in manipulation such, for instance, as the determination of the meridional parts for a given latitude. Later, they will easily pick up tricks for integrating various functions which they may come across in their work, but such tricks are better learnt when the necessity for them arises. As stated at the outset, cadets learn the calculus in order that they may be able to use it, and the object of an elementary course should be to give a boy some general idea of the various kinds of problem to which he will be able to apply it. The best way of attaining this end seems to be to drive home the meaning of the processes by a great variety of examples demanding a very limited knowledge of the manipulative side of the subject. In fact, if a boy can differentiate x^n and perhaps $\sin x$ and $\cos x$, he has, as pointed out above, weapons quite adequate for a host of problems. When he can use these weapons properly, he may be confronted with problems which involve the same fundamental principles, but require the differentiation and integration of other functions, and he will then see the necessity of adding to his stock of standard results and methods of attack. Thus he will gradually increase his power in the manipulation of symbols without losing sight of the main underlying principles.

ALGEBRAICAL GEOMETRY.

About the same time that cadets begin the study of the calculus they do some elementary work in algebraical geometry. No attempt is made to take them through a course of "Analytical Conics." The idea is rather that they shall get hold of a few fundamental principles which can be applied to the case of any curve whose Cartesian equation is given. The Cartesian co-ordinates of a point are looked upon as directed steps, and it is convenient first to consider directed steps in one straight line and see that if the step from O to A

is x_1 and that from O to B is x_2 , the step from A to B is $x_2 - x_1$ [$\overline{OA} + \overline{AB} = \overline{OB}$]. Of course, this conclusion will be arrived at after considering many particular cases. In the same way if the co-ordinates of a point P are (x_1, y_1) , and those of Q are (x_2, y_2) , we can pass from P to Q by taking an x -ward step $(x_2 - x_1)$ and a y -ward step $(y_2 - y_1)$, a positive x -ward step meaning of course a step in the direction OX , and a negative x -ward step one in the direction XO .

The distance PQ is easily found, and also the co-ordinates of a point R which divides PQ in a given ratio.

The next thing is the translation of geometrical statements into algebra, which is obviously an essential part of the subject. This is worth some trouble, and numerous exercises should be done. They need not be difficult, but should be properly understood; *e.g.* :—

(1.) Find the equation of the locus of a point equidistant from $(3, 4)$ and $(-1, 2)$.

(2.) Find the equation of the locus of a point whose distance from $(3, 4)$ is always 7.

(3.) Find the equation of the locus of a point whose distance from $(3, 4)$ is equal to its distance from the y -axis.

Later some harder examples are done which require the elimination of θ from two equations $x = f(\theta)$ $y = \phi(\theta)$.

It is useful, occasionally at least, to plot the locus from the given geometrical conditions and from the algebraical equation independently.

In dealing with these questions on loci a boy incidentally makes the acquaintance of the parabola, the ellipse, and other interesting curves.

The straight line is dealt with in some detail.

1. The facts that $y = mx$ represents a straight line through the origin and $y = mx + n$ a straight line parallel to it are established. Here, as in all cases, the effect of changes in the x and y scales is considered.

The gradient of a straight line can be defined here as the ratio of the y -ward step to the x -ward step from P to Q , where P and Q are any two points on the line.

2. Lines are drawn from given equations and equations found for given lines.

3. The equation of a line through a given point $(3, 4)$ with given gradient 3 is found.

(a) The equation is $y = 3x + n$. Choose n so that the line shall go through $(3, 4)$.

(b) The general equation of a line through (x_1, y_1) with gradient m is $y - y_1 = m(x - x_1)$.

If (x_1, y_1) and m have given numerical values (a) is easier, and the work can usually be done mentally, and the equation written down straight away.

4. The equation of a line through two given points is written down by means of 3, for the gradient is $\frac{y_2 - y_1}{x_2 - x_1}$

5. $y = mx + n$ is parallel to $y = mx + n'$, $ax + by + c = 0$ is parallel to $ax + by + c' = 0$. Hence the equation of a line parallel to a given line and passing through a given point can be written down.

6. It is proved from a figure that $y = mx$ and $y = m'x$ are perpendicular if $mm' = -1$. Hence $y = mx + n$ is perpendicular to $y = -\frac{1}{m}x + n'$, $ax + by + c = 0$ is perpendicular to $bx - ay + c' = 0$, and the equation of a line through a given point perpendicular to a given line can be written down.

7. The point of intersection of two lines with given equations and the angle between them are found, but no general formulae learnt.

8. The length of the perpendicular from (h, k) to

$$ax + by + c = 0 \text{ is } \frac{ah + bk + c}{\sqrt{a^2 + b^2}}$$

This last formula is only arrived at after several particular cases of the problem have been dealt with. The length of the perpendicular from $(3, 4)$ to $2x + 3y + 7 = 0$ would in the first place be found as follows:

(1) Write down the equation of the line through $(3, 4)$ perpendicular to $2x + 3y + 7 = 0$; it is $3x - 2y - 1 = 0$.

(2) Find the intersection of $2x + 3y + 7 = 0$ and $3x - 2y - 1 = 0$;

$$\text{it is } \left(-\frac{11}{13}, -\frac{23}{13}\right)$$

(3) Find the distance of $(3, 4)$ from $\left(-\frac{11}{13}, -\frac{23}{13}\right)$; it is

$$\sqrt{\left(\frac{50}{13}\right)^2 + \left(\frac{75}{13}\right)^2} = \sqrt{\left(\frac{25}{13}\right)^2 (2^2 + 3^2)} = \frac{25}{\sqrt{13}}$$

After solving a few exercises of this kind and noticing that a simplification like that in the last line is always possible, an investigation of the general case is suggested and the length of the perpendicular from (h, k) to $ax + by + c = 0$ is found by the same method. The simple form of the result suggests the existence of some simpler method of proof, and a geometrical proof is found.

When boys have done enough calculus to know that $\frac{dy}{dx}$ gives the gradient of the tangent at any point of a curve, they can establish interesting geometrical properties of curves whose equations are given, the range of operations being determined by the extent of their skill in differentiating.

It seems worth while to devote a little special attention to the circle, as the results obtained can be checked by geometrical

knowledge, and this helps to give confidence in the algebraical treatment. The only really essential thing is to see that $(x-h)^2+(y-k)^2=r^2$ is the equation of a circle with centre (h, k) and radius r , and hence that the general equation of a circle is $x^2+y^2+ax+by+c=0$. This, together with its converse, that $x^2+y^2+ax+by+c=0$ can be written in the form

$\left(x+\frac{a}{2}\right)^2+\left(y+\frac{b}{2}\right)^2=\frac{a^2}{4}+\frac{b^2}{4}-c$ and hence represents a circle with

centre $\left(-\frac{a}{2}, -\frac{b}{2}\right)$ and radius $\sqrt{\frac{a^2}{4}+\frac{b^2}{4}-c}$ is all that need be known about the circle, and it is interesting to deduce some of the elementary properties of the circle by purely algebraical methods.

J. W. MERCER.

APPENDIX.

PASSING-OUT EXAMINATION OF NAVAL CADETS

PART I.

April, 1911.

MATHEMATICS I.

(Three hours.)

[Questions 1, 5, and 9 should be answered by all cadets.

*Any five of the remaining questions may be answered, not more than two being taken from any one section.**Great importance is attached to neatness and accuracy.]*

SECTION I.

1. Assume that to punch a hole of diameter d inches in a certain plate requires a force of $3.4d$ tons; and that the greatest force which can be applied safely to the punching tool is $12.7d^2$ tons.

Draw graphs, figuring scales on the axes from which diameters can be read off in tenths of inches and forces in tons, to show the necessary force and the permissible force for various diameters up to one inch. What is the diameter of the smallest hole that can be punched in the plate?

2. Draw a quadrilateral $ABCD$, the angles at A and B being right angles. Let $AB = h$. $BC = a$. $AD = b$.

Suppose the quadrilateral to revolve about an axis in its plane distant r from AB on the side remote from C and D , thus forming a ring.

Obtain a formula for the volume of this ring by considering it as the difference between a large cone, and a cylinder and smaller cone.

Show how the result may be checked by supposing $a = b$, and by supposing $r = 0$ and $b = 0$.

3. Two positions $A B$ are 21 miles apart.

Two craft $P Q$ are moving in the straight line AB with uniform speeds. P passes A , and Q passes B at noon. They pass one another at 3 P.M., and P passes B one hour and three quarters after Q passed A .

Find the speeds of P and of Q

(i) if they are moving in opposite directions,

(ii) if they are moving in the same direction—from A towards B .

4. If

$$x^2 = x + 1,$$

show by multiplying throughout by x that

$$x^3 = 2x + 1$$

and by continuing the process express x^4 , x^5 , and x^6 in a similar form.

Find the two numerical values of x^6 , to two places of decimals.

SECTION II.

5. Prove the theorem of Pythagoras that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on its sides.

Draw a square $ABCD$, side 3 inches.

With centres A, B, C, D in turn, and a radius of $3\frac{1}{2}$ inches, draw arcs cutting BC, CD, DA, AB in H, I, J, K .

Calculate to two places of decimals the length of a side of the square $HIJK$, using the theorem of Pythagoras. Calculate also the area of this square, and of the triangle AJK ; and from the results found for these areas show how your work may be checked.

6. What is the locus of the centre of a circle of radius $2\frac{1}{2}$ inches

(1) if it touches a fixed circle of radius 3 inches,

(2) if it has a chord of length 3 inches which is trisected at a given point O ?

7. Draw a straight line AOB , and on it as diameter describe a semicircle. Let $AO = a$, $OB = b$. With centre O , radius equal to the radius of the semicircle, describe an arc cutting the semicircle in P . At O erect a perpendicular OQ cutting the semicircle in Q . What are the lengths of OP and OQ in terms of a and b ?

Prove from the diagram that the arithmetic mean of a and b is greater than the geometric mean, and state the condition that these means may be approximately equal.

8. A cylindrical pipe, outside diameter 6 inches, passes obliquely through a horizontal plate, the axis of the pipe making 45° with the horizontal.

Give a *sketch* of the shape of the section cut out of the plate, and find its greatest length AB and its length parallel to AB one inch from AB .

SECTION III.

9. Show that in any plane triangle

$$\frac{\tan \frac{A - B}{2}}{\tan \frac{A + B}{2}} = \frac{a - b}{a + b}.$$

From a ship at A a mark B , bearing $N. 71^\circ E.$, is known to be 6,300 yards distant.

A target C , invisible from the ship, is known to bear $N. 44^\circ E.$ from B , and to be distant 3,400 yards from B . Find the distance and bearing of C from A . Check the work by recomputing the length of AC from the triangle ABC by the formula $b = a \cos C + c \cos A$.

10. A ruler 12 inches long is laid on a desk. The plane of the desk is inclined at 10° to the horizontal, and the ruler makes an angle of 40° with the lines of greatest slope on the desk. What is the difference in vertical height between the upper and lower ends of the ruler? What is the inclination of the ruler to the horizontal?

11. If

$$\tan \alpha = m,$$

(1) what is the numerical value of the expression $\cos 2\alpha + m \sin 2\alpha$?

(2) Express

$$\frac{1}{1 - m} - \frac{1}{1 + m}$$

in terms of α , and hence find its value when $m = .839$.

12. A plane is inclined at $17^\circ 40'$ to the horizontal. It is rotated through an angle of $4^\circ 30'$ about one of its lines of greatest slope.

Draw a sketch diagram of a sphere showing a vertical radius OA , a radius OB perpendicular to the first position of the plane and a radius OC perpendicular to the second position of the plane.

What is the angle B of the spherical triangle ABC ? What angle does the plane in its second position make with the horizontal?

MATHEMATICS II (OPTIONAL).

(Two hours.)

[You may send in answers to any SIX questions, but not more. Questions distinguished thus (*) carry higher marks than the others. Great importance is attached to neatness and accuracy.]

1. In certain cases it can be foreseen that one term of an algebraic expression is much more important numerically than any of the others.

Give an illustration, constructed by yourself, of this statement.

Find an approximate value of x (using tables when necessary) which will satisfy the equations.

$$(1) x^{10} + x = .4$$

$$(2) x^{10} + x = 400000.$$

2. Draw a line OA 1 inch long and erect a perpendicular AB . Take any point P on AB . Join OP . Draw PM perpendicular to OP meeting OA produced in M . At M draw MQ perpendicular to OM meeting OP produced in Q . Let the angle QOM be θ .

By means of similar triangles express OP , OM and MQ in terms of θ , and hence find the equation of the locus of Q for varying values of θ the axes being OA and a perpendicular to OA through O .

3. Find the angle which the tangent makes with the axis of x at the point where $x = 7a$ in the curves whose equations are

$$(1) x^2y = 245a^3,$$

$$(2) 2y^3 - x^3 = a^3,$$

and show that these tangents are at right angles.

4. Draw a semicircle with its bounding diameter AB horizontal, and a chord PQ parallel to AB . On PQ as diameter describe a circle. For what position of PQ will the distance of the highest point of this circle from AB be a maximum?

5. Draw axes Ox , Oy , and mark four points P, Q, R, S , coordinates $a, 0$; $0, a$; $-a, 0$; $0, -a$ respectively, forming a square; and four points A, B, C, D , coordinates $1, 1$; $-1, 1$; $-1, -1$; and $1, -1$: forming another square. In drawing the figure take 5 inches to represent unity and about 2 inches for a .

$ABCD$ is a square of cardboard from which the four triangles APD, DSC, CRB, BQA , are cut away. The four corners $ABCD$ are then brought together, by folding about PQ, QR, RS, SP , forming a pyramid on a square base $PQRS$. Calculate the height and hence the volume of this pyramid in terms of a .

Check by putting $a = 1$ in the result and explain briefly how this gives a check.

Find for what value of a the volume is a maximum.

6. Draw a triangle AOB right angled at O . Take any points P, Q close together on AB and drop perpendiculars PM, QN on OA . Let $OA = r$, $OB = h$, $OM = x$, $ON = x + \Delta x$. Express approximately

(1) The volume of the hollow tube formed by the revolution of $PQNM$ about OB .

(2) the moment of inertia of this hollow tube about OB .

Find by integration the volume and moment of inertia about OB of the cone formed by the revolution of OAB about OB .

7. Find the numerical value of the following integrals

$$(i) \int_1^2 x \ln x dx,$$

$$(ii) \int_1^2 \frac{1}{x} dx,$$

Show that

$$\int x^2 \sqrt{1+x^2} dx$$

may be found by substituting for x by means of the equation

$$1+x^2 = z^2.$$

*8. Draw a rhombus $ABCD$ in full lines and the diagonal AC in dotted lines. Take a point O in AC and join OB, OD in full lines. The full lines represent the six bars of a Peaucelliers linkage. There are hinges at A, B, C, D , and at the fixed point O .

If $OB = OD = b$, each side of the rhombus is of length a , and $AO = x$, $OC = y$, show that

$$xy = a^2 - b^2 \text{ and find } \frac{dy}{dx}.$$

If a pull of W pounds is applied at C in the direction AC and is balanced by a pull of P pounds applied at A in the direction CA , find approximately, ignoring friction, the relation between work done by W in moving a small distance Δy and the work done in moving P a small distance Δx .

Hence express P in terms of W, a, b , and x .

Give numerical results if $W = 100$ lbs., $a = 5$ inches, $b = 4$ inches for $x = 7.5$ inches and for $x = 3$ inches. How does the latter result check the correctness of your formulae?

THE TEACHING OF ARITHMETIC IN SECONDARY SCHOOLS.

In writing this paper the author had in view principally schools of the type of the "Public Schools," as his experience has been almost entirely confined to such schools. But nearly the whole of the paper will be applicable to the state-aided and rate-aided secondary schools, if one allows for two facts in connection with the boys of such schools:—(i) They enter and leave school at a somewhat earlier age; (ii) a larger proportion of them is intended for a business career.

No general minimum standard of Arithmetic can be given for boys entering secondary schools. A large number of schools require that a boy before being admitted shall pass an entrance examination, and Arithmetic is practically always one of the subjects of such an examination. The standard of the entrance examination, however, varies very much with the popularity of the school, and is hardly more than a formality with schools which find it difficult to fill up their vacancies. To give some idea, however approximate, to those unacquainted with English schools, one might say that it is usually expected, as a minimum, that a boy on entrance shall have been taught, (a) numeration, (b) the four rules, (c) the manipulation of compound quantities, (d) the elementary parts of vulgar fractions.

Teachers of Mathematics in secondary schools are met with an initial difficulty. The teaching at many of the preparatory schools which supply the secondary schools is unsatisfactory. At other schools, though the Classics may be well taught, Mathematics may be neglected. Moreover, during the last ten years a great improvement in the teaching of Arithmetic has begun to permeate the schools of this country, and considerable alterations have taken place. The preparatory schools are in all the stages of this development, and the group of boys entering a secondary school at the beginning of a given term will have been taught by a variety of methods, some good, some bad. In this country it is almost impossible to secure uniformity in educational methods. Individuality rather than uniformity is perhaps to be desired in many things, but it adds considerably to the difficulties of the teacher in Arithmetic, when he finds among his pupils a considerable variety of method in performing fundamental operations, *e.g.*, the multiplication of two integers.

In order to appreciate properly the present state of affairs, it is necessary to add that there exists, whether rightly or wrongly, among a considerable section of teachers in secondary

schools, serious dissatisfaction at the results of the teaching of the boys who enter their schools, as far as Arithmetic is concerned.

The difficulty caused by the varied character of the previous teaching of boys entering secondary schools, can be met in various ways which it would be tedious to enumerate. As a matter of fact, I should say that perhaps the commonest solution is to assume that when a boy enters a secondary school his knowledge of Arithmetic is unsound unless the school from which the boy comes is known to teach Mathematics satisfactorily. Then a compromise is effected. It is hardly possible or desirable to start him again at the beginning of Arithmetic. On the other hand, the authorities believe that it is inadvisable to assume that the boy knows the work he is supposed to have done. As a result they start him at some intermediate stage and hope that in his first term or two, while not troubled with new work, he will be the less bewildered by the novelty of method and the more searching nature of the teaching and so will be in a better position to profit by it.

In reading the following description and criticism of arithmetical teaching in this country, it is of the utmost importance to remember that only in a small percentage of schools is the teaching of Mathematics entirely in the hands of properly qualified mathematicians. This is especially true of Arithmetic. In the average school the lower classes will probably be taught by non-mathematicians. In many of the smaller and less wealthy schools there may be only one qualified mathematician on the staff. The non-mathematician who teaches Mathematics may be a good teacher, but, except in rare cases, he does not show, nor can he be expected to show, for a subject with which he has only a limited acquaintance, that enthusiasm which welcomes new ideas and leads him constantly to examine the foundations on which he is building. He is perhaps frightened by supposed difficulties involved in any given change, at the lowest he objects to the additional work that it entails, so that for one cause or another he is strongly conservative. The existence of this class makes it difficult to ensure that the object of a change is properly appreciated and that a reform follows the lines intended by its originators. To take an example from recent history. A considerable amount of work with ruler and compass has been introduced into the teaching of Geometry in recent years. The object of this was to present elementary truths of Geometry in a more concrete form so that they might be more really appreciated. In a vast number of cases this idea was quickly forgotten or was never understood, and there has been in consequence an enormous waste of time over aimless work in what has been called Practical Geometry. The teacher who is not a properly qualified mathematician usually adheres slavishly to the text-book. If text-books were all that

could be desired there might be no very great harm in this, but though they have improved greatly in recent years they are still very far from all being satisfactory, and, moreover, the unsatisfactory teacher so often shows a strong liking for the unsatisfactory text-books. Even when an improved text-book appears, introducing any considerable changes, these are very liable to be misunderstood by the non-mathematical teacher, as, in a text-book it is obviously difficult, from considerations of space, for the author to give adequate explanations of his aims and ideas.

It is difficult to describe, to anyone unfamiliar with the educational progress of this country, the present state of arithmetical teaching, as during recent years it has been in a state of change which, if not rapid, is certainly more rapid than at any other epoch in modern times. Moreover, one can find schools at every stage in this development. There is in consequence far less uniformity than there was even ten years ago. Perhaps the most hopeful plan will be to attempt a short description of (i) the state of affairs before the recent changes became prominent, say 25 years ago, (ii) the kind of teaching which represents a reasonable compromise between the practice and the ideals of the reformer of to-day.

ARITHMETIC AS IT WAS 25 YEARS AGO.

The important thing to notice is that Arithmetic at that period was at the beginning of an important change. Before that time it had developed largely as a commercial subject on the lines most suitable to a boy intended for a business career. The most important exception to this was in the Classical Schools, *i.e.*, schools in which the greater part of the time was devoted to the teaching of Latin and Greek, but there Mathematics was not taken very seriously. The subject was bound to be taught in these schools as in all others, and moreover it was bound to be a difficult subject at the stage at which it was introduced, owing to the inherent difficulties of the English system of weights and measures and to the fact that vulgar fractions rather than decimal fractions were generally used whenever a choice was possible.

The teaching was largely unintelligent. No very serious effort was made to explain the rules; it was usually sufficient if boys learned the rules and were able to apply them. The subject was not regarded as educational.

But with the advent of "Modern" Schools in which Mathematics played an important part in a boy's general education, Arithmetic came to be regarded less as a part of a business training and more as an instrument for further mathematical work. Added to this, as time went on, it found itself faced by the demands of the Science master, especially of the Physics master, who was often handicapped by the bad teaching of Arithmetic. But these developments took place later.

At the period we are considering Arithmetic was treated as a separate compartment, distinct from Algebra and Geometry. Algebraical methods were not allowed in the solution of problems, and the possibility of using geometrical or graphical methods was hardly realised.

I will now proceed to enumerate the various items found in a text-book of the period, adding such discussion or criticism as seems useful.

- (1) *Numeration* (integers only).
- (2) *The Four Rules.*—

(a) Subtraction is just beginning to be taught by the process given below, known as complementary addition.

625	4 and <u>1</u> make 5.
<u>384</u>	8 and <u>4</u> make 12; carry 1.
<u>241</u>	4 and <u>2</u> make 6.

(b) Multiplication is set out as follows, and begins with the unit digit of the multiplier.

236
417

1652
236
944

98412

- (3) *Compound Quantities.*—Tables of English Weights and Measures were numerous and often taught in much detail, even in a first course. Troy Weight (used only for weighing precious stones and metals), Apothecaries' Weight, Cloth Measure, and other tables of limited use were given. The table called "Square Measure" was given as follows:—

144	square inches	make	1 square foot.
9	square feet	,,	1 square yard.
30½	square yards	,,	1 square pole.
40	square poles	,,	1 rood.
4	roods	,,	1 acre.
640	acres	,,	1 square mile.

In the text-book in use in the upper forms this table probably also contained further information, *e.g.*, 10 square chains = 1 acre; 4,840 square yards = 1 acre. But, on the other hand, it was more than likely that the pupil was

not expected to be acquainted with the table actually used by land and estate agents, viz. :

40 poles make 1 rood.
4 roods „ 1 acre.

In this section the processes involved were (1) Reduction, *i.e.*, changing a simple to a compound quantity and *vice versa*. (2) The Four Rules. The first of these, Reduction, introduced at an early stage very difficult work, as can be seen from the following examples:—

(i) Reduce to inches 2 miles, 3 furlongs, 21 poles, 3 yards, 2 feet, 11 inches.

(ii) Reduce to acres, roods, poles, &c., 300,302,596 square inches.

(iii) Multiply 4 tons, 12 cwts, 1 qr., 10 lbs. by 573.

That readers unfamiliar with the British system of weights and measures carried to its full development may be able to appreciate such questions, I append the working of (ii) by the method ordinarily taught at the time we are considering. The table given above will help to explain the method.

$$\begin{array}{r}
 144 \left\{ \begin{array}{l} 12 \mid 300,302,596 \text{ sq. in.} \\ 12 \mid 25,025,216 \quad -4 \\ 9 \mid 2,085,434 \text{ sq. ft.} \quad -8 \end{array} \right\} 100 \text{ sq. in.} \\
 \quad \quad \quad 231,714 \text{ sq. yd.} \quad -8 \text{ sq. ft.} \\
 \quad \quad \quad \quad \quad 4
 \end{array}$$

$$\begin{array}{r}
 11 \mid 926,856 \text{ quarter sq. yd.} \\
 11 \mid 84,259 \quad -7 \left\{ 117 \text{ quarter sq. yd.} \right. \\
 \quad \quad 7,659 \text{ p.} \quad -10 \left\{ \text{i.e., } 29\frac{1}{4} \text{ sq. yd.} \right. \\
 \quad \quad \quad \text{i.e., } 29 \text{ sq. yd., } 2 \text{ sq. ft., } 36 \text{ sq. in.}
 \end{array}$$

p.	sq. yd.	sq. ft.	sq. in.
7,659	29	2	36
		8	100

$$\begin{array}{r}
 40 \mid 7,659 \quad 30 \quad 1 \quad 136 \\
 4 \mid 191 \text{ r.} \quad -19 \text{ p.} \\
 \quad \quad 47 \text{ a.} \quad -3 \text{ r.}
 \end{array}$$

Answer 47 a., 3 r., 19 p., 30 sq. yd., 1 sq. ft., 136 sq. in.

Of course the very difficult examples would be omitted with young boys, but this part of the subject, as it was usually taught, was beyond the capacity of boys at this stage, especially if, as was generally the case, it was taken before the chapter on Fractions.

- (4) *Highest Common Factor and Lowest Common Multiple.*—The former was taught by the long-division method, the latter by the short-division method, that is to say, the method of prime factors was not used as a rule, and certainly was not regarded as the more important method. Not enough care was taken to teach the meaning of the terms factor, multiple, common factor, &c., with the result that boys often learned to perform two processes known as "H.C.F." and "L.C.M." without any clear idea of the object of these processes, or any appreciation of the way in which the processes served that end.
- (5) *Fractions.*—This was regarded as a very important chapter and was treated in considerable detail. In addition to the matter that one would expect it often included questions on proportion and the simplification of complex fractions such as:—

$$\frac{1}{2 + \frac{2}{3 + \frac{3}{4 + \frac{4}{5 + \frac{1}{5}}}}}$$

and $\frac{2\frac{7}{8} + \frac{2}{3}}{117\frac{1}{8} - 106\frac{1}{8}} \times \frac{1\frac{2}{3}}{23}$ of $\frac{2}{3}$ of 21 of $3\frac{3}{4}$.

To such detail was the teaching of vulgar fractions carried that in a text-book of 1886 (and one of the best of its period), in the one chapter devoted to them, there are 43 sets, containing altogether about 1,250 examples. The examples often involved dealing with very large numbers, *e.g.*, in the book mentioned above, near the beginning of the chapter, is the question, "Express in its lowest terms $\frac{114977}{220501}$."

- (6) *Decimal Fractions.*—This chapter contained usually:—
 (a) Extension of the decimal system of numeration to decimal fractions. (b) The four rules. (c) Change from a simple quantity (expressed as a decimal fraction) to a compound and *vice versa* (*e.g.*, express 2·7683 tons as a compound quantity in tons, cwt., qr., lb.) (d) Work with compound quantities, using decimal methods. (e) Change from decimal to vulgar fractions and *vice versa*.

The examples at the beginning of the chapter, explaining and illustrating the decimal notation, were, as a rule, neither numerous nor varied, and dealt with abstract numbers rather than with concrete quantities. Books at this period rarely contained any account of the metric systems of weights and measures in use in other countries,

so that this fruitful source of examples was hardly available at that time.

The examples were usually too heavy (*i.e.*, involved too much merely numerical work), *e.g.*, (i) Find the value of $\cdot474609375$ of 256*d.* in £ *s. d.* (ii) Express $2\frac{7}{8}$ of 1 ton 14 cwt. 99 lb. as the decimal of $5\frac{1}{4}$ of 111 cwt. 70·4 lb.

- (7) *Recurring Decimals.*—This was often a chapter of some length. In quite a good School Arithmetic of 25 years ago there are 13 sets containing nearly 400 examples. Rules for converting such fractions to vulgar fractions (fractions with 7, 11, 13, in the denominator are discussed in detail), the four rules, &c., are given at length. Some of the examples are very heavy, *e.g.* :—

(1) Express as a vulgar fraction in its lowest terms—

$$\frac{1\cdot33285714 + 1\cdot66714285}{1\cdot11571428 + 1\cdot88428571}$$
 (Of course this was merely a trick, and supposed that the pupil could write down $\frac{1}{7}$, $\frac{2}{7}$, &c., as recurring decimals.)

(2) Find the value of—
 $\cdot00401 + \cdot00062 + 72\cdot0111 + 108 + \cdot01$.
 (Answer, 180·0258383564482574.)

- (8) *Practice.*—This was isolated from other methods of computation, and was divided into two forms :—(a) Simple Practice—a method of multiplication, *e.g.*, find the value of 3,165 things at 1*l.* 2*s.* 9*d.* each. (b) Compound Practice—a method by which the value of a compound quantity may be found, when the value of one of its units is given, *e.g.*, find the value of 3 cwt. 1 qr. 14 lb. at 4*l.* 16*s.* 8*d.* per cwt. It was intended to be a rapid method of computation. The rapidity was unduly checked by the method of statement; *e.g.*, take the following question from a current text-book. Find the value of $365\frac{1}{4}$ articles at 4*l.* 7*s.* 9 $\frac{3}{4}$ *d.* each.

	£	s.	d.
The value of $365\frac{1}{4}$ articles at 1 <i>l.</i> each is - -	365	5	0
			4
" " " 4 <i>l.</i> or 4 times 1 <i>l.</i> is	1,461	0	0
" " " 5 <i>s.</i> or $\frac{1}{4}$ of 1 <i>l.</i> is -	91	6	3
&c.			

At the end of the chapter there was sometimes a section on invoices. In it was shown the correct form in which an invoice should be made out. Such questions probably involved the multiplication of compound quantities and possibly questions in compound practice, for the rest they contained nothing but the routine details of a merchant's office.

- (9) *Areas and Volumes* of rectangular figures. — The treatment of this part of the subject differed from more modern treatment in that (i) it did not deal at sufficient length with the new ideas here introduced, by means of preliminary examples involving a small amount of numerical work, but plunged almost at once into the difficulties of compound quantities in the British system; (ii) the examples were sub-divided too much into sets, e.g., a set on finding the amount of carpet required for a floor of a room, followed by a set on finding the area of the walls of a room, and by another on finding the cost of papering the walls of a room; (iii) the data were regarded as rigidly accurate and the answers were worked out with rigid accuracy; thus, in an example in which it is assumed that a cubic foot of iron weighs 450 lb. (this must be cast iron and may vary by some pounds in a cubic foot—wrought iron may weigh as much as 485 lb.), the weight of an iron tank 4 ft. by 3 ft. by 2 ft. 9 ins. and $\frac{1}{8}$ -inch thick (*sic*) is given as $468\frac{8}{15}\frac{1}{8}$ lb.

Appended to this chapter or elsewhere would probably be found a section on "Duodecimals," a method of measuring areas and volumes by a duodecimal system of numeration. A prime (1') is $\frac{1}{12}$ of a foot, a second (1'') is $\frac{1}{144}$ of a prime, and so on. The results were often worked to an absurd degree of accuracy. As the method is confined almost entirely to wood workers and glaziers, the following volume seems unnecessarily accurate:—

450 cu. ft. 3' 3'' 10''' 8'''' 6'''''.

- (10) *Unitary Method*. — Not the least remarkable feature in some of the old books was the position of the chapter on Unitary Method. After the pupil had covered the work sketched out above, he was allowed to approach the following type of question (taken from one of the leading text-books of the day). If 1 lb. of sugar costs 3d., how much will 11 lb. cost? If 4 lb. of sugar cost 1s., how much will 14 lb. cost? The work on direct proportion was followed by a set of examples on inverse proportion. There was no discussion of the meaning of direct and inverse proportion. There was no investigation in any given case as to whether direct or inverse proportion or neither was applicable. The method throughout was a guess-work method; at each step there were two alternatives, to multiply or to divide, and you decided by common sense which of the two was correct. From the point of view of working out examples, the pupil found things much simplified by the fact that in almost every book all the examples on direct proportion were put in one set and those on inverse proportion in another. The prevailing impression left by this part of the subject was

that in an arithmetical argument the principle of direct proportion is of universal application, with two or three irritating exceptions, mostly connected with the ploughing of fields.

The subject was then extended to compound proportion. The following is a question of an advanced type. "If 160 men in $12\frac{1}{2}$ days of 11 hr. each can dig a trench 230 yards long, $5\frac{1}{2}$ yards wide and $1\frac{1}{2}$ yards deep, in how many days of 8 hr. each can 96 men dig a trench 207 yards by $3\frac{1}{2}$ yards by 1 yard, given that the hardness of the ground in the first case is to that in the second as 5 to 7 and that 5 of the former men can excavate as much as 4 of the latter?" This was worked out, not as an example of variation, but somewhat as follows:—Write down $12\frac{1}{2}$ days; if you have 96 men instead of 160 the number of days will be increased, consequently you multiply by $\frac{160}{96}$ and not by $\frac{96}{160}$, &c.

This method was called "Double Rule of Three." The rule as stated in a text-book of 1885 is 12 lines long. It runs as follows:—"Take from the supposition that quantity which corresponds to the quantity sought in the demand, and write it down as a third term. Then take one of the other quantities in the supposition and the corresponding quantity in the demand, and consider them with reference to the third term only (regarding such other quantity in the supposition and its corresponding quantity in the demand as being equal to each other)" and so on.

The method of "reduction to unity" was sometimes employed. This made the working of a question such as that given above very long.

- (11) *Percentages.* — This was connected very closely with questions on Interest; in fact in one of the leading books it forms a part of the chapter on Interest. In another it comes after the chapter on Interest and is concerned almost entirely with commissions charged by stockbrokers, insurance companies, &c.
- (12) *Profit and Loss.* — This was treated as a separate subject, the questions being almost entirely percentage questions.

The subject was elaborated to a considerable extent and the various types of questions taught in detail. Consequently a good deal of time was spent on this chapter.

- (13) *Interest, Discount, Stocks and Shares, Equation of Payments, &c.* — These subjects, which are sometimes grouped together under the title of Commercial Arithmetic

or Financial Arithmetic, were treated at considerable length; they occupy nearly one-fifth of the total space in one of the books from which I have been quoting. This is due to the fact that before this time Arithmetic had come to be regarded as a part of a business training and not as part of the mathematical training in a boy's general education. The general question of Commercial Arithmetic is treated on p. 239.

Interest was treated in much detail. In the section on Simple Interest there would be a set of examples on each of the following:—To find (i) the Simple Interest (given Principal, Rate, Time); (ii) Time; (iii) Rate; (iv) Principal (given Rate, Time, Amount). These would be followed by a set of miscellaneous examples. These questions were sometimes worked by means of formulæ, but, owing to the separation of Arithmetic and Algebra, there was considerable prejudice against it. In any case, as Algebra refused to consider the needs of Arithmetic, boys were probably unfamiliar with the manipulation of the equations $I = \frac{P.R.T}{100}$, $A = P + I$.

Discount was considered to be of two kinds, (i) Bankers' Discount; (ii) True Discount. Take the question: "Find the discount on 500*l.* for 2 months at 4 per cent."

$$(i) \text{ The Bankers' Discount was } 500\textit{l.} \times \frac{2}{12} \times \frac{4}{100}.$$

As such questions could not but be of extreme simplicity (apart from the mere computation), Bankers' Discount was treated as of slight importance; in fact, there seemed to be a certain stigma attached to the use of it.

(ii) To find the True Discount was a more difficult matter. It involved another idea, that of Present Value or Present Worth, *i.e.*, the equivalent sum *at the present time*. Thus in the above question it would be the sum which put out to interest at the present time would amount to 500*l.* in two months, *i.e.*,

$$500\textit{l.} \times \frac{100}{100 + 100 \times \frac{2}{12} \times \frac{4}{100}}. \text{ The True Discount}$$

would be the difference between this and 500*l.*

Connected with this was a section on the discounting of Bills of Exchange involving a certain amount of technical detail. (For a discussion of the relation of True Discount to practical affairs, *see* p. 254.)

Stocks and Shares were also treated in considerable detail. In one book there are eight different sets of examples. Although the subject was headed Stocks and

Shares, in many books shares were not considered at all, and it was assumed that any broken quantity of any kind of stock could be bought. The technical information at the beginning of the chapter was inadequate, owing to lack of space, and nearly always inaccurate in some respects. For example, it was probably assumed that brokerage was always $\frac{1}{2}$ per cent. of the amount of the stock, though this was true of hardly any stock except the British Funds.

- (14) *Square and Cube Root.* — The position of the chapter giving these processes varied very much. In some books it would come at the end of Part I. (Pure Arithmetic). In others it was put very late, often a long way after the chapters on Areas and Volumes.
- (15) *Averages, Exchange, Alligation, Proportional Parts, &c.* — These sections, few of which introduced any new principles, and which added largely to the amount of detail, had become very numerous at this period. The exuberant ingenuity of the text-book writer seemed to find a congenial occupation in adding to them. In one small Arithmetic of less than 200 pages, in addition to the above there are Tare and Tret, Fellowship without Time, Fellowship with Time, Freehold and Lease, Chain Rule, Arbitration and Exchange, Alligation Medial, Alligation Alternate, Single Position, Double Position. Some of these have disappeared so completely that very few people have even so much as heard of them.

The problems were usually grouped into types. There would probably be a section on (a) Problems of Time; (b) Problems of Distances; (c) Pipes, including such examples as: "If one tap will fill a bath in 10 minutes and another in 12 minutes, how long will both take?" A comparatively recent text-book (published about 1905) has produced a separate section on uniformly growing grass. Many of the problems were quite out of place in an Arithmetic and should have been solved by Algebra.

MODERN ARITHMETIC.

On approaching the second part of the subject, viz., the aims and ideals of those who are endeavouring to effect considerable changes in the teaching of Arithmetic, one is met by a serious difficulty. It is possible to find out roughly the views and aims of personal acquaintances, or perhaps even of this or that educational body, but it is exceedingly difficult to feel even reasonably sure that these views are representative of a large section of the teachers of Mathematics in this country.

Many Secondary Schools are independent private institutions. Consequently there is no single body which can claim with any authority to represent them. The Mathematical Association, a

voluntary society of teachers of Mathematics, has, as a matter of fact, taken the lead in the recent reforms. It is practically the only body that has done so in any representative way.

When ideas had been discussed throughout the country for some time and experiments had been made, and after all this preliminary investigation and discussion there seemed to be some results on which there was general agreement, the Mathematical Association summed up these results in a report or syllabus. It has published "Reports on the Teaching of Elementary Mathematics,"* including reports on "The Teaching of Mathematics in Preparatory Schools," "The Teaching of Elementary Mechanics," &c. These reports have had considerable weight and have done a great deal to make such ideas accessible to a wider public. On the other hand the Mathematical Association has representatives from a limited number of schools only. Thus there are many schools which are not touched directly by the new ideas, and reforms filter through but slowly.

Perhaps the most effective propaganda work is done by the text-book; but text-books cannot contain lectures on educational methods, and they often lead to the misunderstanding of new ideas. To take an example. Graphical work was introduced into the Algebra course as an instrument of teaching; it provided an effective means of illustration. But to many text-book writers Algebra is only a loose congeries of miscellaneous subjects thrown together for convenience' sake, and "graphs" is but another added to the collection. It may be confined to a single chapter, and this chapter the last, as is the case in at least one well-known text-book, so that its object is entirely concealed from the teacher who has no other means of information than the text-book.

It is most important in this connection to recall what has already been said, namely, that a great part of the mathematical teaching of this country is in the hands of non-mathematicians. These men are usually ignorant of the changes that are taking place, rely almost entirely on the text-book, and often misunderstand the intentions even of the best writers of text-books.

In endeavouring to give an account of the aims and practice of teachers of Arithmetic in this country at the present time several courses are open. The writer might confine himself to such opinions as he considers to be generally accepted; this would limit him so much (owing to the difficulty of obtaining information) that the result would be of little value. Or again he might give his own personal views and nothing else. As a matter of fact the present writer has adopted a compromise. In the remarks that follow he is not entitled to claim for the views expressed any more authority than for mere personal opinions,

* Reports on the Teaching of Elementary Mathematics, 1902-1908. Reports of the Committee appointed by the Mathematical Association. (London: G. Bell & Sons, 1908. Price 6d.)

but he has endeavoured to give, as a rule, only such opinions as he has found to have some general support. Where he is giving his own unsupported view he has indicated that it is so.

For some time previous to 1902 there had been constantly increasing dissatisfaction with existing practice in teaching Arithmetic.

The reason for this state of affairs has been given on page 225. In earlier times Arithmetic was regarded as a commercial subject, it is now coming to be regarded more as a part of a general scheme of mathematical education. It is a tool to be constantly used, not only in the other branches of Mathematics, but also in Science teaching, especially in connection with experimental work in the Physical Laboratory. It was the failure of the old Arithmetic to meet the demands of the Physics master, that perhaps more than anything else led to the changes in recent years. The attack on Arithmetic from this point of view bred a divine discontent, and led a large number of teachers to examine the foundations of their belief. The whole subject has in consequence been subjected to a somewhat searching criticism.

In 1902 the Mathematical Association considered that the time was ripe for issuing a report stating in detail the changes most urgently needed and the ideas underlying these changes, and a special committee was formed for the purpose.

The report on Arithmetic and Algebra* began with the following general remarks :—

1. The Committee consider that there is considerable danger of the true educational value of Arithmetic and Algebra being seriously impaired by reason of a tendency to sacrifice clear understanding to mere mechanical skill.
2. In view of this the Committee recommend :—
 - (a) that *easy vivá voce* examples should be frequently used in both Arithmetic and Algebra ;
 - (b) that great stress should be laid on fundamental principles ;
 - (c) that, as far as possible, the rules which a pupil uses should be generalisations from his own experience ;
 - (d) that, whenever practicable, Geometry should be employed to illustrate Arithmetic and Algebra, and, in particular, that graphs should be used extensively ;
 - (e) that many of the harder rules and heavier types of examples which examinations alone compel us to retain in a school curriculum should be postponed.

These general principles led the Committee to make various detailed suggestions. Most of those that concern Arithmetic will be incorporated in the following pages. These remarks and suggestions of the Committee have, I think, met with very general acceptance ; at any rate, popular opinion in educational circles is very much in their favour.

* The Teaching of Elementary Mathematics. Reports of the Committee appointed by the Mathematical Association, p. 7. (London: G. Bell & Sons, 1905. Price 6d.)

Of the recommendations given above (b) and (c) embody fundamental principles of wide application in all the recent reforms.

Previously a large part of mathematical work was overmuch divided into independent chapters. At the beginning of a new chapter the pupil learned as rapidly as possible a new set of "rules of the game" and then proceeded to work out in accordance with these rules (and at times the accordance was very arbitrary) a large number of examples of gradually increasing complexity. For example, parts of Dynamics involving principles of great importance became largely an excuse for practice in ingenious problems of Algebra or Trigonometry. So in vulgar fractions the pupil too often was taught arbitrarily without adequate explanation (1) the rules for the fundamental operations, (2) the rules of brackets. He then proceeded to simplify fractions of constantly increasing complexity, till he arrived at such examples as that given on page 229.

The publication of many modern text-books shows that the importance of laying great stress on fundamental principles is more widely recognised.

Another change is foreshadowed in (d), viz., the introduction of graphical work. The advantage of this is more evident in Geometry than in Arithmetic, but even in the latter it can be of great use in making abstract propositions more real, in familiarising the pupil with the decimal notation, and in other ways.

The question raised in (e) has been referred to several times already in the earlier part of this paper. There is, however, a possible misunderstanding in connection with this matter. Examples have been given above which make it evident that 25 years ago Arithmetic was overloaded with much heavy work that had little or no educational value, was of no practical utility, and was often of an artificial character. This work was frequently very difficult, as it involved at the same time the inherent difficulties of the subject, together with the difficulties of computation due to the ponderous nature of the examples. It is the aim of most teachers at the present time to abolish work of this kind, and, by reducing the necessary amount of Arithmetic, to make it possible to introduce Trigonometry, Mechanics, or other subjects into the curriculum of boys who under the old *régime* would have had no chance of learning these subjects. But among these teachers there are many who hold that, though artificial and uselessly ponderous work should be abolished, it is most desirable that in straightforward computation work should not be taboo merely because it is heavy. There is perhaps a danger that Arithmetic may be considered too much merely from the point of view of the Physics master. In the physical laboratory most of the computation may be done by means of logarithm tables to four

figures, but the Arithmetic of the large majority of people is commercial rather than scientific, and for this four-figure logarithm tables are far from adequate. The first and most important aim in Arithmetic should be to make a student accurate, and if possible quick, at straightforward computation work. And it is maintained that so long as the work is straightforward and the questions not artificial there is no objection to dealing with large quantities. "Find the value of $240 \div 49.316$ " is a perfectly reasonable question, and occurs in finding the exchange value of *l.* in dollars. But no one should be asked to reduce 34,512,876 square inches to square inches., square feet, square yards, square poles, roods, acres. Nor is it desirable to waste time in simplifying such a fraction as $\frac{2698703}{54987261}$; the ordinary student does not need to work with vulgar fractions of this size, except for examination purposes.

The author would also like to protest against some of the questions which occur in text-books for a different reason. The following questions, all taken from text-books of repute, will illustrate his point. (*See also* p. 246).

- (i) A fortress is supplied with 92,000 lb. of bread and contains 1,400 men, who consume each 10 lb. per week; how long will the provision last? (*Answer.* It depends on the climate.)
- (ii) A wine merchant has 100 gallons of wine which are worth 8*l.*; find what quantity of water he must add to it so that it may be worth 12*s.* a gallon. (*Answer.* The effect on the value of wine of adding 35 per cent. of water to it depends on the nature of the wine, and the question is insoluble.)
- (iii) If seven boys catch four fish in six days, how long will it take a house of 21 boys to catch one fish at the same rate? (You can but wonder, and at the same time notice how much the literary value of the question would suffer if the words "a house of" were omitted.)
- (iv) A bath is filled by a pipe in 40 minutes. It is emptied by a waste pipe in an hour. In what time will the bath be full if both pipes be opened at once? (This question has its humour, but such questions might be allowed if it were not for the deplorable frequency with which householders are supposed to fill their bath with the waste pipe open.)

Boys have more common sense than Mathematics as a rule, and this kind of thing must surely give them an instinctive, if not a reasoned, contempt for the subject. Surely it is possible in a subject like Arithmetic, which has countless applications in practical affairs, to obtain a sufficient supply of sensible

examples. If no such examples can be found in a particular section of the subject, is not this a sufficient reason for excluding it from the curriculum?

There is one other question of a sufficiently general character to be discussed here, the question of the amount of time that should be devoted to the part of the subject known as Commercial or Financial Arithmetic, including interest, discount, stock and shares, foreign exchanges, &c. This is a matter which is much debated at present, as many of those teachers who are anxious that the average boy should not be kept at Arithmetic, Algebra, and Geometry during the whole of his school career, but should have an opportunity of pushing on to further developments of Mathematics, are of opinion that a considerable saving might be effected in the time given to Commercial Arithmetic. It is argued that the subject as taught at present introduces no new mathematical principles, and that it involves a large amount of technical detail connected with commerce and finance which is very difficult for boys at school who have no experience of the world of commerce, but which they can acquire easily later on. Moreover, the average mathematical teacher cannot be expected to be thoroughly familiar with modern developments of the detailed practice of large commercial houses, banks, the Stock Exchange, &c., so that his description of Commercial Arithmetic is apt to be antiquated and unreal. Certainly the average text-book gives a very inadequate account of the method of procedure of modern business houses.

Against all this it is urged that for a very large proportion of boys Commercial Arithmetic is the only part of their mathematical equipment that they will use extensively in after life, and that of the large amount of time given to Mathematics they can well afford enough to enable them to acquire some effective knowledge of Commercial Arithmetic.

It will, perhaps, be well to consider separately (i) the arithmetical operations involved, and (ii) the language, technical terms, methods of procedure, &c.

As regards the first of these, the business men with whom I have corresponded on the matter would be quite content if a boy on leaving school were reasonably quick and accurate at ordinary straightforward computation and reasonably safe in stating a simple arithmetical argument, *e.g.*, an application of the unitary method. They do not require a knowledge of the special arithmetical work connected with their particular business, as a boy will very quickly acquire this if he has been properly taught the elements of the subject beforehand. As a matter of fact, books of tables have rendered a great deal of this work unnecessary, and calculating machines seem likely to reduce it still further.

As regards the second, *viz.*, a knowledge of the technicalities of modern commerce and finance, there is more difference of

opinion. Some think that, as nearly everyone requires this knowledge, either in managing his own affairs or, as a trustee or executor, in managing other people's affairs, it should be part of a general education. Others regard it as desirable for boys going into business as part of their special equipment.

If some teaching of this kind is given, the question then arises whether it should be regarded as Mathematics. There is one argument against including it in the Arithmetic course. If a boy learns it at the age at which he now learns interest and discount, he will find it very much more difficult than he would later on, not on account of lack of intelligence, but from want of experience of general affairs. Could it not form part of what, for lack of a less objectionable word, is usually called Civics and be put later in the school course?

ARITHMETIC IN THE PRESENT DAY.

I will now discuss the various items which form the subject-matter of a modern text-book of Arithmetic, as I have done on pp. 226-234 for a text-book of a generation ago.

(1) *Numeration* (integers only).

(2) *The Four Rules*.—

(a) Subtraction is taught by the method of complementary addition (*see* p 227).

(b) Multiplication is set out as follows, and begins with the left-hand digit of the multiplier. This gives the most important figures first and makes it possible to teach the formal method of contracted multiplication later on, if it is thought desirable, or to cut off figures by some rough process of approximation.

$$\begin{array}{r}
 236 \\
 417 \\
 \hline
 944 \\
 236 \\
 1652 \\
 \hline
 98412
 \end{array}$$

(c) The Italian method of division is usually given, as well as the old method, but the former has not been received with much favour, owing to the difficulty of checking the work.

(3) *Compound Quantities*.— Tables of English weights and measures are taught in less detail, and the more difficult types of questions omitted, *e.g.*, the reduction of square

inches to a compound quantity, as shown on p. 5. The tables of length and area are sometimes reduced to—

12 inches	-	-	-	-	1 foot.
3 feet	-	-	-	-	1 yard.
1,760 yards	-	-	-	-	1 mile.
144 square inches	-	-	-	-	1 square foot.
9 square feet	-	-	-	-	1 square yard.
4,840 square yards	-	-	-	-	1 acre.

To the latter should be added the land surveyors' table :—

40 poles	-	-	-	-	1 rood.
4 roods	-	-	-	-	1 acre.

The value of a furlong, fathom, chain, &c., can be given separately.

The table of length as it used to be given was really a combination of two tables, one giving the units, inches, feet, yards, the other giving poles, chains, furlongs, miles. The connection between the yard and the pole was an afterthought.

- (4) *Metric System and Decimal Fractions.*—The decimal notation may then be extended to fractions and illustrated by the metric system. It is not necessary at this stage to introduce the latter with more than a small amount of detail, e.g., the table of length is quite sufficient.

The pupil is made thoroughly familiar with the decimal system of numeration by *viva voce* work and simple examples.

Actual measurements with a centimetre rule, or weighing with metric weights are of great assistance.

In teaching this part of the subject in this country one cannot appeal to experience, to the coinage of weights and measures in actual use, in the same way that a teacher on the continent can. The decimal system is beginning to be used. H.M. Customs officers give weights in pounds and thousandths of a pound, and volumes in gallons and thousandths of a gallon. In Lancashire, representative bodies connected with the cotton industry have passed resolutions to the effect that (i) the value of cotton shall be quoted in pence and hundredths of a penny, instead of sixtyfourths; (ii) the weight of cotton shall be given in pounds, 100 pounds, and 1,000 pounds, instead of in pounds, quarters, hundredweights, and tons. But the system of weights and measures in general use is valueless to exemplify the manipulation of decimal fractions.

In many schools, however, there is at an early stage a course of work, usually in connection with the Science teaching, in which the boys are made familiar with the decimal system of length and weight, viz., the metre and

gram, and their sub-multiples. This is very valuable but is usually independent of the mathematical teaching.

There are a few schools in which such work is taught by the mathematical staff, and it is to be hoped that this will become general or, at any rate, will be much more closely connected with the mathematical teaching. Such practical work introduced at the proper stage makes it much easier to teach decimal fractions earlier and to teach them so that the pupil uses them with more confidence and accuracy.

It would be out of place to discuss here the whole question of practical work in Mathematics. The reader is referred to the joint report of the Mathematical Association and Association of Public School Science Masters on "The Correlation of Mathematical and Science Teaching."^{*}

To go back to Decimal Fractions. After the preliminary work on the decimal notation, the processes of addition and subtraction are taught, then multiplication and division by powers of ten.

It is necessary to approach the processes of multiplication and division with great caution. No amount of care in teaching seems to ensure, at this stage, accuracy in placing the decimal point. The writer believes in a preliminary course of rough approximations. Numbers are multiplied or divided mentally or with very little work, and the result is considered accurate enough if within about 30 per cent. of the correct answer. This work tends to develop in a boy the rough checking of results as a constant habit.

Ex.—(1) $47\cdot2 \times 0\cdot086 = 50 \times 0\cdot1 = 5$, roughly.

(2) $0\cdot328 \div 748\cdot7 = 0\cdot003 \div 7 = 0\cdot0004$, roughly.

This work should be very instructive to the inexperienced teacher. He will find, for example, in (1), that at first the ordinary boy has no idea that $0\cdot1$ is slightly greater than $0\cdot086$.

It would be a great advantage if there were some means of ensuring uniformity in such fundamental processes as multiplication and division of decimals, but such an aim would be utopian in this country. Perhaps the most generally accepted methods are the following:—

Multiplication :—

(a) Reduce the multiplier to standard form, *i.e.*, to a number between 1 and 10; *e.g.*,

$$12\cdot6 \times 0\cdot314 = 1\cdot26 \times 3\cdot14.$$

(b) Arrange the work as follows :—

$$\begin{array}{r}
 1\cdot26 \\
 3\cdot14 \\
 \hline
 3\cdot78 \\
 126 \\
 504 \\
 \hline
 3\cdot9564
 \end{array}$$

(c) Rough check.

Division :—

(a) Reduce the division to standard form, *e.g.*,
 $6\cdot824 \div 0\cdot0715 = 682\cdot4 \div 7\cdot15$.

(b) Arrange the work as follows :—

$$\begin{array}{r}
 7\cdot15 \overline{) 95\cdot4 \dots\dots} \\
 \underline{682\cdot4} \\
 643\ 5 \\
 \hline
 38\ 90 \\
 \underline{35\ 75} \\
 3\ 150
 \end{array}$$

7 into 68 goes 9 times, put the 9 above the 8.

(c) Rough check.

At a later stage it is desirable to introduce the practice of multiplying and dividing without consideration of the decimal point, the latter being placed by a rough approximation. This is essential when boys come to use slide rules ; it is desirable as familiarising them with the idea of significant figures in the use of logarithms and in the consideration of errors in approximate work.

The "metric system" of weights and measures (as it is called in this country) *i.e.*, the decimal system in use in France, together with the decimal coinage of other countries, should be taught to some extent in connection with this elementary work on decimals. It can be treated in more detail later and provides a variety of examples illustrating the processes connected with the use of decimal fractions.

In the earlier books the examples were often of an artificial type. A quantity might be given in half a dozen different units, *i.e.*, Multiply 3 Km 5 Hm 0 Dm 7 m 2 cm 5 mm by 13. Moreover, units were used which were hardly, if at all, known to the French themselves, millilitres, deciares, &c. In later books questions are

confined chiefly to those of a simple type that might arise in actual life.

If the chapter on Decimal Fractions is introduced as early as this, it will be desirable to postpone till later further developments. These might form a separate chapter and contain (1) Conversion of decimal to vulgar fractions and *vice versa*, (2) Change from a compound to a simple quantity, in decimals and *vice versa*.

- (5) *Factors and Multiples*.—The meaning of factor, prime factor, multiple, power, index is explained. The process of finding the highest common factor or lowest common multiple is performed in most cases by the method of prime factors. The *idea* of factorisation is important, but elaborate developments are a waste of time.

Rules are usually given for finding at sight whether a number is divisible by 2, 3, 5, &c.

- (6) *Vulgar Fractions*.—This is bound to be a long and important chapter, owing to the complication of our system of weights and measures. It follows very largely the same lines as it did 25 years ago. The chief changes are the following:—(1) The fundamental principles are explained more thoroughly, diagrams being used for this purpose. (2) Where it is necessary to give rules, a more serious attempt is made to explain them. (3) Work on complex fractions is confined to the simpler kinds, such as may reasonably be expected to occur in the natural course of work.

- (7) *Brackets*.—This subject is, as a rule, taught in an exceedingly unsatisfactory manner. It occurs in connection with the simplification of complex fractions and the conventions associated with the multiplication and division of fractions. But it is taught also in the early stages of Algebra in a more elementary and perhaps less arbitrary manner than here. There one usually finds a chapter in which the subject is elaborated to an alarming degree. Such an expression as—

$$\frac{1}{4} \left\{ a - 5(b - a) \right\} - \frac{3}{2} \left\{ \frac{1}{3} \left(b - \frac{a}{3} \right) - \frac{2}{9} \left[a - \frac{3}{4} \left(b - \frac{4a}{5} \right) \right] \right\}$$

is not at all unusual. The process of simplification is performed automatically by means of "rules of signs." Despite all this work few boys understand the purpose that a bracket serves. They are not very sure of the difference between $a - b + c$ and $a - (b + c)$, or why such an expression as the latter is convenient. The subject should be confined to much simpler cases, the convenience of brackets should be explained, and the difference between a rule and a convention or notation should be made clear. In place of the ordinary questions of the Algebra text-

book it is suggested that statements, arguments, &c., of the following types should be written out:—

(i) $a - (b - c)$ means the quantity obtained by subtracting $b - c$ from a .

(ii) If b is subtracted from a the result is $a - b$, if b is increased by c , the result will be diminished by c $\therefore a - (b + c) = a - b - c$.

(iii) Express by means of brackets the change out of a sovereign after buying 3 books, each of which cost 3s. 6d., the discount on each book being $10\frac{1}{2}d.$, and the postage for the three, 5d.

Answer—

$$£1 - \{3(3s. 6d. - 10\frac{1}{2}d.) + 5d.\}$$

(8) *Unitary Method; Proportion.*—In this section the writer is giving his own personal views on many points. He has no alternative, as he has no means of discovering how far there is any considerable agreement on these points among mathematical teachers.

In the old books the more elementary parts of this section consisted almost entirely of questions on multiplication and division; e.g., if 5 lbs. of tea cost 6s. 8d., what is the cost of 1 lb. or of 3 lbs.? Beyond this the whole subject was taught by a guess-work method.

The following will explain the method:—

If a given quantity of food will last 700 men 60 days, how long will it last 300 men?

You assume that the answer must be 60 days $\times \frac{700}{300}$

or 60 days $\times \frac{300}{700}$.

Of these two one is greater than 60 days, the other less. Common sense tells you that the answer is greater than 60 days, therefore it must be

60 days $\times \frac{700}{300}$.

To show how little educational value there is in this guess-work process we might apply a similar process to determine *loci* in elementary Geometry; by it we can save much trouble, if one's main object is to save trouble. All such *loci* are either straight lines or circles, and by finding three points on the *locus* with rule and compass we can determine which of the two it is.

The objections to the method given above are obvious:—

(a) It is unscientific and of a guess-work character.

(b) There is no preliminary discussion as to whether the sets of quantities considered are directly propor-

tional; if not directly proportional, whether they are inversely proportional; if neither, whether they have any other simple relation. As I have said before, one result of the old method was that boys believed that the principle of direct proportion was of almost universal application. And the writers of text-books seemed to share the belief, for one finds such examples as the following:—

(i) If 100 lb. of lead-ore from a certain mine yields 78 lb. of pure lead, what quantity of lead will 17 tons 4 cwt. 3 qr. yield?

(ii) What is the cost of 5 lb. of sugar, if 2 cwt. 2 qr. cost 7*l.*? (Retail and wholesale prices).

(iii) If 28 gallons of beer serve a family of 6 persons 28 days, how long will 34 gallons serve the same family when reduced to four persons?

(iv) If 30 cannon consume 250 barrels of powder in 4 days, how much will be spent by 150 cannon in 12 days?

Notice also the delightful vagueness of the following:—

(v) If wages are *regulated* by the price of wheat, and a labourer earns 1*s.* 9*d.* a day when wheat is 8*s.* per bushel, what ought he to earn when wheat is 48*s.* per quarter?

When one came to other forms of variation, *e.g.*, the relation between the areas of similar triangles and their sides, it meant a long struggle on the part of the teacher before the direct proportion theory was vanquished.

(c) The "guess-work" method broke down in dealing with small fractions. Take the following example:—If $\frac{2}{3}$ of an estate is worth 1,500*l.* what is the value of $\frac{1}{3}$ of it?

This was usually worked either—

(i) by converting $\frac{2}{3}$ and $\frac{1}{3}$ into $\frac{1\frac{1}{3}}{2}$ and $\frac{1\frac{2}{3}}{2}$, fractions with the same denominator; or

(ii) as follows:—Since $\frac{2}{3}$ of the estate is worth 1,500*l.*, therefore $\frac{1}{3}$ is worth $\frac{1,500\textit{l.}}{2}$, *i.e.*, the whole is worth $\frac{1,500\textit{l.}}{2} \times 3 \therefore \frac{1}{3}$ of it is worth $\frac{1,500\textit{l.}}{2} \times 3 \times \frac{1}{3}$.

These methods were obviously unsatisfactory, especially for solving a question such as the following:—

If 1 inch = 0.0254 m., show that 1 cm. = $\frac{1}{4}$ ft. approximately.

The writer would suggest that the more elementary questions usually given in this chapter should be regarded

as examples of multiplication and division, and should be transferred to the chapter dealing with these processes, and that a scientific treatment of proportion should be postponed.

He would then propose a course somewhat as follows:—

(a) A preliminary discussion of the meaning of direct proportion. The test of direct proportion should be applied to sets of numbers, if possible to numbers that have been obtained by experiment, *e.g.*, the circumference and diameter of various circles. In addition such numbers should be plotted on squared paper, so that it can be shown that the plotted points are always on a straight line passing through the origin. It may be worth while pointing out that we have here the simplest instance of *functionality* (of course without using the word).

(b) That specimen examples of the simplest type to which direct proportion applies should be worked out by multiplication and division. It should be shown that such an example involves a statement and a question. The statement contains two quantities, A_1, B_1 , the question only one, A_2 . Of these, A_1, A_2 are of the same kind, while the answer is of the same kind as B_1 . Finally, that the answer is always of the

form $B_1 \times \frac{A_2}{A_1}$. This can be proved easily enough

when A_1 and A_2 are integers. If they are fractions the proof is not so simple, but can be given in special cases by the methods (i) and (ii) of section (c) above.

These results can be generalised by means of Algebra.

(c) Inverse proportion can then be explained in a similar manner and where it is applicable it can be shown

that the answer is of the form $B_1 \times \frac{A_1}{A_2}$.

(d) The guess-work method should be used to check the results found by the methods given above.

Compound Proportion.—See the example on page 232. The writer is of opinion that such questions are in nearly every case either artificial or reducible to questions on simple proportion. He would advance a plea for the omission of any special method of compound proportion at this stage.

Ex.—If 48 tons are carried 120 miles for 21*l.*, how much should be charged for carrying 35 tons 144 miles?

Assuming proportionality and using ton-mile to mean the cost of carrying 1 ton 1 mile, the question reduces to the following example on simple proportion. If

48×120 ton miles = 21L., find the value of 35×144 ton-miles.

But better still, such examples can be left till the chapter on variation.

It may be urged that it is more difficult to teach boys proportion according to the schemes given above than by the more popular "guess-work" method. Of this there can hardly be any doubt. But we are here dealing with one of the really important principles in Arithmetic and it is surely worth a little additional trouble to present a scientific treatment of the subject. The old method was perhaps good enough for Commercial Arithmetic, but now that Arithmetic has to satisfy the demands of Physics and Chemistry some better method of treatment seems needed.

- (9) A chapter on *Variation* would come naturally after that on Proportion, though as a matter of practice it might be taken later.

The treatment of this should be mainly arithmetical, but Algebra can be used where necessary. In a school text-book the only cases considered would probably be $y \propto x^n$, $y = ax + b$, and simple cases of joint variation. The work can be illustrated with diagrams on squared paper.

- (10) *Percentages*.—In the older books percentages were hardly considered except in connection with interest, commissions, &c. They are of such frequent use now and in such varied connections in everyday matters that this section has become an important one. That they are constantly required in scientific work will be acknowledged by everyone. The following extract from a letter sent to me by a business man in a firm which has dealings with foreign countries may be of interest as showing that the use of percentages is not confined to the domain of Natural Science:—" . . . now that everything is becoming more " international people might be taught to think in per- " centages more easily. It is much done on the continent, " and it is a great thing to have a common ground in the " language of figures, so that you may be to some extent " independent of units. One is apt to become very much " bewildered at a meeting where one man thinks in francs " per kilo, another in cents per American gallon, another " in kopecks per pood, and another in shillings per ton."

After the meaning of percentage and the convenience of its use have been explained, *viva voce* examples and a collection of simple miscellaneous examples seem to be all that is necessary. Examples of the ordinary types on profit and loss are nothing but examples on percentages and conveniently form part of this chapter. They are apt to be elaborated too much and so to lead to a waste of time.

(11) *Square Root*.—Little change is needed in this section. There has been much discussion as to whether the division process of obtaining the square root should be explained or whether the rule should merely be stated. As people seem unable to agree, there is probably something to be said for either side, and there seems no reason why each teacher should not follow his own inclination in the matter. It should be made clear that if the square roots of all numbers with the same significant figures are written down there will be two sets of significant figures, *e.g.*, the square roots of 30, 3, 0·3, 0·03, &c. are 5·477 . . . , 1·732 . . . , 0·5477 . . . , 0·1732 . . . , &c. This is important in connection with the use of the slide rule and of tables of square roots.

(12) *Areas and Volumes of Rectangular Figures*.—The fundamental propositions (*viz.*, If the length of a rectangle is a and the breadth b , the area is $a \times b$, &c.) are taught in considerable detail. They are explained with the assistance of diagrams and squared paper. Elementary questions connected with changes of unit (*e.g.*, if 1 m = 10 dm, then 1 sq. m = 10² sq. dm) can be illustrated conveniently by means of the metric system.

The early questions should involve very little numerical work, and here again the metric system is helpful. After the fundamental principles have been taught thoroughly, a miscellaneous set of examples can be given involving applications of these principles instead of separate sets of examples on carpeting the floor of a room, finding the area of the walls, &c.

(13) *Mensuration*.—A further chapter on the mensuration of rectilinear figures, circles, cones, &c., can be added. Some of the formulæ will have been proved in Geometry, most of the others will have to be assumed at this stage.

It is important to see that the results are not given to a greater degree of accuracy than is justified from the data. This is especially the case when the density of a substance is a factor of the answer. Thus in the *Annuaire* published by the French Government the values given for the density of steel range from 7·665 to 7·840. This is perhaps an extreme case. Such values are generally given to three significant figures, it being understood that the third figure is not always to be relied upon. It is consequently absurd to give the result to more than three significant figures. Again, in the mensuration of a circle, if π is taken as $3\frac{1}{2}$, it is not quite correct to four figures, and though the fourth figure of the result may be of some value, it is unlikely to be correct.

(14) *Approximation*.—In the old books on Arithmetic little was said about this part of the subject. In nearly every case a number or quantity was supposed to be absolutely accurate

and the result was worked out on this assumption. Thus, I find the cost of painting the walls of a room given as $3l. 17s. 1\frac{1}{2}d.$ Where approximate answers were required they were always asked to a certain number of decimal places, not to so many significant figures. This is natural, as in Commercial Arithmetic sums of money when approximate are nearly always given to the nearest penny, *i.e.*, in pounds, to three decimal places. But when boys began to work out results from data obtained in the Physical Laboratory, the number of significant figures became more important than the number of decimal places, and the ordinary text-book was found to be deficient.

It is curious to notice how little the ordinary man is alive to this question of the approximate nature of all measured quantities. He seems to have an absolute faith in any number, if it comes from a reliable source. When he is told that the value of the imports for this country in 1908 was 592,953,487*l.* he is very likely to have as firm a belief in the last figure as in any of the others. A very slight acquaintance with the methods of obtaining this value would shake his belief not only in the last but in a good many other of these figures. Scientific works are usually careful not to give an appearance of too great accuracy to their values, and often state the probable error; books of general information have improved in the last few years in this respect, but still there is room for great improvement even in scientific works, and especially in official publications. In the final report of the Royal Commission on Coal Supplies, the available resources of that part of the Cumberland Coalfield which lies beyond 5 miles and within 12 miles of high-water mark were estimated at 854,608,307 tons. The report contained pages of results of this kind. I am told that in a War Office Musketry Manual tables of elevations were given to a degree of accuracy implying that a rifle is able to hit at will the top or the bottom of a halfpenny at the distance of the moon. When the Brooklands automobile racing track was opened in 1907 the length of the oval portion of the track was officially announced as 2·71762547 miles.

Most people are greatly surprised when told that to obtain a physical measurement correct to more than three or four significant figures nearly always requires carefully made scientific apparatus and skilled experimenting. I suppose that few measurements have been made with greater care than the length of the metre in English units. It used to be given as 39·37079 inches; now it is given sometimes as 39·370113 inches, sometimes as 39·370432 inches; that is to say, the sixth figure is quite doubtful. But the length of the Brooklands track is given to nine figures.

In ordinary books of reference one is told that 1 cable = 100 fathoms and that 10 cables = 1 sea mile. This makes a sea mile = 6,000 feet, but the sea mile actually is taken to be 6,080 feet. Whether the books are wrong or measurements are so approximate that an error of nearly $1\frac{1}{2}$ per cent. is not noticeable I have never been able to discover.

In Commercial Arithmetic approximate results are usually worked to the nearest penny, so that in dealing with large sums there seems to be no limit to their accuracy. Thus a sum of money between 10,000*l.* and 100,000*l.* would appear to be accurate to eight figures. But consider the case of a loan of 1,000,000*l.* for 60 days at $4\frac{1}{4}$ per cent. The parties to the loan may have bargained about the percentage to be charged on the tacit assumption that no fractions smaller than eighths were to be considered; *e.g.*, the rate of interest may have been a question of $4\frac{1}{4}$ or $4\frac{3}{8}$ per cent. A difference of $\frac{1}{8}$ per cent. in the rate would make a difference of about 200*l.* in the interest, so that the accuracy of the answer is much more apparent than real.

This subject of errors and approximation is a difficult one to teach systematically, but it is a very important one, both mathematically and as part of a general education. In most cases a theoretical discussion is not worth while at this stage, but the subject has to be faced somehow if the approximate nature of observed values and of results depending upon them is to be brought home to the majority of students. A possible course is outlined very shortly below. In most cases it will be impossible to do much more than illustrate the principles and get the pupil to make use of them—

(1) Meaning of terms, *e.g.*, 'correct to n decimal places,' significant figures, 'correct to n significant figures,' absolute, relative, and percentage errors.

(2) In addition and subtraction work by places of decimals. If possible write down each quantity to one decimal place beyond the last required in the answer.

(3) In addition and subtraction the maximum *absolute* error in the answer is the sum of the maximum *absolute* errors in the given quantities, though where several quantities are added it is exceedingly improbable that it will be anything like so great.

(4) In multiplication and division work by significant figures. If possible write down each quantity to one significant figure more than is required in the answer.

(5) In multiplication and division the maximum *relative* error in the answer is (approximately) the sum

of the maximum *relative* errors in the given factors, when these errors are not large.

From this last we can determine how far to carry an answer (involving multiplication and division only), when we know the accuracy of the factors. For example, take the expression $\frac{(2.67)^2}{48.15}$, each quantity being correct as far as it goes.

The maximum relative error in the answer is—

$$\frac{2 \times .005}{2.67} + \frac{.005}{48.15} = 2 \times \frac{5}{2670}, \text{ approx.}$$

since the last term is small compared with the first.

$$= \frac{1}{250}, \text{ approx.}$$

Thus the max. absolute error

$$\begin{aligned} &= \frac{1}{250} \times \frac{(2.67)^2}{48.15}, \text{ approx.} \\ &= \frac{7}{250 \times 48} = \frac{7}{1000 \times 12} = .0006, \end{aligned}$$

and the answer is roughly correct to three decimal places.

There is considerable difference of opinion as to the advisability of teaching the processes of contracted multiplication and division. If they are taught at all, they will probably come conveniently in this chapter. There is much variety of method in performing these processes. If the method is not to be confined to a mere product or quotient, but is to be used for expressions of the

form $\frac{A \times B}{C}$ or $\frac{(A)^2}{B}$ the writer is strongly in favour of

performing the process without regard to the position of the decimal point and of determining the latter by a rough approximation.

- (15) *Logarithms.*—This subject really belongs to Algebra, but tables of logarithms are used so much now for arithmetical calculations that the subject is found in most modern text-books of arithmetic. The theory and use of the slide rule can be explained as an illustration of the principle of logarithms. Thus with contracted multiplication and division, logarithms, and the slide rule a boy is well equipped for numerical work, *e.g.*, in the physical laboratory.

It is assumed that the pupil has had the Theory of Indices so far explained that—

- (1) He can find the value of 10^3 , 10^4 , 10^5 , 10^6 , 10^7 , &c.

(2) He knows that to add to or subtract from the index the integer n multiplies or divides a power of 10 by 10^n .

From (1) he can draw a graph of $y = 10^x$, for values of x from 0 and 1, and deduce the logarithm curve $x = \log_{10} y$. From this curve and (2) he can find, roughly, the logarithms or anti-logarithms of any number. *E.g.*, if we require $\log 40$ or $\log 0.04$, we find from the curve that $\log 4 = 0.6$, and from (2) deduce that $\log 40 = 1.6$ and $\log 0.04 = \bar{2}.6$.

We can also deduce from (2) the rule that if the first significant figure is n digits to the left of the unit digit the characteristic is n , if n digits to the right of the unit digit, it is $-n$.

The rest of the chapter can be taught in the usual manner.

- (16) *Computation.*—Owing to the fact that we have not a decimal system for our coinage or weights and measures, questions are bound to occur which involve complicated methods of computation. Some of these can be solved most easily by the method of Practice, *see* page 230, but no general method can be given. In fact, the most difficult thing to learn is the choice of the most suitable method in a given case. A few typical questions are given below:—

(a) Find the value of 2,560,000 milreis, given that 1 milreis = 4s. 10½d.

(b) What is 3,765*l.* worth in American money when the exchange is 4 dollars 87.05 cents to the pound?

(c) Find the simple interest on 12,625*l.* for 17 days at 3½ per cent., *i.e.*, find the value of $12,625*l.* \times \frac{17}{365} \times \frac{3\frac{1}{2}}{100}$

(d) A rate of 7s. 1½d. in the pound on a parish produces 3,216*l.* 13s. 7d. What is the rateable value of the parish?

(e) The moon makes a complete revolution among the stars in 27 days 7 hrs. 43 mins. 11.54 sec. Find the angle through which it moves in a day, to a hundredth of a second.

These are doubtless extreme cases, but they are all examples that may occur in actual affairs. It is not desirable that a boy should spend much time on the solution of such questions, but after he has finished doing regular work in Arithmetic an occasional question of this kind may be found good practice. It is suggested that a chapter should be devoted to miscellaneous methods of

computation, that the first section should be given to an explanation of the method of practice, and that this should be followed by a collection of miscellaneous examples involving a variety of methods. The best position for this is perhaps after the chapters on logarithms and approximate work.

- (17) *Simple Interest, Compound Interest, Discount, Stocks and Shares, Exchange Operations.*—For the general question of Commercial Arithmetic see p. 239.

The following suggestions are made on the assumption that it is taught to the same extent as at present.

As far as "Simple Interest" is concerned, it would perhaps be best, from a mathematical point of view, if after the technical terms had been explained, only one set of miscellaneous examples were given. Such questions are either direct questions on percentages or questions to be solved as algebraical problems or by a formula. Similar remarks apply to discount. Some questions on simple and compound interest should be set as examples of arithmetical and geometrical progressions. The factor method of dealing with compound interest should not be neglected. *E.g.*, the amount of 500*l.* in *n* years at 4 per cent.

C.I. is $500\textit{l.} \times \left(\frac{104}{100}\right)^n$, the C.I. is $500\textit{l.} \left\{ \left(\frac{104}{100}\right)^n - 1 \right\}$

&c.

In the section on stocks and shares, the writer finds it convenient to deal only with shares when teaching the meaning of the technical terms. After that, stock can be introduced and with it the further difficulties connected with the ideas of capital and revenue. These are very real difficulties.

The arithmetical side of exchange operations has been neglected in recent years. It seems hardly worth while to load the Arithmetic text-book with an account of the practical details of this subject, which, without being treated very fully, may easily occupy sixty pages (*see* Jackson's Commercial Arithmetic—Macmillan), but it is a fruitful source of examples in computation.

As all this work is only partly mathematical and is in addition an explanation of the details of modern business, it is necessary that the latter should be as accurate as is possible with a given limit of space. To take an example from discount. The meaning of the true discount and bankers' discount of the text-books has been explained on p. 233. Of these two, the former is treated as the more important in nearly every text-book, the latter is dismissed as of slight importance. The writer has made inquiries, and can find no example of "true discount" in actual practice. Discount is the term applied to the sum deducted from

a debt due at a future time in consideration of its payment before it is due. This sum is always calculated by the method of "bankers' discount." Of course, the idea of present value occurs in modern business, *e.g.*, in actuaries' work, but it is connected with the idea of interest, not of discount. "True discount," reckoning compound interest, is still further removed from actual practice. It is also desirable in questions on interest and discount that the data should be reasonable. In a set of examples on discount I find that the average time after which a bill becomes due is over four years! This must give boys a curious idea of the credit system of the country.

- (18) *Graphical Work*.—It is assumed that graphical work has been used throughout as a means of illustration, and occasionally, as in the chapters on proportion and on logarithms, as a means of calculation. It is, however, customary to group together in one chapter examples of various kinds that lend themselves to graphical solutions. Such methods need not be artificial. If a series of values have to be changed from one unit to another, *e.g.*, from feet to metres, and no great accuracy is required, a graphical solution would perhaps be the simplest. The complicated time-tables of some of the railway companies are worked on squared paper and allow the position of a train at any time to be seen at a glance. Many questions of interpolation cannot well be solved by any other elementary method.
- (19) *Problems*.—Little need be said of these, except that their value is much diminished if they are separated into types. This has been discussed on page 234. A great deal of space is devoted to problems in many books. All that seems needed is a set of miscellaneous examples, preceded by a few worked-out examples carefully chosen.

There is a tendency in some modern books to overload the Arithmetic text-book with applications to matters which, though of undoubted interest and value, do not belong essentially to the Arithmetic course. I refer to such subjects as statistics, permutations, surveying, chemical arithmetic.

The writer is of opinion that the sections given above include the essential parts, and that, too great an amount of detail gives a boy unnecessarily the impression that Arithmetic is a more difficult subject than is actually the case.

SUMMARY.

Bringing together the results of the foregoing criticisms and suggestions, we may say that the chief aims which are generally agreed upon by those who wish to make considerable alterations in the teaching of Arithmetic are the following:—The reformers would clear away entirely many parts of the subject which do

not seem essential, and also clear away elaborate and artificial developments of many other parts. This pruning of the subject would enable the student to begin Trigonometry or some other branch of Mathematics at an earlier stage. But part of the time set free by the reduction and simplification of the subject-matter of Arithmetic could be utilised in teaching more thoroughly the important parts. For one thing, a much higher standard of work in straightforward computation is needed throughout the country. The methods now employed are often clumsy and the general standard of accuracy is not high. It is perhaps utopian to expect to secure any uniformity of method even in the fundamental processes, but at any rate considerable improvement of method can be introduced and a higher standard of speed and accuracy secured. It will be a very great help towards this if pupils can be made to acquire the habit of constantly checking their work by rough approximation. In matters beyond mere computation the important principles and ideas (and they are not numerous) should be introduced at first more simply and thoroughly. Rules should not be used until a considerable effort has been made to explain their meaning and, if not to prove them, at any rate to make them seem reasonable by considering particular cases. Sets of examples should be so varied that they cannot be worked by a purely imitative process but must necessarily require thought. New ideas should be made as real as possible by concrete illustration. This should include geometrical illustrations—diagrams, graphical work on squared paper, &c., and also laboratory work in close connection with the mathematical teaching. Although it is not desirable as a rule that any time should be spent on the theory of errors, it is important that the pupil should have some appreciation of the fact that the degree of accuracy possible in an answer depends on the accuracy of the data. He should be able to give an answer roughly to that degree of accuracy that is justified by the data. Questions should also, as far as possible, be taken from the arithmetic of everyday life, and at any rate they should not offend one's common sense.

In all these changes examining bodies can do a great deal to help or hinder. There are many reforms that could be introduced to-morrow if it were not for the demands of examining bodies. The examination papers in Arithmetic of the Civil Service Commission have improved greatly during recent years, and at present not only allow but encourage the teaching of Arithmetic on satisfactory lines. But the majority of the examination papers set at the present time are an obstacle to many changes which nearly all teachers regard as beneficial and are anxious to introduce. So long as these papers remain unaltered, the hands of teachers are tied and progress is necessarily slow.

THE EDUCATIONAL VALUE OF GEOMETRY.

"Every great study is not only an end in itself, but also a means of creating and sustaining a lofty habit of mind; and this purpose should be kept always in view throughout the teaching and learning of Mathematics."—BERTRAND RUSSELL.

The title of this paper has been chosen to indicate that the discussion will not be concerned with the value of Geometry as applied to other sciences or to practical ends, nor even with its place and importance in schemes of mathematical education. The purpose is to state the reasons which appear to have led to the universal acceptance of the subject as a necessary element in education, to ascertain to what extent geometrical teaching in this country can find justification in them, and to give some slight account of experiments in teaching made on this basis by the writer and his colleagues at Tonbridge School. Lest it should be thought, however, that this avoidance of the practical importance of the subject and its relation to other branches of knowledge imposes unreasonable limitations, it may be well to state the reasons for it.

The danger of giving undue importance to considerations of practical utility need hardly be enlarged upon, since it is not proposed to consider Geometry from this point of view. I am more concerned to point out that if the advocates of the subject rest any part of their case on such considerations they at once enter into competition with a host of other interests, many of which have, on such grounds, much higher claims. The parents of a boy who is to adopt a business career will rightly prefer, if his education is guided by his future requirements, that he should spend his time on geography or economics, arguing that surveying and bridge building can have no relevance to his future interests; while those who take a wider but, still utilitarian, view will insist that subjects such as the chemistry of food and civics have stronger claims to a place in the education of every child.

Still more dangerous is the plea that every educated man should have some idea of a subject of such wide utility. Apart from the claims of many other branches of knowledge, this has a further demerit in that the object of teaching the subject is implied to be the acquisition of encyclopædic knowledge, rather than the development of the mental faculties. The old conception of education as the acquisition of information is dead, and it least becomes mathematicians to do anything to revive it. The use of justifications of this type, even though it be only in secondary positions, is likely to defeat the aims of those who advance them and to do much harm to educational ideals.

A discussion of the value of Geometry in relation to other branches of science would be appropriate in a paper dealing with the co-ordination and relative importance of these branches. My object here is, however, to show that the subject has for its own sake a claim to a place in the education of every human being. Such a discussion could, therefore, give only a secondary and relatively weak support to this claim, a support which only becomes valid when the claims of these other subjects to a universal place in education have been admitted. If the view here taken is unjustified, Geometry must then make its own place in such volume of scientific knowledge as may be found necessary to a liberal education. This place would be an important one, especially in view of the now almost universal teaching of Natural Science, but it is hoped that the considerations to be stated in support of the stronger view are such as will meet with agreement among mathematicians, and convince laymen of its truth.

This is, of course, no new claim. Plato inscribed over the entrance to his Academy: "Let no one enter who is ignorant of Geometry," and almost every University now imposes a similar condition. Such recognition of the subject by educationalists who are not mathematicians implies an inherent value which must be expressible in non-technical terms, and it behoves all those who teach it to assure themselves that they appreciate this value and that the education in schools is such as to realise it as fully as may be. In this country at the present time the duty is especially urgent. Educational systems and ideals are changing with some rapidity, and almost every subject in school curricula has been challenged to justify its place, Geometry being one of the few exceptions, if indeed it still be one. It is almost certain that the motives for this forbearance are of utilitarian type, combined, perhaps, with some vague idea that the subject may train a boy to chop logic and hold his own in argument. Thus the lay advocates of the subject, on whom its continuance must depend, base their support on reasons which are open to successful attack from those who take a too material view of education, and are almost beneath attack from those who have higher ideals. Of this weakness they must become conscious; signs are not wanting that this is in process, and unless mathematicians themselves take the initiative in defence they may find the attack developed with some suddenness. If the boy who specialises in science obtains exemption from the study of Greek, his fellow who specialises in classics or history will almost certainly claim exemption from Mathematics, as also will those who intend to devote themselves to subjects such as Law, Medicine, or Music.

The question for mathematicians is then whether they can convince others that the appropriation to the study of Geometry of a portion of the school time of every boy and girl is really expedient. To do this it will almost certainly be necessary,

even though those who are to be convinced have themselves had some portion of their school time so appropriated, to explain in some detail what Geometry really is. The first element in the explanation must be that the subject is based on agreement as to a certain number of cardinal facts, this agreement resting on foundations of general experience common to every civilised human being.* The equality of vertically opposite angles, the angle properties of parallels and those properties of a circle which can be perceived from considerations of symmetry are instances. It is essential to understand that these facts should not depend, even for their elucidation, on numerical experiments made in class-rooms or laboratories. Rough descriptive illustrations there may be, but their only purpose is to recall or intensify conceptions previously formed subconsciously, for it is this universal acceptance of postulates without conscious experiment which differentiates Geometry from Physics or Chemistry. The necessary experiments and inductions are made in infancy without the aid of ruler, compass and protractor; the dog who follows his master by a curve of pursuit fails to make one which is grasped by young children, but not by infants who can only just crawl. They in their apparently aimless wanderings are in the true geometrical laboratory, performing experiments and making inductions.

Trite as this statement may appear to many it is of some importance here. The appreciation of common conviction should, when possible, be the first aim of teacher and pupil as of all those who would make a concerted effort, and here there is found a number of facts of which every pupil can say: "Yes, I know that these things are true."† They feel that they are not using arbitrary rules as in the study of languages, nor dealing with facts asserted by others to be true as in geography or history, nor dealing with experiments put before them by someone who knows what will happen as in Experimental Science; here they rest on their own convictions. I need not enlarge upon the value of this consideration, but I would point out that its importance can be realised by the layman and should influence him considerably. Amid the many arbitrary rules and asserted facts which perforce find place in education the presence of schemes of deduction based on statements which find universal acceptance as descriptions of our space impressions must make for good in the child's development.

The peculiar nature of the premisses on which Geometry is based having been explained, it might appear sufficient to complete the description of the subject by stating that it consists of a series of deductions from these premisses and

* The latter part of this sentence defines the subject as taught in schools. It is, of course, an essentially vicious limitation from the larger point of view.

† The basis for and meaning of this assertion do not enter into the question. The fact that it is made so universally is the point of importance.

supplies therefore a useful training in the art of deduction coupled with the advantage stated above. But this statement would, if not amplified, be so bald as to mislead. The full sequence of processes involved is :—

1. The separation of essential from irrelevant considerations involved in the appreciation of points, lines and planes and their mutual relations.

2. The erection on this appreciation of continuous chains of reasoning, one result leading to another in such a way that each chain can be comprehended as an ordered whole and its construction realised as fully as that of each separate link.

3. A discussion of the interdependence of the various premisses and their precise statement.

Since some such sequence is common to every form of human construction, the educational value of a subject which provides a training in these processes is indisputable; for this purpose Geometry stands alone in that its bases can be appreciated, and the deductions performed, at an age earlier than is possible in the study of any other subject having the same purpose.

There is yet one more consideration, and that not the least important, to be urged in favour of the subject. The appreciation of literary and artistic beauty has of late received increasing recognition as a necessary element in the training of the young. The study of the English language now includes literature; drawing and music advance in importance in schools of every type. But the element of intellectual beauty has not yet received general recognition despite the supreme position ascribed to it by almost all schools of thought from the Greeks downward, and the loss is a great one. The contemplation of unassailable mental structures such as are found in Mathematics cannot but raise ideals of perfection different in nature from those found in the more emotional creations of literature and art. It must induce an appreciation of intellectual unity and beauty which will play for the mind that part which the appreciation of schemes of shape and colour plays for the artistic faculties, or again that part which the appreciation of a body of religious doctrine plays for the ethical aspirations. Fanciful though this may appear to many, I believe that it may be an important factor in determining the retention of Geometry in schools. The conception of a body of truth invulnerable on all sides,* a conception which finds one of its best and most common expressions in the quotation: "Four square to all the winds that blow," appeals to most men, and they will welcome an effort to bring it to their children in one of its purest forms.

Such being the basis on which it is thought that the universal teaching of Geometry may be justified, it remains to develop

* Euclidean geometry is still invulnerable in that it is based on the simplest known description of our space perceptions, and such descriptions form the foundation of most "truths."

the leading principles of a scheme of education, and to examine to what extent teaching practice in this country accords with them.

The division of the processes of Geometry into three classes implies a corresponding division of the period of education into three epochs. In the first, the imagination is stimulated and developed and some general power of reasoning should be acquired without any formal presentation. In the second, ordered systems of reasoning are developed from facts which are now within the scope of the imagination ; and in the third, the true basis for the assertion of these facts is discussed and their interconnection investigated. It is at least doubtful whether this third stage would be a subject proper to a school course if it were within the pupils' grasp ; and it is almost certain that its nature and the problem involved is as much as can be brought home to them. It must, therefore, be assumed that a school course should end with the second of these epochs, and the first two only need be considered in detail.

The increase in power of imagination which is the main object in the first epoch can only be effected by extension from those impressions in which this power is already developed to some extent. A child may know what is meant by a point and a line, and be able to recognise them at sight, and yet not be able to think of points and lines—just as an adult may recognise the whole of a tune when he hears it and yet not be able to reproduce a single note. It is, therefore, inadvisable to commence by drawing points, lines and angles and performing constructions and measurements, because this opens up a new field of unfamiliar ideas having no connection with any of that knowledge which has been so far assimilated as to be a possible subject of imagination. Houses, roads, mountains, islands and the like can all be imagined by a child of ten years of age and his geometrical imagination is developed by stating problems in such terms, but the construction of triangles having given sides leads nowhere at this age and is a mere gymnastic.

It is not hard to devise problems on these lines which stimulate the imagination, excite the spirit of research, and provide exercise in the simpler forms of geometrical reasoning. They may be divided into groups, in each of which one of the following methods is introduced :—

1. Construction of triangles and polygons when lengths only are given.
2. Simple constructions for heights of buildings, ships' courses, &c., depending on compass bearings and angles of elevation.
3. Construction of triangles and polygons when lengths and angles are given.
4. Extension of all the above to problems in more than one plane.

5. Determination of a point as the intersection of two loci or its limitation to lie inside or outside two or more loci. These loci need not be lines or circles and are constructed by actual plotting.

It may be worth while to give a few specimen problems to show what is intended, and it should be stated that these and subsequent suggestions are the outcome of experience and have been tested in Tonbridge School. Each of these problems is well within the scope of a boy of 11 or 12 years of age, after he has received a reasonable amount of teaching in the shape of questions involving the same principles; the concrete terms should, of course, be varied constantly.

a. (*Type 1*).—A straight road runs East and West; a house is 95 yards north of the road and a well is 40 yards north-east of the house. Draw an accurate plan and find on it the position of a cow-shed which is to be built 65 yards from the well, within 70 yards of the road and as far as possible from the house.

b. (*Types 2, 3, 4*).—The height of a tree or chimney may be found from measurements in one or in two planes, which are made by the pupils themselves.

c. (*Types 2, 3, 4*).—A hill rises due north at a gradient of 1 in 10. Find the direction of a road on it which rises to the eastward at 1 in 30.

d. (*Type 5*).—A lightship is 2 miles from a straight shore. A submarine, which is cruising in the neighbourhood, explodes and sinks, the only survivor being the man in the conning turret; he can only say that the wreck is equidistant from the lightship and shore, and the watch on the lightship says it was 3 miles away when the explosion happened. Show on an accurate plan where it must be.

e. (*Type 5*).—Two towns, A, B, are 5 miles apart and a man who lives with his parents at a place 6 miles from A and 7 miles from B, bicycles to A, then to B and then home every day. After a time he wishes to reduce his daily ride to 14 miles. Show on an accurate plan all the places where he can live and indicate those places which are nearest to and farthest from his parents' home.

The first three types require no further explanation, but since the fourth and fifth involve departures from current practice, discussion of these may be of some value. The early introduction of Solid Geometry (examples *b, c*) may cause some surprise, especially in problems of such apparent difficulty. But I am convinced, from the experience of myself and my colleagues at Tonbridge, that boys of 12 or 13 grasp these ideas with much more ease and rapidity than those who have deferred the work for two or three years, and that their whole outlook is improved in consequence. The first problems should concern things which can be seen: *e.g.*, the height of one corner of the class-room can be determined by measurements

taken from the ends of a desk not in line with it, or the diagonal of a block of wood may be found by construction of two right-angled triangles. And a problem such as *c* above is simply illustrated by using one cover of a book half open to represent the hill-side. These aids to imagination are soon found to be unnecessary, fairly complex constructions being undertaken without their assistance, and there is no lack of material from which examples may be constructed.

Since the method described in the fifth type and examples *d*, *e*, is, except for the introduction of other loci, used in the preceding sections, its separate mention may appear redundant, but it is easily seen to involve principles which can hardly receive too much attention. The statement of the laws of selection which define two or more manifolds in such form as to exhibit the common elements (if any) of them all is one of the most common forms of mental activity, and experience has shown that children can understand and perform the process in cases such as those above. In so doing they apply it, not to new and unfamiliar abstractions, but to classes of objects so familiar as to have become possible subjects for mental operations.

Such applications form the second of four stages into which education in logical processes may be divided. In the first they are applied to definite objects, as in the methodical selection from a number of tiles of all those which have given shape, size and thickness; in the second to mental images of classes of objects, as when "a house" is not thought of as any particular house; in the third to abstractions, such as point, line, colour, sense, instinct; while in the last the processes themselves are considered in their utmost purity. For our purpose the first stage is dealt with in the kindergarten and in courses of practical measurement and the third in formal Geometry; the second does not, in my opinion, receive sufficient recognition in the teaching of Geometry at the present time and the exercises above are intended to remedy this deficiency.

The course may also be regarded as an introduction to the idea of a manifold and a function. The importance of these concepts has of late been realised and need not occupy us here, but their difficulty, especially in regard to manifolds, is hardly realised. If a child is directed to mark on paper a number of points such that the sum of their distances from two given points has a given value, he will do so without any idea that all such points form a continuous curve, and if he is told this he will not grasp the fact. But allow him to go on marking points and he will ultimately attain to a conception of the whole and their assemblage along a continuous curve. This cannot be taught; the teacher must wait for the child to reach it, and the value of the work is lost if mechanical means of describing the curve continuously are adopted. Such a process has its own value, but in this connection it obscures the idea of the curve as a manifold of points selected from the manifold which forms the plane.

The early formation of such habits of thought, even though subconscious and unsystematised, has a technical importance which deserves mention despite our general limitation to educational considerations. Recent research has exhibited the whole structure of Mathematics as founded on the processes and conceptions which underlie them and has gone far, with their aid, to perfect a coordination between the different branches of the subject. They are now seen to be essential to the understanding of even Elementary Mathematics and their absence is responsible for most of the difficulties which occur in the introduction to more advanced work. The difficulty of the child who in commencing Algebra says, "Yes, but what *is* x ?" of the boy who cannot solve a simple rider, and the student who finds the calculus puzzling, are now all traced to this common origin. The moral for the teacher is obvious. Until the young child can in some way attain to such ideas in relation to matters within his own experience he has not grasped the groundwork of mathematical thought and so cannot erect the structure on proper foundations, and the present problem for teachers is to bring them to him in as many forms as possible. For this purpose some understanding of the philosophy of Arithmetic, Geometry and Algebra is essential, and I believe that the light which this philosophy throws on problems of elementary education constitutes by no means its least value.

The transition from the preliminary stage just described to a first course of formal Geometry is marked by the introduction of connected reasoning leading from one result to another, and a gradual cessation of the aid to imagination involved in the use of houses, roads, &c. The assumption of postulates as a basis for this reasoning should not be stated formally; unless the preliminary course has failed in its object the pupils will accept them without difficulty as they are required. But it is essential that the choice of these postulates should not be left to chance or dictated by convenience of presentation; it must be guided by the considerations which determine the adoption of the subject as a mental training.

If these considerations be such as are indicated above there is no difficulty in specifying the nature and extent of the assumptions which should be adopted as facts in a first course of Geometry. It must be possible to induce the child to accept them without the aid of numerical measurement of any kind, and every statement for which this is possible should be regarded as a postulate. This definition includes the following:—

- (1) The equality of vertically opposite angles.
- (2) The angle properties of parallel lines.
- (3) Properties of figures which are evident from symmetry.*

* Demonstrations by folding are not proofs as the term is here used. They lack the essential element of deduction from two or more statements already admitted.

(4) Properties of figures which can be demonstrated by superposition, a method which should be once for all discarded as a proof.

The meaning and intention of this definition is best illustrated by reference to recent developments in the teaching of Geometry. There has been a tendency in preliminary courses to associate such ideas with numerical measurement if not, indeed, to profess demonstration by its aid. For example, children may be instructed to draw two straight lines, measure the vertically opposite angles, and state what they observe. It is hard to see what good can be derived from such exercises and they may do much harm. If it be true that the education of the child should follow the development of the race they are condemned at once, for it is inconceivable that these ideas were suggested by such processes or that there was a geometrical Faraday who announced them to the ancient world. They can only be regarded as intuitions from rough experience; a person in whom they are wanting is ignorant of space, and a knowledge of (Euclidean) space is neither more nor less than their full comprehension—a comprehension resting on foundations wider by far than any schoolroom experiments. An attempt to aid its formation by numerical exercises may not unfitly be compared to an attempt to teach music by explaining the mechanism of a piano, and the relation between notation and keyboard, before the pupil has heard a single tune. It is a crippling of subjective growth at its most sensitive stage by the crudest form of materialism. A mind which has ranged over all its experience and has made these intuitions has gained a sense of power and accepted truth which cannot be induced to an equal extent by any substitute for the process.

The statement that all possible intuitions must be taken as postulates is hardly less important than the definition (for our purpose) of a postulate as a statement of common acceptance. The object of teaching Geometry has been stated as twofold: the sense of logical proof is to be developed, and the conception of a chain of proofs is to be formed. There are many ways of doing this; they vary with the selection of the theorems used and the order of their sequence, but each should be subject to the condition that every theorem should prove some new fact which could not have been perceived by direct intuition, symmetry, or superposition. If this condition be violated the pupil's sense of a proof is confused; it appears to arrive with much painful labour at a result which he and everyone else knew before to be true, and the image of inevitable but unforeseen truth is not induced. Take, for example, the theorems concerning bisected chords of a circle: if a boy cannot perceive the truth of these with, possibly, a little stimulus, his knowledge of space is so meagre that he ought not to have commenced formal Geometry. If, on the other hand, he can see their truth there can be few processes more destructive than the elaboration of deductions

from other intuitions with the intention of imparting a sense of proof.

The details of the propositions and the order of their sequence is hardly germane to this paper, but the necessity for presenting them in sequences each of which can be grasped as a whole deserves further mention. If the Euclidean tradition be ignored such sequences can still be found. For example, experience has shown that the angle properties of polygons, arcs of circles, and tangents to circles can be arranged in one sequence, so that the unity of the group can be appreciated by pupils to whom it is the first example of formal Geometry. Another subject is found in the theory of the regular polyhedra; the limitation of their number, the investigation of their shapes and dimensions, and their construction, can be grasped by boys of ordinary ability at an early age. It will probably be agreed that a considerable element of Solid Geometry is essential; the tetrahedron, cone and sphere are other themes which have been found possible and valuable.

It is, of course, true that Euclid presented Geometry in a series of such sequences, and, despite the logical deficiencies in his scheme, his genius is probably still unrivalled. But his theme was the reduction of the number of postulates to a minimum and the introduction of each as late as possible. As soon as it is granted that all possible intuitions should, in a first course, be accepted as postulates, his sequence fails to have value for this purpose, and has, indeed, demerits. It is, therefore, a necessary consequence of the theory of geometrical education here developed that a deliberate effort should be made to replace this scheme by one more suitable. The need is not met by allowing propositions to be treated as intuitions when possible and retaining his sequence, for this destroys the whole meaning of that sequence; his motive enforces the introduction of Solid Geometry as late as possible, while ours demands it as early as possible, and numberless other contradictions arise.

This conclusion induces doubts whether even this residuum of Euclid's propositions, taken in any order, forms the best material for geometrical training of an educative value. Some of it undoubtedly deserves its place, both for its own value and its necessity in other parts of Mathematics; but, stripped of unessentials, this is small in content and less than can be acquired in an ordinary school course. Before including more than this bare minimum it is advisable to ascertain whether in the whole domain of geometrical knowledge there may not be other matter of more educative value within the grasp of ordinary boys and girls. Here I am passing beyond the range of my own experience, but I have, as the outcome of a maturing conviction, made tentative experiments with individual pupils and small classes and deem it worth while to state the result.

If, without regard to the age of the pupils, one asks where the methods and ideals of Geometry are presented in the

greatest unity, simplicity and beauty the answer must be that Geometry of position and projective Geometry have no rivals. Estimated on such standards Euclid's work is dwarfed by these modern creations, and not least so in respect of the ease and generality of their conceptions. Must it be said, in spite of this, that they are to remain the property of professed mathematicians since they are beyond the grasp of the normal adolescent or may it be that—even if only in some comparatively crude form—their power and beauty can be appreciated? Even those, if there be any, who at once deny the possibility will not dispute the importance of the question and the value of success if it can be attained. My own conviction, fortified by such limited experience as I have indicated, is that the elementary concepts and methods of Projective Geometry can be grasped by ordinary pupils; that they would excite a greater interest and fuller spirit of enquiry than any form of Euclidean Geometry, and that their educational value would be far greater. The amount of knowledge required is not so great as might be imagined. A pupil who has had some experience of three-dimensional work can grasp the relations between a figure in one plane and its projection in another plane, including the particular cases wherein a point or line in either plane have their corresponding element in the other at infinity. The theorem of Desargues, with its simpler consequences, is, therefore, within his grasp, and he may so gain some idea of the extent and variety of the results which can be deduced from the axioms of position only and the manner in which they unite in one statement results which he had regarded as disconnected.

It is then natural to enquire whether there is any simple relation between corresponding segments in the two planes. The position ratio $AP : PB$, which defines the position of a point P on a line when two of its points A, B are given, should have become familiar in connection with earlier work, and it is now easy to prove that corresponding position ratios remain in a constant ratio as the points P, P' move along their respective lines. Hence the ratio $\frac{AP}{PB} : \frac{A'Q}{Q'B}$ of two such ratios is unchanged by projection. The metric properties of quadrangles and quadrilaterals (deduced by projection from a parallelogram), and the simple properties of the conic regarded as the projection of a circle, can then be investigated to such extent as may appear possible or desirable. I have myself, with ordinary boys of 18, reached Carnot's theorem without great difficulty.

It may be suggested that this work is unnecessary and that when the minimum of Geometry has been acquired the pupil should proceed to Trigonometry or other branches of Mathematics. As to this I can only say that this paper is in part an attempt to justify the teaching in schools of an amount of Geometry much larger than has hitherto been thought possible,

and this without increase of the time devoted to the subject. My own experience is that an early commencement of Trigonometry can and should be made by some sacrifice of the large amount of time now devoted to Algebra by pupils who are too young to understand the subject. Trigonometry is a valuable stimulus to geometrical thought, but is no substitute for it.

It may be that this suggestion demands too much from the pupils; I do not think it can be thought to err in other respects. If this be so, the responsibility of devising some other more possible scheme as a sequel to the essentials of Euclid remains with any who agree with what has been said above.

It only remains, in considering the scheme of geometrical education, to refer to the interdependence of the postulates which have been adopted. I do not believe that any detailed or systematic discussion of this is possible or advisable at the school age, but if towards the end of this period examples of deduction of some postulates from others were shown it might be possible to lead the pupils to realise the ideal of a Geometry based on a minimum number of assumptions concerning the nature of space. Such considerations would then acquire more reality for them in that they would have some acquaintance with Mechanics and Physical Science and could therefore conceive the general ideal of a minimum of induction and a maximum of induction. And I feel bound to state my conviction that every student of whatever subject, who proceeds to a University education worthy of the name, should gain some slight idea of the nature of non-Euclidean Geometry. The simpler portions of the paper by Carslaw in the "Proceedings of the Edinburgh Mathematical Society for 1910," or the description of Poincaré's well-known illustration given in Young's "Elementary Concepts of Geometry and Algebra" are within the grasp of anyone who has even a slight acquaintance with the geometry of the circle, and their appreciation throws a light on the space-concept in particular and our so-called knowledge in general which can be gained in no other way. It may fairly be said that there are few portions of mathematical knowledge which have more educational value.

The nature of current teaching of Geometry in England is best understood by reference to recent history. Until a few years ago the use of Euclid's text in matter and sequence was universal owing to the regulations of examining bodies. Attempts to secure more freedom had not been wanting. The Association for the Improvement of Geometrical Teaching was formed with this object as early as 1871 and undoubtedly succeeded in awakening interest in the presentation of Elementary Geometry, though no tangible result appeared. In 1901, however, a paper read by Professor Perry before the British Association at Glasgow aroused fresh interest, and a Committee was formed by the Association to consider and report upon the

teaching of Elementary Mathematics. In their report, issued a year later, they advocated a preliminary course of Practical Geometry and stated that it was in their opinion unnecessary and undesirable that one text-book or one order of development should be placed in a position of authority.

Simultaneously the Mathematical Association, a development of the Association founded in 1871, had formed a Committee to consider the subject and this body issued a report in May, 1902. In principle it proceeds on lines similar to the report of the British Association Committee, but it contains a definite statement that 'it is not proposed to interfere with the logical order of Euclid's series of theorems.' In effect it simplifies the development by introducing hypothetical constructions, changes the order of certain groups of propositions, and introduces an algebraic treatment (for commensurables only) of ratio and proportion. As a consequence of the work of these Committees and their supporters the examining bodies yielded one by one, announcing that Euclid's order and methods would not be enforced, and setting questions involving numerical construction and calculation. Finally, the University of Cambridge issued in 1903 a syllabus indicating the amount of geometrical knowledge required for its entrance examination, and stated that the Examiners would accept any proof which appeared to form part of a systematic treatment of the subject. This syllabus, which is still in force, omits portions of the text of Euclid's Books I-IV. and VI. and introduces no additional matter.

Taking these changes in detail we are first concerned with the postulates. The intention appears to have been the increase of their number in order to simplify certain proofs which had presented difficulty; proofs of facts which can be perceived by intuition are still retained, notably in regard to congruent figures and properties of the circle. The present scheme may, therefore, be not unfairly described as Euclid's ideal of the minimum number of assumptions, tempered by consideration for the age of the pupils—a consideration which renders the whole scheme meaningless from his point of view. It may be doubted whether such a compromise is likely to be successful. The broad lines of educational advance should be based on fixed principles and not on expediency. If the pupils can understand the development of a scheme based on a minimum of assumption, and can also understand this motive, let it be adopted as the highest ideal. But if not, as experience has shown with some certainty, the only logical alternative is to allow all possible intuitions as postulates. The educational advantages of this course have already been described, and it has the further advantage of lessening the amount of time and effort required, and so clearing the way for some study of more advanced Geometry, which is not yet represented in any of the examinations referred to. Should any doubt be felt as to the

persistence of Euclidean methods despite the abandonment of his ideal, consideration of the fact that even the elements of Solid Geometry are not included in school courses, except in preliminary work (to which reference will be made), may carry conviction. All examining bodies (the Civil Service Commission excepted) appear to imply by their schedules that sufficient training in schools can be obtained without it, the fact being, of course, that its omission is a survival of Euclid's order.

Thus it may perhaps be said that what is often called the abolition of Euclid must involve the complete abolition of his order and methods and the construction of another theory of development and that in both respects the changes are incomplete. The structure—always of course fallacious in its foundations—has been shattered and we are groping among the fragments.

Turning now to educational methods, there are two important developments; the introduction of preliminary courses and the use of numerical examples. The latter only requires brief reference; it must give greater reality and precision to the results which it is intended to illustrate and there is common agreement that it has done this. The meaning and proofs of propositions are admittedly better comprehended, but the comprehension is too often of isolated results rather than structures of reasoning.

The object of the preliminary course is to enable the pupils to acquire some familiarity with the leading facts and concepts of the subject. For the most part such courses consist of exercises in measurement and construction, coupled (in some cases at any rate) with numerical introductions to or illustrations of the axioms. So long as such exercises are confined to the performance of constructions of known type there is little to be said for or against them in logic or philosophy, though they are, in the opinion of some, as deadening to the intellect as the excessive performance of algebraic simplifications. But when the angle properties of parallels and triangles are introduced by measurement with a protractor, instead of by turning a rod, and when we find an example such as, "Draw a triangle whose sides are 2, 3 and 4 inches and then draw the perpendiculars from each vertex to the opposite side. These lines should meet in a point; see that they do so," the matter becomes more serious, for numerical measurement has been substituted for intuition or demonstration and the impression is hard to eradicate. There is also some general introduction dealing with space-concepts, and here there is usually some allusion to objects in three dimensions; beyond this, Solid Geometry finds as a rule no place in such courses. Its persistent neglect by teachers, examining bodies and writers of text-books is one of the most marked and regrettable features in the developments of recent years.

Apart from this, the main criticisms, from the point of view of this paper, to be directed against such courses are that they are not based on a gradual extension from previous experience and imagination, and that there is a distinct tendency, as has been said, to relate the postulates to numerical processes. Of the second I need not speak further; of the first it may be said that it violates the principle that the development of the powers of imagination, abstraction, and reasoning should be made continuously from experience and knowledge gained in daily life. To construct a preliminary course which consists of work concerning angles, lines, triangles, and circles with, perhaps, a passing reference to a few surveying problems, is to place the child suddenly in a new world where things are replaced by abstractions and to give him an occasional glimpse of his own sphere as from behind bars—bars which are not made thinner by assigning numerical measures to the lengths and angles with which he deals. In the alternative which I have suggested the endeavour is to lead him gradually to this new world of abstract thought and ideal truth, or, perhaps, to present it as an outcome of and one with that more limited world of which alone he is at first cognisant.

The result of this period of freedom has been summed up in a Circular published by the Board of Education (No. 711, 1909), which contains luminous and practical suggestions to teachers, based on the experience of the Board's inspectors. For our present purpose the most striking statement made therein is that the time taken to acquire the matter of Euclid I.—III. varies from one to three years in different schools, and that it is where the work proceeds quickly that it is best, and nearly always where it proceeds slowly it is poor. The difference is ascribed to the manner of treating the earlier part of the work, with which the Circular is mainly concerned. As to this, I will only say that, while its suggestions are far in advance of the matter contained in most text-books, it perhaps hardly lays sufficient stress on the need of development from the child's previous experience, nor does it suggest concrete problems requiring a considerable amount of imagination and reasoning. The importance of Solid Geometry is pointed out with some force, but no very definite suggestion is made as to the time or manner of its introduction in a deductive course, a point on which most teachers are in need of guidance, and especially those who now succeed in covering a matriculation course in two or three years.

The Circular also deals with the general effect of the changes, stating that it has been beneficial. "Unintelligent learning by rote has practically disappeared, and classes, for the most part, understand what they are doing, though they often lack power of insight, and have but a narrow extent of knowledge."

Had the changes already described been the only educational changes during this period, a fairly conclusive inference might be drawn. But it must not be forgotten that the modern secondary school, with graduates of teaching Universities for its teachers (often trained) and a curriculum designed to develop all the pupils rather than to benefit the few of exceptional ability, has during the same time come into being, and the issue is thus confused. Greater comprehension, as due to improved teaching, would have been likely even though Euclid had not been dethroned. The older schools have for the most part been comparatively unaffected by such developments during this time. Speaking as one who has some experience, both as teacher and examiner, of these and other schools, I can but state my opinion that the improvement in the modern secondary schools is far greater than in the schools of other types, including those which have adopted the changes in Geometry most fully. Devoting roughly an equal amount of time to the subject, they obtain better results at an earlier age. If I am right in this—and I believe that most men who have acquaintance with the various types of school in this country will confirm the statement—it follows that most of the recent improvement in modern secondary schools is due to their acquisition of teachers who not only understand the subject but also know how to teach it, rather than to any recent changes in the syllabus of Geometry. Some confirmation of this view is given by an enquiry made some three years ago among the professors and examiners in the modern Universities. They declared themselves as dissatisfied alike with the results of the older and more modern methods, the majority against the modern methods being the larger. The improvement in quality of knowledge was admitted in many cases; it was rather the material and its co-ordination that were criticised.

Finally then, it may be said that improvement in teaching methods and in personnel of the teachers have produced their natural results, and to them must be ascribed much of the admitted improvement. Schemes of geometrical education in this country are lacking in foundation, method and extent, and this because Euclid's scheme—itsself utterly unsuitable as an introduction to the subject—has been so far tampered with that hardly any scheme remains. So long as no attempt is made to devise a connected development based on the many intuitions which are common to all civilised beings before they reach maturity, so long will the subject realise a painfully small proportion of its potential value.

I have endeavoured in this paper to interpret the quotation at its head in its reference to my subject. I do not forget that children will have to do the work of the world and must be fitted for it, and I believe that a training such as is here described will assist them in this. But they will not be less

fitted if their education provides them with widening and inspiring subjects for contemplation when they reach maturity, nor, indeed, is such fitness the sole end of life.

G. ST. L. CARSON.

PARALLEL STRAIGHT LINES AND THE METHOD OF DIRECTION.*

The absence of any direct contact in England between the critical movement of higher Pure Mathematics and the more recent movement for the reform of geometrical teaching in schools has not been altogether favourable to the latter. In no instance has this unfavourable result appeared more widely, and more dangerously to both real reform and true education, than in the various attempts made to replace Euclid's treatment of parallels. So long as such attempts were put forward privately, private criticism seems to have been sufficient on the whole to preserve the reform movement from serious errors; but in England the conditions are no longer private, for in March 1909, the Board of Education issued, under the title "Teaching of Geometry and Graphic Algebra in Secondary Schools,"† a Circular (No. 711), which contains an attempt to base the treatment of parallels on the notion of direction, and thus openly encourages teachers to adopt a method which has been often and rightly condemned.

The exact words of Circular 711 (pp. 3 and 4) are as follows:—

"The next concept, direction, is seldom as well treated, with the result that trouble arises later on with angles and parallels.

"It is impossible to discuss, much more to define, direction in the abstract—just as it would be to discuss colour.

"But just as children gain the general idea of colour from recognising and naming colours—red, blue, green, so they gain the idea of direction (and gain it accurately) by recognising and naming certain standard directions—vertical, north, south, etc.

"At the outset it is essential to keep away from pencil and paper, and the introductory question should be 'Show me a vertical line?' then 'How would you test whether it is really vertical or not?' so introducing the familiar knowledge of the plumb-line. Then "Show me a horizontal line." 'How would you test it?' The test must be independent of the vertical, i.e., the spirit level must be suggested.

"Then, to clear things up further, 'Can you draw a vertical line on the wall, on the desk, on the floor?' and similarly with horizontal lines.

"Then 'How many vertical lines can be drawn through a point?' 'How many horizontal?' so passing on to the different horizontal directions (north, south, east, and west). Boys should know how the windows of their room face and how they themselves face as they sit.

"Then the master may ask, 'Are all vertical lines in the same direction?' 'Are all horizontal lines?' thus leading to a clear conception of parallels as lines in the same direction."

* NOTE.—Owing to unforeseen causes the author was unable to complete this paper in the time intended; and the date at which it was in fact received has rendered it impossible for the Advisory Committee to obtain a contribution discussing the question of Parallels, from another standpoint.

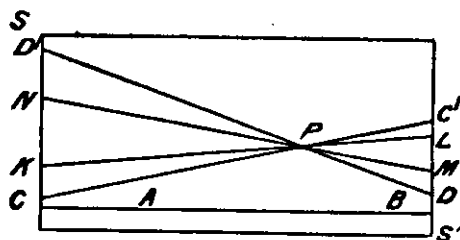
† Board of Education, Circular 711, March 1909. (London: Wyman and Sons.)

A denial may at once be given to the statement of the last paragraph; for it is impossible to give anyone a *clear* conception of parallels as lines in the same direction. Young boys may readily acquire a clear conception of direction, as, for instance, illustrated by the different directions given by the magnetic compass—"the Points of the Compass"—which correspond to the different directions given by straight lines which intersect at a common point. From this clear notion, it is obvious that lines which coincide have the same direction, and lines which intersect have different directions. Neither Circular 711 nor any author before or since its appearance, suggests any possible intuitive way of leading from this clear notion of the different directions of intersecting lines, and of the same direction of coincident lines, to any notion of non-secant lines as lines in the same direction. That there is no logical basis for this attempt has been shown often, and evidence of this is quoted later.

The appeal made to intuition in the Circular is the basis of its suggestion; and the questions referring to horizontal, vertical, plumb-line, spirit-level, imply a reference to physical geography. In these days of correlation in the teaching of the various subjects of school education, is it possible to avoid teaching that plumb-lines point towards the centre of the earth? Is a clear notion of parallels possible to a boy, who is told in the geometry lesson that vertical lines are in the same direction and so are parallel, but who, half an hour later, in a geography or practical mechanics lesson, learns that vertical lines all point towards the same point—the centre of the earth? Will such a boy always avoid thinking that straight lines pointing towards, and so passing through, the same point are in the same direction? The comparison of the spherical surface of an actual ball or globe with the surface of a large blackboard, and in imagination of the earth's surface with an ideal plane surface, is of real educational value to all pupils; but it makes an appeal to geometrical perception quite fatal to the course of procedure advocated in Circular 711. On the surface of a sphere there can, of course, be no straight lines; if great circles (the shortest sailing routes for ocean steamships) are taken as elements replacing straight lines, then there are no parallels. It is a wise step in geometrical teaching to educate geometrical perception to this point at least, that pupils may realise that the existence of a parallel line, whether established by assumption or deductive proof, involves the admission of a new kind of surface where the notions of vertical and horizontal no longer necessarily apply. The failure to make this comparison and contrast between spherical and plane surfaces may long retard the growth of sound geometrical instinct.

Again it is quite necessary in dealing with parallels to make quite sure that some assumptions must be made as a basis for any discussion or proof. As a practical illustration of the

plausibility of the possible untruth (*i.e.*, lack of generality) of Playfair's axiom, we suggest the following:—



Let SS' be a finite plane surface, of any size. AB a straight line drawn on it, and P a given point. It is always possible to draw as many straight lines as desired passing through P , but not cutting AB within the limits of the given surface. Produce AB to the boundary of the surface. Choose points C, D on the boundary nearer P . Join CP, DP , and produce to C', D' respectively. Then as many straight lines as desired, *e.g.*, KPL, MPN , may be drawn through P , not cutting AB within the given surface. A similar construction is obviously possible, however large SS' may be, so long as it remains finite.

The restrictive aspect of Euclid's fifth postulate (or Playfair's axiom) may be illustrated by reference to suitable rules in well-known games, *e.g.*, to the offside rule in Association football, which prohibits play after a forward pass under certain conditions, and so has the ultimate effect of limiting the shots between the posts, finally counted as goals, to those scored in a particular way, and of not considering the others. The game was formerly played without this offside rule, when such successful shots did count as goals. Similarly, Euclid's postulate restricts further discussion on the subject of two parallels, to one only of the logically infinite number of straight lines through P , non-secant to AB ; or, in other words, the parallel to AB through P must be conceived as drawn in a particular way, and the other non-secant lines are not considered. In the Geometry of Lobachewski and of Bolyai such non-secant lines are considered in dealing with parallels.

The notion of straight lines which do not intersect may rightly and ought to be derived from sensory experience; but a clear concept of parallels is impossible without comparison and analysis of such geometric perceptions, and without some appeal to infinity.

Even as a tentative method of approach for the youngest pupils, this suggestion of Circular 711 seems without definite value; for no simple practical test is given to determine when any two straight lines are in the same direction. Indeed, no one has succeeded in stating a criterion for sameness of direction without introducing (usually by suppressed assumption) some property of parallels already known, because proved by Euclid.

Early History.—The controversy with regard to parallels is at least as old as Aristotle; possibly even this very fallacy of “direction” is referred to in the passage*: “for they unconsciously assume such things as it is not possible to demonstrate if parallels do not exist.” In later times, according to Heath, Killing has traced the origin of the direction theory to Leibniz. However, there is no doubt that Gauss brought his great influence to bear against the fallacy. In the extracts from the correspondence between Gauss and Schumacher, edited by Houel, no specific mention of the “direction method” occurs, although Gauss enters minutely into the other unexpressed assumptions of Schumacher. But this passage from Heath’s Euclid† seems to be decisive:

“The idea of parallels being in the same direction perhaps arose from the conception of an angle as a difference of direction (the hollowness of which has already been exposed); sameness of direction for parallels follows from the same ‘difference of direction’ which both exhibit relatively to a third line.”

But this is not enough. As Gauss said (Werke, iv., p. 365):

“If it [identity of direction] is recognised by the equality of the angles formed with *one* third straight line, we do not yet know without an antecedent proof whether this same equality will also be found in the angles formed with a *fourth* straight line,”

(and any number of other transversals); and in order to make this theory of parallels valid, so far from getting rid of axioms such as Euclid’s, you would have to assume as an axiom what is much less axiomatic, namely, that—

“straight lines which make equal corresponding angles with a certain transversal do so with any transversal.”‡

De Morgan.—The opinion of this great teacher was adverse to the direction method, and was strongly expressed in a review (“Athenæum,” July 18, 1868) of Wilson’s Geometry, 1868. Large extracts from this review were reprinted in Appendix II. of Dodgson’s “Euclid and his Modern Rivals”; and from this source we quote the following:—

“ . . . There is in it one great point, which brings down all the rest if it fall, . . . That point is the treatment of the angle, which amounts to this, that certain notions about direction, taken as self-evident, are permitted to make all about angles, parallels and all, immediate consequences.

“What ‘direction’ is we are not told, except that ‘straight lines which meet have different directions.’ Is a direction a magnitude? Is one direction greater than another? We should suppose so; for an angle, a magnitude, a thing which is to be halved and quartered, is the ‘difference of the direction’ of ‘two straight lines that meet one another.’ . . .

“Parallels, of course, are lines which have the same direction. It is stated, as an immediate consequence, that two lines which meet

* Taken from Heath’s Euclid, Vol. I., p. 191.

† *Op. cit.*, p. 194.

‡ Dodgson: “Euclid and his Modern Rivals,” p. 128. (Macmillan & Co., 1885.)

cannot make the same angle with a third line, on the same side, for they are in different directions. Parallels are knocked over in a trice. There is a covert notion of direction, which, though only defined with reference to lines which meet, is straightway transferred to lines which do not. According to the definition, direction is a relation of lines which *do* meet, and yet lines which have the same direction can be lines which *never* meet . . . How do you know, we ask, that lines which have the same direction never meet? Answer—lines which meet have different directions. We know they have; but how do we know that, under the definition given, the relation called *direction* has any application at all to lines which never meet?"

De Morgan's criticism was never overcome, and in later editions of Wilson's *Geometry*, the direction-method was dropped, presumably because it is unsatisfactory and erroneous.

*The Association for the Improvement of Geometrical Teaching.**
—At the general meeting of this Association, held January 17th, 1891, Mr. E. T. Dixon gave a short account† of his book on the "Foundations of Geometry," which introduced the direction-method in a form modified to meet some of the objections raised by De Morgan. There was a discussion in which the author failed to meet the just criticism of the chairman (the late R. B. Hayward, F.R.S.). The chairman said: "He nowhere found it defined when the direction from A to B was the same as that from C to D. What was meant in this case by 'in the same direction'?" The method failed, just as Circular 711 fails, to provide any criterion, practical or logical, of the notion of "same direction." Not only are we asked to accept "direction" as an undefinable, but also "same direction" may be used without explicit statement as another undefinable, whenever convenient!

United States.—We are indebted to the Bureau of Education, Washington, for a copy of Circular No. 3, 1890, on the Teaching and History of Mathematics in the United States, by Prof. Cajori. In the section "On Parallel Lines and Allied Subjects," p. 376, we are given a criticism of the many erroneous attempts made to replace Euclid's treatment of Parallels. In particular, on pages 382, 383, we have an able examination of the direction-method which results in a conclusion quite adverse to its adoption in teaching. From p. 383 we quote the following:—

"One of the many objections to all attempts to found the elements of geometry on the word 'direction' is stated by Professor Halstead (in a letter to Prof. Cajori) in the following manner:—Direction is a common English word, and in Webster's Dictionary, our standard, the only definition of it in a sense at all mathematical is the fourth: 'the line or course upon which anything is moving . . . ; as, the ship sailed in a south-easterly *direction*.' Direction, to be understood in any strict sense whatever, posits and presupposes three fundamental geometric ideas, namely, straight line, angle, parallels. After the theory of parallels founded upon an explicit

* Now the Mathematical Association.

† Association for the Improvement of Geometrical Teaching, Report for 1891, p. 22.

assumption has been carefully established, a strict definition of direction may be based upon these more simple concepts, and we may use it as Rowan Hamilton does in his *Quaternions*."

If additional evidence is required, reference may be made to Dodgson's "*Euclid and his Modern Rivals*"; pages 97-157 contain detailed criticisms of three Geometries by Wilson, Pierce, and Willock respectively, which employ the direction-method for parallels. No answer to these criticisms can be found.

Though the subject of this paper is distinctly narrow and special, the author begs all teachers of geometry to give it their earnest consideration. Nor do we hesitate to appeal to mathematicians of the highest rank to give some consideration to this discussion, for Gauss has by his example led the way for all. The question at issue is not the old one, now settled, of the retention of Euclid's *Elements*, for it is admitted generally that this work is not suitable for beginners in geometry. But the important principle at issue is whether a fundamental part of geometry should be introduced in such a way as to confuse the later logical development. There is a widespread feeling that it would be wise in elementary teaching to increase the number of explicit assumptions; agreement with this feeling does not require retention of the direction-method, for the explicit assumptions, built upon suitable illustrations and practical work, may surely be arranged otherwise. In this respect we may call attention to the methods advocated by Méray, Poincaré, Hadamard, Borel, and others in France, with the intention of using a method for parallels which in an elementary way is a first introduction to the notion of a group.

Borel's *Géométrie* contains an account prepared for young beginners; while Hadamard's *Leçons de Géométrie*, in Note B (p. 278), *Sur le Postulatum d'Euclide*, contains a reasoned discussion of the method in its relation to non-Euclidean geometry. But we mention this method merely for illustrating a point in the argument of the paper—that any method adopted must provide for the whole of Euclid's treatment of parallels, while leaving it possible to refer without contradiction to the work of Lobachewski, Bolyai, Gauss, Beltrami, and others in a wider sphere of work.

T. JAMES GARSTANG.

THE ALGEBRA SYLLABUS IN THE SECONDARY SCHOOL

I.—INTRODUCTORY.

The object of the present paper is to examine the aims proper to the teaching of algebra at school and to discuss what subject-matter should be comprised in the course in order to attain the aims in view.

As a preliminary step we must adopt some rough classification of the pupils who study algebra; that which follows is quoted from a recent report of the Mathematical Association,* a report that may be read in connection with the subject of this paper.

“It is possible, and in many secondary schools customary, to divide mathematical pupils broadly into a few main classes:—

- “I. Boys reading for mathematical honours† and examinations of similar standard.
- “II. Boys looking forward to an engineering career, army candidates, and others who need mathematics as one of the main subjects of their education.
- “III. Boys studying mathematics as part of a general education—an education which University entrance examinations and other examinations of similar standard are designed to test.”

Classes I. and II. may conveniently be described as specialists, Class III. as non-specialists.

It may be assumed that specialists form an important minority among boys, and a minority numerically insignificant among girls.

One of the main contentions of the present paper is that the course of algebra teaching usual in this country sacrifices the interests of the non-specialist to those of the specialist; further, certain remedies for this state of affairs are suggested. But let it be granted at the outset that no remedy is satisfactory which reverses the alleged injustice, and sacrifices the interests of the specialist to those of the non-specialist. This consideration must be and has been kept steadily in view.

* Report of Mathematical Association Committee on the Teaching of Elementary Algebra and Numerical Trigonometry, 1911. (London: G. Bell & Sons. 3d.)

† *i.e.* With a view to the study of Higher Mathematics at a University.

When the interests of two groups of students appear to diverge, the first remedy to hand is to separate the groups and teach them in separate classes. This course is more feasible in an English school than in the schools of many foreign countries, inasmuch as the practice of providing different courses for different classes of students in a single school is very familiar in this country. But this is anything but an ideal method. In the first place it is not easy to distinguish future specialists from future non-specialists at an early age: bifurcation cannot well be resorted to before the age of 16 or so. Secondly, it complicates organisation, and tends to impair the solidarity of a school.

A better way is to devise some curriculum which will carry all students together up to a certain stage. This elementary course must be designed to occupy the attention of the average non-specialist up to the close of his mathematical education at school.* Further, it must be a suitable introduction to a subsequent specialist course, in order that the specialist student may pass normally through the elementary classes studying side by side with the other students. The specialist will, as a rule, be endowed with some degree of mathematical ability, and will pass more or less rapidly through the elementary classes,† emerging finally from the highest elementary class with a year or more of his school life in hand to devote to special study of mathematics. This scheme is in accordance with the system of English schools, and corresponds closely to the arrangement very generally adopted; the only modification needed is to frame an elementary course that shall be for the non-specialist a fairly self-contained and harmonious whole, while serving as a suitable basis for a specialist superstructure.

Mathematics appears to hold in some respects a favoured position in the classical schools of this country; the time given to the subject in English classical schools is in excess of that given in the corresponding schools in France or Germany. The reason of the continued existence of mathematical teaching in the classical departments of the great schools is partly to be found in the fact that the Universities of Oxford and Cambridge demand elementary mathematics as a necessary preliminary to a degree; so long as they continue to make this demand, public schools must comply with it.

It is not altogether certain, however, that the demand will continue to be made. When the two ancient Universities cease to require Greek of students in mathematics and science (and

* In English schools classical specialists often drop mathematics for the last two years of their school-life. It is hoped that this tendency may be arrested by the adoption of such a syllabus as that now proposed.

† Foreign readers will understand that no attempt is made in most English schools to carry all boys of the same age through the same course; the principle is rather that each boy progresses at the rate determined by his ability and industry.

the time cannot be far distant), it is by no means impossible that they will cease to require mathematics of classical students. It is doubtful if compulsory mathematics rests, at present, on a consensus of educated opinion that mathematics is a necessary part of a liberal education. Among the headmasters of public schools there are probably many who would deny that mathematics, as taught in their schooldays, was of perceptible value educationally, though no doubt there are many who hold the contrary opinion very strongly.

It appears to be a matter of pressing importance that this question of the non-specialist mathematical curriculum shall be taken in hand at once. What we need is a curriculum of admitted educational value, a value that shall be admitted not merely by mathematicians, but by the great body of educated men. This is not a question that can be settled by mathematicians alone, in a high-handed way. We must carry public opinion with us; we have to convince people that the curriculum can, and will, be recast in such a form as to provide an indispensable element in education. If we can do this, mathematics will no longer be dependent upon the protection afforded by a precarious University regulation.

There can be little question that, if the compulsion applied by the Universities were removed to-morrow, the time given to mathematics at many public schools would tend gradually to be reduced to a vanishing point. Public schools are governed, in the main, by classical men, and many classical men regard the time they spent over mathematics as time wasted. "Many of us still smart with indignation at the hours which we were compelled to spend in learning by heart the first two books of Euclid, and though he has since been deposed from the eminence which we always grudged him, the tyranny is by no means overpast. We are repeatedly told that there are young scientists whose early life was blighted by having to read the Apology in a crib, but why is no sympathy ever expended on those upon whose early years there fell the transient but blighting shadow of $x + y$?"* This is the opinion of many able men of literary tastes. That such an opinion can be held widely points to the conclusion that "something must be done."

We who believe in the value of mathematical education have not the slightest doubt that something can be done, and we have clear ideas as to what this something should be. The average boy is perfectly ready to take an interest in mathematics if it is taught with some appeal to the imagination, with free use of intuition, and above all, if he is allowed to see *that it is leading him somewhere*. In the system of teaching that awakens such bitter memories for the headmaster of Shrewsbury, the imagina-

* Letter to the "Times," December 29, 1910, by the Headmaster of Shrewsbury School.

tion was never touched, intuition was proscribed as destructive of "logical training," and the pupil never had the sense of advancing to new discoveries and applications, for indeed he was not allowed to advance till he was perfect in algebraic manipulation, and to this perfection he could never attain.

If we are to justify the time given to mathematics, we must manage in some way or other to introduce boys to an altogether wider range of ideas than is now opened to them. Every boy who is not mentally deficient ought, by the time he leaves school at 19, to have a working knowledge of numerical trigonometry, and a grasp of the simplest principles of mechanics taught experimentally. Those of us who are optimistic by nature would go further, and would urge that the fundamental ideas of the infinitesimal calculus should not be withheld from the average boy. There is no originality in this view; these ideas enter already into the official curricula for non-specialists in France and Austria. But we are a nation of little faith, and the mere suggestion of revealing these mysteries to the common boy is enough to alarm the Tories of education.

To begin with, then, let it be made clear that I am not proposing to set boys down to struggle with the ordinary books on the calculus, on which the present generation of teachers were brought up. What the detailed proposals are will be explained at a later stage in the argument; but let us for a moment look at the matter in a broad light.

The infinitesimal calculus has now been before the world for two and a half centuries. It is the fundamental form in which mathematics is applied to the affairs of modern life. We must recognise as a law of development in educational affairs that matter which in one century occupies the attention of the foremost philosophers finds its way in a subsequent century into the elementary curriculum. If the infinitesimal calculus was the high-water mark for the 17th century, so was the Euclid's geometry to the 3rd century B.C., the Arabic notation (in Europe) for the 12th century A.D., the method of long division for the 15th, and so forth.

The quality of the human brain does not alter, presumably, from one century to another; how then has it been possible to make the school boy of one generation assimilate matter that has puzzled the best brains of earlier times? * The answer is simply this: teachers come to know more, simpler methods of presentation are discovered, and a clearer view is attained of how the curriculum can be disencumbered of the obsolete and the unessential. No one who has taken the trouble to acquaint himself with the world-movement in education will doubt that one of the tasks of the 20th century is to find a way of import-

* Wallis, Newton's forerunner, was very doubtful whether he might write $\sqrt{8}$ as $2\sqrt{2}$.

ing the notions of the infinitesimal calculus into the ordinary school curriculum.

The present paper is a contribution to the question—how may the curriculum be disencumbered so as to make room for the new matter? The process of disencumbering has already made good headway in geometry; England has at last followed the rest of the world and dispensed with the unessential parts of Euclid, and at the same time of necessity has abandoned the particular sequence adopted by Euclid. In arithmetic much remains to be done. But what I am specially concerned to inquire is—how may the algebra course be purged of superfluities and irrelevancies? At present it is like a sinuous river, the navigation of which is impeded with shoals and sandbanks. We have to clear away the sandbanks and straighten the course of the river; by such means only shall we make a serviceable waterway to the rich country that lies beyond.

Algebra grows out of arithmetic and leads on to the calculus. The natural course of study then will be determined partly by the mode in which it grows and partly by the object to which it tends. If we can determine this natural course by considering the origin and the object, we shall then be in a position to see what is essential and what is superfluous. This determination will occupy us in the next section.

II.—ALGEBRA IN THE SECONDARY SCHOOL CURRICULUM.

The opening years of the 20th century revealed a very healthy amount of commotion and disagreement in the educational world; and, in particular, in that portion of it inhabited by mathematicians and those who depend on the work of mathematicians. The old ideals of the mathematical teacher seemed no longer to fit the times. The mathematical teacher was no longer allowed to go on ploughing his lonely furrow unregarded; other people—engineers, physicists, chemists—were beginning to take an interest in that furrow and the crop it produced. This was, in a way, encouraging for the mathematician; and, in a way, embarrassing. So long as no one watched the way he taught or cared much what came of it, the mathematician might choose his own methods and ideals; and these ideals were expressed broadly by the words "mental discipline." But the engineers, physicists, &c., without having very strong views about the mental discipline question, wanted something of more immediate applicability to their own problems.

Here were the makings of a very pretty dispute between two sets of teachers. Happily the dispute never defined itself in this way. The mathematicians had been leavened with a wider view of their duties; a majority of them saw their way to accept and even to press for modifications of the curriculum, and were satisfied that they could move with the times without violating their "mental discipline" axioms.

It is not proposed to give any detailed description here of the nature of the changes introduced ; it will be enough to note that, in so far as they concerned elementary work, they applied mainly to geometry and arithmetic ; and that there was a strong movement in favour of breaking down the barriers between the various mathematical subjects.

Such changes as found their way into the algebra course were to a great extent beneficial, but did not appear to arise from any clearly defined principle. It is true that the introduction of "graphs" into algebra work was accomplished ; but even in this case it is doubtful if those who followed the fashion were agreed as to the object of graph work in algebra ; and in the absence of an accepted guiding principle, graph work led to various fantastic and ill-balanced developments.

In attempting to define the educational properties of algebra, and the aims that should control the selection and presentation of material, I am not disposed to lay any great stress upon particular mental faculties that algebra may be presumed to develop. I am inclined to think that our knowledge of the psychology of education is hardly advanced enough to make any such line of argument very satisfactory. No doubt, algebra is well suited to provide practice in clear thinking, concentration, and so forth ; but so, we hope, are other parts of the curriculum.

We are, I believe, on firmer ground if, from a school point of view, we characterize algebra as an essential mode of thought, a useful instrument of research, and an outlook subject. An essential mode of thought—I shall try to show how algebra grows necessarily out of arithmetic ; a useful instrument of research—this aspect will provide much of the stimulus to the pupil ; an outlook subject—the outlook consisting partly in an appreciation of the value of this instrument in modern life, partly in a certain mature habit or complexion of thought in regard to number and quantity. This habit or complexion of thought will contain such elements as the idea of generalised arithmetical results and the idea of functionality. Such ideas will gradually and increasingly colour the mental outlook as time goes on ; they will not dawn in one lesson or in ten, but are the work of years.

One thing at any rate is clear ; the appeal that algebra may make to the schoolboy will not be identical with the appeal it makes to the mature mathematician. To the latter, the interest may be connected with the task of basing a science on the necessary and sufficient fundamental assumptions ; or again, the interest may be that which, for our present purpose, is sufficiently suggested by the word "algebras." "Algebras" are not for schoolboys ; equally I am convinced that a scientific deduction of algebra from stated fundamentals is work for the university rather than the school. These preoccupations, historically, come late in the growth of mathematical thought ; they are an afterthought. For the schoolboy the question will

be—to what use can I put this new piece of algebra I am now studying? Or rather, the use must precede and suggest the need of the piece of algebra. He must continually be confronted with situations that demand an enlargement of his arsenal of operations. The motive, in fact, will not be scientific, theoretical or deductive; it will be, in a broad sense, utilitarian.

Algebra is essentially a more abstract and mathematical subject than either arithmetic or geometry. It *may* be brought into touch with real life; but on the other hand it may be immeasurably remote. At times one is tempted to wonder how algebra of the remote type can ever have found a place in a co-ordinated system of secondary education. But perhaps it is not cynical to say that the system of secondary education with which we are most familiar is not a co-ordinated system; we may also admit that it was not generally demanded in this country during the 19th century that the subject-matter of education should be in touch with real life.

An attractive method in discussing educational problems is to lay down some general principle, *e.g.*, “the object of education is to correlate the individual with his environment and the age in which he lives,” and to deduce the practical conclusions. But this method of reasoning does not commend itself to English habits of thought, which are suspicious of general principles, and prefer as a rule to test principles in the light of the conclusions to which they lead. I shall therefore not entrust my argument, at the outset, to the dictation of a set of axioms. Perhaps this fear of axioms is inconsistent with the complaint expressed above that the algebra curriculum was not based upon clear principles. How far this is a real inconsistency must appear in the sequel.

The question—should algebra enter into the secondary curriculum?—is almost an academic question; for algebra, in some form or other, can hardly be kept out. The use of letters for numbers is a mental economy, a mental shorthand, of which we avail ourselves naturally. It is worth while to examine in detail how this economy suggests itself.

If history is to be trusted, the desire to solve arithmetical problems is a desire, almost an instinct, that arises early in the development of thought. Whether or no it can be described as an instinct, it resembles some other instincts in having a definite blossoming time, after which it seems to die away. Mature mathematicians are not specially interested in ingenious and tricky problems, though in some cases (*e.g.*, “Lewis Carroll”) the taste is preserved and developed by practice. But philosophers in the dawn of mathematics, Delian oracles, and modern schoolboys undoubtedly delight in numerical problems. The dullest class of 14-year-olds are roused to animation by a think-of-a-number problem. The instinct seems to be purely disinterested and non-utilitarian; these “diverting questions” seldom have any actual relation to the pursuit of bread and

butter. Perhaps there have been jealous husbands ever since the institution of matrimony; but they have never had occasion to make practical use of the classical problem of the "three jealous husbands." Problem-solving is a naïve form of mental game to which the mind betakes itself naturally at a certain stage, very much as the body betakes itself to ball games or skipping. We will assume that the instinct in each case is a correct instinct, and makes for the development of mind or body. Arithmetical problem-solving does *something* for our cerebral convolutions at a certain stage.*

Amid the uncertainties of educational theory, we are fortunate if we can find something like an instinct on which to build. As the teaching of English may be based upon the instinct to describe things seen, so the teaching of algebra may be based upon the instinct to propose and solve diverting problems. The passage from arithmetical problems to algebra is familiar; the problem-setter soon gets the upper hand of the problem-solver, as soon in fact as he begins to set inverse problems. The game now becomes too hard, and the defence is broken down till it adapts a new weapon, viz. x .

It would be interesting, though fairly obvious, to trace how the problem opening will lead on to equations, notation, &c. But this would lead us from the present task, which is to justify the statement that the use of letters for numbers is a step that the human mind insists on taking.

The problem approach is one avenue by which our steps are led from arithmetic to algebra. Now for another approach.

In any conceivable system of arithmetic teaching the pupil is introduced to certain generalisations, e.g., that the sum of a number of fractions of the same denominator is a fraction of the same denominator whose numerator is the sum of the numerators. This may be presented in a bad way, in the form of a rule, or in a good way, inductively. But in one way or another such generalisations will enter. The reader will see at once how they lead to algebra:—

$$\frac{3}{11} + \frac{4}{11} = \frac{7}{11}, \quad \frac{3}{d} + \frac{4}{d} = \frac{7}{d};$$

$$\frac{a}{11} + \frac{b}{11} = \frac{a+b}{11}, \quad \frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}.$$

The last-named identity expresses in a compendious form the same information as the generalisation quoted above. It expresses indeed a good deal more, for a , b , d , need not be positive integers. Should this trouble a teacher at this stage? In my opinion, no. The time for discussing the identity more

* Most arithmetic text-books reduce the whole business to dust and ashes by classifying problems and assigning an appropriate dodge to each class. Examiners play into the hands of authors by setting old types.

fully is when the pupil points out (or can be induced to point out) the difficulty; the degree of strictness in mathematical statement should not be far in advance of that which the pupil's mental development demands. This, however, is a side issue: the main question is whether this use of letters as mathematical shorthand, as a means of compendious expression, can be described as an unforced, spontaneous development.

To a trained mathematical mind it seems natural enough, but we have to put ourselves in the position of the learner.

The stage when a boy first learns that $\frac{3}{11} + \frac{4}{11} = \frac{7}{11}$ is probably not the stage for introducing him to symbolic generalisation; on the principle "one thing at a time" generalisation without symbols must come before generalisation with symbols. When the mathematical generalisation has been assimilated, an opportunity will be found to reveal the "shorthand" way of writing the rule. Experience shows that symbolism, introduced with discretion and at the psychological moment, is accepted without any mental jar.

Any algebraic identity embodies a general statement of an infinite set of arithmetical results; by means of algebraic identities the mathematician can pack into his brain an indefinite number of arithmetical generalisations. In fact, he can go further still in the simplification of his mental furniture; instead of carrying vast numbers of identities he can carry the creative principles of all identities, namely, the laws of algebra. Mathematics stands for economy of thought.

To return to the pedagogical problem, any teacher will readily think of other cases suitable for symbolical generalisation.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}; \frac{1}{x/y} = y/x.$$

$$x^m \times x^n = x^{m+n}; \sqrt{x^{2m}} = x^m.$$

All the elementary operations of algebra are instances of symbolic generalisation; probably this is the very best way of introducing these elementary operations to the student.

It has now been shown that primitive algebra is firmly rooted in arithmetic; and the two main roots we have traced are problem-solving and symbolic generalisation. At this stage of the argument I must state a guiding principle; the principle, namely, that the mathematical curriculum is to be viewed as an organic growth, each new subject or topic growing out of what has already been mastered. If algebra grows from arithmetic in the two modes suggested, this genesis will control the development of the subject in its earlier stages. As development proceeds, the subject will gain further motive power from its own inherent vitality, and, again, from subjects such as geometry and physics, with which it may be correlated.

And, lastly, just as a plant grows towards the light, so will the curriculum be influenced in its mode of growth by the attraction of the infinitesimal calculus. The desire to prepare for the calculus will guide the teacher's course. With this end in view, he will use every suitable opportunity of bringing in that aspect of mathematics which the Germans call *Funktionsdenken*—"function-thinking"—we have no accepted phrase to describe it. "Graphs" form one aspect of "function-thinking," "variation" another; let us agree to use the phrase "the idea of functionality."

I must entreat readers not to shy at the introduction of such an idea as functionality into elementary mathematics. The word is an awkward one, I admit, and this is a pity; but when once the word is swallowed the worst is over. There is no necessity to inflict the word "function" prematurely upon our pupils, though, if we look at it dispassionately, it does not seem much more difficult than "expression" or "co-efficient." Least of all shall we attempt to give a definition of function; most of us could not.

The fact is that we have been teaching functionality for years, whether we have realised it or not. Every schoolboy now learns to plot graphs; this is nothing but the study of functionality in its visible form. But, to get full value out of graph-plotting, we must make the boy realise that he is not only finding points in the curve $y = x^2$, but that he is *studying graphically the function x^2* ; y may sometimes be left out of the question with advantage. And again—we must emphasize not only the graphic but also the *cinematographic* aspect; we don't want the boy to think merely of a set of spots on the paper, and a nicely drawn curve—a pattern just sitting quietly on the paper with no life in it; we want him to think dynamically rather than statically; to think of the x as changing, or flowing, continuously, and the x^2 (or $\frac{1}{x}$, or whatever the function may be) flowing *consequently*; to move his pencil point along the curve, and watch the x and y waxing and waning according to the law of their functional relationship.

But indeed we ought to get away from paper and black-board altogether, and weave the idea of functionality into the pupil's conception of the outside world. This should be easy, for everything is in motion; and probably any two measurable things we like to name are connected functionally. At any rate, there is no difficulty in choosing undoubted cases of functionality from the objects around us. The barograph shows the pressure of the atmosphere as a function of the time; the algebraical form of the function is unknown; if it were known we could predict the weather. All infallible prediction, in fact, depends on a knowledge of the mathematical form of the controlling functions. A boy would like to know what function his height is of his age; but of this he must remain

ignorant. But we can tell him his expectation of life as a function of his age; not perhaps as an algebraic function, but rather as a function in the form of a table or a graph. The daily papers supply plenty of examples of this type. For algebraic functions we must resort to simpler matters. The distance you walk at 4 miles an hour is a function of the time (simple proportion; why not draw an occasional graph as well as work sums on this subject?); the time needed to finish a job is a function, *ceteris paribus*, of the number of men employed (inverse proportion); given the principal and rate of simple interest, the amount is a function of the time (neither simple nor inverse proportion, but a new kind of functional relation); the length of the mercury column in the thermometer is a simple function of the temperature; the angle turned through by the pointer of a weighing-machine is a function of the weight; the petrol consumption per level mile of an automobile is a function of the speed; the amount of cake you have left is a function of the amount you have eaten; the circumference and area of a circle are functions of the radius, &c.

We live in an atmosphere of functionality. When the study of physics is begun, we have to inquire what is the functional relationship between the length of the spiral spring and the suspended weight, between the pressure and volume of the enclosed gas, between the inclination of the plane and the force needed to support the weight, between the attraction and the distance apart of two magnetic poles; all these and many other opportunities arise out of correlating physics with mathematics *via* functionality. In a large number of simple cases the functional relationship is an instance of so-called "variation"; variation, in fact, is the technical term used to cover the functional relation $y = kx^n$. It is on the functional aspect of variation treated both graphically and algebraically that the stress should rest; too often text-books are silent on this point, and concerned mainly to find the value of k .

The main point throughout is to insist not only on the statical but the dynamical aspect of quantitative relationships. Arithmetic and algebra have in the past been too narrowly preoccupied with the statical aspect; the problem has always been to *determine values* satisfying given conditions. This, of course is a very necessary thing to do, and is never likely to fall into neglect. This determination of values I am calling "statical," in contrast to the "dynamical" discussion of the *relationships between variables* connected functionally. The latter type of work is difficult for the teacher, but this is true of all the best work in teaching. It needs judgment and discretion. It also needs careful preparation of the subject beforehand, though my own experience is that the happiest opportunities for a word in season have a way of occurring unexpectedly. In fact the word in season on the subject of functionality is very

preferable to the set lesson. No one wants to import into mathematical teaching the lecture method—the long-winded teacher and the passive class. This is quite unnecessary; anyone who tries the experiment will find that boys take very readily to the idea of functionality; they are soon on the look-out for functional relationships; the conception is so fundamentally inherent in human experience that it is easily developed and rendered articulate; the attainment of it seems to broaden the whole mental horizon.

It is needless to multiply instances of functional relations, for in the nature of things it is impossible to avoid such relations. It may be noted, however, that geometry (in which I include trigonometry) provides many good instances, especially if we make free use of "motion" in our geometry teaching. The arc of a given circle is a function of the angle of the centre. I was dealing with this point on one occasion in connection with variation, when a boy (of 14) raised the question: What function of the angle is the *chord*, or as he put it, "What is the formula for the chord" (I had not used the word "function"). This struck me as encouraging, for it showed that he had assimilated the root-idea of functionality. It suggested a refreshing variant of the stock question: Given the angle, find the chord.

The view that the idea of functionality should form a main connecting-thread of the mathematical curriculum is a view very widely held at the present day. In the French official programme for secondary schools the following forms part of the work set down for boys of 16–17, who are later on to specialize in mathematics or science; "*Classe de Seconde, C et D*" (they do only five hours a week of mathematics). "Notion de la dérivée; signification géométrique de la dérivée. "Le signe de la dérivée indique le sens de la variation; applications à des exemples numériques très simples et en particulier aux fonctions étudiées précédemment." These boys are to acquire the fundamental notions of the calculus, and have been prepared for it by a study of functionality in preceding years; remember that they have never given more than five hours a week to mathematics, and in most years only three or four hours. From the official programme it also appears that specialists in linguistic studies may study the elements of the calculus, though naturally at a later stage of their school career than the specialists in mathematics and science.

In the new Austrian programme of 1909 for the gymnasium (a classical secondary school),* the idea of functionality is named explicitly as the central idea of the mathematical course. Boys

* A translation has been published by the Board of Education, under the title of "Syllabus of Mathematics for the Austrian Gymnasium." (Board of Education. Educational Pamphlets, No. 22.) London: 1910.

of 12 are to study "magnitudes directly and inversely proportional *as the simplest introduction to functional thinking.*"* At 13 "further allusion to functional thinking; variation of linear dimensions, surfaces and volumes of similar figures and solids according to first, second, and third powers of determining lengths, variation of linear dimensions according to square root of surfaces; to cube root of volumes." More work of this type from 14 to 16. At 17 the differential coefficient. In the "Remarks" appended to the programme further stress is laid on the idea of functionality.

In the present programme for Prussian secondary schools it is directed that the pupils are to acquire a thorough familiarity with the idea of function. It does not seem, however, that the spirit of this general instruction is realised in the details of the programme; there is a strong movement for the recasting of the programme, headed by Professor F. Klein, perhaps the most prominent among the leading mathematicians of our day who have devoted themselves to the improvement of mathematical education. Klein says,† "The elements of the infinitesimal calculus, treated properly, provide a far more suitable material for mathematical education at school than that heterogeneous and lifeless subject-matter which now-a-days is so repellent to all boys who have no definite turn for mathematics. The calculus would automatically arise from a fundamental and fruitful treatment of the idea of function, an idea admittedly of high importance and deserving of a central position. Further, the calculus is indispensable for a clear comprehension of numerous physical phenomena; and, from the point of view of mental training, is an essential element of mathematical education."

III.—DETAILS OF A NON-SPECIALIST CURRICULUM.

Before examining in detail how the algebra curriculum may be relieved of excrescences, I must deal with the cause that in my opinion is at the root of the excessive demands that algebra now makes on the time available for mathematical study. It is, I am afraid, a fact that in this country the main preoccupation of teachers is to impart to their pupils a high degree of mechanical manipulative dexterity in handling algebraic expressions. Now this statement does not imply that teachers try to make boys manipulate algebraic expressions without understanding what they are doing. I am not speaking of ordinary bad teaching; but I am saying that a great number of very competent teachers are inspired by a wrong ideal. They want boys *to understand in order to manipulate correctly*; whereas

* Italics are mine.

† F. Klein and R. Schimmack.—Vorträge über den mathematischen Unterricht an den höheren Schulen, Teil I., p. 115. (Leipzig: B. G. Teubner, 1907.)

their ideal should be just reversed. The ultimate aim should be, not manipulation, but understanding and outlook.

English education is dominated by examinations. Examiners cannot test outlook and they can test understanding only by testing manipulation; teachers have to supply what examiners demand; it is not then surprising that many teachers have mistaken the means for the end. A vast amount of time is spent on purely mechanical work; highest common factor, fractions and factors beyond the types needed for practical purposes, needlessly heavy equations; together with all sorts of artifices and elegancies, which are to the average boy as pearls to swine.

If a boy is certainly destined for a career in which he will be bound actually to make use of mathematical manipulation, a case might possibly be made out for drilling him at school to a high degree of dexterity in the technique of algebra; just as a student who aspires to be a professional pianist must devote an astonishing number of hours to the technique of playing the piano. But, for the moment, we are considering the case of the general student of mathematics, the non-specialist. We may assume that the average man, not connected with any mathematical or scientific profession, finds practically no occasion in the affairs of life to enter into the details of an algebraic calculation; it is even more certain that, if such an opportunity presents itself to him exceptionally, the opening is declined, in spite of (perhaps because of) the heavy drill that has darkened his school days. On the other hand, those of us who believe in a mathematical training are profoundly convinced that such general mathematical ideas and modes of thought as may be wrought into the mind by a suitable course of instruction are an asset of permanent cultural value. It is these ideas and modes of thought that we regard as a necessary element of a liberal education; manipulative dexterity on the other hand we look upon as a specialised technical accomplishment. It is a specialised technical accomplishment just as much as dexterity in glass-blowing, or Latin verses, or machine-drawing, or playing the piano, or shorthand; all of them very excellent things in the right place, but by no means essential to a liberal education. If there were no other way of using mathematical time than in giving this technical skill, I should say at once: Cut off some time and give it to English, or Science, or something else of general cultural importance. But naturally I do not take this view; on the other hand I am persuaded that a drastic abatement of this juggling with algebraic symbols would free enough time to put every ordinary boy in possession of the fundamentals of trigonometry, mechanics, and the calculus. Mathematics might gradually become, for the nation at large, a thing of real significance; and we should no longer hear the leaders of educational thought speaking bitterly of "the transient but blighting shadow of $x + y$."

These views may at first meet with some opposition. Some people will say that I am proposing to make mathematics too easy by cutting out every difficulty that forces a boy to think ; I would ask these objectors to reserve their objections till I have discussed in detail the omissions I suggest. Others will fear that I am making mathematics too hard, by making it philosophical ; to these I would say that a philosophical discussion of the foundations of mathematics is the very last thing I would introduce.

The central objection is not so easy to meet, and that is to the following effect : Ideas and modes of thought are all very well, but what we value in mathematical education is rather the mental and moral discipline it affords. We think that such a discipline is gained from the present curriculum, and we are not at all sure that it would not be sacrificed by the changes proposed. The average boy needs a certain sameness, a certain amount of merely mechanical drudgery in his work ; it is a mistake to supply him continually with new ideas : he must have time for mental digestion.

This is the typical conservative attitude, and as such it is deserving of all respect. The answer is that the experience of those of us who have tried the new course is reassuring. No doubt a certain amount of mechanical drudgery is needed, and we find that the new course supplies it. But the drudgery is vivified by purpose and by a sense of progress. Mechanical drudgery at heavy algebraic fractions is not on the same footing as mechanical drudgery in solving a problem of trigonometrical surveying, whose data have been derived from actual observation. It is not for a moment proposed to eliminate manipulation from the course ; but the manipulative skill should be acquired not for its own sake, but *as a stepping-stone*. The test should be as follows : Each topic leads on to another, *e.g.*, multiplication leads on to factors, factors lead on to quadratics, &c. Multiplication, then, should be practised till the boy is sufficiently at home in it to verify his factors ; factors should be practised till they no longer present an impediment in the solving of quadratics, and so forth.

In reality, of course, the line of division is between the mental-and-moral-discipline teachers, and the school who, consciously or unconsciously, follow Herbart. This is not the place for, nor would I care to undertake, a discussion of this large question. The mental-and-moral-discipline school have been in possession of the field for a long time, but their star is no longer in the ascendant. The doctrine could not have had such influence had it not been based upon a solid foundation, but its system of *formal training* was carried too far, alike in linguistic and in mathematical teaching. It appeared to be indifferent to the intrinsic worth of the subject-matter of instruction, and therefore failed to enlist the interest and active co-operation of the not specially gifted pupil. Teachers are, as a rule, recruited

from a more or less gifted class, which fact helps to explain their adherence to the system. But the formal training theory does not appeal to the mass of mankind, and it is dangerous for teachers to lose touch with outside opinion. The theory is essentially psychological, but seems now-a-days to be forsaken by professional psychologists; at any rate, it can no longer be regarded as the fundamental motive in education. It has to be supplemented, and in part replaced, by the motive of enriching the mind with an assimilated mass of fruitful ideas, based upon and related to the circumstances of life. The reaction against excess of formal training we see at work in classical education; for instance, the view now prevails that Greek should be taught for the sake of the ideas it presents and the bearing of these ideas upon 20th century life; the old view was that Greek and Latin alike were given to mankind for the formal training of youth, and that the fortifying effect of the ancient languages arose rather from the process of acquiring them than from their value when acquired.

Without proposing then to throw overboard the formal training theory—educational psychology has not perhaps gone to this length as yet—I shall assume that it will no longer be the touchstone in choosing topics for a mathematical course; that competent teachers of to-day will succeed in extracting as much formal discipline from a lesson on modern subject-matter as did their predecessors from a lesson on subjects whose interest has now evaporated. If any particular topic can be shown to be irrelevant to the development of the subject, its retention cannot be justified by considerations of discipline; it will be regarded as an impediment to progress.

In attempting to define the essentials of an algebra course for non-specialists, I shall regard the course as growing out of arithmetic through problems and symbolic generalisation. These lines of advance will be followed until the original impetus is exhausted; in the meantime the stress will gradually be transferred to the idea of functionality, which will lead straight on to infinitesimal calculus. The whole body of doctrine thus developed will be viewed not as a self-sufficing system, but as an instrument for the elucidation of the physical sciences—geometry, physics, and so forth.

We shall not, of course, begin with a list of definitions and fundamental laws. Formal definitions are even less educative in algebra than in geometry; and anything like a thorough discussion of fundamental laws is altogether out of place at this stage. As for notation, the proposal is to adopt the method familiar to modern language teachers. They are accustomed to introduce grammatical rules, not in a mass, but gradually, as they are needed in the course of reading; for “grammatical rules” and “reading” substitute “algebraic notation” and “problems.” The simplest problems become amenable to algebraic treatment with a minimum of notation; probably all

that is needed for a start is the conventional suppression of the sign of multiplication: $2x$ for $2 \times x$. The difficulty of negative numbers will be avoided at the outset by a suitable choice of problems.

Some problems will necessitate the use of brackets, $4(x + 3)$; again, a problem leading to such an equation as:—

$$2x - (x - 5) = 10$$

will serve as an excuse for a lesson on the combination of signs:

$$-(x - 5) = -x + 5.$$

In these, and other ways, many of the fundamental notations and operations will emerge naturally from openings presented by a skilfully arranged set of problems, and will therefore be meaningful from the start.

It will be necessary to practise each new operation as it is learnt; the problem will serve as an introduction to a set of exercises on the operation, and when the operation is mastered it will in its turn be applied to a suitable set of problems.

All this will have to be worked out very carefully, and for practical purposes must be in book-form, as most teachers have not the time to do these things for themselves. It may be contrasted with the procedure of most standard text-books, which usually begin with notation and substitution, brackets, &c., and go on to problems when the fundamental operations are presumed to be well in hand. This procedure is abstract in the highest degree, and mystifies a stupid boy from the start; he can attach no meaning to the work he is doing.*

The problem and simple equation will be a peg on which to hang the following types of exercise:—

- (1) Addition (and subtraction) of like terms:—

$$3a + 5a - 7a.$$

- (2) Simple brackets:—

$$3(y - 7) - 5(y - 6) \text{ or} \\ (a - 2b + c) + (2a + b - c) - (3a - b + 2c).$$

The Chinese puzzle type of brackets sum may be dispensed with $-5a + (4b - 3[c + 4\{a + b\}])$.

It is natural to class with the removal of brackets the inverse operation—the insertion of brackets:—

$$2x - 8 = 2(x - 4):$$

here factors make their *début*, but need not be developed to any extent for the present.

(3) Various applications of negative numbers to quantity; and such questions as:—If I walk $(2x + y)$ miles N. and $(x - 3y)$ miles S. where do I find myself?

* "I attempted Mathematics The work was repugnant to me chiefly from my not being able to see any meaning in the earlier steps in Algebra." (*The Life and Letters of Charles Darwin, including an autobiographical chapter, edited by his son, Francis Darwin, Vol. I., p. 46. London: John Murray. 1887.*)

(4) Substitution of numerical values in algebraic expressions. This will hang naturally on to the equation, for the solution when found must be verified by substitution.

The next step may be to introduce the simplest type of algebraic fraction, that in which the denominator is numerical. Following our usual procedure, we shall set a problem leading to a fractional equation, solve the problem, then take the opportunity of studying this type of fraction, and finally apply the acquired mastery of fractions to the solution of more problems.

Incidentally this plan will serve to break up the heavy subject of fractions into manageable bits; for the present the high-water mark may be such simplifications as:—

$$\frac{x-4}{6} - \frac{2x-5}{8} + \frac{3x-2}{12}$$

and it will be noted that such work as this is almost pure arithmetic.

Everything so far is, in fact, arithmetical; there has been little to direct the attention to algebraic form. We have not necessarily had occasion to introduce the product of two literal symbols, or any index notation; though we may very likely have been tempted to pass from—

$$2(x-4) \equiv 2x-8 \quad \text{to} \quad a(b-c) \equiv ab-ac.$$

Now it would be easy to make good further progress with this limited algebraic vocabulary, in fact to the solution of simultaneous equations. But there is reason for thinking that this is a convenient point at which to extend the notation. For simultaneous equations must be taught in connection with the graphical illustration; and it is not desirable that all the first illustrations of graphing functions should be taken from linear functions. There is perhaps no serious danger of giving the impression that all graphs are linear; but, after all, a straight line is a dull thing, and a few strange curves tickle a boy's fancy and make him keen.

Here then shall enter the index notation, and it might be a pedantic stressing of our principles to fight shy of the general product notation any longer.

On the other hand, it would not be fantastic to introduce x^2 by such a simple problem as "One side of a rectangle is 11 inches longer than another, and the area is 242 square inches. What are the sides?" This leads to a quadratic; and the desire to solve quadratics some day may reconcile the boy to a preliminary study of an extended notation. The index notation may be familiar already, from prime factor work in arithmetic; some easy types of exercise will come in

usual routine of $x^3 \pm a^3$, $a^4 + a^2b^2 + b^4$, up to $a^3 + b^3 + c^3 - 3abc$. But from our stand point this is merely marking time, wandering out of the main stream into a backwater. The boy will never get to his journey's end at this rate. Any manipulative skill thus acquired is not necessary to progress, but technical; probably, however, it is less a matter of acquiring skill than of charging the memory.

Factorising is the operation inverse to multiplication, and this is the place for discussing "long" multiplication and division (with vertical addition and subtraction). If these rules are to be taught at all, it seems unnecessary to bring them in till we are ready to consider factors. And factors can hardly be said to demand the use of the long rules; horizontal multiplication will verify that $(x + 5)(x - 2) \equiv x^2 + 3x - 10$. Long division is on rather a different footing, for we need it as soon as we have to divide by a binomial. But do we need to divide by a binomial, in elementary work? The problem: Given one factor, find the other, needs long division; but in practice this problem does not occur except in the unpractical type of exercise already criticised: "reduce the degree of an equation of which one root is known."

The "long" rules must be studied if we are to retain the alternate division method for Highest Common Factor, but later on I shall attempt to prove that this method may be dropped.

Apparently long multiplication and division are not necessary stepping-stones, and probably they have been included traditionally in the algebra course as a reminiscence of arithmetic. And perhaps this suggests the only sound reason for retaining the rules. They are an immediate extension of arithmetic, and from this point of view not uninteresting. But the arithmetical analogy ceases as soon as we go beyond polynomial functions of one variable, and homogeneous polynomials in two variables. Such exercises as

$$(x^3 + 8y^3 - z^3 + 6xyz) \div (x + 2y - z)$$

lead to infinite series in unskilful hands; they should certainly not be set to non-specialists. On the whole I am inclined to sum up in favour of omitting long multiplication and division.

From factors the course will proceed to quadratics in one unknown, with illustrations from the graph of the quadratic function.

The solution of the general quadratic will involve skill in "completing the square," in identifying perfect squares of trinomial form, and in finding by inspection the square roots of such perfect squares. To carry "Square root" further leads us nowhere. The rule for square root in algebra (figuring under the distinguished name of "evolution") has been upheld as an extension of arithmetic; but the analogy is imperfect. The rule is not of general application in algebra. It is not

applicable to polynomials of odd degree. Again, if it is applied to a polynomial of even degree, not a perfect square, the result is an infinite series, whose convergency is conditional; but the conditions are, of course, not stated. To apply the rule to

$$\sqrt{4x^4 + 3x^3 + 4x + 1}$$

is not at all the same thing as to apply it to $\sqrt{4341}$; in the latter case the result is an infinite series necessarily convergent. If the rule is to be applied only to perfect squares, it becomes unnecessary, for the square root of a perfect square can easily be found by inspection. The example—

$$\sqrt{4x^4 - 12x^3 + 29x^2 - 30x + 25}$$

is taken at random from a widely-used text-book; what need is there of any elaborate "rule"? The square root, if it exists, must be of the form $2x^2 + ax \pm 5$; and it does not require much ingenuity to determine the value of a and the sign before 5. This is the common-sense and instructive way of finding square roots, if they must be found; but I imagine most mathematicians other than teachers will admit that in the course of their experience they have never had occasion to find the square root of anything worse than a trinomial. Why cumber schoolboys with problems that the experience of men shows to be pointless? Such anachronisms cannot be defended except on the dangerous ground of mental discipline, and what educational abuse is not defended on this ground? There are as good fish in the sea as ever came out of it; and as good discipline can be got out of the essential elements of mathematical progress as from the obsolete survivals of pre-Newtonian days.

Many quadratic problems lead in the first instance to fractional expressions with literal denominators, a type of fraction hitherto almost untouched; the only fractions studied as yet have been those with numerical or monomial denominators. Now the "study" of fractions as the term is understood in England does not mean the study of the

behaviour of a function of fractional form such as $\frac{ax + b}{cx + d}$;

it means the addition and multiplication of fractions. The multiplication of fractions—an operation unfamiliar in higher mathematics—reduces itself to factorising and cancelling; it does not take up very much time, and, if useless, is only harmful in so far as it ministers to a boy's passion for cancelling.

Addition of fractions, on the other hand, yawns with pitfalls, and, perhaps for this reason, is very dear to a certain school of teachers. It runs away with an enormous amount of time, is entirely void of concrete meaning, and for the most part has no bearing on future work; in integration, for instance, the adding together of fractions would only increase the difficulty;

what is needed is the inverse operation. The excessive amount of time spent on fractions is responsible, I believe, more than any other single cause, for the barrenness of much of our mathematical teaching. And yet the pressure of examinations makes it almost impossible to economise time; the fraction question is inevitable, and few teachers venture to defy this compulsion.

There is no remedy till the powerful examining bodies agree to some limitation of fraction questions; what limitations can be proposed to them?

It is not practical politics to ask for the suppression of fraction questions, and there is difficulty in drawing a clear line of limitation; there must be something arbitrary about the line. And yet, till we have a line, teachers are bound to prepare their pupils for the worst.

I suppose that there can be no objection to the principle of a line of limitation; even now it is agreed tacitly that some questions are beyond the line; *e.g.*, continued fractions. The Mathematical Association Committee has attempted to draw a line scientifically by the following test: "The types of fractions and factors that a boy must master thoroughly are such as present themselves in the solution of reasonable problems, including problems occurring ordinarily in physics, trigonometry, &c."*

Now a test of this kind has in it as little of the arbitrary as we can hope for, and it will be found to lead to a very useful limitation in the types of fractions admissible. It is, in fact, very difficult either to discover or to invent a "reasonable" problem which introduces a fraction with a quadratic or higher denominator. Plenty of reasonable problems lead to fractions with binomial denominators, as in $\frac{300}{x+5} = \frac{300}{x} - 2$. Binomial denominators then must be faced, but we may draw the line at this point. We shall retain exercises such as $\frac{5}{x+2} - \frac{4x}{2x+3}$; but we shall postpone to the specialist stage such forms as—

$$\frac{2x}{x^2+4x+3} - \frac{3}{x^2-1}, \text{ or}$$

$$\left(\frac{2}{x-2} - \frac{3}{x+3}\right) \div \left(\frac{3}{x-2} + \frac{2}{x+3}\right);$$

these do not arise out of reasonable problems.

The above limitation of algebraic fractions would be a great relief to teachers and taught, and, what is more, it would be found to involve logically relief in another direction. I shall

* Report of the Mathematical Association Committee on the Teaching of Elementary Algebra and Numerical Trigonometry, 1911. (London: G. Bell & Sons. 3d.)

attempt to prove that under the circumstances contemplated, Highest Common Factor may be omitted entirely.

The "alternate division" process of finding Highest Common Factor is not a process that one would willingly introduce into school teaching, as it is mechanical and difficult to understand; its claim to survive must depend upon its serviceableness. For what purposes is the process needed? It is needed to calculate Sturm's functions in the theory of equations, and to test for the existence of multiple roots. These problems are for the specialist; when he needs the Highest Common Factor method, then is the time for studying it. For the non-specialist, the only application is to the case of two or more fractions whose denominators cannot be factorised at sight, but are suspected to have common factors. The only reason for such a suspicion would be familiarity with the ways of examiners and text-books; if such a case presented itself to a mature mathematician as a real problem he would know that in the nature of things the odds against the existence of a common factor are $\infty : 1$, and would not seek for such a factor. But the known habits of examiners must be respected, and so long as a boy is liable to have to add together two fractions with cubic denominators, so long must he be prepared with a rule for Highest Common Factors. A limitation of denominators even to the quadratic form would dispense with Highest Common Factor; the factors being obvious at sight, only their Least Common Measure would be required.

It appears, then, that Highest Common Factor is no longer needed for use; it becomes a superfluity; is it to be retained for its own sake? Is there any intrinsic interest in finding the Highest Common Factor of two expressions? The factors of each expression may be of interest, but is it interesting or important to inquire what there is common to the two sets? The contents of Brown's bookshelves may be interesting, and so may be the contents of Smith's, but is it interesting to know what common elements they have?

I am satisfied that algebraic Highest Common Factor is not worthy of a place in the non-specialist course either on the ground of intrinsic interest or of importance in application. Should it then be studied as an extension of a known arithmetical process? I should say no; for I am equally convinced that arithmetical Highest Common Factor is a superfluity. If I have to deal with such an expression as:—

$$\frac{2467}{1289} + \frac{296}{541}$$

or to decide whether $\frac{123456}{654321} >$ some other fraction, I should

naturally reduce each fraction to a decimal, working to an appropriate number of decimal places. No mathematician would think of reducing the two to a common denominator.

But the utility of Highest Common Factor depends on the assumptions (1) that it is necessary to reduce heavy fractions to a common denominator, (2) that there is sufficient probability of the large denominators having a common factor to justify the labour of seeking for it, neither of which assumptions can be maintained if we have in view such fractions as might present themselves in the course of a real piece of work. It is always simpler to substitute a decimal for vulgar fractions, unless we are concerned with quite simple vulgar fractions whose denominators are products of a few small prime factors; and in these cases the Least Common Measure of the denominators is found without any use of Highest Common Factor.

If the above reasoning is sound, there is no scope for Highest Common Factor in connection with fractions; and the difficulty of inventing unforced concrete applications of arithmetical Highest Common Factor is sufficient evidence that the affairs of life can be carried on without any such rule. It is in fact a relic of pre-decimal times.

By this time the original impetus of the problem-opening must be almost exhausted. The first freshness of equations has disappeared; the time is ripe to break new ground. And yet at this very stage most schoolboys are driven into the wilderness of simultaneous quadratics and irrational equations, to wander there for the rest of their mathematical lives.

If it is considered impossible to drop simultaneous quadratics from the non-specialist course, at least they should be confined to the type connected with the intersection of a conic and a straight line; it might not be amiss to work a few exercises of this type:—

$$\begin{aligned}y &= 3x - 4. \\y &= 2x^2 - 3x + 5.\end{aligned}$$

mainly for the sake of the graphical interpretation.

Irrational equations seem to possess no special interest or importance; the same may be said of special types of equation such as $(x^2 + 4x)^2 - 3(x^2 + 4x) + 2 = 0$; as Bacon would say, "They be but toys."

The practical use of logarithm tables is now general in English schools; and if logarithms are approached through indices, there must be a preliminary treatment of fractional and negative indices. Formal proofs of index laws, as of all fundamental laws in algebra, pertain to the advanced specialist course; the laws $x^m \times x^n = x^{m+n}$ and $(x^m)^n = x^{mn}$ for positive integral values of m and n will be inferred by a rough induction. The aim in index exercises is to familiarise the pupil with the interpretation of fractional and negative indices, and not to make him expert in simplifying such combinations as:—

$$\left\{ \sqrt[3]{(x^{-\frac{1}{2}} y^{\frac{1}{3}})^2} \right\}^{-\frac{1}{2}}$$

More instructive practice is obtained from formulæ in physics and engineering that involve indices, which may

be set both in the form of numerical evaluations and also in the form of "changing the subject of a formula" (see p. 299).

Surds also should be dealt with lightly; the non-specialist need not enter into elegancies such as the rationalization of denominators, or Euclid's problem of $\sqrt{a + \sqrt{b}}$; any surd expression he is likely to meet with may be evaluated by means of square-root tables.

The confident use of the term ratio is of fundamental importance, and should have been acquired at a much earlier stage, in arithmetic. Ratio needs no special study in algebra; the simple properties met with in connection with similarity may be mastered as they are needed, in the geometry lesson. We may omit all such technical terms as antecedent, consequent, sub-duplicate ratio, invertendo, componendo, &c.; and formal theorems such as:—

$$(1) \frac{a}{b} > \frac{a-x}{b-x} \text{ if } a > b \text{ and } x > 0.$$

$$(2) \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n} \right)^{\frac{1}{n}}$$

Progressions form an instructive piece of work, but there are the following objections to their inclusion in the non-specialist course.

(1) The most instructive aspects of geometric progression cannot be included, as the difficulties of convergency could not be faced.

(2) It is the experience of examiners that no variety of questions can be set on progressions; hence this becomes a favourite cram subject.

(3) It is an isolated subject, a cul-de-sac, for those who will never proceed to a general study of series and convergency; it does not contribute to the development of thought in the direction of functionality and the infinitesimal calculus; any side-issue of this kind means distraction of aim and dissipation of energy. Perhaps this argument must not be pressed too strongly, for it may be argued that a good introduction to limits is by way of geometric progression.

The remainder theorem (*i.e.*, the theorem that if a polynomial $f(x)$ is divided by $x - c$ the remainder is $f(c)$), is important in the approximate numerical solution of cubic and higher equations (Horner's method), but the only case with which the average boy can be concerned is the particular case $f(c) = 0$; and the general theorem is not needed for the application of the particular case to quadratics.

Imaginary and complex numbers are, for the specialist, a subject of vast importance and interest, from the point of view of both pure and applied mathematics. They have never been regarded as belonging to the non-specialist course; and,

for the general reason suggested under (3) on page 305, their inclusion cannot be recommended here.

Other topics, on the border-line of higher mathematics, which may now be classed as definitely beyond the non-specialist course are the following: theory of quadratics, permutations and combinations, scales of notation, binomial, exponential and logarithmic series. To these may be added: artificial and "elegant" manipulations and devices.

It has fallen to my lot, in the preceding pages, to press for a considerable measure of destructive reform of the non-specialist algebra course; and, in defence of such drastic pruning, I have pleaded repeatedly the necessity of finding time for a wider range of less abstract subjects than those provided by algebra. But, before closing the discussion of algebra, I should like to urge that variation deserves a far fuller and deeper treatment than it usually receives. It is not a side-issue subject, but very much on the main line. Variation should be regarded from the "functionality" standpoint; in fact, it is unscientific to consider variation as a topic by itself separate from function and graphs; it amounts to little more than the addition of a little verbal nomenclature. Still, variation is an established heading in the text-books, and many teachers who would be frightened by the term "function" may be willing enough to teach variation. It must be correlated with physics; teachers of physics often complain that the boys coming to them have no grasp of the idea "varies as." This is due to boys being allowed to take refuge in algebraic symbols before they have soaked themselves in the functional ideas of variation; it is at bottom the same trouble as arises from premature use of symbols and formulæ in learning dynamics.

As this paper deals with the subject-matter of algebra rather than the way of teaching it, a long account of the methods I have found helpful in teaching variation would be out of place. But I may give a brief description. The more algebraical part of the subject, and especially the determination of constants, should not enter till the *idea* of different forms of variation is as familiar as the idea of multiplication. The first step may be the consideration of a set of related values, such as loads and consequent extensions of a spring. These values will naturally be subject to the errors of experiment; purely arithmetical methods might not reveal the law, but plotting the values will suggest it; we find a relation that may be expressed naively as "If you double x , you double y , &c." More precise statement, verbal and algebraic, will be worked out step by step; numerous other physical and geometrical illustrations are available; and gradually the pupil should attain a conception of the simplest case of variation, the complete conception involving (1) a knowledge of the arithmetical relation between two pairs of values; (2) a mental picture of the graphical

appearance of this case of functionality; (3) an associated knowledge of the algebraic form of the function.

In a similar way, the other common forms of variational functionality will be studied *one at a time*. In each case the instruction is incomplete till the pupil can say without hesitation, "In this kind of variation, if you double x , you double (quadruple, halve, quarter, &c.) y "; it is incomplete till the graphical and algebraic forms are so firmly associated as to suggest one another.

With variation may be studied such common forms of functional relationship as $y = a + bx$; strictly not a case of variation in the usual sense of the word, but in its essence forming part of the same chapter of thought.

I am not sure what to say about cases of functionality involving two independent variables, cases suggested by the phrase "varying jointly as." Perhaps for the average boy the case of one independent variable is as far as we need go; other cases might be taken if and when they are needed in physics. Graphical representation of $f(x, y)$ would hardly be helpful at this stage. I am not even quite sure that the language of variation is really useful any longer, *e.g.*, in dealing with a concrete problem involving $z \propto xy^2$, do we really advance matters by saying that z varies jointly as x and the square of y ? Has not the time come for relying frankly on the formula?

In any case, it is quite clear to me that the language of variation as applied to $z = f(x, y)$ is on a different plane of difficulty from anything connected with the $y = f(x)$ case. One independent variable is quite enough for a first study of variation.

I have now completed the review of the algebra course. I have asked for more time to study variation; on the other hand, I have suggested the possibility of very considerable economies, by the omission of the following from the non-specialist course:—

Formal proofs of fundamental laws.

Factors beyond the quadratic type.

Fractions (except those with monomials or linear denominators).

Highest Common Factor.

Long multiplication and division.

Simultaneous linear equations in three unknowns.

Literal equations (except in connection with formulæ).

Square root of polynomials.

Progressions.

Formal proofs of index laws.

Complicated exercises in fractional and negative indices and surds.

Simultaneous equations in which both equations are of quadratic or higher form.

Remainder theorem.

Imaginary and complex numbers.

Theorems on ratio and proportion.

Theory of quadratics.

Permutations and combinations.

Scales of notation.

Binomial, exponential and logarithmic series.

Artificial and "elegant" manipulations and devices.

I have urged that the above omissions will in no way hinder the general student's progress to trigonometry, mechanics and the calculus; that he will lose no essential element of a liberal education; and that, even on the side of algebraic manipulation, what he loses will be a skill that may be classed as a technical accomplishment, while he will still be expert enough to pass easily step by step to further topics.

To the time gained from algebra we must add a further saving which may be effected by a reconstruction of the arithmetic course on well-understood lines; but of this I need not treat in the present paper.

Let us now see what may be done with the extra time we shall have at our disposal. This subject I will deal with briefly, for I suppose that most teachers, whether they are mathematicians or not, would desire that the average boy should be brought nearer to the actual applications of mathematics to living problems, if only they can satisfy themselves that this can be done without encroachment on the time devoted to subjects other than mathematics. The following three subjects can be introduced:—

1. Numerical Trigonometry.
2. Mechanics.
3. Infinitesimal Calculus.

Numerical Trigonometry, as applied to geometry, surveying, mechanics, &c. This will include a numerical and graphical study of the tangent, sine and cosine; the solution of right-angled triangles, at first without logarithms; the solution of general triangles, at first by splitting into right-angled triangles,* and finally by means of the two formulæ:—

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

and $a^2 = b^2 + c^2 - 2bc \cos A$.

Throughout the course, applications to concrete problems; the data in some of the earlier cases to be derived from actual observations in the field or class-room; the solutions, again in

* The case of three sides given does not lend itself easily to the splitting method.

some of the earlier cases, to be checked by drawing to scale. No formulæ or identities, except the above, and :—

$$\tan A = \frac{\sin A}{\cos A}, \quad \cos^2 A + \sin^2 A = 1.$$

This needs perhaps two hours a week for two terms.

Mechanics, namely statics. Experimental discovery of the facts about equilibrium under three forces, resolution and composition of forces, moments, centre of gravity, friction; measurement of work together with velocity-ratio and efficiency of simple machines, the idea of conservation of energy (but no calculation of kinetic energy). The pupil finally should be able to work out straightforward numerical or graphical applications of the principles: the aim of the work should be grasp of principles, not facility with problems.

Such a course as this is actually mastered by naval cadets of 13½, who give to it four hours a week for one term (with revision at a later stage); presumably boys of 16 could do it in less time.

Infinitesimal Calculus.—Mr. J. W. Mercer, of the Royal Naval College, Dartmouth, has recently published an excellent "Calculus for Beginners," the most significant feature of which is that (in development of a suggestion originally due to Professor Perry) all the main applications of differentiation and integration are exemplified without using any function more abstruse than x^n . The first 259 pages are given to x^n , and in the course of these pages the reader learns how the calculus bears upon velocities and accelerations, maxima and minima, relative errors, definite and indefinite integrals, areas, Simpson's rule, volumes, centres of gravity, moments of inertia, work done in stretching strings and by expanding gases, mean values.

In all this work the manipulation is slight, but the value for mental enlightenment is immense. The importance of these pages is that they prove that a wide field of applications is thrown open by the very simplest tools that the calculus provides. The *ideas*, then, of the calculus, and a feeling of the extraordinary power of this new instrument, are accessible to a student with a modest degree of manipulative skill in algebra. It is shown that it is not necessary to tread for years the weary paths of Highest Common Factor, fractions and the like, before becoming worthy to enter this rich territory.

Now, it is not proposed that the calculus, in so far as it belongs to the non-specialist course, should cover the first 259 pages of Mr. Mercer's book, or anything like this amount of work. The

non-specialist cannot integrate $\frac{1}{x}$, for this involves e . Many of

the applications enumerated above will be beyond his range. For all this, there is much that he can do. He will be like the prospector picking up the first nuggets on a new goldfield.

Having been trained to think functionally, he will have little difficulty in grasping the idea that the gradient* at any point of a curve (say $y = x^2$) is a function of x . On a copy of the graph, lithographed on paper ruled in inches and tenths,† he may draw tangents with a ruler, measure off the gradients by means of transparent squared paper, and plot these gradients as ordinates; this will show him that the gradient is a function of x , and probably he can now see what function. After a suitable amount of preparation in this style, he may proceed to determine the gradient analytically. He is now in possession of the root-conception of the calculus, and the next step will be to space-time diagrams and velocities. Next may come the differentiation of x^3 , x , e , $\frac{1}{x}$, and further applications; maxima and minima will

present no insurmountable difficulties. Next, integration; first, perhaps, indefinite integration as the inverse of differentiation;

then the definite integral $\int_0^x x dx$ and $\int_0^x c dx$ can be found as areas,

and thus comes the notion that $\int_0^x f(x) dx$ is a function of x .

With these materials the link may be made between the

indefinite and the definite integral. Finally, $\int_0^x x^2 dx$, $\int_0^x \frac{dx}{x^2}$, and

then a whole range of applications; areas, volumes, centres of gravity, work. This will be the coping-stone of the non-specialist course.

That this range of work is within the powers of the average boy at an English secondary school is proved, not only by the testimony of English schools that have made the experiment, but also by the universal experience of the classical sides of French lycées. The time given to mathematics by these classical boys is shown in this table:—

**MATHEMATICAL HOURS PER WEEK IN FRENCH LYCÉES—
CLASSICAL SPECIALISTS.**

Ages - - -	8	9	10	11	12	13	14	15	16	17	18
Hours - - -	3	3	4	4	2	2	2	3	2	4	2

* The term "gradient" is taken to mean "the tangent of the slope."

† Such lithographed figures cost 8s. per 1,000.

The limit of the work undertaken in the highest class is shown by the following extract from the official programme* :—

“Notions de la tangente et de la dérivée. Exemples de tangentes obtenues géométriquement comme limites d'une sécante (cercle, parabole). Coefficient angulaire de la tangente; applications à quelques cas simples :—

$$y = x^2, y = x^3, y = \frac{1}{x}.$$

“Notions sur l'usage de la dérivée pour reconnaître le sens de la variation d'une fonction.

“Evaluation approximative de l'aire d'une courbe tracée sur du papier quadrillé en comptant les carrés contenus à l'intérieur de la courbe; limite de l'erreur fournie par le nombre des carrés que traverse la courbe; cette erreur peut être rendue très petite en employant un quadrillage très fin.

“Aire du triangle obtenue comme la limite commune de deux sommes de rectangles dont l'une est inférieure, l'autre supérieure à l'aire cherchée. Aire de la parabole. Problème inverse de la recherche d'une dérivée. Aire d'un triangle, ou d'une parabole, obtenue par la recherche d'une fonction dont la dérivée par rapport à x est ax ou ax^2 .

“Application de la méthode infinitésimale à l'évaluation des volumes ou des surfaces des corps considérés en géométrie élémentaire.”

The students in question have studied the elements of trigonometry in an earlier year. On the other hand, mechanics is not taught in mathematical time. But the marked contrast between the amount of time given to mathematics in France and England leaves us with no doubt that English boys can reach the above limit, even if mechanics is included with mathematics; probably English boys give twice as much time to mathematics as do their French contemporaries.

The boy of no marked literary ability usually carries on his mathematics, in England, to the end of the school life; but classical specialists not infrequently are released from mathematics in their last two years. Partly this is due to the pressure of competition for specialist scholarships in classics. But it is due also, and fundamentally, to the belief of both boys and headmasters that mathematical studies during these two years would not be a valuable element of education. And if a non-specialist mathematical curriculum is to be confined to the cramped limits now customary here, the belief is well-founded. A recognition that a wider field of ideas is accessible to the general and the classical student would eventually raise mathematics to its true position in the appreciation of those who control our great schools.

C. GODFREY.

* Plan d'Études et Programmes d'Enseignement dans les Lycées et Collèges de Garçons. (Paris: Delalain Frères. 1 fr. 50 c.)

THE TEACHING OF ALGEBRA IN SCHOOLS.

The objects of this paper are as follows :—

- (i) By quotations from some half dozen of the best known and most recent text-books on elementary algebra, to illustrate the usual methods of exposition.
- (ii) To compare the present system of teaching algebra with a system based on modern ideas relating to number.
- (iii) To criticise briefly Mr. Godfrey's views as expressed in his paper on "The Algebra Syllabus in the Secondary School."*

Classification of Students.—It is convenient to group mathematical pupils as follows :—

- (a) *Ordinary Students, i.e.*, those who study mathematics as part of a general education.
- (b) *Practical Students*, among whom may be included army candidates and those who intend to specialise in science or engineering.
- (c) *Mathematical Specialists, i.e.*, students who, after the age of 16–17, give the greater part of their time to mathematics.

Chief Aims.—For the "ordinary" student the chief aim will be mental culture, and for the "practical" student utility. Neither of these objects can be secured if algebra is taught as a mere bundle of rules; nor can success be achieved by the working of stereotyped examples. Attention must be paid to principles and, from beginning to end, the course must be an exercise in clear thinking.

In practice, it is necessary to frame an elementary course to suit (so far as possible) all classes of students. Here it is particularly important that nothing should be learnt which will have to be unlearnt later on.

Text-books.—Most of the text-books on elementary algebra contain a great deal of loose reasoning. One cause of this is that the writers do not always explain the exact meaning of words used in a technical sense. Difficulties are often avoided by the use of ambiguous phrases. The chief fault, however, lies with the present system, which does not appear to lend itself to clear exposition.

As a result, the text-book is used chiefly for the sake of the examples which it contains.

* See p. 280 above.

Now many teachers of Mathematics are not professed mathematicians; for these (at any rate) it is necessary that the principles should be properly explained in the text-books.

For the beginner, most of the teaching should be oral; it is therefore advisable that the book work in the early part of the course should be written chiefly as a guide to the teacher.

Under various headings, I will now proceed to the points indicated in (i) and (ii).

Words and phrases used in a technical sense should have definite meanings assigned to them; further, these meanings should be in accordance with modern usage. In connection with a definition, the *idea* is all important; often this can be most readily conveyed by referring to particular instances.

Much slovenliness in the text-books arises from inexact definitions and from the fact that, as a rule, definitions of equality and inequality and of the fundamental operations are omitted.

Sometimes an author appears to think that the meaning of difficult words can be revealed merely by the use of italics. For example—

“If two integral expressions are *identical*, they differ only in *appearance*, and not in substance; hence if, in the case of each separate expression, we remove all brackets and collect like terms, the *final residuum* obtained will, in each case, be the *very same*.”

Of course it would be difficult to say exactly what is meant by *the appearance*, *the substance* or *the final residuum* of an integral expression, or to say what degree of sameness is denoted by the *very same*.

Further, the writer falls into the common error of using “integral expression” to mean “rational integral expression.”

The following are instances of faulty definitions:—

“A number which has the sign + prefixed to it is called a *positive number*. A number which has the sign — prefixed to it is called a *negative number*.”

A more misleading statement can hardly be imagined. No idea is conveyed as to the double use of + and —. Further, it is implied that if 2 is preceded by —, then 2 (and not — 2) is a negative number. That there may be no doubt on this point, later on we are informed that

“ $6 + (-2)$ indicates the operation of *adding* the negative number 2 to the positive number 6.”

Again,

“An identity asserts that two expressions are always equal for all values of the letters involved in it.”

This is of course true, but it misses the point, namely, that either of the expressions can be *transformed* into the other by the rules of reckoning.

Here is a curious definition of a surd:—

“If a quantity is such that its root cannot be exactly obtained, the root is called a surd.”

What is a quantity and what is *the* root of a quantity? Webster says that in mathematics "quantity" means anything to which mathematical processes are applicable. Apparently, then, a circle is a quantity. Has a circle a root? Is -1 a quantity? Is $\sqrt{-1}$ a surd? Many writers wrongly say that it is.

A little further on, the same author says,

"Similarly, dealing with algebraical quantities, $\sqrt{a+b}$ is a surd."

Here the writer falls into the very common error of confusing an "irrational function or expression" with an "irrational number." $\sqrt{a+b}$ is an irrational function of a and b , but its *value* is not necessarily a surd.

Here is an explanation of graphical representation:—

"It has been shown that there are two ways of representing a quantity graphically; lines may be drawn horizontally, or vertically. By using both methods, we can represent two variables in the same figure. *Ex.*—To show in a figure that when $x = 4$, $y = 7$."

If the two ways of representing quantity described here are essentially different, it follows that there are many other ways by drawing slanting lines. The example appears to be devoid of meaning.

Here is another instance of confusion of "function" and "value." The writer is explaining functional notation:—

"If the symbol $f(x)$ be used to denote *some function of x* , then $f(a)$ denotes the *same function when x has the value a* ."

This is entirely wrong, for $f(a)$ is the *value* of $f(x)$ when $x = a$.

Most writers confuse "sequence" and "series," thus:—

"A series is a succession of quantities which are connected by some definite law, called the law of formation."

Here the word "sequence" should be substituted for "series" and it should be explained that the law of formation must be such that the terms of the sequence can be calculated in succession.

Useless or obsolete terms are often introduced, such as "affected quadratics," "arithmetico-geometric series." So far as algebra is concerned, "sub-duplicate" and "triplicate" ratio, "compounding ratios" belong to this category. This is apparently the meaning of the following curious statement:—

"Ratio and proportion can be treated independently of fractions; consequently, there are several expressions in common use the meanings of which must be known although the need for them no longer exists."

I give a few more instances of the slovenly language which is far too common :—

“One of the equations can be resolved into linear factors.”

“A fraction bears the same relation to division that brackets do to multiplication.”

“If both sides of an equation are divided by nothing, the result obtained is not necessarily true.”

The thoughtful reader will be in a dilemma, for, if nothing means zero, to divide by it is an impossible operation. If to divide an expression by nothing is not to divide it at all, then the original equation remains unaltered and we conclude that it cannot be true to start with.

Methods of Reasoning.—(a) For the beginner and the “practical” student, many truths will have to be based directly on experience. This does not justify loose reasoning or slovenly language. In cases of this kind, the exact nature of the argument must be pointed out. Take, for example, the following proposition :—“The graph of a linear equation in x , y is a straight line.” The argument should run as follows :—

As a particular instance, consider the equation $y = 2x + 4$. Give to x a number of particular values, find the corresponding values of y and plot the corresponding points. Observe that these lie on a straight line. Now verify the following statements :—

- (i) Every point whose co-ordinates satisfy $y = 2x + 4$ lies on the line.
- (ii) If the co-ordinates of a point do not satisfy the equation, the point does not lie on the line.
- (iii) The co-ordinates of every point on the line satisfy the equation.

In the same way, we can show experimentally that the graph of any equation of the form $y = mx + n$, where m and n are given numbers, is a straight line.

(b) Much caution is required when an appeal is made to the *principle of continuity*. Take the following instance, in which the writer is explaining the notion of a limit :—

“By the principle of continuity we are led to say that the value of the function $(x^2 - 4)/(x - 2)$ when $x = 2$ differs infinitely little from the value $(x^2 - 4)/(x - 2)$ has when x differs infinitely little from 2.”

To use “continuity” in explaining “limit” is to put the cart before the horse, for the definition of a continuous function presupposes the idea of a limit. The thoughtful student is not to be cajoled in this way. He will have discovered that to divide by zero is an impossible operation and he will see that the function has *no* value when $x = 2$. The point of view is

also misleading in this respect:—*it is implied that a limit of a function is necessarily a value of the function.*

Why make a mystery of a matter which can be explained in plain language, *when the ground has been suitably prepared?* (See pp. 332–334.)

(c) *Cases in which an argument fails* should receive careful attention. With most writers on elementary algebra, a letter stands for “an algebraical quantity.” The question as to whether a statement is true for all values of the letters concerned is too often disregarded. For example, in assigning a meaning to a^0 , the reasoning should be as follows:—

If the law $a^m \cdot a^n = a^{m+n}$ is to hold when n is zero, we must have

$$a^m \cdot a^0 = a^{m+0}, \therefore a^m \cdot a^0 = a^m.$$

If a is not zero, dividing each side of the last equation by a^m , we have

$$a^0 = a^m/a^m = 1.$$

For all values of a except zero, we therefore *define* a^0 as 1. If a is zero, a^m (which is also zero, except when $m = 0$) cannot be used as a divisor. We are, therefore, unable to assign a definite meaning to 0^0 .

The case in which $a = 0$ is usually disregarded in the textbooks.

What is Algebra?—Some writers avoid this question and content themselves by periodically asserting “In arithmetic we do this, in algebra we do that.”

Sometimes we have a general statement, of which the following is an example:—

“Algebra may be divided into two divisions, Arithmetical Algebra and Symbolical Algebra. In Arithmetical Algebra letters are used, but the letters are supposed to have such values as make the operations arithmetically intelligible”

The distinction is not a happy one, for 1, 2, 3, are symbols as well as a, b, c, \dots and “symbolical” applies equally to arithmetic and algebra.

The points to insist on are these:—Arithmetic and algebra are parts of the science of number. Algebra grows out of arithmetic by a series of steps, and it is hard to say where arithmetic ends and algebra begins. The first step is the use of letters to denote the numbers used in arithmetic. Later on, we find that these numbers are insufficient for our purposes and we invent new classes of numbers.

(A large part of algebra has to do with *equations* and *functions*; some writers consider that true algebra begins at this point.)

Notion of Number.—The chief faults of exposition in the early part of the course are due to these causes :—

- (i) It is not explained that letters used to denote quantity stand for (abstract) numbers and for nothing else.
- (ii) The notion of number is founded on *measurement* and not *directly on counting*, so that the reasoning proceeds from the concrete to the abstract.

Now a child counts before he can measure ; by counting he soon acquires the notion of (abstract) number. To add 3 to 5, he counts *five* and then *three more*. The simplest definitions of the fundamental operations are based directly on counting, and it is imperative that such definitions should be given if the subsequent reasoning is to be sound.

A Logical Course.—*Stage 1* is a generalised arithmetic in which letters are used to represent natural numbers. In this stage, the student is introduced to all the elementary processes of algebra, including factors, H.C.F. and L.C.M., the simplification of easy fractional forms, graphical representation of number. Problems and the solution of the equations on which they depend are introduced at the very beginning and recur at suitable intervals.

In Stages 2 and 3, negative numbers and fractions are introduced. For each of these new classes of numbers it is necessary to *define* equality, inequality and the fundamental operations, and to consider the relation of these numbers to measurement.

The subsequent stages follow the usual lines. Theoretically and from the point of view of utility it is important to consider carefully the subject of *approximations* and the calculation of *limits of error*.

The great advantage of this scheme is that difficulties are introduced *singly*.

“The lateness of occurrence of any particular mathematical idea is usually closely correlated with its intrinsic difficulty” (Encyc. Brittan., Vol. I., page 600). Negative numbers were first introduced by Girard, Vieta, and Descartes, about 1600, whereas a generalised arithmetic in which letters were used to denote numbers had been used for centuries before this. Further, the notion of negative numbers is not required for the solution of many classes of problems and the repetition in *Stages 2 and 3* of part of the work done in *Stage 1* is very useful for purposes of revision.

Finally, the scheme has proved very satisfactory in actual practice.

Fundamental Laws.—In “*Stage 1*,” for all except fairly advanced students, experience in arithmetic will afford sufficient justification. Proofs should, however, be given for the teacher

and for the fairly advanced student. At this stage, the reasoning is not always sound. For example,

“We know that $6 \times 4 = 4 \times 6$, In the same way $xy = yx$ and $xyz = xzy = xzy$.”

Now it does *not* follow that if $yz = zy$, then $xyz = xzy$ (that is, if the order of operations is from left to right). In order to prove this, the distributive law must be used.

Another writer, assuming that $ab = ba$, proceeds as follows:—

$$“bac = b \times (ac) = b \times ca = bca.”$$

In removing brackets, he assumes, without any justification, that to multiply by (ac) is the same as multiplying by a and c in succession.

The reasoning should be as follows:—

Ex.—The order of operations being from left to right, use a general method to show that—

$$4 \cdot 3 \cdot 2 = 4 \cdot (3 \cdot 2) = 4 \cdot 2 \cdot 3.$$

We have

$$\begin{aligned} 4 \cdot 3 \cdot 2 &= (4 \cdot 3) \cdot 2 && \text{(By hypothesis)} \\ &= 4 \cdot 3 + 4 \cdot 3 && \text{(Def. of multiplication).} \\ &= 4(3 + 3) && \text{(Distributive Law).} \\ &= 4 \cdot (3 \cdot 2) && \text{(Def. of multiplication).} \end{aligned}$$

In the same way we can show that $4 \cdot 2 \cdot 3 = 4 \cdot (2 \cdot 3)$.

Now $4 \cdot (3 \cdot 2) = 4 \cdot (2 \cdot 3)$, for $ab = ba$, whence the result follows.

Negative Numbers.—The following quotations illustrate the unsatisfactory character of an argument from the concrete to the abstract.

“*Negative Quantities.*—Arithmetically we cannot subtract 6 from 3; *i.e.*, the expression $3 - 6$ has no arithmetical meaning. In algebra, however, such an expression has an intelligible interpretation. This is best seen by considering a few examples. If a farmer buys 7 cows, and sells 4 cows, he has 3 *more* than he had at the start. On the other hand, if he buys 4 cows, and sells 7, he has 3 *less* than at first.

“We express this algebraically thus:

$$\begin{aligned} 7 \text{ cows} - 4 \text{ cows} &= + 3 \text{ cows} \\ 4 \text{ cows} - 7 \text{ cows} &= - 3 \text{ cows}, \end{aligned}$$

&c.”

Here, the subject of “cows” is (perhaps wisely) dropped. An opportunity has, however, been given to the “intelligent” youth to put some very awkward questions, as, for instance:

You said that the farmer bought 4 cows and then sold 7. How did he manage this? If he had none to start with (and you do not say that he had any) I shall be obliged to conclude that he stole some. Did you ever see or count $- 3$ cows? Is “ $- 3$ cows” a quantity? Does the minus sign denote subtraction? If so, what are the 3 cows to be taken from? If the $+$ and $-$ in your answers do not denote addition and

subtraction, then I suppose that + 3 cows is a positive quantity and - 3 cows is a negative quantity, so that + and - denote opposite *qualities of some kind*. What kind of property is possessed by a negative cow? What is the average price of one?

The necessity for *defining* the operation of adding a negative number will appear if we consider the method adopted in the following extract:—

“Again, $8 + (7 - 5)$ means that we first take 5 from 7 and then add the result to 8, *i.e.*, we add to 8 the excess of 7 over 5. If we first add 7 to 8, we have added 5 too much; hence to obtain a correct result, we must now subtract 5,

$$\text{i.e., } 8 + (7 - 5) = 8 + 7 - 5.$$

“In exactly the same way in algebra, since symbols represent numbers.

$$a + (b - c) = a + b - c.”$$

The “intelligent” youth puts questions: May I give to a, b, c any values I please; for instance, may I put $a = 3, b = 3,$ and $c = 5$? If so, I am to conclude that just as $8 + (7 - 5) = 10$, in the same way $3 + (-2) = 1$. Is the following definition of the expression $3 + 2$ correct? $3 + 2$ denotes the last number used in the following process of counting: Start with the number 3 on the natural scale, and count two *more* (4, 5). Can you give any definition of addition which is not equivalent to this? To find the value of $3 + (-2)$, I must therefore start with the number 3 on the scale, and count (-2) numbers forward. How am I to count (-2) numbers? I could not count (-2) cows. Perhaps you mean I am to count two symbols back, but I only know one definition of addition, and this tells me to count forwards.

It will now be obvious that $3 + (-2)$ cannot be found by *counting*. Neither can it be found *geometrically* (for this method means that units of length or area are to be *counted*), nor by reference to any concrete magnitude, unless some assumption is made which is foreign to ideas connected with abstract number.

The nature of such an assumption will be evident from the following illustration:—

If a gain of 3*l.* is denoted by + 3, then a loss of 2*l.* is denoted by - 2. It *seems reasonable* to denote a gain of 3*l.* taken with a loss of 2*l.* by $3 + (-2)$. If we agree to do this, then $3 + (-2)$ means the result of subtracting 2 from 3, and we conclude that $3 + (-2)$ *ought* to mean $3 - 2$.

This example exhibits the method often employed to *prove* that $3 + (-2) = 3 - 2$. The words in italics are omitted, the assumptions are glossed over, and our notions with regard to pounds, shillings and pence are used to *prove* a property of abstract number which, logically, is incapable of proof.

Here is the correct method: the expression $3 + (-2)$ is *defined* to mean the same as $(-2) + 3$, in order that the

equality $a + b = b + a$ may hold good for all numbers, whether positive, zero, or negative.

(Observe that $(-2) + 3$ can be found by counting 3 places forward from -2 on the scale

. . . $-2, -1, 0, 1, 2, 3$. . .)

Even Prof. Chrystal (to whom the present writer, in common with most teachers of algebra, is vastly indebted) does not deal satisfactorily with the addition of a negative number. In his "Text-book of Algebra" (1886), p. 5, he proposes to prove that $a + (b - c) = a + b - c$ without the limitation $b > c$; he says that it follows from the associative law for addition that

$$+ a + (+ b - c) + c = + a + \{ + (+ b - c) + c \}.$$

He does not remark on the fact that when $b < c$, he is applying a law, which has been investigated only for one class of numbers, to numbers of a totally different class. To justify this, a new operation must be *defined*.

The operation of multiplying by a negative number and by zero will be defined in such a way that the equality

$$(a - b)(x - y) = ax - bx - ay + by$$

may hold good for all values of a, b, x, y , whether positive, zero, or negative.

Unless this simple method is followed, the explanation is likely to be very confused, as, for instance:—

"The $-$ sign put before a number indicates that we have now to take that number in the reverse direction"

This statement is unintelligible. What operation is indicated by *taking* a number? When a number has been *taken*, what has *direction* to do with it? In this case, reference had just been made to the number of steps, and the omission of the words "of steps" renders the statement meaningless. This is a good illustration of the confusion which arises from an attempt to reason from the concrete to the abstract. The writer continues—

"When we come to $+ 4 \times - 3$, we find that $+ 4$ taken $- 3$ times has no meaning. We have already seen that $-(+ 4)$ means the reverse of $+ 4$ or $- 4$. Thus $-(+ 4)$ may be regarded as $+ 4$ taken once reversed."

Here an attempt to reason from the concrete (number of steps) to the abstract (number) leads to further difficulties, for in an attempt to explain that $4 \times (- 3) = - 12$ we are told that

" $+ 4 \times - 3$ will mean $+ 4$ taken 3 times reversed."

If it is hard to understand the meaning of (the number) 4 taken once reversed, it is three times harder to gather the sense of "4 taken 3 times reversed." Presumably the writer had in mind the taking of steps.

Now to take 4 forward steps reversed is an awkward expression ; if we assume that it means to take 4 steps backwards, then 4 forward steps taken twice reversed must mean 4 steps forward.

If any meaning at all can be gathered from the method, it appears to be that

$$4 \times (-3) = -12 \text{ and } 4 \times (-2) = +8.$$

Fractions.—In order that a value of x may exist which shall satisfy the equation $xb = a$, when a is not divisible by b , we invent a new class of numbers called fractions. For these numbers, it is necessary to *define* equality, inequality, addition, &c.

If a is divisible by b and x by y , we know that

$$\frac{a}{b} \begin{matrix} > \\ \equiv \\ < \end{matrix} \frac{x}{y} \text{ according as } ay \begin{matrix} \geq \\ \equiv \\ < \end{matrix} bx.$$

In order that the rules for equality and inequality may be the same for all numbers, the above statement is taken as the *definition* of equality and inequality for fractions.

It is easy to see that this rule assigns a definite place to every fraction in the scale with integers and other fractions.

Similar reasoning leads us to *define* addition, multiplication, &c., in the case of fractions by the formulæ

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \quad \&c.$$

which are known to hold when the divisions indicated can be performed.

With these definitions, it is evident that the fundamental laws continue to hold.

For the beginner, it may be better to define the fraction $\frac{a}{b}$ as *the measure* of a magnitude obtained by dividing the unit into b equal parts and *taking* a of these parts. He will be already familiar with this idea and the definition leads immediately to the above definitions of equality, &c.

In every case, the definition of a fundamental operation for numbers of a new class is obtained by extending one of the fundamental laws to these numbers.

Instead of adopting this uniform plan, the text-books resort to all kinds of artificial methods, as, for example:—

“To multiply one number by a second is to do to the first what is done to unity to obtain the second. To multiply $\frac{5}{4}$ by $\frac{3}{4}$, we must do to $\frac{5}{4}$ what is done to unity to obtain $\frac{3}{4}$.”

The “intelligent” youth speaks: I can obtain $\frac{3}{4}$ from 1 by subtracting $\frac{1}{4}$ from 1: apparently then, to multiply $\frac{5}{4}$ by $\frac{3}{4}$, I must subtract $\frac{1}{4}$ from $\frac{5}{4}$, for this is what I did to 1 in order to obtain $\frac{3}{4}$. Thus $\frac{5}{4} \times \frac{3}{4}$ is the same as $\frac{5}{4} - \frac{1}{4}$.

The only conclusion is that a definition so incorrectly expressed should never find a place in a mathematical treatise.

Equations.—In nearly all modern text-books, the treatment of equations is illogically based on Euclid's axioms. The writers fail to remark that (i) the process called "solving an equation" is in reality a *search* for a solution. Unless every step in this process of search is reversible, a verification by actual substitution is necessary.

A suitable introduction to this subject, stated roughly, seems to be as follows:—

All processes in algebra depend on

- (i) the laws which govern the fundamental operations;
- (ii) various theorems called Rules for Equality and Inequality.

Rules for equality are as follows:—

- (a) If $a = b$, then $b = a$.
- (b) If $a = b$ and $b = c$, then $a = c$.
- (c) If $a = b$, then $a + x = b + x$, &c.

The converse of each of these theorems is true, provided that zero is not used as a divisor.

Ex.—Search for a solution of the equation $2x + 3 = 17$.

We begin by *assuming* that a value of x exists which satisfies the equation, and that x has this value—thus

$$\begin{aligned} 2x + 3 &= 17 \\ \therefore 2x &= 14 \text{ (Rule for Equalities)} \\ \therefore x &= 7 \text{ (Rule for Equalities)}. \end{aligned}$$

Hence, if a solution exists, then 7 is the solution.

Again, every step in the above process is *reversible*. (This follows from the fact that the converse of each of the "Rules for Equalities" is true.) Thus the steps can be taken in the reverse order, and 7 is a solution, and is the only solution.

A similar method can be applied to simultaneous equations.

Simultaneous Quadratics.—The lack of system in the usual treatment of these equations is bewildering.

Nearly every writer wrongly asserts that the *general method* is to eliminate one of the unknowns, say x , and solve the resulting equation in y .

The student finds that, in practice, this method is used only for the case in which *one equation is linear*, and that examples of other types are worked out by miscellaneous methods which apparently have nothing in common.

Now the general method of solving two equations in x, y , both of which are of the second or higher degree, is as follows:—

Denote the given equations by $X = 0, Y = 0$. By a series of reversible steps, try to obtain an equation of the form $PQ \dots = 0$, where P, Q, \dots are linear functions of x and y

The given equations are equivalent to the following sets of equations (one of which is linear)—

$$P = 0, X = 0; Q = 0, X = 0; \&c.$$

Instead of inserting a general statement of this kind, it would be better to illustrate the principle by working out examples such as—

Ex. 1. $(x - y)(x - 2) = 0, (x + y - 6)(y + 3) = 0.$

Ex. 2. $x^2 = 4y^2, (x + 3)^2 = (y + 1)^2.$

Ex. 3. $x^2 + y^2 = 5, xy = 2.$

[Equivalent to $(x + y)^2 - 9 = 0$ and $(x - y)^2 - 1 = 0.$]

Ex. 4. $-3x^2 - 5y^2 = 7, 3xy - 4y^2 = 2.$

[Equivalent to $3x^2 - 5y^2 = 7$ and $(2x - 3y)(x - 2y) = 0.$]

Approximate Values.—Both theoretically and in practical applications of Mathematics, the following problem is of extreme importance:—*If a, b, c, . . . are the observed values of numbers whose actual values are denoted by A, B, C, . . . , and it is known that a, b, c, . . . are subject to errors whose numerical values are less than $\alpha, \beta, \gamma, . . .$, respectively. It is required to find an upper limit to the error in taking $f(a, b, c, . . .)$ as the value of $f(A, B, C, . . .)$, where $f(a, b, c, . . .)$ is a given function of a, b, c,*

Considerations of this kind are ignored by the writers of text-books, who fail to see that the statement “ x is nearly equal to X ” is useless, unless we are prepared to say *how nearly* this is the case.

It is absurd to say that simple cases of the above problem are too hard for the fairly advanced school student.

If such questions are ignored, how is it possible to convey correct ideas relating to limits, or to approach the calculus.

The fundamental theorems are as follows:—

If a, a', b, b' are given numbers and A, B are any numbers such that

$$a < A < a' \text{ and } b < B < b',$$

then

(i) $a + b < A + B < a' + b'.$

$$a - b' < A - B < a' - b$$

and if all the numbers are positive,

(ii) $ab < AB < a'b' \text{ and } a/b' < AB < a'/b.$

These theorems apply directly to many important types of examples. The following are instances:—

Ex. 1.—The error in representing $\frac{1234567}{3456789}$ by $\frac{12}{35}$ is less than 0.05.

Ex. 2.—The approximate lengths of two sides of a rectangle are 1.8 in. and 0.7 in., the error in each case being

numerically less than 0.05 in. Show that the area differs from 1.8×0.7 sq. in. by less than 0.13 sq. in.

Ex. 3.—The observed values of A , B are 20 and 16, and each of these values may be in error by as much as 3 per cent. of the corresponding true value. Show that the value 4 of $A - B$ may be in error by as much as 36 per cent. of the true value.

Ex. 4.—If x is small, show that $\frac{1+ax}{1+bx}$ is nearly equal to $1 + (a-b)x - b(a-b)x^2$; also, if $a = 5$, $b = 2$ and $x < 0.1$, the error in this approximation is less than 0.01.

Irrationals.—The beginner requires (i) a general idea of the character of these numbers; (ii) to understand that for purposes of practical reckoning, no matter how exact, an irrational may be represented by a rational number.

Meaning of $\sqrt{7}$.—For practical purposes, we may put the matter thus:—By the square-root process, we are led to the endless decimal $2.6457 \dots$. If $a_n = 2.6457 \dots$ to n places, by taking a sufficiently large value of n , we can make $7 - a_n^2$ less than any positive number we may choose, however small.

If then we regard $\sqrt{7}$ as an abbreviation for $2.6457 \dots$, where the decimal is to terminate after a large number of figures, we may say that for all purposes of practical reckoning, $(\sqrt{7})^2 = 7$.

What is virtually the Dedekind definition may be presented to the fairly advanced school student as follows:—

Def.—If a , a' are positive rationals such that $a^n < A < a'^n$, then a , a' are called *approximate values of the n th root of A* , with errors, respectively in defect and excess, less than $a' - a$.

An irrational is defined by some rule which gives it ordinal rank with the rational numbers.

In finding approximate values of the square root of 7, we are led to the endless decimal $2.6457 \dots$, whose figures are calculated by a definite rule. This decimal is such that 7 is greater than all the numbers

$$(2.6)^2, (2.64)^2, (2.645)^2, \dots,$$

where there is no last number.

Also 7 is less than all the numbers

$$\dots (2.646)^2, (2.65)^2, (2.7)^2$$

where there is no first number.

We invent the symbol $\sqrt{7}$ to denote a number called the square root of 7, and we place it among the numbers on the rational scale, so that it follows all the numbers 2.6 , 2.64 , 2.645 , \dots , where there is no last number, and precedes all

the numbers ... $2\cdot646$, $2\cdot65$, $2\cdot7$, where there is no first number.

This rule cannot fail to determine whether $\sqrt{7}$ is to follow or to precede any rational which we may choose. For the decimal $2\cdot645$... is endless and does not recur, hence $2\cdot645$... cannot be identical with the decimal representation of any rational, for such a decimal either terminates or recurs.

In dealing with surds and irrational functions, the chief faults of the text books are—

- (i) The meaning of a surd is not properly explained.
- (ii) Irrational functions are almost invariably confused with irrational numbers (*see* p. 314). The writers apparently do not know that in the classification of algebraical functions (or expressions) “the terms *rational, integral, &c.* refer exclusively to the way in which the variables occur and have nothing to do with the nature of the coefficients.”

Thus we are informed that “ $x^2 - 7$ has no rational factors”, “ $x^2 + x + 1$ cannot be broken up into linear factors.”

- (iii) The method of search for a meaning of a fractional, negative or zero index is not properly described.

The following points should be noticed:—

The fundamental index laws are—

- (i) $a^m \cdot a^n = a^{m+n}$;
- (ii) $(a^m)^n = a^{mn}$;
- (iii) $(ab)^n = a^n b^n$.

Whatever meaning may be assigned to a^n , when n is not a positive integer, in order that such a meaning may be *useful*, it must be such that these laws continue to hold.

We *invent* meanings for a^n when n is (i) a fraction, (ii) zero, (iii) negative, by assuming that *one* of these laws holds: we then enquire if, with these new meanings, the laws (i)–(iii) hold for all rational values of the indices (deferring the consideration of a^n when n is irrational). (*See* p. 316.)

Ratio and Proportion.—The usual explanations are unsatisfactory, for example,—

“The ratio of one quantity to another of the same kind is the quotient of the one divided by the other. Thus the ratio of x to y

is $\frac{x}{y}$.”

The fact that most students who read the chapter on ratio and proportion, in a text-book on algebra, have a preconceived notion of what is meant is no excuse for failing to put the matter properly. Explanations in algebra should clear up difficulties in arithmetic.

To talk of dividing 5 ducks by 3 ducks is as absurd as to say that the (abstract) number 5 is to be divided into 3 equal parts.

Again,

“The relative magnitude of two quantities, measured by the number of times which the one contains the other, is called their ratio. The ratio of a to b is expressed by the notation $a : b$.”

Here (as usual) the difficulty lies in passing from the concrete to the abstract: a and b are intended to denote abstract numbers. How many times does 5 contain 3?

The essential points seem to be as follows:—

(i) *Equality and Inequality of Concrete Magnitudes.*—In considering how a magnitude or a quantity of any kind is to be measured, at the outset, we must *define* what is meant by saying that “ A is equal to B ,” “ A is greater than or less than B ,” where A and B are quantities of the kind under consideration.

The student should reflect on the meaning of equal straight lines, equal angles, equal forces, &c.

(ii) *Ratio.*—We assume that a magnitude B can be divided into any number (n) of equal parts. Denoting one of these parts by C , B is said to contain C (exactly) n times, C is called the n th part of B , and we write

$$B = nC \quad \text{or} \quad C = \frac{1}{n} B.$$

If A contains the n th part of B exactly m times, we denote this by writing $A = \frac{m}{n} B$.

Def.—If A and B are magnitudes of the same kind, and if integers m and n exist such that A contains the n th part of B exactly m times (that is, if $A = \frac{m}{n} B$), the magnitudes A and B are said to be *commensurable*, and the *ratio of A to B* (written $A : B$) is defined as the number m/n .

For *convenience*, m/n is often written in the form $m : n$, and is called the *ratio* of the number m to the number n .

By considering the side and diagonal of a square, it is easy to explain what is meant by the ratio of two incommensurable magnitudes. It should be pointed out that in practice, the ratio of two incommensurables is represented by a suitable approximate value.

Logarithms.—The usual plan is to begin with a definition,—

“The logarithm of a number to some chosen base is the *index* of the power to which the base must be raised in order to be equal to the number.”

At this stage such a definition is useless, for it implies that a meaning has been assigned to a^x when x is irrational.

The following is a rough sketch of a reasonable explanation:—“Choose some number a , greater than unity, as a

base or number of reference. If x and N are numbers connected by the relation $a^x = N$, then x is called the *logarithm of N to the base a* . This is shortly expressed by writing $x = \log_a N$."

Taking 10 as the base of the *common* system, $\log N$ is an abbreviation for $\log_{10} N$. Whence easily $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, &c.

The method by which we determine the logarithms of 1, 10, 100, . . . does not enable us to answer the question "What is meant by $\log 2$?" In fact it is easy to show that no rational x exists such that $10^x = 2$.

It can, however, be shown that "a rational x can be found such that 10^x differs from 2 by as small a number as we please. In practice, such a number is taken to represent " $\log 2$."

This explanation will probably suffice, for the beginner. A useful exercise would be to draw the graph of 10^x by methods suggested in the text-books, using this to determine logarithms graphically.

The fairly advanced pupil should make a further study of the function a^x . He should be acquainted with the following theorems (the proofs of which are easy):—

If $a > 1$ and x is rational, then—

- (i) $a^x \begin{cases} > \\ \geq \end{cases} 1$ according as $x \begin{cases} > \\ \geq \end{cases} 0$;
- (ii) a^x increases steadily with x ;
- (iii) If n is a large positive integer a^n is large, a^{-n} is small and both $a^{\frac{1}{n}}$ and $a^{-\frac{1}{n}}$ are nearly equal to unity.

The meaning of $\log 2$ can now be explained as follows:—
"By comparing integral powers of 2 and 10, we can find a number x such that 10^x is nearly equal to 2. For by contracted multiplication, we can show that—

$$\begin{aligned} 2^{10} &= 10^3 \times 1.024 \\ 2^{100} &= 10^{30} \times 1.26765\dots \\ 2^{1000} &= 10^{301} \times 1.071\dots \\ 2^{10000} &= 10^{3010} \times 1.9\dots \\ \therefore 10^{3010} &< 2^{10000} < 10^{3011} \\ \therefore 10^{0.3010} &< 2 < 10^{0.3011} \end{aligned}$$

This process can be continued to any extent and gives rise to the endless decimal $0.3010\dots$, which is such that—

$$10^{0.3} < 2 < 10^{0.4}, \quad 10^{0.30} < 2 < 10^{0.31}, \quad 10^{0.301} < 2 < 10^{0.302},$$

and so on without end."

By means of the preceding theorems, we can easily show that if $x = 0.3010\dots$ to n places, by taking a sufficiently large value of n we can make $2 - 10^x$ less than any positive number we may choose, however small.

If then we regard $\log 2$ as an abbreviation for $0.3010\dots$, where the decimal is to terminate after a large number of figures, we may say that for all purposes of practical reckoning $10^{\log 2} = 2$.

The following inadequate definition occurs in several books:—

“The integral part of a logarithm is called the characteristic and the decimal part the mantissa.”

For practical purposes, it is important that some idea should be conveyed of the degree of accuracy obtained by the use of (say) four-figure tables. For this purpose it is useful to work out an example of the following type:—*Ex.*—Use four-figure tables to find an approximate value of $(3.944)^6$; show that the only exact information as to the value of this expression obtainable from such tables is that $953.1 < (3.944)^6 < 954.6$.

G.C.M. and H.C.F.—The rule for G.C.M. in arithmetic is of little use in practical reckoning. Its real use is in connection with the theory of numbers and indeterminate equations.

The “long rule” for H.C.F. properly belongs to the theory of polynomials, the theory of partial fractions, &c., subjects which are beyond the scope of an elementary text-book. This rule should therefore occur late in the course.

It is astounding that certain writers do not appear to understand the essential points of distinction between the arithmetical theory of G.C.M. and the algebraical theory of H.C.F. For example, one writer finds what he calls “the H.C.F. of 782 and 5451.” He then says:

“This method depends upon the fact that if any two numbers have a common factor, the remainder, when one is divided by the other, has the same factor This principle being true for arithmetical numbers must also be true in algebra, since the symbols stand for numbers.”

This statement implies that the two theories are identical. Now it is obvious that this is not the case, for every pair of numbers has a G.C.M., but two polynomials may have no H.C.F. It is merely another instance of the confusion of *function* and *value*.

In describing the algebraical process, the following explanation is necessary:—

If R_1, R_2, R_3, \dots are the remainders, each of these after the first is of lower degree than the preceding. The process can therefore be continued until a remainder is found which either vanishes identically or is independent of x . If a remainder vanishes identically, the preceding remainder is the H.C.F. If a remainder is independent of x , the given functions have no H.C.F.

The corresponding explanation for the G.C.M. is entirely different.

As an immediate consequence of the G.C.M. process, it is useful to give the following theorem:—"If a and b are any numbers which are prime to each other, positive integers " x and y can be found such that $ax - by = 1$."

Indeterminate Equations.—The following method, which depends directly on the G.C.M., is simple and expeditious:—

Ex.—Find the general solution in integers of the equation

$$20x - 47y = 5 \quad \dots (a)$$

Give also three least positive integral solutions.

For convenience, let $a = 47, b = 20$.	$b = 20)47 = a(2$
The reckoning on the right shows	40
that	—
$7 = a - 2b,$	$7)20(2$
$6 = b - 2(a - 2b) = 5b - 2a,$	14
$1 = (a - 2b) - (5b - 2a) = 3a - 7b.$	—
Thus $1 = 47 \cdot 3 - 20 \cdot 7$ and a can	$6)7(1$
be written in the form,	6
$20x - 47y = 5(47 \cdot 3 - 20 \cdot 7);$	—
$\therefore 20(x + 35) = 47(y + 15).$	1

Here it is assumed that x, y denote integers. Now 47 is prime to 20, $\therefore 47$ is a divisor of $x + 35$;

$$\therefore x + 35 = 47m \quad \text{and} \quad y + 15 = 20m,$$

where m is an integer or zero, and the general solution in integers is

$$x = 47m - 35, \quad y = 20m - 15.$$

The three least positive integral solutions are obtained by giving to m the values 1, 2, 3, in succession as in the table below.

m	1	2	3
x	12	59	106
y	5	25	45

Partial Fractions.—The general theory is out of place in an elementary book. The method described in the following examples is applicable in all cases and has special advantages.

Ex.—Show that values of a, b, c, d, e can be found such that

$$\frac{x^4 + 4x^3 + 7}{(x - 1)(x^2 + 1)^2} \equiv \frac{a}{x - 1} + \frac{bx + c}{x^2 + 1} + \frac{dx + e}{(x^2 + 1)^2} \dots (a)$$

Assume for a moment that such values exist. Multiply each side of (a) by $x - 1$, and put $x = 1$ in the result* ;

$$\therefore a = \frac{1^4 + 4 \cdot 1^3 + 7}{(1^2 + 1)^2} = 3.$$

Proceed as follows :

$$\begin{aligned} \frac{x^4 + 4x^3 + 7}{(x-1)(x^2+1)^2} - \frac{3}{x-1} &= \frac{x^4 + 4x^3 + 7 - 3(x^2 + 1)^2}{(x-1)(x^2+1)^2} \\ &= \frac{-2x^4 + 4x^3 - 6x^2 + 4}{(x-1)(x^2+1)^2} \\ &= \frac{-2x^3 + 2x^2 - 4x - 4}{(x^2+1)^2} \\ &= \frac{(-2x+2)(x^2+1) - (2x+6)}{(x^2+1)^2} \\ &= -\frac{2x-2}{x^2+1} - \frac{2x+6}{(x^2+1)^2} \\ \therefore \frac{x^4 + 4x^3 + 7}{(x-1)(x^2+1)^2} &\equiv \frac{3}{x-1} - \frac{2x-2}{x^2+1} - \frac{2x+6}{(x^2+1)^2}. \end{aligned}$$

Thus $a = 3$, $b = 2$, $c = 2$, $d = -2$, $e = -6$.

It will be seen that, as we proceed, the work verifies itself; moreover, the reasons for assuming the special form are rendered obvious.

Theory of Equations.—The school text-book should contain as much of this subject as is required in other branches of elementary mathematics.

In establishing the *relations connecting the roots and coefficients of an equation*, the remainder theorem is often applied in an unwarrantable fashion. For example,—

“Let $\alpha, \beta, \gamma, \dots$ be the roots of the equation $f(x) = 0$ where

$$f(x) \equiv p_n x^n + p_{n-1} x^{n-1} + \dots + p_0.$$

“Since $f(\alpha), f(\beta), f(\gamma), \dots$, each equal nothing, $(x - \alpha), (x - \beta), (x - \gamma), \dots$ are all factors of $f(x)$.”

“Hence

$$\begin{aligned} p_n x^n + p_{n-1} x^{n-1} + \dots + p_0 \\ \equiv p_n (x - \alpha)(x - \beta)(x - \gamma) \dots \text{,} \text{ } \&c. \end{aligned}$$

The last step is not an *immediate* consequence of the remainder theorem; moreover, it is tacitly assumed that $\alpha, \beta, \gamma, \dots$ are all different.

* Previous examples would suggest this procedure.

† The division of $-2x^4 + 4x^3 - 6x^2 + 4$ by $x - 1$ is as follows :

$$\begin{array}{r} -2 \quad +4 \quad -6 \quad +0 \quad +4 \\ \quad -2 \quad +2 \quad -4 \quad -4 \\ \quad \quad +2 \quad -4 \quad -4 \quad +0 \end{array}$$

∴ Quotient = $-2x^3 + 2x^2 - 4x - 4$; Remainder = 0.

‡ $-2x + 2$ is the quotient and $-(2x + 6)$ is the remainder in the division of $-2x^3 + 2x^2 - 4x - 4$ by $x^2 + 1$.

A preliminary explanation of the following kind is necessary :—

“*Equal Roots.*—If the left-hand side of an equation is divisible by $(x - a)^m$, it is said to have m roots equal to a . This is also expressed by saying that a is an m -multiple root.

“*Fundamental Theorem.*—Every equation has one root. Assuming this theorem (the proof of which is beyond our present scope), it is easy to show that Every equation of the n^{th} degree has n roots.

For let $f(x) \equiv x^n + p_1x^{n-1} + \dots + p_n$ and let a be a root of $f(x) = 0$. By the Remainder Theorem, $f(x)$ is divisible by $x - a$; we may therefore assume that

$$f(x) \equiv (x - a) (x^{n-1} + q_1x^{n-2} + \dots + q_{n-1}) \equiv (x - a) \cdot \phi(x).$$

Let β be a root of $\phi(x) = 0$; as before, $\phi(x)$ is divisible by $x - \beta$, and we may assume that

$$\phi(x) \equiv (x - \beta) (x^{n-2} + r_1x^{n-3} + \dots + r_{n-2});$$

$$\therefore f(x) \equiv (x - a) (x - \beta) (x^{n-2} + r_1x^{n-3} + \dots + r_{n-2}).$$

Proceeding in this way, we can show that

$$f(x) \equiv (x - a) (x - \beta) \dots (x - \lambda),$$

where there are n linear factors on the right. Hence $f(x) = 0$ has n roots $a, \beta, \gamma, \dots, \lambda$, and no others.”

The general theory of the algebraic solution of the cubic and biquadratic would be out of place, but a *working* knowledge of the theory can be imparted by such examples as the following :—

Ex. 1.—(i) Show that $a + b$ is a root of $x^3 - 3xab - (a^3 + b^3) = 0$. (ii) The equation $x^3 - 6x^2 - 6 = 0$ can be put into the above form by choosing a and b so that $ab = 2$ and $a^3 + b^3 = 6$. Determine a and b from these equations, and hence find the real root of $x^3 - 6x - 6 = 0$.

Ex. 2.—(i) Show that the equation $x^4 - 4x^3 - x^2 + 6x + 3 = 0$ can be written in the form

$$(x^3 - 2x + \lambda)^2 - \{(2\lambda + 5)x^2 - 2(2\lambda + 3)x + (\lambda^2 - 3)\} = 0,$$

where λ has any value whatever.

(ii) Write down the equation giving the values of λ for which the expression

$$(2\lambda + 5)x^2 - 2(2\lambda + 3)x + (\lambda^2 - 3)$$

is a perfect square. This equation in λ has a rational root; find it.

(iii) Hence express $x^4 - 4x^3 - x^2 + 6x + 3$ as the difference of two squares, and solve the equation

$$x^4 - 4x^3 - x^2 + 6x + 3 = 0^*.$$

* In this way, the solution of any equation of the fourth degree can be made to depend on the solution of a cubic equation.

The study of a Polynomial in x for small and for large values of x is very important from the practical point of view; it is also a necessary preliminary to the study of *limits*.

First, the meaning of certain phrases should be explained and illustrated, as for instance,—

“ to say that x is small implies that we know or can determine some positive number ϵ such that $|x|$ is always less than ϵ .

To say that x is large implies that we know or can determine some positive number N such that $|x|$ is always greater than N .

To say that x tends to zero is to say that x varies in such a way that its numerical value becomes and remains less than any positive number that we may choose, no matter how small. This is expressed by writing $x \rightarrow 0$.

If $x - a$ tends to zero, where a is constant, we say that x tends to a ($x \rightarrow a$).

To say that x tends to infinity ($x \rightarrow \infty$) is to say that x becomes and remains greater than any positive number that we may choose, however great that number may be.

In this case we also say that $-x$ tends to $-\infty$.

Thus, as x tends to zero, $1/x$ tends to ∞ and $1 - 1/x$ tends to $-\infty$.

The following points should then be considered :—Relative importance of terms,—Approximations,—Graphs near the origin and near “infinity.”

All this can be done in a simple and interesting manner, and, if it were done as a matter of course, writers would not be driven to make a mystery of “infinity” or to convey false notions with regard to limits.

Take, for example, the following statement, taken from one of the best known of the books (published as late as 1910)—

“ A quantity less than any assignable quantity is called zero and is denoted by the symbol 0. A quantity greater than any assignable quantity is called infinity and is denoted by the symbol ∞ . Hence we may now say briefly when $x = 0$, the value of a/x is ∞ .”

It is really amazing that any one can speak seriously of *infinity* as a quantity or a value, or of zero as being something which is not zero.

Limits.—After the preliminary work described above, it is easy to explain the nature of a limit correctly and yet in language which the dullest cannot fail to understand.

It will be advisable to explain separately the meanings of—

$$\lim_{x \rightarrow 0} f(x), \quad \lim_{x \rightarrow a} f(x), \quad \lim_{x \rightarrow \infty} f(x).$$

The following explanation is suggested as being suitable:—

“Limit of $f(x)$ as x tends to zero.—Consider the function

$$f(x) \equiv \frac{(1+x)^3 - 1}{x}.$$

When $x = 0$, $f(x)$ has no definite value, for it then takes the form $0/0$. In other words the function $f(x)$ is *undefined* for the value zero of x . For every value of x except zero, we have

$$f(x) = (3x + 3x^2 + x^3)/x = 3 + 3x + x^2.$$

Thus, if $f(x)$ is nearly equal to zero, $f(x)$ is nearly equal to 3: further, by making x small enough, we can make the difference between $f(x)$ and 3 as small as we please. This is roughly what is meant by saying that 3 is the limit of $f(x)$ as x tends to zero.

In the above statement, the reader will note that the exact meaning of “nearly,” “small enough,” “as small as we please” is by no means clear. The matter may be put precisely as follows:—Choose any positive number ϵ , no matter how small. If $|x| < 1$, we have

$$|f(x) - 3| = |3x + x^2| < 4|x|.$$

$$\therefore |f(x) - 3| < \epsilon \text{ provided that } |x| < \frac{1}{4}\epsilon.$$

Thus, as x tends to zero, $|f(x) - 3|$ becomes and remains less than any positive number ϵ that we may choose, no matter how small. This is precisely what is meant by saying that 3 is the limit of $f(x)$ as x tends to zero.

Def.—If $f(x)$ is a function of a variable x and a constant l can be found such that, as x tends to zero, $|f(x) - l|$ becomes and remains less than any positive number that we may choose, no matter how small; then l is called the *limit of $f(x)$ as x tends to zero*. This is expressed by writing $\lim_{x \rightarrow 0} f(x) = l$.

The above definition is equivalent to the following:—If to any positive number ϵ , no matter how small, there corresponds a positive number η such that

$$|f(x) - l| < \epsilon \text{ provided only that } |x| < \eta,$$

then l is called the *limit of $f(x)$ as x tends to zero*.

Observe that, in these definitions, *nothing is said about the value of $f(x)$ when $x = 0$* . Such a value may exist or, as in the above illustration, there may be no such value. In any case, we are not concerned with it.

Ex. 1. Prove that $\lim_{x \rightarrow 0} (2 + 3x - 4x^2 - 5x^3) = 2$.

Let $f(x) \equiv 2 + 3x - 4x^2 - 5x^3$. Choose a positive number ϵ no matter how small. If $|x| < 1$, we have

$$|f(x) - 2| = |3x - 4x^2 - 5x^3| < (3 + 4 + 5) \cdot |x|.$$

$\therefore |f(x) - 2| < \epsilon$ provided only that $|x| < \epsilon/12$.

In the same way it can be shown that

$$\lim_{x \rightarrow 0} (a + bx + cx^2 + \dots + kx^n) = a.$$

The student, who has read the preceding explanation carefully, will see that this is *not* a round-about way of saying that when x is zero, $a + bx + \dots + kx^n = a$.

Continuity.—Everyone has a pretty clear idea as to what is meant by a continuous curve. For all except the mathematical specialist it is enough to say that a function of x is regarded as *continuous from $x = a$ to $x = b$* , when we have reason to believe that the corresponding part of its graph is a continuous curve.

It is time that such "definitions" as the following should cease to have a place in any book:—

"A function is said to be continuous when for an infinitely small change in the value of the variable, the change in the value of the function is also infinitely small."

Or again,

"A tangent to a curve is a straight line which meets it in two consecutive points."

The dullest can see that there is no such thing as "consecutive points" on a continuous curve, for continuity implies that if A and B are two points on the curve, however close, a point can be found between them on the curve.

Infinite Series.—As might be expected after the treatment accorded to "limits," the usual explanations relating to "infinite series" are very unsatisfactory. For example,—

"When a geometrical progression has an infinite number of terms, we speak of it as an infinite series, and of the sum of such a series as the sum to infinity of the series."

This "definition" does not reveal the fundamental fact that "the sum (to infinity) of a convergent series is essentially a *limit*, and is not a sum in the sense described in the definition "of addition."

A suitable preliminary explanation would seem to be as follows:—

"*Definitions.*—Let u_n be a function of n , which has a definite value for all positive integral values of n . An expression of the form

$$u_1 + u_2 + u_3 + \dots + u_n + \dots,$$

in which every term is followed by another term, is called an *infinite series*. This series is often denoted by the symbol Σu_n and the sum of its first n terms by s_n .

As n tends to infinity, there are four distinct possibilities: s_n may tend to a finite limit, to infinity or to minus infinity, or it may do none of these things.

If s_n tends to a finite limit s , the series is said to be *convergent*, and s is called its *sum to infinity*. Thus s is defined by the equation

$$\lim_{n \rightarrow \infty} s_n = s \text{ or briefly } \lim s_n = s.$$

This is also expressed by writing

$$u_1 + u_2 + u_3 + \dots = s \text{ or } \sum_1^{\infty} u_n = s.$$

If s_n tends to ∞ or to $-\infty$, the series is said to be *divergent*. If s_n tends to no limit, whether finite or infinite, the series is said to *oscillate*.

Series which diverge or oscillate are often said to be *non-convergent*."

In establishing the elementary tests of convergence, nearly every writer spoils the argument by tacitly assuming that the series is convergent.

Thus in considering the convergence of $u_1 - u_2 + u_3 - \dots$, the usual method of procedure is as follows:—

"By writing the series in the forms

$$u_1 - u_2 + (u_2 - u_3) + (u_3 - u_4) + \dots$$

and

$$u_1 - (u_2 - u_3) - (u_3 - u_4) - \dots$$

we see that, since each term is less than the preceding, the sum of the series"

Here it is assumed that the series has a definite sum.

It would be an improvement to insert examples of the following types in the exercises:—

Ex. 1.—Find a value of n such that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} > 100.$$

Ex. 2.—Criticism the following reasoning:—

$$\begin{aligned} 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots\right) &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\ &= \left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right), \\ \therefore \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots &= 1 + \frac{1}{2} + \frac{1}{4} + \dots, \end{aligned}$$

which is absurd, for $\frac{1}{2} < 1$, $\frac{1}{4} < \frac{1}{2}$, and so on.

Ex. 3.—If $0 < x < 1$, prove that the error in taking $x + x^2/2$ as the sum of the series $x + x^2/2 + x^3/3 + x^4/4 + \dots$ is less than $x^3/3(1 - x)$.

$$\begin{aligned} \text{The error} &= \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \dots < \frac{x^3}{3}(1 + x + x^2 + \dots) \\ &= \frac{x^3}{3(1 - x)}. \end{aligned}$$

Binomial Theorem for fractional or negative Index and Exponential Theorem.—For each of these theorems, a sound proof involves considerations which are quite beyond an elementary text-book. At this stage, it is better to boldly assume the results and to give a chapter on the *use* of binomial, exponential and logarithmic series rather than to give the so-called proofs.

The Algebra Syllabus.—There can be no doubt that the school course requires revision. Too much time has been spent on unfruitful topics, and the subject has not been brought into sufficiently close contact with the problems of real life.

In revising the syllabus, the chief points to insist on appear to be as follows:—

- (1) The course should be constructed chiefly in the interests of those pupils who are rather above the average.
- (2) The various steps by which algebra grows out of arithmetic should be dealt with singly.
- (3) The subject should be taught in such a way as to lead up to the Calculus as quickly and as naturally as possible.

In his paper in this series, "The Algebra Syllabus in the Secondary School," Mr. Godfrey insists on our regarding the Calculus as the ultimate end of the teaching of algebra, and, so far, most teachers will agree with him. I propose to consider certain points in this report which will not meet with universal approval.

Mr. Godfrey refers contemptuously to those who regard mental discipline as the chief aim of mathematical education. Yet it is difficult to assign any other reason for teaching Mathematics to the ordinary boy.

That reasoning is to play but a secondary part in his scheme is obvious, for he says "A scientific deduction of algebra from "stated fundamentals is work for the University rather than "the School."^{*}

On second thoughts, he will probably admit that, however elementary the teaching, the fundamentals must be stated clearly and deductions must be made from them in a scientific manner. For the beginner the fundamental assumptions will be many and for the advanced student comparatively few. Presumably Mr. Godfrey had in mind the deduction of algebra from the least possible number of fundamentals.

Again, the syllabus is designed "to occupy the attention of "the average non-specialist up to the close of his mathematical "education at school"[†] and to lead up to the Calculus.

I believe the course laid out to be totally inadequate for either of these objects. I am convinced that a dull boy could

* See p. 285 above.

† Op. cit., p. 231.

easily cover the ground by the age of 16. Further, I know by experience that, by the age of 19 and with the usual allowance of time, the average boy can learn vastly more algebra and at the same time acquire a fair knowledge of geometry, trigonometry and mechanics.

To adopt Mr. Godfrey's proposals is to invite the classical headmaster to restrict the time given to Mathematics, a proceeding which would certainly be contrary to the wishes of the average parent.

Like many other reformers, Mr. Godfrey is at no pains to consider whether his scheme is practicable under existing conditions. He will hardly deny that a much higher degree of competence on the part of teachers would be required than at present. Is he sure of a sufficient supply of competent teachers?

S. BARNARD.

THE TEACHING OF ELEMENTARY MECHANICS

In considering the question of the teaching of Elementary Mechanics in this country, a brief glance must be taken at the history of its development. Engineering and Physics, art and science, practice and theory, have advanced on independent lines, and have only recently amalgamated, and the growing importance of Mechanics as a school subject is due to this amalgamation. Shipbuilding, to take one instance among many, was formerly an art calling for much expert knowledge and applied experience both from the designer and the workman. It has become a science in which experiment and theory are continually working in double harness, and the subdivision of labour, along with the standardization of materials, has taken much of the responsibility from the workman, and laid it on the naval architect. The same thing is true in bridge building, carriage building, and all kinds of mechanical engineering. In addition the applications of electricity have produced a new company of electrical engineers. It would seem that, roughly, one boy in five wants to be an engineer, and it is recognised that a severe mathematical training is wanted for all such boys. The training is provided by Engineering Colleges and Technical Institutions; but many of these require their students to pass an entrance examination in which Elementary Mechanics is a subject. We have therefore a large number of boys who require a course of Mechanics at school with direct application to the requirements of their future profession. There exists also the comparatively small class of boys who are working for mathematical scholarships, and a somewhat larger class of Army Candidates. For these Mechanics is merely part of their general education, a branch of Applied Mathematics. But the movement of approach between practice and theory has manifested itself on both sides, and while the practical men have become more mathematical, the mathematicians on their side have in recent years made a determined effort to make the study of their subject more practical.

The practical teaching of Mechanics in schools is almost entirely a modern growth; Professor Worthington's Physical Laboratory at Clifton was started not more than 30 years ago. Now a large number of schools possess physics laboratories where some mechanical work is done; but even now there is very little connection between it and the mathematical teaching. To quote from a report, published in 1909, made by a Joint Committee of

the Mathematical Association and the Association of Public School Science Masters* :—

“ In no subject is the want of co-operation between the mathematical and the science masters so apparent as in Mechanics. In the large majority of schools Mechanics is taught by the mathematical master as a part of the regular mathematical work ; no experiments are made, and in very few cases are practical difficulties mentioned ; this instruction is confined to the senior boys, with the result that the science master finds the boys who come to him are ignorant of such conceptions as the principle of moments, or the parallelogram of forces, and unable to distinguish between ‘ force,’ ‘ work,’ and ‘ power,’ and is obliged to put most of his younger boys through a short and necessarily inadequate course in Practical Mechanics, in order that they may be able to follow his lectures. There is no connection between the lessons given by the two masters, and it too frequently happens that the better mathematicians leave school without having done any practical work in Applied Mathematics, whilst many promising science boys are deprived of the valuable training afforded by the mathematical treatment of this subject.

“ The requirements of the Army Entrance Examination have caused a certain amount of practical work to be done during the past few years, but even now rather less than half the public schools (which are those chiefly affected by the regulations of the Civil Service Commissioners) report that practical work forms a part of the regular teaching in Mechanics.”

The main defects in organisation appear to be briefly these :—

- (1) The very short course of Practical Statics which a teacher of Physics has to give as a preliminary to the study of Magnetism and Electricity has no connection with the mathematical teaching ;
- (2) Boys working for mathematical scholarships not only frequently miss even this short course (a comparatively trifling matter), but they also have few opportunities of acquiring a practical acquaintance with more advanced mechanical ideas.

The matter of organisation will probably right itself in time, more especially as mathematical teachers are becoming more and more accustomed to laboratory work. The present paper is concerned with the actual details of teaching rather

* The Correlation of Mathematical and Science Teaching. Report of a Joint Committee of the Mathematical Association and the Association of Public School Science Masters. (London, G. Bell and Sons, 1909.)

than with the question whether Mechanics is to be taught as part of Mathematics or of Physics.*

It is assumed that whether practical work and theory are taught by the same person or by different persons, the work will all form part of the same scheme, and for the purposes of this paper it is taken that the laboratory work is under the control of the mathematicians, as in the case of Professor Henrici's Mechanics Laboratory at the Central Technical College. This institution, as a place of training for engineers, and as a college and not a school, comes outside the scope of this paper. But as the inferior limit of age is 16, and as the first year's course for all students includes Elementary Mechanics, it may be as well to mention how that course proceeds. Before any lectures on Statics are delivered, *i.e.*, during the whole of their first term, students spend three hours during one afternoon in each week at practical work in the Mechanics laboratory. Then, when the time for lectures on the subject comes round, they are all familiar with the *things* signified by the *words* employed by the lecturer. The same plan is adopted in the teaching of Kinetics, which follows that of Statics. Naturally there must be great variety in the ability and previous knowledge of the students, many of whom are as old as 19, and all of whom have passed the entrance examination in Mechanics; and their attitudes towards the laboratory work are as various as their attainments; but they all go through the course, and it produces a marked effect even on those whose book knowledge is already considerable. It should be noted that the laboratory in question is mathematical only, under the direct control of the Professor of Mathematics, and quite separate from the engineering laboratories to which students proceed later in their courses.

Before we consider the problem presented to schools of the ordinary public school type, there remain two other institutions providing for a special class, from which much may be learnt. The Naval Colleges at Osborne and Dartmouth carry on the training of naval officers between the ages of 12½ and 16½.

Before the end of the first two years, spent at Osborne, cadets have begun Trigonometry and have been through a course of Practical Statics, devised with the purpose of giving these young boys "a certain number of ideas on the subject of Mechanics, and breaking the ground for a more complete treatment of the subject at Dartmouth." The work is almost entirely statical, though notions of potential and kinetic energy are treated in general terms and without formulæ. Special features of this elementary course are the detailed treatment of pressures and tensions by means of spring balance experiments, and the measurement of work with particular reference to the efficiency of tackles, screwjacks, Weston's pulley, &c.

* A pious hope may here be expressed that before many years are past the two subjects will be taught by the same man.

This simple practical course is now followed at Osborne by a more detailed mathematical treatment, by which the elementary notions of the subject are "rubbed in."

At Dartmouth, where the second two-year period is passed, a more complete course of Statics is followed by Kinetics treated inductively and experimentally throughout. Newton's laws of motion are not assumed, and the rest deduced, as in the ordinary mathematical course; but at each step experimental work is done as a preliminary. The use of formulæ is not encouraged, except in the case of the more able mathematicians; but in all cases an effort is made to ensure that the physical facts lying at the base of the calculations are grasped by the students who make those calculations.

In the case of the ordinary public or secondary school, it may be said that, apart from the short course already referred to, Mechanics does not form part of the general education of all boys. This may be regarded as unfortunate, inasmuch as a reasoned Mechanics is one of the intellectual possessions by which the modern world is distinguished from the mediæval; but the cause is not far to seek. The average boy does not carry his study of Mathematics to the point at which Mechanics is usually introduced, and unless universities and other examining bodies begin to regard it as a necessary part of a general education, it may safely be prophesied that the average boy will not reach that point. It would be possible for teachers of Mathematics to advance the stage of beginning Mechanics by shortening the course of Algebra and introducing the elements of Trigonometry along with quadratic surds and ratio, the laboratory course of Mechanics being carried on simultaneously. The whole question is, for schoolmasters, complicated by the requirements of examinations, and as long as Elementary Algebra is demanded as a subject by itself, divorced from its applications, much time must be occupied in a process similar to that of teaching swimming in a dry gymnasium. The schedules of examining bodies fall outside our discussion, the scope of which may be outlined by the following headings: (1) The age of beginning the subject; (2) its relation to Trigonometry; (3) the question of taking Statics or Kinetics first; (4) the amount of practical work desirable from the mathematical point of view; (5) the need of a laboratory. It is at once obvious that under existing conditions the different classes of boys already referred to must be considered separately in discussing these points. But in view of future possibilities it seems more fitting to range the boys in two classes, not from the examination standpoint but from that of mental capacity. We will not call them (*a*) boys who are going to an engineering college and (*b*) boys who are working for a mathematical scholarship, although this is what may actually be the case. We will rather call them (*a*) boys of a practical turn of mind without conspicuous mathematical ability, and (*b*) boys who

find Mathematics easy. The first of these classes is very much larger than the second, and it is with it that the first head of our discussion is particularly concerned.

The age for beginning Mechanics.—We assume that a course of elementary physical measurements has been worked through by all the boys whom we are now considering, and that length, area, volume, mass, density, and Archimedes' principle have been dealt with in the way which has become almost universal in schools of the type in question. The practical work should have served the purpose of making mathematical work more of a reality, and if the course adopted is similar to that outlined in the Report of the Joint Committee already referred to, recommendations 1-6, it will have done this. Recommendation 7 runs as follows:—

“The term ‘measurements’ as used above should not be understood as including any form of Mechanics; *e.g.*, even simple experiments upon the principle of moments or the parallelogram of forces should be postponed until the pupils have gained some acquaintance with the elements of Trigonometry.”

Up to this point the “measurements” course has been intended for all boys, whatever their mathematical ability. A stage is reached where an Elementary Science course for all boys should begin. The Elementary Physics course usually includes Heat, Light, and Magnetism and Electricity. *Should it include Mechanics?*

The Physics course now takes two different aspects. The science teacher must perforce regard it as of primary importance, and Mathematics as its handmaid. The mathematician's standpoint reverses these positions. For the present let it be granted that we are still using our practical work to help on the Mathematics. The boys have been helped to recognize the advantage of algebraic formulæ for mensuration purposes, they have mastered the decimal point and approximate methods. Before they get much further they must learn indices and logarithms. Elementary Practical Mechanics is not of much use here. A subject like Heat presents more practical applications of the use of logarithms, and Mechanics may very well wait until the beginnings of Trigonometry are approached. After Heat the Physics course very likely takes Magnetism and Electricity. It is here that the science teacher finds the need for a little Trigonometry and Mechanics. Probably the boys have not done any yet, and they have to be put through “a short and necessarily inadequate course in Practical Mechanics.”

Is this a bad thing? Ought the mathematical master to hurry the boys on, so that he may provide a groundwork for the teaching of magnetism? To the writer it seems that no harm will be done by this short and inadequate course, provided that a sufficient understanding exists between the teachers of Mathematics and Physics to make the Practical Mechanics

course, when it does come, supplementary and confirmatory to the previous brief course. Moreover, the Statics required for Voltaic Electricity may become negligible if moving-coil instruments are substituted for tangent galvanometers, and the electrolytic unit of current for the electro-magnetic one.

Our view, then, is this. Mechanics need not form part of the ordinary Elementary Science course except in so far as it is required for the explanation of magnetic and electrical measurements. It is impracticable to debar boys from the study of electricity because they have not made sufficient progress in their Mathematics to have reached the parallelogram of forces in the ordinary course. On the other hand it is important that the study of Mechanics in the mathematical course should be begun by a careful laboratory course, which should, if possible, be conducted by the teacher of mathematics himself, or at least should be drawn up with his assistance and concurrence. Here recommendation 10 of the Joint Committee may be quoted with advantage:—

“ 10. The aim of Mechanics teaching in the earlier stages should be to familiarise the pupil with mechanical principles rather than to enable him to solve mathematical problems. At first, therefore, he should be made to rely upon experimental and graphical methods. As the course advances Trigonometry should be introduced into the work, and in order that this may be ready to hand, a short course of Trigonometry, as recommended above, should precede any instruction in Mechanics.

“ It is advised that Statics be begun in the lower part of the ‘Upper School’ as part of the regular mathematical teaching, *i.e.*, it should be taught by the mathematical master during mathematical hours. The first term should be devoted to practical work in the laboratory, and comparatively simple numerical and graphical applications of the principles arrived at by experiment, the more formally mathematical portions of the work being left to be dealt with in subsequent terms.”*

In considering the details of the laboratory work we are stopped at the outset by (3) “ Statics or Kinetics first ? ” This much-debated question is becoming gradually settled as laboratory teaching of Mechanics becomes more general. They are quite clear on the point at Osborne and Dartmouth. There

* The necessity of laboratory work has been assumed, but there is some truth in the objection that work with specially devised apparatus is apt to become unreal and meaningless. Dr. Routh used to urge his pupils to ponder on mechanical problems and laws in their walks abroad. Bicycles and motors would have added more fields for practising his advice. Practical work without thought is almost as useless as paper work without thought. But even for thoughtful mathematicians the laboratory may furnish material for thought and means of arriving at true notions which common outdoor life does not always afford.

Statics comes definitely first. But it may be urged that the Osborne boys begin Mechanics earlier than would be the case in ordinary public schools, and that for boys starting Mechanics at the age of 16 or thereabouts, which is perhaps the average age indicated by the preceding pages, *Kinetics first* may be the best order. To the writer it seems that to arrive by laboratory experiments at the conclusion, "Force *is* (or is measured by) rate of change of momentum," without employing statical notions of force, is beyond most of us. Kinematics, the study of pure motion, may easily precede work in Statics; but as soon as we attempt to connect movements of masses with their dynamical causes we have to shape in our mind concepts of pressure or tension which could not possibly get there without numberless experiments in Statics performed consciously or unconsciously.

It may be granted that notions of velocity and acceleration may be acquired before or concurrently with notions of moments, centre of gravity, triangle of forces and the like. But for our purpose we will follow the more common practice of taking Statics first. We will also at the same time consider our fourth heading, *the amount of laboratory work desirable from the mathematical point of view*, bearing in mind first the requirements of the average boy.

Experimental work in Statics frequently begins with a verification of the parallelogram of forces. Then follow the triangle and polygon of forces, and various ways of verifying the law of moments. Without in any degree deprecating work of this kind I should like to point out that other practical work of greater importance ought to come first. Before anything else is done, a number of quite simple experiments are necessary to give sound notions of the transmissibility of force and the way in which a force can be measured. The properties of a loaded spring ought to be studied, using, to begin with, spirals first of copper, then of steel wire. Then one of the spring balances to be employed should be tested with weights to find out the degree of accuracy which may be safely ascribed to it. Next two or more spring-balances "in series," and "in parallel," and, if possible, similar work with compression spring-balances, will help to clear ideas on the nature of tension and pressure.

The principle of action and reaction may here be illustrated in many different ways, and it is worth while for boys to carry out a verification in the case of fluid pressure, finding the loss of weight of a body when it hangs in water, and afterwards finding the downward pressure of the body on the water. It is astonishing, but by no means rare, to find boys, and men, who can state Archimedes' principle correctly, and yet imagine that a glass of water counterpoised on a balance will be unaffected when a stone, hanging by a thread from an independent support, is lowered into it.

Again the alteration of the direction of a force by a pulley should be tested for a possible alteration in the magnitude, and the tension of a horizontal string, passing over two pulleys with equal weights hanging from its ends, should be measured by a spring-balance. Ask any average person who has not tried this what the tension will be, and even if he gives a correct answer he will not give it confidently. All this work is not waste of time. It may be helpful even to boys who are quite good mathematicians. Such boys have no difficulty in assimilating the parallelogram of forces and are apt to regard its experimental verification as a bore, unless a high degree of accuracy is insisted on, and shown to be possible; but they are often helped by such preliminary work as has been described.

In the parallelogram of forces the less able boys are very liable to confuse the resultant with the equilibrant, and the parallelogram with the triangle. For such boys the practical work must not be hurried and the polygon of forces should be taken both ways, *i.e.*, a closed vector polygon should be obtained from a set of forces in equilibrium, and conversely, any closed vector polygon being drawn, a corresponding set of forces should be set up, and found to be in equilibrium.

Next we come to the notion of *moments*. Professor Minchin, in a paper read to the Association for the Improvement of Geometrical Teaching (A.I.G.T., Fifteenth Annual Report, January 1889), holds "that the equilibrium of a particle should be thoroughly treated before the notion of a *moment* is introduced at all. This latter notion has no necessary relation whatever to the equilibrium of a particle and it should not be introduced until the equilibrium of a body of finite extension is treated of."

It is at this stage that practical work again becomes of immense importance. The particle has been hitherto represented by a small ring, to which strings have transmitted the forces under investigation. Now if the ring be replaced by a piece of board to which three or more strings are attached in different places, the student finds that another condition is necessary besides the formation of the closed vector triangle or polygon. But he is not likely to discover the second condition for himself in the case of the polygon. A series of practical exercises on the lever and on moment bars and discs must come first.

In the writer's opinion the treatment of parallel forces should follow moments, and a very useful practical exercise can be given by means of a bar laid across two compression spring-balances, and loaded at different places with suitable weights. Centre of gravity experiments naturally follow. When these have been done, the question of the equilibrium of an extended body, such as the board before mentioned, can again be tackled, and will be found comparatively simple. Once again it may be

said that this practical work is indispensable for the average boy, and a great help to the better mathematicians.

Friction, Work, Machines.—Direct friction experiments are not very satisfactory, and time may be wasted if the friction is measured by means of a weight and string passing over a pulley. If on the other hand a spring-balance is employed, both the limiting friction and the sliding friction can be determined quickly and with moderate accuracy for varying loads, and a valuable exercise in averaging results by squared paper may be obtained. But a much more satisfactory exercise for beginners can be obtained from any simple machine, *e.g.*, a screw-jack or a set of pulleys. The notion of work, a difficult one, or rather a meaningless one, unless approached practically, becomes in this way greatly simplified. The velocity ratio of the machine is easily measured, the actual load and the theoretical load are put in parallel columns, and from the difference the work lost in overcoming friction is calculated. The increase of the friction with the load can be clearly demonstrated, and the student's respect for the mathematical laws of friction correspondingly raised.

Meanwhile the student of Mathematics will have been making progress in Trigonometry, and with the equipment provided by the laboratory course he should be capable of mastering any course of Elementary Statics. Kinetics, however, present a more difficult problem.

At the outset we have Kinematics; in other words the relations between space and time in the case of a single moving point.

For those who, like the writer, believe in the historical order as the best, because the most natural, in approaching any scientific subject, Galileo's experiments should form the introduction to any study of Kinematics.

It is a matter of regret that no handy English edition of Galileo's Dialogues is available for school work. But his experiments with inclined planes are very simple and easy of performance. Professor Henrici has combined Galileo's water-clock method of measuring time with an electrical control; but without any expensive apparatus, in fact with nothing except a smooth groove and steel balls, such as are employed in motor car bearings, many of Galileo's results can be verified. A paper by the writer which appeared in the "School World," September and November 1905, contains a description of some of these experiments.

Professor Minchin, in the address already quoted, speaks of the erroneous language of the text-books in common use. Since the time at which the address was given there has been a notable change in this respect. It would be hard to find a text-book in common use in which answers to problems speak of velocities or accelerations as so many *feet* without allusion to time. But all the same, boys still find considerable difficulty in

realizing that space, velocity, and acceleration are measured in different units. The nomenclature employed by Mr. J. B. Lock has not been generally adopted, though, to quote Professor Minchin, "when a *coulomb* and an *ampère* are mentioned, the "very names instantly fix the mind on the two physical notions "involved, notions exactly analogous to those of *length* and "*velocity* in dynamics; the names help the understanding." The difficulties of the student are not diminished by a long course of transformation of units from one system to another, and, although such extreme instances as those quoted by Professor Minchin are not often found now, there is still a good deal of work of this kind which is rather a hindrance than a help.* In the writer's opinion the new physical notions should be assisted by mathematical expression, but in quite a different manner.

The notion of a variable velocity and of the velocity at a given instant have to be acquired. The attempt to express the measure of a variable velocity at any given instant led, as far as we know, to the invention of the differential calculus. What is more natural than to use this opportunity for approaching the calculus in a simple way? Squared paper is cheap enough, and often used unnecessarily. Here is a case in which it should be freely employed. A space-time curve can be plotted from any railway time-table in which the mileages are given. This leads naturally to the measurement of speeds by gradients, and a very little further explanation is necessary to give a meaning to the phrase "velocity at a point," even if the actual symbol ds/dt be omitted. The notion of acceleration as rate of change of velocity follows naturally with the help of squared paper, and severe and valuable exercises in units can be obtained from work of this kind. It is at this stage, and when we are beginning to approach the subject of Kinetics, that we may receive much help from a piece of apparatus of great simplicity and ease of manipulation. It was described in the "School World" of May 1904 in a paper by its inventor, Mr. W. C. Fletcher. It is an adaptation of the old tuning[†] fork and smoked glass experiment, and consists of a strip of steel firmly clamped at one end and carrying an inked paint brush at the other. A long trolley carrying a strip of paper moves underneath this paint brush in the direction of the length of the steel strip. Suppose the strip to vibrate 10 times a second, then the paint brush will trace a wavy line on the moving paper and 10 waves will represent the distance travelled in one second. If the trolley is accelerated by a weight hanging over a pulley, or by tilting the plane to which the

* Mr. Ashford says: "Rigidly insisting on having the units stated "always, and plenty of numerical examples, in my experience entirely solve "the difficulty of getting boys to state and realise the meaning of the " 'answer,' and that is a long step to their thinking in the concrete while "doing the Mathematics."

apparatus is fixed, the acceleration is made obvious to an entire class by the increase in the wave length. A series of records can be taken very quickly and served out to the class to be measured up. It is easy also to adjust the weight employed so as just to overcome friction and thus to obtain a constant velocity. In this way boys quickly realize Newton's First Law, and see that a constant velocity implies equilibrium of the external forces acting on the body. From this it is an easy stage to the measurement of the accelerating force and the resulting acceleration.*

In the opinion of many experienced teachers Newton's Second Law is incapable of experimental verification. The writer does not share this view. He regards the law as stating the direct proportion between acceleration and force rather than as supplying a definition of what force is. But whichever view is adopted there is very little question that boys are in a better position for realising the definition of the unit force after a series of experiments with this apparatus, and subsequent work becomes much simplified. The much debated question of the employment of the poundal as unit force need not be discussed here.

Two trolleys may be employed for momentum experiments and a trolley may be made to set an inertia bar in rotation, thus providing a set of experiments in angular momentum which are useful in giving to students a notion of the actual quantity which enters into their calculations. In fact with all this experimental work although a fair degree of accuracy may be obtained with a little care, a large part of the value lies in the demonstration of the *things* for which words are merely symbols.† The amount of practical work necessary varies with different students. For the best mathematicians it may be sufficient to see the thing done once. Their imagination will grasp the laws quickly, and very little practical work is wanted. With others it may be necessary not only to let them see the experiments but to let them perform the practical work themselves. It is perhaps needless here to enumerate the different portions of the subject in which practical experiment will be found of service. The conversion of potential into kinetic energy, the energy of a flywheel, simple harmonic motion illustrated by the oscillation of a spiral spring, the pendulum, Young's modulus, and the modulus of torsion are a few out of many. Here again we may quote Professor Minchin: "There is nothing so good for

* Successful efforts have recently been made at Dartmouth with the "speedometer" of a motor car to take the place of the vibrating spring in demonstrating acceleration to a class.

† There is a tendency on the part of mathematical examiners to set problems in which the difficulties are mathematical rather than mechanical. There seems to be a need for an intermediate class of questions involving a sound knowledge of mechanical principles, in which the Mathematics are simple enough not to obscure the issue, but are not altogether absent, corresponding, in Trigonometry, to the setting of easy triangles for solution.

“ calling the student’s attention to the question of units as numerical calculation, and yet the tendency of most text-books is to run to symbolism. Mere symbols, unless they are often checked by numerical application, are apt to be devoid of *reality* for the beginner.” Two very valuable exercises may be mentioned as affording good tests of a student’s knowledge of units, the determination of Young’s modulus for a given wire, and the determination of the moment of inertia of a flywheel from an energy experiment.

With regard to the use of symbols, I may perhaps quote some remarks of Mr. C. E. Ashford. “ When a boy of some mathematical ability has obtained equations of motion by induction and subsequent mathematical analysis, and has worked out examples of all types from first principles, and has then applied these equations to the same examples, he can afterwards apply them directly to new examples with a reasonable hope of comprehending the mechanical principles involved throughout the mathematical work ; he may be able to think mechanically while he works in symbols. . . .

“ But the case is different with a boy of small mathematical ability As soon as he reaches the equations he stops thinking (if allowed), clings blindly to them and is quite willing to learn any number of ‘ formulæ ’ and apply them by rule if the master will show him how to do so. He thereby learns to pass examinations, but learns no more Mechanics, and our object is to teach him Mechanics, not to connive at misleading the examiner and the Admiralty.”

In the foregoing pages the quantity of practical work necessary for a mathematical boy has been indicated as varying inversely with his ability. With regard now to the need of a special laboratory, again we must consider the calibre of the boys composing the class. No laboratory is necessary for *demonstration* of most of the various experiments to which allusion has been made. The trolleys take up very little room ; and an apparatus like the Ribbon Atwood machine can be fixed against the wall of an ordinary classroom. It is possible also to fix round the walls of a classroom frames for holding drawing boards and pulleys for statical experiments ; so that a small class of not more than 12 might use the room as a laboratory. With larger numbers and less able boys a special laboratory is a necessity. As has already been said, Professor Henrici has adopted Galileo’s inclined plane for his laboratory work, and uses water-clocks started and stopped electrically for the measurement of time. Such apparatus takes up a good deal of room, but is wonderfully simple and accurate. The question of expense comes in here ; but it may be pointed out that good mechanical apparatus will last for a very long time with reasonable care, and the expense of maintenance is very small indeed.

It has been remarked that a reasoned Mechanics is one of the intellectual possessions which separate the modern from

the mediæval. Kepler was mediæval, Newton the first of the moderns in the region of Celestial Mechanics. But the elementary work which we have been considering is, from a mathematical standpoint, largely pre-Newtonian. The new intellectual vista does not open for any student who fails to reach Newton's demonstration of Kepler's Laws. At a time when many at Oxford are urging the inclusion of Elementary Science as a necessary part of the education of a classical scholar, it may be pointed out that the sister University used to demand from candidates for Classical Honours an effort to attain the wider outlook. The demand is no longer made, and perhaps the cause of learning has suffered. But a warning note may here be struck. The inclusion of Mechanics as a compulsory subject in Responsions or the Previous Examination must mean more than two hours a week of one term spent with pulleys and spring balances if it is to ensure that the student can appreciate the ideas which we owe to Newton.

W. D. EGGAR.

A SCHOOL COURSE IN ADVANCED GEOMETRY.

There is no branch of Mathematics which occupies so large a place in the school curriculum as Geometry, and there is scarcely one which avoids contact with it. On this account it is highly important to secure as complete a coordination as possible between the various forms which Geometry assumes. This is a problem of considerable complexity; so far from there being any general consensus of opinion, it is not far from the truth to say that no serious organised attempt has been made to grapple with the situation.

While the tendency of recent years has been all in the direction of extending the school course in each branch of Geometry, little effort has been made to secure a mode of development which exhibits the fundamental unity of the subject. No one will doubt that this involves real educational loss; not only is it valuable to illustrate one side of a subject by another or to compare one mode of procedure with another, but the student may fail to acquire a comprehensive view of the subject in its entirety, and regard the ground he has covered as so many distinct regions of thought, bearing but little relation to one another, and only for convenience grouped together under a single heading. This is not merely an abstract possibility. The ordinary boy welcomes a strict partition of subjects; and any suggestion to break down the barriers causes him real mental discomfort.

But there is another important consideration. A lack of cohesion in the presentation of the subject involves an unnecessary amount of repetition, and so leads to a waste of time which can be ill afforded. So much is now expected of the candidate for a mathematical scholarship at the Universities, that every master finds it a formidable task to cover the ground in the time at his disposal. The truth of this statement, as far as Geometry is concerned, is rendered evident by a glance at the syllabus with which he is confronted. This may be stated in a brief form as follows:—

- (a) The substance of Euclid, Books I–IV., VI.
- (b) An informal course on the elementary principles of solid geometry and the fundamental ideas of practical solid geometry.
- (c) More advanced work on the properties of the triangle and circle, and the theory of ranges and pencils of the character typified by Casey's sequel to Euclid.
- (d) Analytical Geometry—cartesian and polar coordinates and some acquaintance with homogeneous coordinates.

- (e) Geometrical Conics, based on the focus-directrix definition.
- (f) Projection—orthogonal and conical.
- (g) Duality.
- (h) Homography and Involution.

If each of these sub-divisions is to be regarded as a separate subject requiring development along isolated lines, and to that degree of thoroughness of which it is capable, it is clear that the time which can be allowed at school is wholly inadequate. The consequence is that some of these groups will either be entirely omitted, or else hurried over in such a fashion that the student fails to derive any permanent benefit from them. It is probably true to say that few boys, other than specialists, ever touch on sections (f), (g), (h). Whatever may be said about (g), (h), the omission of (f) is greatly to be regretted, for it gives an entirely new outlook to the student, furnishes him with fresh ideas, and places in his hands an instrument of remarkable power and interest. But, under existing conditions, coming as it does at the end of most school courses in which the earlier parts have been unduly elaborated, there is no time to do justice to the subject.

The practical question it is here proposed to consider is, whether a re-arrangement of the order and the subject-matter will secure within reasonable limits of time a more comprehensive training in Geometry. At present, the ordinary boy, if he covers the course, travels over the same ground in his investigation of the properties of the conic no less than three times by three different routes, as indicated by (d), (e), (f). This is no doubt due to the successive introduction of these subjects into the school curriculum, which corresponds to and probably results from the historical order of their development. Even at the present time, projective geometry is considered by many to be suitable only for specialists, and for them only at the close of their school career. Everything no doubt turns on the manner in which these subjects are presented. The recent custom of making the elementary treatment of analytical geometry mainly numerical and of basing projective geometry on analysis, and illustrating the results graphically does much to smoothe away the difficulties, caused by the novelty of the line of thought. But the success which has attended this change of treatment makes it clear that there is no insuperable objection to introducing these ideas at an earlier stage.

It is necessary to make two alterations in order to secure an earlier treatment of projection than has hitherto been customary. In the first place, the analytical work on conics should be taken simultaneously with and used for simple geometrical properties, a brief discussion on elements at infinity and imaginaries should be included, and the idea of transformation, treated both algebraically and geometrically, from one curve to another should be introduced. And secondly, an exhaustive study of

what is known as "geometrical conics" (*e*) should be postponed until the more elementary results of projective geometry (*f*) have been established.

The amount of time devoted to (*e*) is out of all proportion to the profit derived from it. No one will deny that it provides good practice in rider-work and furnishes the student with a sound and substantial knowledge of the metrical properties of the conic; but the absence of novelty in conception and execution detract largely from its educational value. The methods are precisely those with which he should have already become fully acquainted from his earlier geometrical work, although no doubt the tools he is using are now more numerous. But his outlook is no more extended and the form of the investigation tends to obscure the organic structure of the characteristic features of the conic. There is nothing stimulating in its discrimination between the three species of conics and nothing to suggest the reason for the existence of such elements as the centre, foci and asymptotes. On the other hand, a projective treatment offers numerous opportunities for originality. Indeed, its generalisations and breadth of thought mark a definite stage of intellectual development, as much in the history of the individual as in the evolution of the science. The Principle of Continuity is well called by Dr. Taylor "the vital principle of modern Geometry"; and any scheme of work which fails to attach adequate value to this idea robs the pursuit of Geometry of one of its most precious possessions.

It is, of course, a matter of dispute as to how far purely geometrical results should be allowed to rest on an analytical basis, and still more to what extent the two methods of procedure should be jointly used in the building up of the science. The present writer holds that the dominating consideration is simplicity. Any result which can be more easily obtained by analysis should be deduced in that way. A valuable economy of time is thereby effected; and further, it is desirable that the student should be encouraged to consider, whenever he tackles a problem, what means is most likely to bring success. Of course it may be objected that the ordinary boy is at the present time in danger of having recourse to analysis, just because it affords him a method of attacking a difficulty in a mechanical and unintelligent fashion, that he is too ready to write down equations and wade through sheets of analysis when some simple appeal to Geometry would lead to a solution in a few lines. This is a danger which the teacher has to guard against; and it can only be dealt with effectively by insisting on the importance of geometrical methods when rightly applied. It does not however warrant their use when the object sought may be obtained more simply otherwise. Such a plan merely commits the same error in the opposite direction. Because algebraic methods may be and often are abused, it does not follow that they should not be used when suitable, and the object of

the teacher should be to frame a course which by uniting the two exhibits their relative advantages. It should not be regarded as a merit to offer a geometrical solution of a problem which is capable of a shorter algebraic method; just as it is rightly considered a mark of bad style when a long analytical proof is given of a simple geometrical rider.

Again, by keeping analytical and geometrical methods rigidly apart, the student is apt to fail to interpret the equations he works with. An unintelligent manipulation of symbols is valueless. If, however, the two processes are taken together, there is abundant opportunity for illustrating one from the other. If analytical work is to be educative, it is essential that the student should realise the significance of his analysis. An admirable instance of this is afforded by Dr. Salmon's analytical proof of Pascal's theorem, where the solution is obtained by a translation into geometrical language of the equations employed. Another example is the transference of imaginary elements from analysis to Geometry, and in particular the interpretation of the equation of a circle by means of the circular points at infinity. Great care must of course be taken to ensure the student understanding precisely what is meant by the geometrical usage of ideal and imaginary elements; and at a first reading it is quite certain that analysis provides the only possible base for the admission and justification of at any rate imaginary elements. The purely geometrical theory, remarkable and fascinating as it is, is quite unsuitable for beginners. The student must be taught to consider such elements as merely a convenient mode of expressing in geometrical terms certain analytical phenomena. For example, the statement that the necessary and sufficient condition for a conic to be a circle is that it passes through the circular points at infinity should be regarded as a translation into geometrical form of the analytical phenomenon that the necessary and sufficient condition for the Cartesian equation of the second degree to represent a circle is that the terms of the second degree are $a(x^2 + y^2)$. Or, again, the statement that the tangents to a conic from anyone of its foci pass through the circular points at infinity is a convenient way of stating the fact that the Cartesian equation of a conic referred to its focus as origin is of the form—

$$x^2 + y^2 - (ax + by + c)^2 = 0.$$

The standpoint which this mode of thought leads to is most instructive and illuminating. There is much, then, to be said for a fusion of geometrical and analytical methods. It has its dangers of course; but what scheme is free from them? And few who have put it into actual practice will deny that the advantages which accrue from it outbalance the defects which are likely to appear unless proper precautions are taken.

Among those, however, who acknowledge the general utility of combining geometrical and analytical methods, there will

still remain differences of opinion as to its detailed application. The introduction of elements at infinity is likely to be one of the subjects of discussion. On the one hand, it is urged that the method of projection suggests an interpretation of ideal elements so clear and forcible as to require no additional illustration. It shows that such elements are introduced in order to secure a generality of statement of theorems which otherwise would involve a separate discussion of special cases; and it explains the sense in which all elements at infinity in a plane may be said to lie on a straight line. The geometrical method is clearly most valuable and illuminating. But on the other hand, it may be urged with equal validity that the analytical method is highly suggestive, and may profitably be employed to enforce from a different direction exactly the same ideas which emerge from Geometry. The use of a very simple form of homogeneous coordinates is all that is required to establish the same continuity of statement in analysis as was needed in Geometry. So far from finding their introduction confusing, the student takes a real interest in tracing out the sense of completeness now obtained. Moreover, just because it is so easy for a wrong impression to be formed of the meaning of ideal elements and because the correct attitude of mind is of real educational value, it seems wise to seize the double opportunity which Geometry and analysis offer. Few of those who have tried it in class-work will be inclined to give it up.

Another point on which opinions differ is the degree to which it is desirable to employ analysis for establishing the fundamental theorems of projection. This must ultimately be reduced to a matter of experience, and the present writer has no wish to appear to dogmatise on the subject. In his opinion, however, for school purposes, the preliminary difficulties are greatly reduced by admitting analytical proofs for the fundamental theorems, so that it is then possible to take the subject with boys who have little natural mathematical talent. This is an important practical consideration. For purposes of organization, it is highly advantageous to frame a course which suits equally well the specialist and the boy who is reading Mathematics as part of his general education. The longer it is possible to keep the two together, the better it is for both.

It is, of course, true that some of the initial difficulties may be lessened by laying emphasis on the graphical work; this is certainly valuable and it should find some place in whatever course is chosen. But the real obstruction, if analytical methods are excluded, is the identification of the projective with the focus-directrix definition of the conic. To supply this link, it is necessary to employ either involution or some form of Delbalat's method of proof of the converse of Chasles' theorem, which although short is so difficult as to be quite unsuitable for the ordinary school-boy.

The following course has been drawn up to secure three objects:—

- (1) To economise in time by avoiding needless repetition.
- (2) To bring within the range of the average boy those stimulating ideas which make higher geometry so educative a subject, such as continuity, projectivity and the notion of transformation.
- (3) To frame a scheme which, while suitable for the boy of no marked mathematical talent, will at the same time form a satisfactory preparation for the specialist.

In order to achieve this purpose, analysis is employed wherever it appears preferable to geometrical methods: and the subject known as "geometrical conics" is treated in less detail and at a later stage than is customary.

After reading the substance of Euclid I.-IV., VI., and XI., which now usually includes a very simple course of plan and elevation work, it is customary for the student to proceed to the analytical treatment (at first mainly numerical) of the straight line and, simultaneously with this, it is natural to take an elementary course on the properties of ranges and pencils, exhibiting the results trigonometrically and analytically. With the analytical geometry of the circle, it is clearly suitable to combine the geometrical theory of the radical axis, orthogonal circles and polar properties. At this stage the conception of imaginary and ideal elements should be introduced. This must be done with great caution if a proper appreciation of their significance is to be ensured. Every opportunity should be taken to impress on the student the essential principle that these new ideas are brought into play to secure a continuity of thought and expression, and that they provide a convenient means of avoiding detailed consideration of a number of special cases of what is intrinsically a single fact. The general solution of the quadratic affords a simple illustration; without the use of imaginaries it is impossible to enunciate the general theorem that every quadratic has two roots. So also in the consideration of the coordinates of the points of intersection of a straight line and circle, it is natural to extend the meaning of the term "point" so as to identify it with a pair of coordinates, real or complex, and thus obtain the general theorem that every straight line meets every circle at precisely two "points." Similarly it appears that the radical axis of two circles is in all cases the straight line through their points of intersection; the theory of pole and polar contributes another example of the same character.

The extension of the meaning of a "point" leads to the identification of a "straight line" with the aggregate of "points" which satisfy an equation of the first degree. Some interesting numerical work may be done in connection with conjugate imaginary pairs of points and lines; such questions

as "does an imaginary line contain any or several real points?" are stimulating and instructive. It is also important to examine the analytical results for lengths, angles and cross-ratios in cases where imaginaries occur, with special reference to the lines $y = \pm \sqrt{-1} x$. The analysis is of so simple a character that, with the aid of a few directions and suggestions, the student can make for himself the necessary investigations which should include the properties of pencils where two conjugate rays are isotropic lines. All who have had experience of this work will agree that it arouses remarkable interest in virtue of the singularly surprising and elegant results.

Turning to ideal elements, the mathematical meaning and use of the word "infinity" must be clearly and carefully stated, and the fallacy of regarding it as a *number*, inexpressibly large, must be exposed. Here, again, abundant illustration and discussion is essential; it is fatal to hurry; the logical value of the work to follow will be grievously impaired unless every precaution is taken to secure the correct attitude of mind. On such an occasion as this, an informal class-debate is most effective.

For analytical purposes the easiest plan is to make use of a very simple form of homogeneous coordinates. Suppose that the Cartesian coordinates of a point are $\frac{x}{z}, \frac{y}{z}$ where x, y, z are connected by the invariable relation $x + y + z = 1$. The equations of any two parallel lines then assume the form $y = mx + cz, y = mx + dz$; and this leads to a discussion of the locus $z = 0$, and its identification with an ideal line to be called the line at infinity. Too much emphasis cannot be laid on the fact that this line is ideal or fictitious in the sense that it has no quality of direction or geographical position and is introduced merely for the sake of continuity. It is clearly impossible within the limits of this paper to consider the details of the discussion on this conception which must necessarily follow. The first stage of the work is then concluded with the exposition of the existence and meaning of the circular points at infinity, deduced from the joint use of the equations $x^2 + y^2 = a^2 z^2$ or $x^2 + y^2 + 2gxz + 2fyz + cz^2 = 0$ with $z = 0$, and illustrated by simple examples, treated analytically, such as:—concentric circles have double contact at the circular points at infinity, &c.

It is probably best to start the second stage by an investigation of the equations of the ellipse and hyperbola referred to their principal axes, deduced from the focus-directrix definition. The fundamental formulæ, such as the equations of the tangent, normal, polar, &c., should then be developed simultaneously for the two curves. If possible, the methods of the calculus should be employed when obtaining the equations of tangents. The writer is in favour of taking an elementary calculus course, of the nature of that in use at Dartmouth, as soon as the student

has obtained a firm grasp of the principles of numerical trigonometry. It has been found in actual practice that work of this kind can be fully appreciated at that stage, and naturally the student makes use of this knowledge when he comes to analytical geometry. To illustrate the use of the principal formulæ, certain simple metrical theorems may be taken, and the following results should also be included; the identification of the centre with the pole of the line at infinity, the existence of asymptotes viewed as tangents from the centre, the important analytical theorem that a tangent which passes through a focus must be an isotropic line, and that the directrix is the polar of the focus. This work should be completed by obtaining the standard equation of the parabola, $y^2 = 4ax$ and the analytical formulæ required for the investigation of its properties. It is important to emphasise the fact that the parabola may be considered as the limiting case which unites the ellipse and hyperbola, not only from the idea of eccentricity, but also in its relation to the line at infinity.

The next step is to explain the process of orthogonal projection. First come the fundamental theorems on the disposition of points, lines, ranges, pencils, tangents, polars, &c., which are true for general projection. These should be used to illustrate the one-to-one correspondence between a figure and its projection. Next should be taken the series of properties peculiar to parallel projection such as the transmission of parallelism and ratios of areas and of segments of the same or parallel lines. It is natural to give geometrical proofs of these results in the first instance, but there should be no delay in passing on to the consideration of the very simple form of analytical transformation $x = cX, y = Y$, which determines the relation between a figure and its orthogonal projection. And it is important for the student to realise, by direct substitution, that the previous geometrical theorems can be deduced with extreme simplicity by the analytical transformation just mentioned. This transformation

is next used to show that the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be projected

orthogonally into a circle; and in order to make the process of transmission as clear as possible, it is well to show how the relation must be chosen in order that the process can be actually visualised. Then calling attention to the fact that there is no logical necessity which requires the constants of transformation to be real, and emphasising it by pointing out that the theorems connecting a figure and its projection are all capable of analytical proof—in fact discussing informally the meaning of the Principle of Continuity—it is easy to show that the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can also be projected orthogonally into a circle.

The path is now clear for deducing from the circle a wide group of properties which can be transmitted by orthogonal projection to the ellipse and hyperbola. Without attempting

to enumerate these results, it may be worth while to classify them.

- (1) Polar properties; existence of conjugate points and lines; harmonic theory.
- (2) Existence of diameters; conjugate diameters regarded as conjugate lines.
- (3) Connection with the auxiliary circle.
- (4) Apollonius' theorem on the rectangles of the segments formed by two intersecting chords; numerous deductions (*e.g.* Carnot's theorem).
- (5) Other metrical results and theorems on areas which can be obtained by the transmission of ratios.

It is now instructive for the student to apply these results to the parabola by regarding it as the limit of an ellipse or hyperbola, when the centre moves off to infinity. This will lead him to discover on his own account, the theory of diameters of a parabola, and to enunciate a wide group of rectangle properties.

Having obtained a thorough understanding of the way in which one figure can be transformed into another by an investigation of the very simple example afforded by orthogonal projection, the student is now ready to approach the more general process which conical projection supplies. He has come to see that a given law of correspondences between two figures, whether in the same or different planes, involves the transmission of certain characteristics from one to the other, so that properties in the first have their analogues in the second. The generation of the second figure from the first may be a graphical process or it may be expressible only in analytical terms; that is immaterial. The essential fact is that the correspondence, point by point and line by line, of the two configurations demands a certain similarity of structure which enables properties of the latter to be deduced from known properties of the former system.

It is, therefore, necessary first of all, to examine the effect produced by conical projection on a given system of points and lines and to determine what relations are unaffected by its action. This involves a discussion on the cross-ratio of ranges and pencils, tangents, &c. Metrical properties do not submit to direct transmission, and the student requires exercise in expressing such facts in cross-ratio form; the treatment of parallel lines also demands attention and supplies the teacher with an opportunity of illustrating the geometrical conception of the line at infinity. The next step is to obtain, by a very simple piece of work, an analytical transformation representing the geometrical process of conical projection. The best choice of axes gives this in the form $x = \frac{pX}{Y+p}$, $y = \frac{qY}{Y+p}$. It is instructive for the student to use this relation to obtain analytical solutions for such results as—a system of parallel

lines project into a system of concurrent lines, &c. A moderate amount of numerical work, involving this transformation, is desirable, in order that the student may have a precise idea of the nature and consequences of general projection. As soon as he realises that the operation may be regarded entirely from an analytical standpoint and that the geometrical form assumed is merely a convenient mode of expression, the validity of imaginary projection is established.

The conic is now defined as a curve obtained by any projection of a circle; and the different species of conics—the ellipse, parabola, hyperbola—are discriminated according to the nature of their points of intersection with the line at infinity. In the previous work, the conics considered were obtained from the circle by orthogonal projection. It is therefore necessary to show that no new curve arises from the general process, if we wish to make use of the properties already established. This can be effected by analysis. In the first place, it follows by direct substitution in the equation of the circle that every conic can be represented by an equation of the second degree. And secondly, it is known that the general equation of the second degree can be reduced by a change of axes to one of the forms $ax^2 + by^2 = 1$, $y^2 = 4ax$, which represent the curves previously discussed. This identification can also be made by purely geometrical methods, after the theory of the foci has been developed.

Before advancing to the application of projection to the geometry of the conic, one fundamental theorem must be established, viz., any conic can be projected into a circle and at the same time any given line into the line at infinity. This is effected without difficulty, if the axes are properly chosen, by means of the transformation already mentioned. The student should then prove for himself that the necessary and sufficient condition for a conic to be a circle is that it passes through the circular points at infinity. The theorem just mentioned is then seen to be equivalent to the statement that any two points can be projected into the circular points at infinity, which may be verified independently.

With this machinery, it is now possible to discuss the geometry of the general conic. It is impossible to do more in this paper than indicate the general nature of the work which naturally follows.

- (1) The harmonic theory of poles and polars, conjugate points and lines, the inscribed quadrangle and circumscribing quadrilateral.
- (2) Application of (1) to special cases: the centre, conjugate diameters; consequent properties peculiar to the parabola and rectangular hyperbola; theory of foci deduced from circumscribing quadrilateral; existence of principal axes; focus-directrix property; equations of conics in standard form.

- (3) Standard theorems:— Chasles' theorem on cross-ratio property of conic and its numerous consequences; Brianchon's theorem; Pascal's theorem; properties of systems of conics through four fixed points, including Desargues' theorem; etc.
- (4) A large number of miscellaneous theorems and riders.

As soon as the student understands the general method of procedure, he is in a position to attack a group of problems of remarkable variety. He acquires a sense of power which stimulates and educates his mind and encourages him to pursue the subject and explore on his own account the domain to which he now has access. To this end, it is important to impress on him the necessity of viewing all the elements of a problem from a projective standpoint. Frequent practice in generalising elementary properties of the circle is both interesting in itself and admirably adapted to the purpose of securing insight into general principles. Another matter of practical importance is the value of constructing the generalised figure, whenever possible. This often indicates a method of procedure which is obscured by the special conditions of the original problem. Thus, for example, it is often suggestive if the asymptotes are represented as actual tangents or parallel lines as concurring on a line corresponding to the line at infinity.

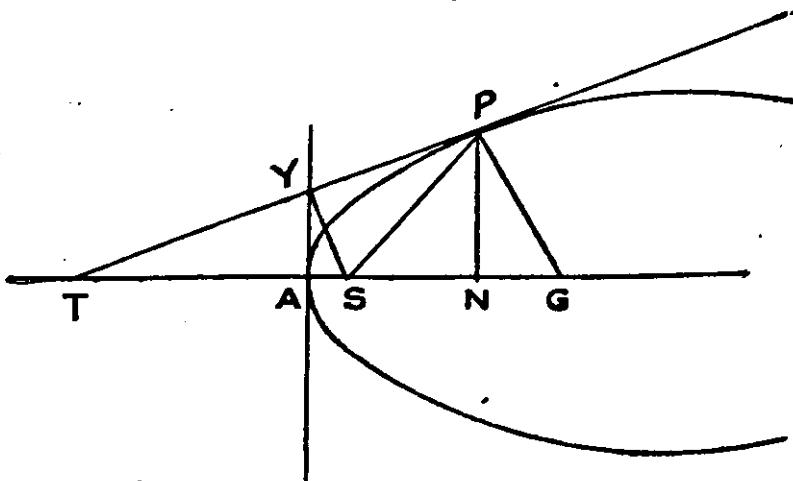
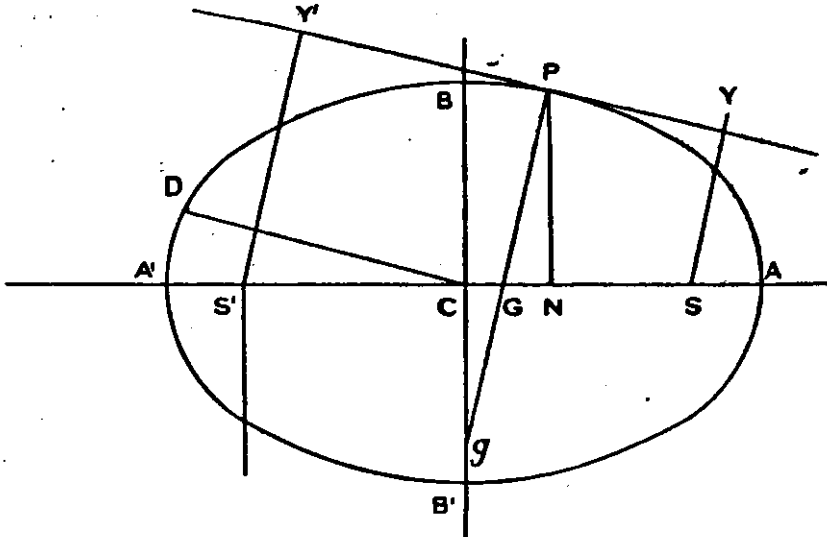
It is well for the student to make a list for himself of the different ways in which properties can be transmitted by projection. This should include the following:—

- (1) Metrical facts must be expressed in cross-ratio form; e.g., $\frac{AB}{CB} = \{ABC \infty\}$; angle $ABC = 90^\circ$ is equivalent to $B \{AC; \omega\omega'\} = -1$ where ω, ω' denote the circular points at infinity; angle $ABC = \theta$ may be represented by $B \{A\omega C\omega'\} = \cos 2\theta + i \sin 2\theta$.
- (2) A circle is a conic through ω, ω' .
 A parabola is a conic touching $\omega\omega'$, or the line at infinity.
 A rectangular hyperbola is a conic with respect to which ω, ω' are conjugate points.
- (3) If C is the centre, C is the pole of the line at infinity.
 If S is a focus, $S\omega$ and $S\omega'$ are tangents. The directrix is the polar of the focus. Conjugate diameters are conjugate lines through the centre.
 The asymptotes are the tangents from the centre.
 Parallel lines are lines which meet on the line at infinity.
 Concentric circles may be regarded as conics having double contact at ω, ω' .
 Confocal conics may be regarded as inscribed in a common quadrilateral, of which ω, ω' are one pair of opposite vertices.

It is obvious that experience alone can show the student the best way of choosing the mode of projection. But it is

astonishing how rapid is his progress in this respect. He is not baffled by the novelty of the idea, but enjoys it to the full. The impression that the subject requires either a marked maturity of mind or conspicuous ability is responsible for its place at the end of the specialist's school course. Experience shows how groundless this opinion is, if the theory is developed by analytical methods.

It is necessary to turn next to that group of theorems which are best treated without any reference to projection. In such cases the choice between Geometry and analysis should be made solely from considerations of simplicity. It is not worth while for any, save the specialist, to devote much time to exercises on these results, which should themselves be set in the first instance as riders, but it is important that the student should be sufficiently familiar with their substance to be able to quote them when required. They are not very numerous, and it will not be out of place to enumerate them, using the English standard notation, as illustrated by the adjoining figures.



The Ellipse and Hyperbola.

(1) $SG = e \cdot SP;$

(2) $CG = e^2 \cdot CN;$

$$GN = \frac{b^2}{a^2} \cdot CN,$$

$$PG = \frac{b}{a} \cdot CD,$$

$$Pg = \frac{a}{b} \cdot CD;$$

- (3) The tangent is equally inclined to the focal radii through the point of contact;
- (4) If U, I are the feet of the perpendiculars from any point O on the tangent at P to SP and the directrix, then $SU = e \cdot OI$;
- (5) Tangents subtend equal or supplementary angles at the focus;
- (6) The feet of the focal perpendiculars on any tangent lie on the auxiliary circle;
- (7) $SY \cdot S'Y' = CB^2$;
- (8) The property of the director circle.

The Parabola.

(a) The limiting forms of (1), (3), (4), (5), (6), (8)

(β) $SP = SG = ST;$

(γ) $AN = AT;$

(δ) $NG = 2 AS;$

(ε) $SY^2 = SA \cdot SP;$

(ζ) the exterior angle between two tangents equals the angle either subtends at the focus;

(η) the circumcircle of the triangle formed by three tangents passes through the focus.

It is suggested that (2) should be proved by analysis.

The student has now acquired a tolerably comprehensive view of the geometry of the conic. It has been found in practice that two hours a week for four terms is required to cover the scheme outlined above for sections (γ), (ε), (ζ), (η), with part of (δ); this, however, allows for rather more practice in analytical manipulation than the above course suggests.

The final stage is now reached. Frequent opportunities have already presented themselves for introducing the idea of duality, notably in connection with the harmonic theory of the quadrangle and quadrilateral, and certain polar properties of the circle. This work can now be generalised by investigating the theory of reciprocation with any conic as base. This is a comparatively simple matter, but the theory of point reciprocation presents greater difficulty. Limitations of time will probably make it necessary to restrict the book-work to the reciprocation of a conic with respect to its focus, and of an angle formed by two straight lines. In addition to the ordinary metrical proof of the

first of these, it is highly instructive to examine the descriptive method which is based on the fact that the tangents from the focus are isotropic lines, and to discuss in the same way the case of a confocal system. With this very limited material, it is possible to make numerous applications, and the student should be encouraged to utilise this method to formulate new theorems for himself.

Similarly it is best to avoid much detail in discussing homography and involution. The important thing is for the student to realise the nature of the one-to-one correspondence set up by the fundamental relation $axx' + bx + cx' + d = 0$ and the consequent cross-ratio property. This is best achieved by a series of numerical and graphical examples which can be arranged to illustrate almost all the chief theorems; these exercises should also include the analytical tests for point-pairs in involution, expressions for the double points, Desargues' theorem and some mention of homographic and involution ranges of points on a conic. But it would be beyond the scope of this paper to offer any detailed syllabus.

In conclusion it may be well to summarise briefly the contentions which are made in support of the above proposals.

- (1) Considerable time is saved by welding together the analytical, geometrical and projective theory of the conic into one subject; a clearer view is obtained of its unity; and a firmer grasp of its fundamental principles is ensured.
- (2) The change of standpoint adds to the interest which the student takes in the subject, gives him an added sense of power, and encourages him to investigate its developments on his own account.
- (3) The study of projective geometry and the new sphere of thought bound up with it is no longer restricted to the specialist.
- (4) Although further experience will suggest modifications, yet the course in its present shape is not merely a paper-scheme, since it has been in actual operation for the last few years.

C. V. DURELL.

THE CALCULUS AS A SCHOOL SUBJECT.*

Students of the calculus in English schools fall into three classes. At one extreme (A) are boys aged 17 to 19 who will in due time compete for University scholarships and in all probability proceed to take an Honours course in Mathematics.

At the other extreme (B) are found, at any rate in paper schemes, boys of 12 to 14 who are studying the elements of the subject.

Lastly, (C), in many schools boys (aged 16 to 18) who have reached the higher mathematical classes take the calculus as a regular item of school Mathematics.

This paper is mainly concerned with Class C, and the other two groups will be discussed very briefly.

Class A.—As regards scholarship candidates, the traditional syllabus cultivated a very considerable power of manipulation before any suggestion was made that a differential coefficient had any geometrical or physical significance. The first application was to questions of maxima and minima, and this was followed by the discussion of curves given by algebraic equations of degree higher than the second: for from work in analytical conics the calculus was rigorously excluded. To these higher plane curves very much time was allotted.

The fact that most of these candidates would proceed in due course to such subjects as Analytical Mechanics was undoubtedly a partial justification for the time spent in procuring additional weapons instead of using those most ready to hand. But, especially with the less able candidates, the effect produced by so general and extensive a course may be indicated by the criticism that they found it much easier to "find the equation of the tangent to $y=f(x)$ " than to "find the angle which the tangent to $3y=8x^2$, at the point where $x=\frac{1}{2}$, makes with Ox ."

The changes during recent years have been in the direction of—

- (i) Laying increased stress on soundness of demonstration.

During the interval which usually elapses between December, when a boy gains his scholarship or exhibition, and the end of his school career in July, special attention is often devoted to this matter. It may be a question whether more attention to soundness of reasoning at quite an early stage in school Mathematics would not be better repaid.

- (ii) Diminishing the extent of work on higher plane curves and introducing some, at any rate, of the more direct applications of the calculus to Mechanics.

* The writer desires to express his gratitude to Professor Love and to Mr. W. J. Greenstreet for most generous assistance. He is also deeply indebted to many members of the London Branch of the Mathematical Association for information.

- (iii) Replacing the study of the more advanced and difficult theorems of the differential calculus by that of the simpler processes and applications of the integral calculus.
- (iv) Guarding against mere formal acquaintance with generalities by demanding numerical applications.

A good idea of the nature of the work of this class of candidate will be gathered from other papers in this series, and from the examination questions appended to the paper by Dr. Macaulay and Mr. Greenstreet.

Class B.—It is familiar knowledge that Professor Perry has on several occasions forcibly and eloquently urged that the calculus should be an element in the ordinary course of Mathematics for quite young boys of 12 to 14.

Events appear to indicate a movement in this direction, but at present attempts to begin the calculus before the age of 16 are relatively infrequent, for at the very outset a doubt arises.

Infinite labour has been expended on teaching Greek to small boys. Some skilful and experienced schoolmasters are confident that the knowledge so acquired with pains to all concerned might have been gained more easily, and with less exhaustion of the motive power of the victims, had the commencement of the study been postponed for three or four years, to the age of 16.

Infinite labour has been expended on teaching the beginnings of formal arithmetic to children under the age of seven. It now appears that psychological theory, and careful practical observation of results, alike lead to the conclusion that better effect is obtained with less injury to the victims by postponing the attempt for another couple of years.

Mr. Jackman at the Imperial Education Conference in April, 1911, observed that much time was wasted in trying to teach children between seven and nine years old what they would learn easily at 10 or 11.

Nothing but careful and conscientious observation in the class-room by skilful and broad-minded investigators with ample opportunity of comparing results can decide the question at what age the calculus can really be profitably commenced. Inquiry of several teachers of experience appears to indicate that, according to the opinion of the majority, an ordinary boy is definitely unable to take in the ideas* of the calculus until the age of 15 or thereabouts. Whether, however, possible methods of instruction have yet been so adequately thought out and tested as to render even an expression of opinion anything but premature may be a fair question.

*. The differential calculus is the science of rates and its peculiar subject is quantity in a state of change."—Buckingham, "Calculus" (Chicago, 1881), quoted by Haldane, "The Pathway to Reality."

Class C.—The attempt to include the calculus as an element in the ordinary syllabus of the upper classes is being made in a considerable number of schools at the present time.

Two important public examinations—that for entrance to the Army, and that for admission to the second division of the Civil Service—have had a marked effect both on the introduction of the subject and on the lines which its development has taken. The examination for the Army has a considerable influence on the work of many public schools. The “junior appointments” competition attracts some of the ablest pupils from the secondary schools with a normal leaving age of 16 to 17. The syllabus of these examinations—so far as relates to the calculus—is as follows:—

“A working knowledge of the notation and fundamental principles in so far as they can be illustrated graphically, with simple applications to the properties of curves, to turning values, and to easy mechanical and physical problems.

“Further discussion of geometrical applications to be treated under co-ordinate geometry. A working knowledge (without rigorous fundamental demonstrations) of the elementary infinite series for $(1 + x)^n$, e^x , $\log(1 + x)$, $\sin x$, $\cos x$, $\tan^{-1}x$, and their use in approximate calculations, especially in finding the slope at a given point of the graph of a function. In co-ordinate geometry the methods of the Calculus may be used freely.”

Specimens of typical examination papers will be found in the appendix to Mr. Hawkins' paper in this series, and these, with the questions printed in the Appendix to this paper from the Junior Appointments Competitive Examination, will give some notion of what is attempted.

Though there are considerable variations in the extent and mode of treatment of the subject, it is believed that on the whole the results produced under this syllabus have been very satisfactory. The better students undoubtedly acquire a sound grip of principles, and can make simple and direct applications of their knowledge.

This being a brief statement of the present position, which is admittedly changing before our eyes, the question evidently arises, “What are the reasons for including the calculus in a school course of mathematics?”

A discussion of these reasons may supply material for answering the questions of when to begin, how to begin, with whom to begin, which beset the practical teacher.

(1) First may be named the importance of the subject as a part of the history of human thought, and the intellectual beauty of its main ideas. No branch of knowledge possesses a longer or nobler history: Greece and Italy, France, Germany and England, and many another country too, can each point to names in the long roll of workers who gradually laid that foundation firm and strong on which the structure of modern science rests.

In subtlety and beauty of ideas the subject hardly yields even to pure geometry, while its human interest is incomparably greater.

It may be that at present hardly enough attention is paid to the history of the subject, and at any rate there is little or nothing available in the way of recorded experience in teaching mathematical history to ordinary students.

The difficulties of avoiding obsolete notations and side tracks are obvious. Nor is it easy to maintain a due sense of proportion and relative value. Someone has remarked that children see little difference between the period of Elizabeth and that of Cæsar. Each lived a long time ago. We do not wish to produce the impression that Newton, Neil, and Gregory, each the proprietor of a theorem, are stars of equal magnitude.

(2) *Utility of the Calculus.*—No man, unless he is free from intellectual anxiety of any kind, can be guaranteed against a wish that he knew more Mathematics. The most cursory glance through the files of a scientific journal will reveal the fact that writers on biology, on experimental psychology, on statistics, on physics of apparently the most definitely experimental type, have invoked help from the processes and from the notation of the calculus.

Sixty years ago, in a passage* which has been often quoted, Sir John Herschel, writing an introduction to astronomy, observed: "Admission to the sanctuary, and to the privileges and feelings of a votary, can only be gained by one means, sound and sufficient knowledge of mathematics, the great instrument of all exact enquiry, without which no man can ever make such advances in this or any other of the higher departments of science as can entitle him to form an independent opinion on any subject of discussion within their range."

To-day the chemist, the electrician, the botanist, the psychologist, the statistician, the soldier, the physician, are all feeling the truth of Sir John Herschel's statement.

(3) Even with reference to the ordinary subjects of mathematical instruction at school a very considerable saving of time may be effected if the methods of the calculus can be used with confidence and facility. For instance:—

(a) Much algebraic and trigonometric drill will be carried out unconsciously during the study of the calculus, and in consequence preliminary work in algebra may be to some extent lightened.

(b) In analytical geometry, instead of painfully working out the equation of the tangent to each different form of conic, with sub-divisions for each type of equation, by algebraical trickery appropriate to each special case, a general method is available. At least seven

* *Outlines of Astronomy* (London: Longmans, 1850).

distinct "pieces of bookwork" are thus replaced by one.

- (c) In mechanics it is impossible to deal in any satisfactory manner with varying velocity and with acceleration without the ideas of the calculus, though these ideas may be more or less successfully concealed: and though, as will be mentioned later, much may be done by a graphical substitute. In particular, the widespread impression among students that acceleration is invariably uniform is due to attempts to dispense with the ideas of the calculus.

As statements of principle the foregoing reasons appear to be generally accepted; but there is no such agreement as to the practical action which should follow, and certain details must be considered more closely. In the first place, as to the amount of algebraic facility required, we must ask whether our main object is utilitarian or æsthetic. It is quite true, if the latter is our aim, that the notion, for instance, of the tangent, as pointed out by, and yet as distinct from, the line joining two points on the curve taken arbitrarily near together, may be studied just as effectually from the equation $y = x^2$ as from the equation $y = f(x)$, where $f(x)$ is a function of some defined amount of generality.

The subtle paradox of the impossibility of motion may be indicated with none but the simplest algebraic operations on $s = \frac{1}{2}ft^2$.

Even so, however, Mach's remark that a competent view of the world is not to be got as a gift must not be overlooked.

Not merely the practical necessities of the class-room, but a sound pedagogic instinct, compel the teacher to make his pupils work for their ideas.

If, on the other hand, our object is mainly utilitarian—to put a powerful weapon into the hands of the future physicist or engineer—a different degree of algebraic skill is necessary. It is true that many practical problems may be solved with exceedingly simple analytical machinery by one who knows how. But here a vicious circle may easily be set up.

It is just the reasonably skilled algebraist who can detect the scheme of notation, and the slight transformations, which will avert complicated algebraic reductions.

Some years ago a boy would learn to find the n^{th} differential coefficient of $e^{-ax} \cos bx$ before having any idea that the velocity of a body which fell $16t^2$ feet in t seconds could be found by differentiation. As has been remarked by Mr. Eggar, such a course gave the impression of a cinematograph film whose time-sequence had been accidentally reversed. Professor Perry in his well-known treatise on the calculus initiated a reaction, by showing how much could be done by or for a student who could differentiate and integrate x^n .

But the reaction against excessive preliminary cultivation of algebraic skill threatens to become itself excessive. A valuable caution will be found in Professor Gibson's "Elementary Treatise on the Calculus," at p. 296: "Much of the difficulty beginners find in integration is due to a deficiency of power in algebraic or trigonometric manipulations. . . . The student who has not a thorough mastery of elementary algebraic or trigonometric transformations . . . will have to struggle with difficulties that are due not to the nature of the calculus but to his own deficient algebraical training."

The engineer whose only possession in the calculus is the differential coefficient of x^n is better off than one whose stock in trade of clear conceptions is nil, but he will be apt to find that the weapon he relies upon breaks in his hand when tried upon any actual problem.

It may be admitted that the greatest danger a schoolboy mathematician runs is that his memory may be allowed to do the work of his imagination. But does not this very danger show that the memory may be given work to do, and do well, which would be intolerably irksome later?

Another remark, which, as a general statement is universally accepted, is that difficulties must be dealt with one at a time. This principle not only indicates that the requirement of some algebraic dexterity before commencing the calculus may be defended, but has another bearing.

There are at the present time many university students whose primary interests lie it may be in engineering or in physics but who find some acquaintance with the notation and ideas of the calculus indispensable or very much to be desired.

Text-books and courses of lectures intended for such students have been written and may be profitably consulted, but with caution; for the young man, even the somewhat dull and ordinary young man of 20, differs from the schoolboy, even the brilliant schoolboy, of 16, in his greater intellectual maturity and greater breadth of view.

Thus an illustration, drawn from mechanics or electricity, of a piece of pure mathematics may be helpful and luminously clear to the undergraduate physicist but may merely place another obstacle in the schoolboy's path.

Previous Knowledge requisite.—The calculus must not be considered as a separate subject; the true problem is to frame a coherent and systematic course which in Geometry,* Algebra, Trigonometry and Mechanics shall develop the elementary ideas (without the notation) of the calculus, hence lead naturally to a formal treatment of the calculus, and thus constitute in its entirety an ordered body of doctrine. Thus, for instance, the slope of a chord will be dealt with in simple cases in algebra;

* The use of "limit" proofs in Elementary Geometry is not here contemplated

both by formulæ and by graphical work. The notion of a rate will receive careful illustration in arithmetic from examples on death rates, rates of growth, cost per hour or per mile of coal for a ship, wages per week or per piece, relative importance of errors of measurement, &c., &c.

In mechanics the difficult concept of variable velocity will be discussed thoroughly, with constant reference to facts.

By this means a considerable knowledge of the ideas and principles of the calculus will have been gained and ingrained before any of the formal theorems are approached. According to this view, the question, "What amount of mathematics must be known before beginning the calculus?" is badly framed.

The true questions would seem to be:—Accepting the view that the calculus is desirable and attainable for the boy of fair average ability who continues at school until 17 to 18—

- (1) How can a scheme of work be drawn up which will steadily lead up to the required results?
- (2) What portions of the traditional work can be omitted as both of small value in themselves and incapable also of forming part of such a scheme?
- (3) How shall the line be drawn between results which are needed for setting up the calculus, and results which are more easily obtained by using the calculus?

It is in connection with the last question that the greatest differences of opinion exist. On the one hand it is acknowledged that familiarity with the thing must precede acquaintance with the name. A student must have met with specimens of limits and have a good working notion of what a limit is before he is confronted with the name or with any special notation.

It is desirable that a student should have some notion of finding the volume of a solid by breaking it into slices or elements before being confronted with the formula for the limit of a sum.

In the past this sound principle has unquestionably been abused. Heavy and rigorous summations, effected by ingenious algebraic devices, for finding the volume of a sphere, or the distance moved by a falling body in t seconds, called for an unreasonable amount of time and energy. Moreover, the student's attention was so engrossed in the manipulative details that he usually failed to realise the presence of a principle. He dug for the gold and had some benefit from the digging, but he did not find the gold.

At the present moment the tendency is perhaps the other way. To introduce generalities without sufficient preparation from particular cases has always been the besetting sin of mathematical teachers, and the more mathematical the teacher perhaps the greater the likelihood of the offence. Moreover, it is possible to deal in generalities without indicating that generalisation is the process by which mathematical knowledge

develops. While it is generally agreed that the ideas of the calculus must be prepared for and led up to by simple illustrations and "sketchy" methods, there is at present, and is for some time likely to be, considerable variety of opinion as to the amount of calculus work without calculus notation which is desirable and the degree of rigour which should be attempted.

In brief, we all condemn "calculus-dodging" but we do not all agree as to what constitutes dodgery.

The same differences of opinion exist as to the details of the second question. For instance, in a paper in this series Mr. Godfrey advocates the omission of progressions from the Algebra course. The majority of teachers appear to be of a contrary opinion, for the reason that boys find the topic easy and interesting and that the teachers regard it as usefully leading up to notions of summation and relative magnitude, with sufficient incidental work to give the pupils a run for their ideas.

To give one specific illustration of the sort of doubt which continually arises: Should pupils study the maximum or minimum values of a few expressions of the forms $a + bx \pm x^2$ graphically and by completion of the square, as a preparation for the calculus method; or is the time spent on such exercises unremunerative?

In the remainder of this paper a few detached topics, as to which opinions are much divided, will be briefly noticed.

Rigour of Demonstration.—The modern demand for logical rigour has not as yet had very much effect on elementary work. The establishment of the differential coefficient of x^n illustrates this clearly.

It appears that the majority of teachers obtain the result that, if $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$ (n being constant) by quoting the binomial theorem.

To this there are two objections:—

- (1) If room is to be found for the calculus the binomial theorem may have to be thrown overboard.
- (2) It is certain that the pupil has not proved the binomial theorem in a satisfactory manner: and consequently the whole structure is in the air.

How is it that the old and easily avoidable method lingers? For this reason: the more rigorous and elementary methods involve the discussion of three if not of four cases:—

- n a positive integer,
- n a negative integer,
- n a rational fraction, and perhaps
- n irrational.

Now, there is nothing the ordinary boy dislikes so much as what he calls proving the same thing four times over. Moreover, the boy has, at the present time, almost certainly "done"

the binomial: and it seems a pity not to use it. How far a sounder and more logical treatment (which is much needed) of the elements of Algebra will lead to a better appreciation of logical accuracy is an interesting and important question, of a satisfactory answer to which there seems no reason to despair.

The solution ultimately arrived at will probably be that which is being slowly groped towards in Geometry. Assumptions must be clearly indicated; care must be taken to show, first, that there is some justification for them and, secondly, that they are assumptions and not axioms or deductions; and, subject to these precautions, assumptions must be freely made when required. But while assumptions are numerous, deductions when they are undertaken should be rigorous. The importance of rigour in a school course of Mathematics must not be undervalued. A person whose education has been on other lines and whose standard of precision of thought and execution has been formed may want to pick up the notation of the calculus as he might wish to pick up a few sentences of Russian in view of a prospective journey. But if the calculus is to be part of the medium by which a schoolboy's standard of reasoning is to be formed, the foundations must be sound. Nor does this demand for logical correctness proceed from the teacher only. A dislike amounting to disgust at seeing theorems produced from the sleeve is by no means rare among pupils and should be respected.

The Exponential Theorem and Logarithms.—Perhaps at the present moment the question of “*e*” occasions more trouble than any other.

There are four alternatives each of which finds advocates:—

- (1) Obtain, more or less carefully,—

$$L^t \left(1 + \frac{1}{n} \right)^n = e,$$

$$n \rightarrow \infty$$

and then differentiate $\log. x$ from first principles.

- (2) Obtain by some geometrical device the slope of the graph of $y = a^x$ at some assigned point and choose “*a*” so that the slope at $x = 0, y = 1$, may be 1 in 1, or so that the tangent at the point $x = 1$ may go through the origin.
- (3) Assume an infinite power series for y in terms of x and determine the coefficients so that $\frac{dy}{dx}$ may be equal to y .

- (4) Define $\log. x$ as $\int_1^x \frac{1}{t} dt$.*

* Hardy, “Pure Mathematics.” (Cambridge University Press, 1908.)

Methods of the second type should, it is thought, be definitely abandoned. They are artificial, and difficult for the pupil to recollect, and they do not conciliate the rigorist.

No one would advocate (4) for the class of pupils under consideration. This is a serious objection. To set up separate "practical" and "academic" courses in the calculus will deprive each party of valuable material and will repeat a mistake from the effects of which the teaching of mechanics has not yet recovered.

The fact that method (1) is capable of almost indefinite graduation in point of rigour is a strong argument in its favour. Moreover, even if in the first instance weak places are left in the argument, it is easy to point them out, and thus bring home to the pupil, in a manner most satisfactory and direct, the good sense and necessity of subsequent rigorous treatment.

Graphical Calculus.—An attempt is being made by what is known as "graphical calculus" to convey the ideas of the calculus without its notation.

A boy most certainly can acquire some facility in finding the velocity of a moving object from a space-time diagram drawn to scale, or in interpreting the area of a resistance-displacement curve as representing work done.

How far the principle of the method enters into his mind—so that he could reconstruct his rules after a six months' holiday—is a question upon which opinions differ.

Experience seems to show that in many cases the boy relies on his memory for the fact that the distance travelled is represented in a velocity-diagram by the area between the time axis and the graph, and not by the area between the velocity axis and the graph.

In other words, the diagram for the time being is meaningless to him, though it may not always remain so.

In considering the value of this graphical method the distinction already alluded to between pupils of the schoolboy type and of the undergraduate type is of great importance.

A person who has some considerable knowledge of a branch of physical science, let us say of steam engines, has subconsciously acquired, or at any rate is ready for, many of the ideas of the calculus. He has little difficulty in apprehending that the area of a (p.v.) diagram represents work or that the slope of a velocity-time graph represents acceleration. To such a student a course, on the lines of the pamphlet* by "F. R. S.," to the effect that the calculus gives a short way of expressing familiar ideas, may have considerable value. To such a student, again, the graphical calculus will appeal strongly: not only from his familiarity with graphical work, but because the graphical calculus

* "Calculus made Easy." (Macmillan & Co., 1911.) A translation in Spanish has been published.

can deal with cases which are not amenable to analysis and yet are of practical importance.

It is open to any one to contend that the schoolboy of 16 should be in the same position. Some promising educational experiments are tending to put him into such a position, but he has not yet reached it.

Notation.—A very vexed question arises: Should a beginner be taught to use differential coefficients, differentials, or derived functions?

A popular text-book by Professor A. Lodge uses *differentials*. One argument in favour of this course is that the differential, say of x , being an increment, is the same sort of quantity as x , whereas $\frac{dx}{dt}$ is a different quantity.

To this it is answered that familiarity with the notion of a rate, whether gained arithmetically or from the study of experimental mechanics, is an essential preliminary to a study of the calculus.

The old objection to the use of differentials was, to put it crudely, that the statement

$$\text{“If } y = x^3, \text{ then } dy = 3x^2 dx \text{”}$$

was either false or else illusory. This objection has lost much of its force (and this must be put down to the credit of rigour) from the disappearance of many somewhat metaphysical discussions about “infinitesimal quantities.” Definite integration is probably decidedly easier to a student trained on differentials. On the other hand, experience seems to lead the majority of teachers to the conclusion that the method of differentials involves trouble when second differentiation becomes necessary, and generally that it is ill suited for complicated cases. Though there has been a decided increase in the number of students using the “differential” notation it is still true that the majority work with $\frac{dy}{dx}$, the differential coefficient of y with respect to x .

Two remarks may be made:—

(1) It is a decided inconvenience to a lecturer to have in the same room people who have learned to use differentials, though not easily, and people who have learned to use differential coefficients, though not easily; for to such students an unfamiliar notation will often be the “last straw.”

(2) The suggestion of fractional character made by $\frac{dy}{dx}$ is unfortunate, $\frac{dy}{dx}$ being just not a fraction but the limit which the fraction $\frac{\Delta y}{\Delta x}$ never attains as a fraction. The notation $f'(x)$ or y' , is free from this objection.

On several other subjects, *e.g.*, whether the best way of beginning the integral calculus is by consideration of areas, or, as the reverse process to differentiation; what is the proper relative sequence of learning for the differential and the integral calculus; whether a naive 18th-century treatment of infinite series should be accepted and gradually discarded, or whether such series are altogether out of place in an elementary course and should be postponed until Taylor's theorem comes to be faithfully dealt with; it must suffice here to say that on these questions, too, judgment has certainly not yet been given by the voice of the profession.

To sum up :—

1. The teaching of the calculus to the average boy of 16 to 18 is in an experimental stage. The great need above all others is for increased opportunities for teachers to compare their views and to arrange for the systematic testing of alternative methods.

2. The introduction of the calculus, with abundant simple applications, as part of the work of boys of 16 of more than average intelligence, has been very successful. Their appreciation of mathematics and their willingness to attempt a new problem instead of looking at it has increased immeasurably.

3. How far the same effect can be obtained by attempting to teach the same work to clever boys of 13 or 14, or all boys of 16, must be regarded as an open question, as to which *some* evidence is available in favour of a negative answer.

4. New ideas must be led up to by preliminary work, which must not, however, take up so much time that a student is tired before he gets to the real thing. A class likes to get to business.

5. Applications to Mensuration, Mechanics, and other concrete matters should be made prominent throughout the course, always with a due regard to taking difficulties singly and not illustrating *obscurum per obscurius*. Geometrical applications should hold a rather subordinate place. The mensuration of volumes is helpful as an introduction to integration. As soon as possible the thing that is being led up to by concrete illustration should be itself introduced, and immediately after the formal introduction of the theory or process its utility for similar and other concrete purposes should be illustrated.

6. Any thing which has a name, *e.g.*, Function, Limit, Differential Coefficient, Integral, should be known by concrete examples before it is named, but the names should not be concealed when the things are known. For example, the notion of a limit is essential, and the name should be given to it.

7. A certain degree of rigour both in definitions and demonstrations is essential. Students are really quick to detect that an argument is not sound although they may be unable to point out the fallacy. Sometimes the uneasy feeling that there is something wrong prevents them from making further progress.

8. Extreme rigour is apt to retard progress by seeming to create unnecessary difficulties and to be hunting for trouble. The wisest plan may be to give an incomplete proof and insinuate the doubt *at the time*. It should be explained that the proof, not the result, is in doubt.

9. All conventions require abundant explanation. Conventions of sign are often perplexing, and so also is the custom of making the same symbol or letter stand sometimes for a number and sometimes for a quantity measured by the number in terms of certain units.

10. It is very desirable that the teacher should know something of the history of the subject, but there does not seem to be any experience by which to estimate the value of attempts to include some details, or a general outline, of the history in the course of the instruction. It would seem to be obvious that historical treatment would enhance the value of the subject as a part of general education.

11. There is comparatively little scope for intuition in calculus work.. The difficulty in general is what to do in the absence of intuition, how to detect the absence of this inner light, and how to correct an erroneous intuition. In practice an assertion by the teacher will often have to pass for an intuition by the pupil. This consideration weighs with some who appreciate fully the superiority of a genuine intuitive conviction over acquiescence in the result of a deduction, but who hold that the way to attain at last to the intuition is through the deduction.

C. S. JACKSON.

APPENDIX.

QUESTIONS FROM THE OPEN COMPETITIVE EXAMINATION FOR
 "JUNIOR APPOINTMENTS," &C. IN THE CIVIL SERVICE,
 DECEMBER, 1911.

MATHEMATICS. CLASS III. PAPER 1.

2. Give a definition of the base e , explaining how the necessity arises of introducing this base in certain applications of the Infinitesimal Calculus.

The rate at which a substance dissolves in water at any instant is proportional to the quantity of the substance which is undissolved at that instant. What percentage of the whole will be undissolved in one hour (1) if 10 per cent. of the whole is dissolved in the first tenth of an hour, (2) if 1 per cent. is dissolved in the first 0·01 of an hour, (3) if it dissolves *initially* at the rate of 100 per cent. per hour?

3. Find the slope of the tangent to the curve $y^n x^m = 1$ at a point on the curve whose co-ordinates are given to be x' y' , and deduce the formula for the differential coefficient of $1/\sqrt[n]{x^m}$ in terms of x .

With $m = 1$, $n = 2$, find the *angle* which the tangent makes with the axis of x at the points $(-2, \frac{1}{2})$ and $(-1, 1)$. Draw a graph of the curve and verify roughly.

With the same values of m and n prove that the segments of the tangent at any point $(x' y')$ intercepted between the axes are in the ratio of 2 to 1, and test by measurement on your figure.

5. A solid homogeneous cylinder rolls a certain distance down an inclined plane in 1 second; a cylindrical tube takes 1·1 seconds to cover the same distance. Express the radius of gyration of the tube and its internal radius as decimals of its external radius.

MATHEMATICS. CLASS III. PAPER 2.

Time allowed, 3 hours.

1. An airship starts from the ground at a point P distant one mile away from an observer O. It rises at the rate of 15 miles per hour at an inclination of 35° to the horizon in a vertical plane inclined at an angle of 45° to the line OP, the motion of the airship being away from the observer.

Find by drawing or by calculation the angle of elevation of the airship as seen from O at 1, 2, 5, 10, 15 . . . minutes from starting, and draw a graph showing the relation between the angle of elevation and the time measured from the start.

What is the asymptote of this graph?

2. Transform the integral $\int_{-1}^1 f(x) \sqrt{1-x^2} dx$ by the substitution $x = \sin \theta$ and evaluate it when $f(x) = x^2$.

Transform the integral $\int_1^a \frac{dx}{x}$ by the substitution $x = y$.

From the latter transformation deduce that if the area of the curve $xy = 1$ between $x = 1$ and $x = a$ be denoted by $f(a)$ then $f(a^n) = nf(a)$.

3. A strip of sheet metal 1 foot wide is to be bent into the form of a rain-water gutter of triangular section, one side of the triangle being vertical and attached to the wall of the house and the other side projecting outwards and upwards, while the third side of the triangle is open and horizontal. Find, by direct calculation, the angles and sides of the triangle when the area of the cross-section of the gutter is a maximum.

Examine whether the result could be inferred from first principles by supposing two such gutters to be placed one above the other so as to form a pipe of triangular section.

4. Four forces in equilibrium act along the sides of a trapezium (a quadrilateral with a pair of opposite sides parallel). Prove that the four forces are proportional to the four sides, the forces in the two non-parallel sides corresponding to the sides along which they act, and the parallel forces corresponding to the sides opposite to those along which they act. If the quadrilateral is formed of jointed bars to which the forces are applied, and is kept in shape by a fifth bar along the diagonal, show that the stress in this is proportional to its length.
5. Explain how the differential calculus gives a measure of the velocity v and acceleration f of a point moving in a straight line, whose distance s from a fixed point in its path is a given function of the time.

Deduce the equation $f = \frac{d}{ds} (\frac{1}{2} v^2)$

If s is proportional to t^n prove that

$$v = \frac{ft}{n-1} \quad s = \frac{vt}{n} \quad s = \frac{ft^2}{n(n-1)} \quad v^2 = \frac{n}{n-1} fs$$

and if the average velocity in any interval, starting from rest, is equal to one-third of the final velocity, use these results to find to what power of the distance from the starting point the acceleration is proportional.

6. A tube of thin glass the area of whose cross section is k , and whose length is l , has its upper end closed and it is lowered into a cylindrical jar partly filled with mercury, till it just touches the liquid. It is then depressed through a distance s . Supposing the area of a cross-section of the jar to be K , and the height of the mercury barometer to be H , find equations giving the distances of the mercury surfaces inside and outside the tube below and above their original level.

Take $H = l = 76$ cm., $s = 38$ cm., $k = 1$ sq. cm., $K = 4$ sq. cm. Obtain numerical results, and find the work done in displacing the mercury, neglecting the work done in compressing the air in the tube, and assuming 1 c.cm. of mercury to weigh 13.6 gm.

MATHEMATICS. CLASS III.

LABORATORY WORK.

Time allowed, 1½ hours.

You are advised to read through the paper before beginning to work.

Your work will be valued according to its accuracy, the completeness of your record of it, your care of the apparatus, and the suitability of the methods you adopt.

Should the time in the laboratory prove insufficient for the completion of your record and your calculations, half an hour more will be allowed in another room.

On the sheet of drawing paper provided draw a straight line AB of length 20 cm. On opposite sides of AB construct a circular segment ACB of height 8 cm. and a parabolic segment ADB of height 12 cm., CD the perpendicular bisector of AB being an axis of symmetry for the whole figure.

Draw in the figure a series of chords parallel to AB, the distance between consecutive chords being one centimetre.

Measure the lengths of these chords and deduce approximate values for:—

- (i) The area of each segment,
- (ii) The distance from AB of the centre of mass of each segment and of the whole figure,
- (iii) The moment of inertia of the whole figure about AB and about CD.

So far as time permits, obtain these results also by direct calculation without using your measurements.

HIGHER MATHEMATICS FOR THE CLASSICAL SIXTH FORM.

INTRODUCTORY.

It is to many of us a deplorable fact that the exigencies of preparation of boys in the Sixth Forms of Public Schools, for Entrance Scholarships at the Universities, actually or supposedly necessitate the abandonment of Mathematics by these competitors, at least during the term or terms immediately preceding the scholarship examination; and it is true of many schools that a boy drops Mathematics entirely on entering the Upper Sixth, or at any rate devotes to the subject only such time and energy as will ensure his passing the standard required for entry to the University. Most mathematical masters must have frequently seen with keen regret pupils arrive at the most profitable and interesting stage of their work only to be compelled to abandon it abruptly on their reception into the Sacred Band of the Classical Upper Sixth.

The consequence is that every year large numbers of intelligent boys—on the whole the best educated and most capable of all—leave the Public Schools with a knowledge of Mathematics just sufficient to take them safely through “Smalls” or “Little-go,” and for the rest of their lives give no more attention to the subject. To them the Higher Mathematics is a distant and inaccessible region in which they feel little interest. They have never been led near enough to its boundaries to know what it contains, and in most cases the desire to explore even its outlying fringes has never occurred to them. They believe either that it is an arid and inhospitable tract devoid of features to attract the ordinary traveller; or else that it is sundered from them by a gulf which they cannot hope to cross. If this state of affairs is not inevitable, and if it can be remedied without hurt to the classical side of the boy's education, it is surely our duty to make some attempt at improvement.

It is no doubt true that just as there are persons to whom music is meaningless, and others for whom poetry differs from prose chiefly in being “left ragged at both edges, or sawn off straight on one side only,” so there are children and adults who are congenitally incapable of appreciating or profiting by any mathematical instruction. But such persons are probably much “rarer birds” than is commonly supposed, and need not be considered in any arrangements made for the welfare of the general community.

At all periods and in all countries, through the manifold varieties of curriculum, there are, probably, only two subjects which hold an undisputed place in the weekly, if not daily, time-table. They are the Mother Tongue and Mathematics. Now proficiency in the Mother Tongue may be acquired indirectly through home environment, through reading incident to literary studies, especially in the classical training in which the Public Schools are admittedly strongest. It has been charged against these schools that they neglect the study of English, and it is certainly true that the ordinary public school boy's acquaintance with his native language is deplorably deficient. But the members of the Sixth Form are not ordinary boys. Their classical work has taught them to express themselves, as a rule, with facility and correctness and appreciation of literary form. Their wider range of general knowledge has given them a maturity of outlook which differentiates them sharply from other boys of the same age. In short, the knowledge and wisdom of the Classical Sixth makes it a peculiarly fertile ground for educational seed.

But if the classical scholar of the Sixth Form has acquired, by the wayside, as it were, the greater part of whatever skill he may possess in the use of his own language, it is surely not possible for him to derive mathematical knowledge thus by devious channels and unconscious absorption. If he drops Mathematics when he reaches the University Entrance standard, the odds are long against his ever taking further interest in the subject. And so it comes about that a large proportion of the pick of the Public Schools—boys who go on to win their Firsts and Fellowships at the Universities in History and Classics and Philosophy, to make their mark in the Civil Service, in Law, in Journalism, in Politics—are doomed to remain equipped for life with only the meagre rudiments of arithmetic, algebra, and geometry, and perhaps a smattering of trigonometry.

Now it sometimes happens, of course, that even this limited stock of elementary knowledge has been perfunctorily acquired and imperfectly assimilated, but much more frequently the boys in question have their groundwork soundly established, and have arrived in addition at a relatively mature intellectual stage where their mental outlook and appreciation of facts is markedly in advance of their technical knowledge of Mathematics. Their high standard of general culture forms an admirable equipment for an introductory survey in outline of some of the higher departments of Mathematics, for which purpose the power of clear and intelligent thought is of more importance than mere technical facility or specialised knowledge.

WORK OF THE SPECIAL SET AT TONBRIDGE.

When Mr. G. St. L. Carson some three years ago assumed command of the Mathematical Department of Tonbridge School,

he reorganised the teaching of Mathematics throughout the school. The aim of this readjustment of methods and subject matter was to give the boy from the beginning a clearer sense of the relation of his mathematical work to the facts of his everyday experience, without sacrificing in any way his training in clear and accurate thinking; to foster a greater elasticity, handiness, and originality in his work, and at the same time to retain all the benefits given by the logical rigour of the old methods of pedagogic practice. In this contrast I avoid the terms "practical" and "mental gymnastic," because the latter covers a multitude of sins of dullness and servility, and the former too often connotes no more than countless "graphical" solutions of equations in algebra, and in geometry such "proofs" as the inference that the angles of a triangle are equal to two right angles from the fact that the mean of half a dozen measures of actual triangles with the protractor amounts to, say, 179° . I do not wish to arrogate any special virtue for Tonbridge Mathematics, but I mention this rather definite attempt to extract the fertilising phosphates from the dry bones of the subject, because the experiment which forms the excuse for this paper is in close relation to the general scheme.

It was in September 1909 that arrangements were made for four hours a week of mathematical work for those members of the Upper Sixth who were free for the ensuing year from the immediate imminence of the claims of University Entrance Scholarships. The boys, who numbered some half dozen, were denominated the Special Set, and I was entrusted with their instruction. It is on the results and experience of a year's work with these boys that the present paper is based.

The conditions under which we began were not of the most favourable. None of the boys, except one, had done anything but quite elementary work; most of them had done no Mathematics at all for a year or so; all were sadly out of practice; and finally the average of mathematical ability in the set was, as it happened, decidedly lower than the general standard of boys passing into the Upper Sixth would be in an ordinary year. The average age of the set at the beginning of the year was 17, and though the boys were mainly of the plodding type, the general level of common-sense and intelligent appreciation of facts of everyday experience was as high as one would expect in the younger members of a Sixth Form.

We began with no definite scheme or schedule of work. The aim was to open out suggestive lines, to prepare the ground for new and fruitful ideas concerning the meaning of problems and the method of tackling them, rather than to study exhaustively any special field either from an abstract or practical point of view. We proposed especially to deal with simple statistics, of the type that give rise to everyday problems in business, or politics, or science, and thus to gain some insight into the rudiments of mathematical research. Even in cases where

questions that interested us were too difficult or too complex for actual solution with our limited knowledge, it would be no small gain to realise, even somewhat hazily, the broad and general lines on which such problems might be tackled and the nature of the tools employed.

It was, of course, necessary at the outset to give the class some simple practice in the plotting and reading of curves to remove the rust of disuse and to promote quickness and facility in the more mechanical side of their work. Further preliminary effort was spent on such matters as the choice of suitable scales in graphical representation; methods of tabulation; arrangement of work; meaning of statistics of various kinds; and the degree of accuracy possible with the data available. This preliminary work was carried out mainly by problems selected to illustrate particular points in tabulation, or graphical presentment of data, or interpreting of given curves, wherever it happened that the boys discovered difficulties or showed misconception and confusion of ideas. With the groundwork fairly established, we proceeded to consider the idea of a "rate" by means of problems involving that idea in one of the simple forms in which the boys would already possess the notion as part of their ordinary experience. The Sixth Form boy, who has done nothing but elementary mathematics, has doubtless acquired the conception of a rate, but only in a limited degree. He has not met with the idea in any other form than that of a time-rate, derived from cases involving speed or growth. Even such a comparatively simple illustration as the "Birth-rate" was, I found, by no means clearly grasped at first. The chief difficulty at this stage was in selecting problems which would enable the boys to extend their rudimentary notion of rate to cover a wider and more varied field where time did not enter into the case. I found that statistics of population provided problems more effective for this purpose than anything else. By the end of the first term the boys had attained a fair proficiency in dealing by graphical methods with statistical questions of a varied character, and with simple commercial and scientific problems, most of which would have been meaningless to them a few months before. They had acquired in some degree the notions of a limit and a function; their conception of average and rate had gained greatly in precision and value; and they were familiar in a practical and inductive way with the elementary ideas of the calculus. When this stage was reached, they were introduced to a more abstract consideration of the subject-matter, and devoted a few periods to the algebra of simple differentiation. We did not spend much time on this deductive treatment of the subject, but I believe that it is important for the pupil to realise that in simple cases, at any rate, he can deduce his results from first principles, independently of graphical representation. Our programme so far had been varied occasionally by digressions

into questions of Algebra, Geometry, and Trigonometry, arising out of the work in hand. Among other things we had studied the geometry of the regular solids, and before beginning our final revision of the year's work we undertook the calculation of the areas and volumes of the pyramid and sphere by the method of infinitesimal "strips" and "steps." This work called for some excursions into the field of progressions and series, thereby providing useful algebraic exercise, and at the same time offering valuable illustrations of such fundamental ideas as those of convergence and approximation and limiting value, which had already arisen in the previous work of the Special Set.

The foregoing outline of our general plan of operations is, I fear, rather vague, and for those who may be interested in the details of the course pursued I append the three papers set at the Summer Examination on the year's work.

These papers illustrate faithfully and copiously our choice of subject matter and our handling of it, and they will give a better idea of our general policy and programme than any detailed description.

TONBRIDGE SCHOOL.—July 1910.

SPECIAL SET.

Mathematics I.

1. A tall cylindrical vessel has a uniform inside cross-sectional area of 20 sq. cms. It contains water which is allowed to escape from a tap at bottom into a graduated vessel, so that the volume that has escaped up to any moment can be read off the scale directly.

The time ($=t$ secs.) that has elapsed since the turning on of the tap is noted at the instants when the total amount of water that has run out is 50, 100, 150 cub. cms., &c. ($=V$ c.c.).

The results are tabulated thus:—

t secs. = time elapsed since tap was turned.	0.76	1.53	2.28	3.13	4.05	5.01	6.06	7.20	8.48	9.99
V c.c. = total volume of water that has run out.	50	100	150	200	250	300	350	400	450	500

(The second decimal place in the values of t is not very trustworthy.)

Plot a graph from these data and hence construct a table and graph showing the rate in cms. per second at which the surface of the water in the jar is descending. Hence answer the following questions, giving your reasons in each case:—

- (i) Is the rate of descent of the surface uniform?
- (ii) If not, is it changing uniformly?
- (iii) What can you find out about the acceleration of the descent, if any exists?
- (iv) At what rate is the surface descending when $t = 5.7$ secs.?
- (v) How can you check the accuracy of any portion of your "rate of descent" curve from the original data?

2. How many regular solids are there? What are their names? What conditions limit their number? Draw a perspective sketch of each; and give in tabular form the number of the faces, edges, and polyhedral angles of each, the shape of the faces, and the number and size of the plane angles that compose the polyhedral angles of each.

3. Draw the curve $y = x^3$ between the limits $x = -1\frac{1}{2}$ and $x = 1\frac{1}{2}$, taking 1 inch as unit both horizontally and vertically, and plotting the value of x^3 for each change of 0.1 in the value of x .

Then calculate the value of $\frac{d(x^3)}{dx}$ for any value of x , and plot the curve $y = \frac{d(x^3)}{dx}$ on the same scale to the limits of your paper. Hence:—

- (i) Show that the tangent of the curve $y = x^3$ at any point where the value of x is positive is parallel to the tangent at the point where x has the corresponding negative value.
- (ii) Make the closest approximation you can to the area contained between the x axis and the portion of the curve $y = \frac{d(x^3)}{dx}$, between the limits $x = -1.2$ and $x = 1.2$.
- (iii) What meaning can you attach to this area? What check can you apply to the value found for it?
- (iv) Point out any features in one curve that are related to features in the other, and explain the connection.

4. ABC is any triangle, whose sides are a , b , and c . A is an acute angle. The length of the perpendicular from A on the side a is p . Hence the area of the triangle is $\frac{ap}{2}$. If x is the

length of the projection of AC on BC, note that $p^2 = b^2 - x^2$. Also note that $x = b \cos C$. Given that in any triangle $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, p may then be expressed in terms of a , b , and c only.

Hence show that the area of a triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{a+b+c}{2}$$

TONBRIDGE SCHOOL.—July 1910.

SPECIAL SET.

Mathematics II.

The American lake-ferry steamboat "Navajo" is 240 feet long. She draws 10 feet of water at normal draught. She is similarly shaped at both ends, being designed to travel in either direction without turning. The cutwater at both ends is vertical.

The table gives the half-width of the ship (b feet) at different heights above the keel for various distances (s feet) from either end:—

Distance of section from } cutwater = s feet =	0 feet.	10 feet.	20 feet.	30 feet.	40 feet.	60 feet.	80 feet.	100 feet.	120 feet.
Half-width of vertical section at 1 foot above keel, keel = b feet =	0.1	0.4	0.8	1.6	5.4	9.0	11.0	13.2	13.2
At 2 feet above keel, $b =$	0.1	0.9	1.4	3.7	8.2	11.6	13.7	15.6	15.6
At 4 feet above keel, $b =$	0.1	2.0	3.3	7.3	11.5	14.3	15.9	17.1	17.1
At 6 feet above keel, $b =$	0.1	3.2	5.8	9.8	13.5	15.6	16.7	17.4	17.4
At 8 feet above keel, $b =$	0.1	4.8	9.0	12.0	14.7	16.2	17.0	17.4	17.4
At 10 feet above keel, $b =$	0.1	6.4	11.2	13.4	15.3	16.3	17.0	17.4	17.4

Use the table to plot the shape of the *vertical half-section* of the vessel at the different distances from the end for which the data are given. Use a scale of 2 ft. to the inch. Hence find the area of the *whole* vertical section at these points, and from your results draw a curve showing how the area of the cross-section varies with the distance from an end.

Hence compute the volume of the submerged portion of the vessel.

The weight of a floating body being equal to that of the volume of the liquid it displaces, find the tonnage (displacement) of the vessel.

(1 cub. foot of water weighs 1,000 oz. 1 ton = 2,240 lbs.)

If you have time, plot on a smaller scale, say, 20 feet to an inch, the *horizontal* sections of the vessel at 2, 4, 6, 8, and 10 ft. above the keel. Compute the area of each of these sections. Draw a curve to show how the area of the horizontal section varies with the height above the keel, and hence compute the volume of the submerged portion of the vessel, thus checking your former result.

TONBRIDGE SCHOOL.—July 1910.

SPECIAL SET.

Mathematics III.

1. A cylindrical closed tin is required by a canning firm to hold a gallon of canned tomatoes. If one gallon of the tomatoes ready for canning occupies a space of 256 cubic inches, find the most economical dimensions for the tin. That is to say, find the height and diameter of a gallon tin that will require the smallest area of sheet tin to manufacture. How much tin will be required for each can?

(Hint:—If h = height of tin, r = radius, and k = capacity, then $\pi r^2 \times h$ = volume of tin = k , i.e., $h = \frac{k}{\pi r^2}$. Hence area of tin required can be expressed in terms of r only, and a graph can be drawn to show how the area varies with r .)

2.—(i) Calculate the volume of a sphere by considering it to consist of an infinite number of discs of equal thickness.

(ii) Using your result and the fact that the volume of a pyramid is one-third of the product of the height and the area of the base, deduce the area of the surface of a sphere.

(iii) From your second result deduce the connection between the area of the surface of a sphere and that of the curved surface of a cylinder of equal height and radius.

(iv) Prove the connection deduced in the last part of your answer by some independent method.

3. A patent chemical manure costs 1s. 6d. per lb. to manufacture. Records were kept of the average weekly sales at different prices, with the tabulated results:—

Price	-	-	-	-	1s. 9d.	2s. 0d.	2s. 6d.	3s. 0d.	3s. 8d.	4s. 4d.	5s. 0d.
Lbs. sold	-	-	-	-	4,000	3,900	3,650	3,350	2,800	2,000	1,000

- (i) What was the most profitable price?
- (ii) What other price would give the same profit as 2s. 10d.?
- (iii) At what (high) price do you infer that there would have been no profit or loss?
- (iv) At what price would the raising of the price by 1d. have produced an increase of 10l. in the weekly profits?
- (v) At what price would the lowering of the price by 1d. have produced an increase of 15l. in the weekly profits?

4. If a triangle is divided by lines parallel to one side into n sections of equal width (the distance of the opposite vertex from that side being h), and if its area is taken as equivalent to the sum of the areas of n rectangles of the same width as the sections, what will be the error in excess when the "stepped" figure so obtained has its "steps" outside the triangle, and what will be the error in defect when the "steps" are inside the triangle? Is the one error equal to the other? Is the percentage error the same in both cases?

The marks obtained in these papers were not high because the work shown up was short in quantity and marred by arithmetical blunders. The ordinary boy is a slow worker at the best and his want of the knack which is born of experience in tabulation and symbolic representation often hampers him sadly in dealing with problems even where he understands what is wanted and recognises the method that will lead him to a solution. When he fails to sketch out a labour-saving plan at the outset, it is not surprising that oversights and blunders disfigure his needlessly complicated arithmetic. Nevertheless the results were on the whole quite satisfactory. An intelligent appreciation of subject-matter and method was shown and every question was answered thoroughly and correctly, though not always "elegantly," by one or more boys. In short, the results of the year's work as well as the interest shown throughout by the boys themselves amply justified the experiment.

A few suggestions and reflections arising out of the year's experience may be worth brief mention. In the first place there is the question of the selection of suitable problems. It is, I believe, most important that the subject-matter of the problem should be thoroughly familiar to the pupil. If he does not clearly visualise the material ground on which he is at work, if there is any blurring of the goal in view or of his stages of approach to it, he is apt to lose his bearings and

flounder ineffectively. Again and again I have traced the origin of some difficulty solely to misunderstanding of the subject-matter. A very large field of problems, otherwise eminently suitable, is thus closed to classical boys whose knowledge of Science is negligible. Furthermore only one or two of the boys in question had done any elementary mechanics. Accordingly we were debarred from a prolific source of questions dealing with force and work, readily accessible or easily manufactured. In future years this difficulty will not arise at Tonbridge, for easy mechanics now finds a place in the timetable of every boy in the upper half of the school. In any case the difficulty is not a serious one. The manufacture of problems is, surely, an occupation not without profit to the master, and if time or imagination are lacking, such a book as Saxelby's "Course in Practical Mathematics" will provide a rich mine of material.

Another question concerns the introduction of abstract and deductive treatment of the subject. I do not believe it should be avoided entirely. The boy, as a rule, has a good appetite for a rigorous proof, though his taste may be easily deceived by a plausible counterfeit. Rather does the difficulty lie in persuading him that for practical purposes rigorous exactitude, besides being generally impossible, is no more valuable than an approximation within a certain degree of accuracy. The recurring decimal still exercises a deplorable tyranny over younger boys. The best plan will probably be to give the class just so much of abstract "book-work" as lies within their capacity, and no more than they are willing to digest.

SUMMARY.

This particular year's course has been described more or less in detail as typical of what can be done with Sixth Form boys, but there is, of course, no implication that it is the only programme, or the best. My object is simply to urge that the Sixth Form boy should be given a glimpse of more advanced Mathematics than he has hitherto met with. For example, if the boys have done no mechanics, I am very strongly of opinion that a term at least might be devoted to an elementary treatment of the subject. Whatever the direction of a man's interests in the world to-day, it is scarcely possible that mechanical ideas should fail to bulk largely in his daily environment. Can we, for example, call anyone educated who is ignorant of the meaning of such a universal term as "horse-power"? Indeed it would be hard to find a more suitable subject for such boys as we are considering than the fundamental conceptions of mechanics. And there are other fields in which a course might be selected. Solid geometry provided us with a few weeks' work during the year under review, and in the current year co-ordinate geometry has been given a place in the time-table.

Such work as this can hardly fail to be a source of profit and pleasure to the teacher. His class is small. Problems can be worked out by master and boys together in a way that is impossible with larger classes. Interest is easily aroused and maintained. The simplicity of the fundamental ideas of the calculus, of mechanics, and of the methods of more advanced algebra comes as a pleasant surprise to most boys. The initial difficulties encountered in the elementary stages of arithmetic and algebra are, I believe, relatively far greater than any presented at the frontiers of more advanced work. The further a boy progresses in Mathematics, the easier does still further progress become.

The benefits which an intelligent boy will derive from such a course as here described do not consist solely in his wider mathematical range. The varied problems that can be selected as bearing on questions of political, or commercial, or scientific interest will be handled with a logic and rigour that the daily newspaper never applies to their discussion. The passion for investigation which is characteristic of almost every boy, and the awakening interest of the adolescent citizen in a wider circle of ideas, will provide a mutual stimulus that, properly directed, will find expression in efforts of an inestimable training value.

And what of the æsthetic aspect of the subject? Is there any intellectual cult which worships at a nobler shrine than Mathematics, whose Goddess is Truth, in all its austere and essential purity, and the rubric of whose ritual consists in the ultimate laws of Thought. In Mathematics the standard of beauty is Truth, and the criterion of efficiency is Logic. Plato and Aristotle are its sponsors. "But the mathematicians do not read Plato, while those who read him know no Mathematics, and regard his opinion on this question as merely a curious aberration." I quote from an article entitled "The Study of Mathematics," by the Hon. Bertrand Russell, in the first number of "The New Quarterly" (November 1907), where the æsthetic value of the subject is specially discussed. It is an article which should be read by every teacher of Mathematics. Many, no doubt, while admitting the inspiring qualities of the subject for the advanced mathematician, will maintain that the school-boy cannot yet appreciate such things. I should prefer to say that he rarely gets the chance. He abandons the subject just when the quality of the work itself and his own mental development promise to react upon each other most profitably. It may be granted that it is in most cases some form of literature, music, painting, or the like, which awakens in the growing boy the earliest stirrings of that sense of exaltation that attends the contemplation of the highest achievements of mankind in Art or Science. Without doubt the revelation of beauty in Mathematics belongs to a later and an intellectually higher stage of appreciation. But even if we cannot hope to make the schoolboy drink of these waters, we can at any rate lead him to the brink ;

and if in later years thirst comes to him, he may remember where it can be quenched.

In conclusion I would repeat with emphasis two facts. In the first place—as has been pointed out again and again in recent years—the last half-century has seen a greater enlargement of the encyclopædia of human knowledge than any previous period of history, and there has been no abatement of the rate of increase in the last decade. That enlargement has been almost entirely due to the expansion of the frontiers of science, not merely in new territory of her own, but also in the imposing of her suzerainty of scientific method on old-established departments of human culture. And in the second place we have the Classical Sixth Form boy—the most cultivated product of the Public School—passing out into this altered world with an educational equipment almost unaltered from that of fifty years ago. Cannot we give him some share in this new heritage of culture? Mathematics has been justly called the Mother of the Sciences, and certainly Scientific Method is only another name for Mathematics. Under present conditions, at the time when his mathematical work is on the point of widening out into new channels of living interest and far-reaching import, when the boy's mental outlook is reaching that maturity that would specially help him to appreciate and profit by some study of the higher branches of the subject, just at this point—more inviting and fruitful, if he only knew what awaited him, than any previous stage of his career—he turns his back on Mathematics for ever. Mathematics has its "classics" no less than Literature. Surely the classical authorities will not grudge a few hours weekly for devotion to the science which has called forth the greatest achievements of the human intellect.

W. NEWBOLD.

PRACTICAL MATHEMATICS AT PUBLIC SCHOOLS.

INTRODUCTION.

Changes in recent years.—In recent years great changes have been introduced into mathematical teaching. It has been realised that the methods formerly followed, however suitable they might be for the training of “born mathematicians,” were not suited to the needs of the average boy; and the teaching has been re-modelled in an earnest endeavour to meet those needs. Although it is too early to pronounce a final opinion upon the results, the success of the venture so far is highly encouraging; the boys enjoy their work and learn with alacrity instead of with reluctance. They may fail to surmount, or even to encounter, some of the more difficult lessons of the older regime, to which importance has been hitherto attached; but it is quite possible that this importance was overrated, and in any case the losses may not be sufficient to set against the undoubted gains in other directions. A brief survey of the situation is rendered possible by the appended reports from four different Public Schools.

“Born” mathematicians.—Before proceeding to consider what has been done I may recur for a moment to a remark made in the first paragraph about the “born mathematician.” A Congress of Mathematicians will naturally be anxious for his welfare, and alert against any short-sighted sacrificing of his interests: and I hope I may claim to share this anxiety, as one who owes everything to the care lavished on boys who showed mathematical promise under what may be called the old regime. But I say with some confidence that the new regime is not likely to differ from the old in this respect. I have not seen the slightest tendency to relax vigilance in looking for mathematicians or care in training them when found. On the contrary, I firmly believe that there is on the modern plan an increased chance of finding the boys with special ability, since the net is spread much wider than before. Before he is recognised as such, a boy may possibly learn his Mathematics in a manner which the older school might think slipshod; he might, for instance, learn his geometry without the thoroughness of Euclid’s demonstrations; but this stage is, from the nature of the case, not likely to last long, and when the boy of ability is once detected he can be, and is, treated specially. In my opinion the great gains to the average boy have not been accompanied by any loss to the specialist worth mentioning.

Leading idea of new Methods.—The leading idea which runs through the new methods is stated by Mr. Brewster of Oundle School thus :—

“It is found that a boy will understand and remember a mathematical process with much greater ease if it is introduced to him in connection with a problem of practical value.”

So long as our attention is fixed on this principle alone we should probably all accept it as reasonable enough ; it is only when it comes into contact with other principles that objections arise. The main conflict is with the principle of logical order. Euclid presented his geometrical propositions, not as they were arrived at historically, nor as they might arise from practical problems, but in an arrangement which aimed at logical sequence. And he has been followed in this aim by the great majority of subsequent writers of text books on Mathematics. To adopt the principle above enunciated was therefore to throw over the existing methods in some degree. How far should they be abandoned? The answers have varied greatly, straggling the whole distance from complete adherence to complete abandonment. Those who prefer to keep as close as possible to the old methods have made a minimum concession by introducing a few models and a little squared paper into their class rooms ; the thorough-going reformers have boldly left the class-room altogether and installed themselves in the workshop. “An attempt is made to look upon the class as a staff of workmen,” writes Mr. Sanderson of Oundle, “actually engaged in some ‘live’ work . . . Mathematics come in incidentally and are learnt as need arises. The various mathematical principles and methods are thus acquired indirectly by continually applying them, and the boy learns his elements of Mathematics in much the same way that he learns to walk. At a later stage he may put his knowledge into a logical framework.” In other reports it will be found that less drastic measures have been adopted. Special mathematical laboratories have been organised, where Mathematics are learnt practically by weighing and measuring. Or without setting apart special rooms for this purpose, physical laboratories have been utilised for infusing practical knowledge into the theoretical ; or, as above remarked, models and apparatus have been used in the class-room, with but little departure from the old traditions.

Keen interest of the boys.—But in whatever degree practical work has been introduced, it seems to have been successful in arousing the interest of the boys and stimulating the teaching. It would almost seem as though those who had gone furthest in this direction wrote with the greatest conviction ; but it would be rash to draw the obvious inference, or at any rate to act upon it at present. Rather may we welcome the probability of a period of varied experiment and comparison of results. Such periods may involve strenuous exertion, but the exertion is likely to be of a stimulating and not necessarily of an exhausting

kind. That at any rate is the impression produced by my experience of the keenness of discussion in the Mathematical Association, and among the Science Masters whom I have met in connection with it.

Meanwhile we may note with satisfaction the awakening of interest which seems to characterise the new methods throughout. "This work is the boy's first experience of experimental science," writes Mr. Bell of Winchester, "and he enjoys it He makes a good deal of noise over it, and a master with nerves—or a classical division in the room below—may be worried by the dropping of weights and water; but the boy is keen on the work, and there is no difficulty about discipline." It is surely significant that the discipline should be good even in the presence of *noise*, for nothing is more conducive to disturbance.

The interest shown crops out at various points. Mr. Siddons of Harrow refers to the increased interest in Arithmetic, *i.e.*, in getting sums actually right instead of merely showing knowledge of method. As one who has lectured on Astronomy for nearly 20 years, I can well appreciate the point. Mathematicians, especially those of considerable ability, have often been curiously unable to get a correct numerical result, chiefly through want of attention. It has been my custom to draw the attention of my classes to the importance of correcting this habit by the example of Nautical Astronomy. "If you get your sum wrong, you may sink the ship." If interest is aroused in the result for its own sake, there is little difficulty in getting the sums correctly done.

Mr. Brewster of Oundle mentions yet another point. "The secondary aim is to foster an interest in the higher parts of Mathematics at the earliest stage possible." That this aim can even be stated complacently says much for the new methods. Older generations fixed the limit of their interest in Mathematics at a very short distance from zero.

Co-operation.—It will be seen from the reports that the new system is distinguished from the old by an extensive use of the principle of co-operation. Boys like working together, and in former times this desire has been an inconvenience to be repressed: for it led to "copying," or to one boy doing the work for others. But in the modern system co-operation has positive advantages. They appear most prominently in the Oundle report where the whole plan of work depends on co-operation. "In the workshop method of teaching, the boys are regarded as forming a body of workmen engaged in carrying out some of the regular work of the shop. Each boy is a member of the staff and has his definite part of the work to do—and he finds that whatever this work may be it is of importance to the whole"

In mathematical laboratories there is not of course so complete a co-operative scheme, but the boys work in pairs or

threes. "There is one obvious disadvantage in working in pairs," writes Mr. Siddons, "the possibility of a pair being very unequal in ability, so that the clever boy is kept back and the weak boy scores marks for work mostly done by his partner; to avoid such cases occurring it is sometimes necessary to rearrange pairs. But this possible danger, which is easily obviated, is immaterial compared with the advantages gained by the boys in discussing the experiments together and frequently succeeding in doing an experiment which neither could do alone. Of course all calculations are done by each partner and the results compared," another valuable lesson in the proper method of doing calculations, well known to astronomers.

Wholesale co-operation need not, however, be confined to the workshop. In the laboratory it has been found of great service in the construction of models, which would be far too serious a labour for an individual, but becomes easily manageable for a number working in concert. If, for instance, each boy constructs a plane section of some solid figure, when the whole is built up the excellence or defect of individual sections will teach the lesson (above mentioned in connection with the workshop) of the importance of each definite piece of work to the whole.

Imagination.—With practical work it is much easier to set a boy to think out a thing for himself. Teachers have long realised the difficulty of providing problems which call for spontaneity and not merely for the remembering of some rule. There were, for instance, Euclid "riders," but the good ones soon got tabulated and new ones were either too hard or too easy. In practical work it is only necessary to vary the dimensions of some of the apparatus in an experiment to call for new ingenuity from a boy who may have just done almost the same experiment.

Expenditure of Time.—In one point the new methods compare, at any rate in the first instance, unfavourably with the old. Progress is not so rapid, or at any rate not so obvious. It will be seen from the reports, however, that the advocates of practical methods claim compensating advantages in sureness and reality.

"It is true that the number of mathematical questions or examples [a boy] will get through in the hour or two hours 'lesson' may be very few, but the questions arising in a concrete and visible form will make a more vivid and abiding impression on the mind, and both the methods and results will become an integral part of his being. On the other hand, although a boy will be able to work through many more examples and problems in a class-room, yet it may safely be asserted that the impression left by each one is comparatively feeble . . ." (Oundle report.)

This characteristic of practical work—that it takes more time—is indeed well known. Against it are to be set, not only the considerations of thoroughness above mentioned, but the possibility of according actually more time for work. The

number of hours spent by a boy in the class-room at our public schools has hitherto been rather small, chiefly because the work was of an exhausting kind to both boys and masters. Practical work need not tire the boys in the same way, especially when they are keenly interested. It is in many respects little different from the carpentering and similar work with which some boys were and are accustomed to employ part of their leisure. Hence we may contemplate without anxiety a possible extension of school hours into those hitherto given up to mere amusement.

Examinations.—Allied to the characteristic just mentioned is the objection that practical work cannot readily be tested by examinations. "Results as tested by an examination may be disappointing," writes Mr. Bell. "It is hard to devise an examination which is a fair test, but this does not condemn the work. The boy understands what he is doing, as shown by the fact that he can make use of previous experiments, and make correct deductions from them; but it needs a much higher standard of training before he can show an examiner that he understands."

This difficulty of examinations is met with elsewhere. To take a very different instance which happens to have come within my experience:—The working classes have recently appealed to Oxford and other Universities for Higher Education. The appeal was not made by large numbers, but on behalf of a very small percentage indeed, who are of special earnestness and (as the event proved) of special ability. Looking back on what has happened, we might perhaps have foreseen that when twenty people out of (say) 20,000 presented such an appeal, they would be men out of the common run: but the ability shown certainly took us by surprise. After only a few months' training (by a capable teacher) these men were able to produce essays of sterling merit—comparing favourably with essays written by first-class men at Oxford. But they were, and are, almost wholly unable to face an examination; it would be difficult to devise an examination which would bring out their merits in any adequate degree. To give another instance somewhat nearer the matter which concerns us:—I have been accustomed for many years to examine the students who have attended University Extension lectures in Astronomy, and am familiar with what may be expected from them. One of the lecturers has recently taken a rather new departure by encouraging his students to do practical observing for themselves at their homes instead of taking their facts from him or from books—to note the positions of the stars, and the variations in the height of the tides and trace them to their causes. I have been able to satisfy myself in a variety of ways that his students have thoroughly appreciated the new departure and have gained much from it: some of them have even gone on to do astronomical work of real value. But the examination results are disappointing. It seems difficult to make sufficient allowance for the time taken to work out a problem of

this practical kind, under stress of examination, even when it is on familiar lines: and there is great risk that those who have merely committed some facts to memory will appear to advantage in examination compared with those who have a much sounder working knowledge.

Experience of this kind suggests that too much importance must not be attached to examination results. If possible we must suit our methods of examination to the methods of teaching, though it is admitted that there may be serious difficulties in doing this. The difficulties may be surmountable; but even if they are not, it would seem to be undesirable to alter sound methods of teaching to suit examinations. And meanwhile we must be prepared for anomalies wherever examinations come into contact with the new methods. An instance may be given in illustration. To pass "Smalls" at Oxford (which has become a kind of entrance examination into the University) it is necessary for the candidate to show a slight acquaintance *either* with geometry *or* with algebra. Most boys of any ability qualify for this examination while at school, and it is only the weaker brethren who are left to face it on coming up to Oxford. Seeing that the new methods of geometrical teaching were designed to help the weaker brethren, it was expected that when they were introduced geometry would be preferred to algebra. But exactly the contrary happened; geometry was deserted and nearly all the candidates chose algebra. On making inquiry I found the reason to be that for the purposes of passing an examination it is undoubtedly easier to cram a dull boy with a few facts in the old way (which is still unreformed in algebra) than to teach him how to *reproduce* his knowledge of geometry. The difficulty does not arise in getting the knowledge into him, but in getting it out again in examination. Many of the boys in question formerly learnt their Euclid by heart, and this resource has been taken from them. They now learn a few rules in algebra by heart. Whether this does them much good is another question.

Relations of Teachers to one another.—With much more diffidence I remark on one other outcome of the modern methods of teaching. They have, if I am not mistaken, broken down, or at any rate are breaking down, the barriers between the teachers of Mathematics and of Science (especially Physics) which have seriously embarrassed education hitherto. In my early experience as inspector of some public schools in Mathematics on behalf of Oxford University, only half a dozen years ago, I could not but remark how little notice was taken by some of the teachers of what was being done at other schools, or even by their colleagues. I venture to think that a great improvement is being effected in this respect. In many schools a definite system of co-operation between the mathematical and scientific staffs has been already organised: and doubtless others will adopt this plan in the future. The joint meetings of the

Mathematical Association and the Association of Science Masters will help to further this improved understanding.

Conclusion. — In view of these considerations it seems reasonable to conclude that the methods of practical mathematical work recently adopted in our public schools have justified themselves hitherto. They have aroused the interest of the boys, have made learning easier, co-operation possible, and have stimulated imagination. More time is perhaps required, but there should not be any grave difficulty in assigning it if necessary. There are more serious difficulties in suiting examinations to the new methods, but these must be faced and not allowed to be a hindrance. There are also signs, which can scarcely be mistaken, that the new methods have been a welcome stimulus to the teachers as well.

H. H. TURNER.

PRACTICAL MATHEMATICS AT CLIFTON COLLEGE.

As far back as 1875, Professor Worthington, then a master at Clifton College, organised a course of practical work in elementary mensuration and hydrostatics which ever since has formed an introduction to the practical science taught to all boys on the modern side and to some boys on the classical side of the school.

The methods of his text book, the first of its kind, have since been adopted throughout the educational world, and the development of his scheme at Clifton has been so successful that the new demand for what is called Practical Mathematics seems to be met quite adequately, in the lower part of the school at any rate, by what has been done here for many years.

Course of Practical Physics.

The time at present allotted to this work in the laboratory is as follows:—

Modern III.	-	1½ hours per week.	Mensuration.
Classical III.	-	1 hour	"
Modern IV.	-	1 " "	Hydrostatics and Heat.
" V.	-	2 hours "	Heat and Mechanics.

After this preliminary course the sets proceed to the practical work in Light, Magnetism and Electricity, &c.

The experiments are divided into Sets and now follow the order of Rintoul's "Introduction to Practical Physics," written by the present Head of the department at Clifton.

Reference to this book will show that sufficient explanation is given to enable the boy to perform an experiment without much assistance, and the educational value of the book lies in the fact that the boy is led to deduce the successive stages of the argument by a series of questions, the answers to which are to be written in such a way as to indicate the nature of the question which is being answered. For instance, the text book gives the following directions for finding, by the principle of Archimedes, the volume of any solid that sinks in water.

"Counterpoise your balance as before, and hang to the short pan a glass stopper. Weigh the stopper while hanging in air. Then immerse it in a beaker of water and weigh again while in water.

"What is the upthrust of the water on the stopper?

"What is the weight of water displaced by the stopper?

“What then is the volume of water displaced by the stopper, and therefore the volume of the stopper? Calculate the density of the stopper.

“In comparing this method of finding the volume of a solid with the method of displacement of water, which do you think capable of giving you the more accurate result?”

Perhaps the most valuable part of the work lies in the way in which the account of the experiment is written out, and it is regarded here as most important to take advantage of the opportunity thus provided for training the boy to express himself clearly and concisely. When satisfactory, both as regards accuracy and intelligibility, the boy's work is signed by the master, and the fact recorded on a board in the laboratory on which is fastened a tabulated list of the experiments of the Set.

It thus follows that the members of a given Form or Set are working quite independently of one another, and that each boy performs a logical sequence of experiments during his career at the school.

The system also allows a boy, who has for some reason dropped the subject for a time, to take it up again where he left off without interfering with the work of the particular block of the school of which he may then be a member.

As the Physical Laboratory is almost continuously occupied by Sets in Physics it is at present impossible to make arrangements for mathematical masters to take their Sets into the Laboratory to do practical work in combination with their theoretical work. Since, however, Mechanics is taught as a branch of experimental Science in the Physics course, most boys go through a course of Practical Mechanics even though they may not advance so far in their Mathematical Sets as to take it as a part of their mathematical work.

Practical Mathematics on the Military and Engineering Side.

At present, the Practical Mathematics now demanded in the Army Examinations is done in the Laboratory out of school hours, and is confined to practice in performing experiments similar to those set in previous examinations. It is hoped, however, that it may be possible shortly to provide accommodation for the Army Side so that their practical work may be done in close connection with their theoretical work.

On the Military Side, during the lessons in Mechanics, experiments are performed by the master in the presence of the class.

One room is provided with apparatus of a simple type sufficient for elementary experiments in Statics and Dynamics. The cost of the apparatus scarcely exceeded 2*l.*, and most of it is kept in the drawer of the table on which the experiments are performed. The most useful articles are the Rintoul dynamo-

meter (supplied by Messrs. Harris of Birmingham), and a board which can be clamped to the table and used for the demonstration of such theorems as the triangle of forces. When more elaborate apparatus is required such as Fletcher's trolley, it is generally possible to make arrangements for the Set to be taken in a Physics Lecture Room.

The class draws a diagram of the apparatus, the necessary measurements and readings are taken by the master and written down by the class who then make the calculations and deductions, after which a formal proof of the theorem is discussed. The advantage of introducing the apparatus into the Class Room was especially marked in the improvement which took place in the drawing of diagrams, and there is now a complete absence of the representation of bodies in equilibrium under impossible conditions.

Practical work to illustrate the application of the Calculus is only valuable when the boy is capable of working out theoretically the formula which is to be applied. There is very little educational value in an experiment which merely requires the substitution of numerical results in a formula which the boy is incapable of obtaining from his theoretical knowledge unless he is asked to make some deductions from the results which he has found. This seems to be a weak point in some of the questions set in the Army Examinations, but the Civil Service Commissioners are doing valuable work in insisting on practical work in the Competitive Examinations, and it is to be hoped that mathematical masters, whose training has been purely theoretical, will soon be brought to recognise the value of such practical work as part of a boy's mathematical education.

R. C. FAWDRY.

PRACTICAL MATHEMATICS AT HARROW SCHOOL.

Harrow is one of the great English Public Schools—corresponding to the German Gymnasium and Realgymnasium, and the French Lycée; there are about 530 boys in the school, their ages ranging from $13\frac{1}{2}$ up to 19 years. The boys are admitted to the School after they have passed an entrance examination for which they are prepared at various Preparatory Schools; a large number of them go on to the Universities, or into the Army.

Mathematics is taught to all boys, with the exception of about 16 or 18 at the top of the school, who are specialising in literary subjects.

The school is divided into three parts—the Classical Side (corresponding to the Gymnasium), the Modern Side (corresponding to the Realgymnasium), and the Army Class. The Classical and Modern Sides, which are about equal in numbers, are likely to be united in the near future for the teaching of Mathematics, though at the moment of writing they are entirely independent of one another. As the arrangements for practical work on the Modern Side are in advance of those on the Classical Side, the Modern Side alone is considered in the following notes. The Army Class is a separate organisation, consisting of about 40 or 50 boys all in the upper half of the school. These boys are preparing for the examinations for entrance into the Army, so that their work is of a special nature and need not be considered here; suffice it to say that they do a good deal of practical work in the mathematical laboratories, and that it is owing to the necessity for such work that the laboratories were originally equipped.

For the teaching of Mathematics the whole of the Modern side is rearranged in *divisions*.

The number of mathematical lessons (about 50 minutes each) on the Modern Side is six per week, but in the upper part of that Side some boys, who show special aptitude or who need Mathematics for their future careers, devote an extra $2\frac{1}{2}$ hours per week to the subject.

The time given to preparation is about four hours per week, but that is likely to be reduced in the lower part of the school.

The mathematical laboratories are two rooms measuring about 26 feet by 16 feet and 21 feet by 18 feet; the former is used entirely for practical measurements, and the latter for practical mechanics. There is also a small room used as a workshop and store-room adjoining the laboratories. There is accommodation in each laboratory for 16 boys working in pairs,

so that it is necessary to limit to that number the size of those divisions which do practical work (the usual number in other divisions is 16 to 18).

In considering the work done in the mathematical laboratories, it is natural to divide it into two parts: (1) practical measurements, and (2) practical mechanics.

PRACTICAL MEASUREMENTS.

In 1896 Mr. C. E. Ashford, now Headmaster of the Royal Naval College, Dartmouth, arranged a course of experiments for use in the science laboratories at Harrow. He started with measurement and weighing, and the work done in the mathematical laboratories follows the lines he laid down; but the work was not done under the supervision of the mathematical masters, and it was impossible at that time to bring it into relation with the mathematical work of the boys. Some six or seven years later, at the time when great changes were being made in the teaching of Mathematics, some drawing and measuring was begun in the ordinary mathematical lessons, but this was quite independent of the work referred to above, which was still being done in the science schools. About 1906, the course of measurements and weighing previously done in the science schools by the science masters was begun in the mathematical laboratories under the supervision of the mathematical masters; but until 1910, owing to the difficulties of organisation, the boys did not do this work with the same master with whom they did the rest of their Mathematics. However, at the beginning of 1910, it was arranged that it should become part of the ordinary work, so that it could be closely related to the rest of the mathematical work.

Method of Work.—Each boy has a notebook* of the experiments to be done; in the early part of the book, a description of the experiments is printed and a specimen table for the results of observations is set out, blanks being left where actual figures would occur; this table may be made out in another note-book, with or without assistance from the printed book, either for the actual experiment given or any other illustrating the principle involved; this teaches the boy how to arrange observations and deductions therefrom. Later on, the experiments are described, but the boy is left to make his own table of observations, &c. At a later stage he has to do everything himself, write a description, and make his own table.

The maximum mark to be assigned to each experiment is printed in the book at the side of the experiment, so that the master can mark with the greatest facility; so far as possible the work is marked in the course of the lesson, this enables any

* Siddons and Vaseall: "Practical Measurements." Cambridge University Press.

mistakes to be discussed by master and boy, in this way a great deal of the teaching is done, the master having time to criticise every detail of the boy's work: experiments that need correction are marked with an asterisk, and a boy is not allowed to go on to a new experiment till all corrections are made, and the asterisk initialled by the master. Apart from this individual work it is essential for the success of the course that there should be some class-teaching.

As we have said above, the boys *work in pairs*; so far as possible the arrangement of the pairs is left to the boys themselves. There is one obvious disadvantage in working in pairs, but it is more than outweighed by the advantages: the disadvantage referred to is the possibility of a pair being very unequal in ability, so that the clever boy is kept back and the weak boy scores marks for work mostly done by his partner; to avoid such cases occurring it is sometimes necessary to rearrange pairs. But this possible danger, which is easily obviated, is immaterial compared with the advantages gained by the boys discussing the experiments together and frequently succeeding in doing an experiment which neither could do alone. Of course all calculations are done by each partner and the results compared.

It is not desirable that boys should be taken through the practical course at once; there are several advantages in taking only a part of it each term, the interest undoubtedly raised by the work is spread over a longer period, and it is easier to correlate the practical and theoretical work.

Relation of Practical and Theoretical Work.—A collection of arithmetical and algebraic examples is being gradually made in which the data are of a physical nature and in some cases have to be obtained in the laboratory. By degrees it is hoped that this work will replace a great deal of the artificial arithmetic work, which has heretofore taken up so much of a pupil's time. It has not been found necessary to collect geometrical examples, as many of the modern text books are well supplied with such examples.

Sketch of the Practical Course in Measuring and Weighing.

(1) Measurement of length.

(a) Straight lines—the connection between the inch and the centimetre.

(b) Curved lines—the experimental determination of π .

(2) Measurement of area.

Area by counting squares—hence the experimental determination of the rules for finding the areas of rectangles, triangles, and circles.

(3) Measurement of volume.

(a) Volume by building up solids with inch cubes—hence the experimental determination of the rules for finding the

volumes of rectangular blocks and solids of uniform cross-section.

(b) Capacity by means of graduated cylinder and burette.

(c) Volume by displacement—experimental determination of the formula for the volume of a sphere.

(4) The use of the balance.

(a) The connection between the ounce and the gram.

(b) Area by weighing.

(c) The weight of unit volumes of various substances by direct methods.

(d) From (c) lead on to the idea of specific gravity and its determination by the specific gravity bottle.

The course then goes on to Archimedes' principle, &c., but this part of the work is done by the science masters and, at the time of writing, no arrangements have been made for correlating any arithmetic with that practical work.

A detailed list of apparatus is printed in the notebook used (*see p. 404*), so it is not necessary to give such a list here; but it may be pointed out that the models, &c. used for measuring are simple and familiar, and, so far as possible, reduplication of models is avoided so that the master may be better able to remember the constants for the models.

What are the Aims and Objects of the Course?

(a) A clearer conception of length, area, and volume.

(b) Some acquaintance with the units of measure and weight, and some slight power of guessing rough approximations to lengths, areas, volumes, and weights without the use of instruments.

(c) A practical knowledge of how to measure length, area, volume, and weight.

(d) Clearer ideas as to the trustworthiness of results and the folly of giving results to too many significant figures.

(e) Enormous increase in interest in arithmetic, &c. resulting in increased powers of concentration.

(f) Training of hand and eye.

It is essential that the teacher should keep these aims and objects perpetually in mind, otherwise the boys will not get the advantages hoped for, and there will be danger of the practical work becoming an additional burden instead of a real help to the Mathematics.

Is there any Waste of Time?

Teachers of a single subject like Mathematics are apt to consider innovations entirely from the point of view of their own subject, instead of from the point of view of the pupil.

We may first consider this from the pupil's point of view. Before the mathematical masters began taking boys through this practical course it was being done by the science masters

as a necessary training preliminary to other science work; so that the recent change, so far as the pupil is concerned, consists merely in the change of master with whom the work is done; and, as the new arrangement correlates that work with the other mathematical work, there is nothing but gain from the pupil's point of view.

Now we must consider the mathematical master's point of view. Assuming that there is no increase in the time at the mathematical master's disposal, does not the introduction of practical work take away time that ought to be devoted to other branches of Mathematics? All that is necessary is to reduce the time devoted to arithmetic; the author of this paper has no hesitation in saying that such a change is beneficial, and that the standard of the arithmetic goes up because of the interest gained through the practical work.

Could more than 16 Boys be taken at the same Time?

The answer to this is undoubtedly YES, especially if there is a laboratory assistant to give out apparatus, &c.

At Harrow the number in a division must be limited to 16, because the available room does not provide accommodation for a larger number. There are advantages in not having more; the boys can be treated as individuals, which is more necessary as all boys do not do the same experiment at the same time.

PRACTICAL MECHANICS.

Up till 1904 mechanics was taught in the two top mathematical divisions on each side of the school, but less than 10 per cent. of the boys ever reached these divisions, and no experimental work was done. The science masters taught mechanics, theoretical and practical, to 6 or 8 boys reading for Science Scholarships; it was put late in their course partly because correlation with the mathematical master was at that time impossible.

About 1904, owing to changes in the Army entrance examination, it became necessary to teach practical mechanics to boys in the Army Class. It was then that the mathematical laboratories were equipped and some mechanics was first taught from the experimental point of view.

From 1906 onwards the second and third divisions on the Classical Side have done a little practical work in statics, and since 1908 a division about one-third of the way down the Modern Side has devoted about 3 hours a week to an introductory course of practical statics. Last year, 1911, the course was extended, so that two divisions now take it.

It is hoped in the near future to extend this work so that all boys on both sides of the school will go through a course of practical statics; and a large number through a short course of practical kinetics, though most of the kinetics will

probably be in the nature of lecture experiments on account of difficulties of apparatus.

Method of Work.—The course of practical statics has not yet developed sufficiently to be printed in book form, but no doubt that will be desirable in the near future; at present it is in manuscript, so that it is being tested and can easily be modified as experience dictates.

As in the case of practical measurements, the boys work in pairs, this arrangement being more necessary in practical statics because of the difficulty of setting up apparatus and the larger amount of room needed.

The course is arranged at present for two terms' work, but there would be some advantage in spreading it over three terms and decreasing the number of lessons given to it each week (the interest would be spread over three terms and other subjects would be developed concurrently).

Relation of Theoretical and Practical Work.—The theoretical work is taken in class at exactly the pace that the practical work dictates, and it consists only of those parts of the subject which are closely related to the practical work, though the principles discovered are applied to cases which it is impossible to test in a laboratory. Later on the theoretical work is carried further.

Outline of the Course.—One term is devoted to the principle of moments, the lever, the wheel and axle, and other machines, bringing in the idea of work and efficiency; the next term is given to the parallelogram of forces and the triangle of forces, and examples thereon, leading to resolution, &c. These two parts are taken in alternate terms, and they are arranged so that either part may be taken first, this arrangement being convenient in the case of boys who stay more than one term in the same division.

Aims and Objects of the Course.

(a) The experimental determination of the principles of mechanics.

(b) Familiarity with the principles and their application to real problems (1) which can be verified by actual experiment in the laboratory, (2) which arise out of practical engineering work, &c., but cannot be verified in a laboratory.

(c) Increased interest in mechanics and its applications by making the questions that arise much more concrete and real—in former days the problems of the ordinary text book on mechanics had little or no relation to actual conditions, and involved a minimum of mechanics and a maximum of algebra, trigonometry, &c.

(d) Training of hand and eye—opportunity frequently occurs for the display of ingenuity in the setting up of apparatus.

One of the great difficulties of the practical work, in both the measurement and the mechanics courses, is to allow boys to go at their own pace and yet to keep them sufficiently close together to make class-teaching possible. To get over this difficulty it is convenient to have a minimum course mapped out and to have additional experiments for the better and quicker boys; this enables the master to work up the main principles in class and, besides that, to take some of the more important side issues with the abler boys.

It may be that the practical work at present causes greater strain on the master, but there can be no doubt that it has helped Mathematics; it has made great principles clearer, it has added interest to the work, it has helped the weaker boys to go further than they could otherwise have gone, and it has enabled the better boys to go on to higher Mathematics at quite as early an age as is desirable, with a freshness and keenness that was often blunted in the past by the dullness of the old text book grind.

A. W. SIDDONS.

PRACTICAL MATHEMATICS AT OUNDLE SCHOOL.

Preliminary Information as to Numbers and Organisation.—There are 350 boys in the school, 150 on the Classical Side, 160 on the Science and Engineering Side, and 40 in the Junior School.

For Mathematics there are three groups :—

- (1) The classical and junior school with 10 classes: an average number in each of 15.
- (2) The three sixth forms on the Science Side divided into four sets: average of 16.
- (3) The remainder of the science and engineering forms, divided into seven sets: average 14.

With three exceptions (on the Classical Side), the mathematical sets are taken by men with either mathematical or science degrees.

Workshops and Laboratories used in the Teaching of Practical Mathematics.

(a) The Workshops—use being made in the early stages, with very young boys, of the workshop benches and tools.

(b) The Power Room and the Engineering Laboratory and Testing Room.

(c) The physical laboratories and the junior chemical laboratory.

(d) To a limited extent the mathematical class rooms.

The Forms and Classes taking Practical Mathematics.

(a) The three lower forms of the Classical Side and the Junior School—116 boys—do practical work in the workshops and laboratories. The time given is from two to three hours per week.

(b) The Science and Engineering Sides spend two, three, or in some cases five hours per week in the laboratories.

(c) The higher mathematical sets on the Classical Side give a period (1½ hours) of their mathematical time to experiments in the laboratory—chiefly in experimental dynamics and statics.

(d) The laboratory space being available, only very simple practical mathematics is done in the mathematical class rooms—chiefly of a drawing office or calculating office character.

Summary of the Methods of Teaching Practical Mathematics.—There may be said to be two methods of teaching practical mathematics. There is the indirect method which is associated

with the practice of workshops, which may therefore be called the workshop method. There is also the direct method of "Practical Measurements," consisting of a series of experiments and measurements of a simple character arranged in some kind of logical sequence. Both methods are followed at Oundle. An account of the first is given below, and a brief account of the second follows on page 414.

The Workshop Method of Practical Mathematics.—In the workshop method, boys are set to do work in the shop, to investigate the working of a machine, to carry out experimental tests, and to be responsible for the "power" station. An attempt is made to look upon the class as a staff of workmen actually engaged in some "live" work. In the course of this work, investigation, or test, use must be made of Mathematics. Mathematics come in incidentally and are learnt as need arises. The various mathematical principles and methods are thus acquired indirectly by continually applying them, and the boy learns his elements of Mathematics in much the same way that he learns to walk. At a later stage he may put his knowledge into a logical framework.

I. In carrying out this method of teaching practical mathematics three rooms are used:—

- (1) The workshop itself, with the usual plant, including woodwork tools and benches, screw-cutting lathes, planing, drilling, and milling machines; power being supplied either by separate motors or through shafting.
- (2) The engine-room, containing a steam engine and turbine, gas and oil engines, and dynamos. This plant is used to supply the electricity for school purposes, and is also designed for experimental work.
- (3) An engineering laboratory and testing room. In these rooms are instruments for the testing of gauges, the determination of the calorific value of coal, gas, and petrol, the testing of ammeters and voltmeters. It may be added that the workshops do the repairs of the school and boarding-houses, provide the electric lighting, undertake the wiring and laying of cables, and make some of the machines, instruments, and tools which are required. The main object of this is that the workshop may be of a practical and business character, giving continuous employment to skilled workmen.

II. The following lists of experiments from amongst those provided for will show the nature of the work done by (1) boys in the junior school, and (2) boys in the fourth forms:—

- (1) *Junior School Boys, Ages between 10 and 13*—
 - (a) The working of geared wheels.
 - (b) The action of a screw-cutting lathe.

- (c) Investigating various pulley tackle.
- (d) Finding the speed and the horse-power of a motor.
- (e) Measuring up the benches in the shop.
- (f) Investigating the shafting.

These experiments involve the working of simple examples in the elementary rules of arithmetic, introduce decimals and fractions, bring in the idea of ratio at an early stage, and lay the basis for a knowledge of the facts of geometry.

(2) *Fourth Form Boys from the Ages of 12 to 15—*

The following are some of the experiments which can be done:—

- (a) Test the readings of a pressure gauge, using a Crosby gauge tester.
- (b) Find the efficiency of an electric motor.
- (c) Find the calorific value of coal.
- (d) Find the calorific value of gas.
- (e) Gas engine trials, indicated power, electric power : cost of gas per H.P. hour.
- (f) Steam engine trials.
- (g) Experiments with an electric fan : its efficiency : the air pressure.
- (h) Characteristic curve of a Manchester dynamo.
- (i) Photometric bench, with standard lamp and testing-room photometer.
- (j) Ohms law—using a bank of lamps, voltmeter, ammeter.
- (k) Pulley tackle—Weston pulley tackle ; compound differential tackles to lift 30 cwt.

III. *Some Advantages of the "Workshop" Method are:—*

In the workshop method of teaching, the boys are regarded as forming a body of workmen engaged in carrying out some of the regular work of the shop. Each boy is a member of the staff, and has his definite part of the work to do—and he finds that whatever this work may be it is of importance to the whole, and whether it is simple or complex, easy or difficult, it is equally essential—and that only bad work which could not be used will be scrapped. Such, at any rate, is the attitude of mind of master and boy.

Now, the experiments, tests, measurements, designs, which form the regular part of the work in the shops involve the use of a wide range of Mathematics, and in carrying them out it will be found that a boy will learn his Mathematics as he learns the use of tools by constantly applying them. It is true that the number of mathematical questions or examples he will in this way get through in the hour or two hours "lesson" may be very few, but the questions arising in a concrete and visible form will make a more vivid and abiding impression on the mind, and both the methods and results will become an integral part of his being. On the other hand, although a boy will be

able to work through many more examples and problems in a class-room, yet it may safely be asserted that the impression left by each one is comparatively feeble, and that experience of the continual working of similar exercises shows that the succession of impressions only tends to blur each other, and that the general effect is to create weariness and dullness.

These advantages may be summarised :—

- (a) The method promotes alertness and zeal.
- (b) Each problem is presented in a concrete form—so that the method of calculation is suggested by the problem itself.
- (c) The mind dwells for a longer time on each stage of the process involved.

Another and perhaps the most important advantage of the method is that scientific phenomena and theories are studied “in parallel” and not “in series,” and in this respect the method is the direct opposite of the mathematical and “practical measurements” method. Both these treat the subject in some kind of logical order in which facts and principles are only introduced as they find their proper place in the order. But this process is not the way in which ideas have taken root in the minds of boys. Experience and observation (with information incidentally obtained from others) have filled the mind with the small beginnings of many ideas at an early stage, and, like the seeds sown in the soil, these ideas must be encouraged to grow up simultaneously, and teaching should be directed towards this end.

Experiments of the type described above provide this kind of teaching. For example, the experiments which can be done by the very young boys will be found to provide simultaneously simple addition, subtraction, multiplication, and division. Decimals and simple fractions quickly follow, whilst the ideas of average, ratio, percentage—the facts about areas, circles, and the other figures—the methods of calculating proportional parts and rates of change—make an easy appearance in simple practical forms. Formulæ in finding lengths, areas, volumes; the well-known formulæ $\frac{(W_1 - W_2) CN}{33000}$ equals HP, or HP equals

$\frac{P.L.A.N.}{33000}$, have to be given—and they provide the best introduction to algebra.

These experiments have this further value; the calculations will suggest further developments in Mathematics. An example can be given. Three or four boys are set to work with an electric motor. They are given a speed counter and are told to find the revolutions in, say, 3 minutes, and thus find the revolutions per minute. This they will do two or three times. But suppose a mistake is made, and instead of getting the readings for the revolutions in 3 minutes, they are got for

$3\frac{1}{2}$ minutes. Here is the opportunity for development. Some alert boy may see that the observations need not be thrown on one side. He may see that the half minute may be taken as a unit, and thus the number of revolutions per minute obtained. He may learn to divide by $3\frac{1}{2}$, and to know what such a division means. The choice of unit is prominent in experiments of this type, and fractions depend upon the choice of unit.

It is a necessary feature of the method that full working instructions should be provided with each experiment, and at first sight this seems to make the experiments of little educational value, yet it is found that there is ample scope within the instructions for the exercise of the logical faculties and for the display of a boy's individuality and originality. It will be found that the fears naturally entertained are not justified in actual practice.

The experiments require careful reports of some length. These reports should contain tables of observations and readings, graphs, drawings, photographs, and in some cases some historical introductions—the writing up of which is especially valuable for boys of distinct scientific tastes, and should form an essential part of the work of a science form.

It may be added that it is a useful training to gain exercise in following out detailed written instructions.

The students are classified as follows:—

1. *Junior Boys, Ages 10 to 13.*

These boys are taken in forms—the number in each form varying from 10 to 20. A mathematical or science master is in charge, and a workman set free to help. Each lesson lasts one hour.

2. *Classical Side. Fourth Forms. Ages, 12 to 15.*

The fourth forms on the Classical Side are taken in forms (the average number in a form being 20). The period assigned is one hour. The form is divided into groups—five or six groups according to the nature of the experiments, as some machines require two boys, some four or even six. A mechanic is set free to help.

3. *Engineering Side. Lower Forms. Ages, 14 to 16½.*

These forms average 20 in each, and the period given is 2 hours. The experiments are arranged in accordance with the ordinary physical or chemical laboratory experiments. Only a small number of boys, six or eight, are taken at a time from the physical or chemical laboratory, and are in charge of a master. The experiments are of a fuller character than those done by the forms given above. It is possible, however, to work a class of 15 to 20 boys in an engineering laboratory.

The "Practical Measurements" Method.—There are organised courses for all the forms in elementary practical measurements,

hydrostatics, and mechanics. The experiments are done in forms, and the time given varies from 1 to 3 hours. They are done either in a physical laboratory or in the junior chemical laboratory—the fitting of the latter being very suitable for many of the experiments.

The experiments are of a simple character, each one being designed to isolate some principle or property and make it prominent. They can therefore be arranged to follow and illustrate a logical development of the subject, and as all the boys in a class can be working the same experiment at the same time, the laboratory course can form the basis of a strictly logical treatment in the class-room.

This method has now been in working order in schools for 25 years or more, and experience suggests that without some elasticity and freedom from syllabuses it will become as great a slave to "logic-chopping" as anything ever was or can be in a mathematical class-room.

The following observations can be made :—

- (1) The elementary apparatus and material used in the experiments should as far as possible be of a practical character. Spheres, cylinders, cones, which are only for measuring are apt to be dull, and might on occasions be replaced by actual parts of machines or tools.
- (2) In the study of a property, such as specific gravity, the logical sequence may be departed from for a time, and the class be set to find the specific gravities of a number of common substances by methods suitable for each substance.
- (3) That "all the class working at the same experiment at the same time" can be carried to the point of monotony and dullness, and that there should always be a certain number of boys at every lesson engaged upon some practical work which requires the use of more complex and more accurate apparatus.

In the Appendix are given a number of "calculation" questions and an examination paper, with an introductory memorandum by Mr. Brewster, Senior Mathematical Master.

F. W. SANDERSON.

APPENDIX.

DIRECTIONS FOR TYPICAL EXPERIMENTS AT
Oundle School.

Practical Mathematics in the Mathematical Class Room.—At Oundle the mathematical class-room is not used for experiments, its function is more that of a drawing or calculating office. Hence in the class-room experience is gained in practical methods of carrying out calculations. Problems which may occur in the workshops and laboratories are worked out, and exercises of a practical character suggested by these problems are set. This involves the use at an early age of logarithms, slide rules, and the other mechanical aids. Arithmetic, algebra, geometry, and trigonometry are studied or applied simultaneously, and one freely used with the others. In the lower sets the formal logical order of sequence is openly abandoned.

One lesson per week at least is devoted to doing some continuous piece of calculation. The class-room is converted into a calculating office. A problem is set which may involve arithmetic, algebra, geometry, trigonometry and, in higher forms, co-ordinate geometry, calculus, dynamics. The data for the work is supplied from the school shops, testing rooms, or fields. These may be obtained by the boys themselves, or may be supplied to them from a central office. Five or ten minutes is taken up by the master in talking about the subject and explaining the nature of the calculations involved; the boys set to work and are given help and information as need demands. When the calculations have been made and tested the boy can write a report on the work, in which his results will appear in their proper place. The calculations and results and the report are of some permanent interest, and need not be scrapped.

In comparing the results obtained with those gained by following more traditional methods, it is found that a boy will understand and remember a mathematical process with much greater ease if it is introduced to him in connection with a problem of practical value.

There can be no question that the scheme assists the secondary aim of introducing new methods and processes in a natural manner, and of fostering an interest at the earliest stage possible in the higher parts of Mathematics.

CALCULATIONS TO BE PERFORMED IN THE
MATHEMATICAL CLASS ROOM.

SUGAR BEET.

Four tons fifteen cwt. of roots were taken up from the School experimental plot this year. The soil being damp clung to the roots. To find the true weight of the crop 18 lbs. 8 ozs. of roots were washed, after which they weighed 16 lbs. 7 ozs. Calculate the true weight of the crop.

The area of the plot was four-tenths of an acre. Calculate the crop per acre.

The estimated average yield of sugar beet for the whole of Prussia this year is 5.94 tons per acre. Find by how much per cent. our crop is better than this.

The actual yield of sugar beet for the whole of Prussia last year was 11.90 tons per acre. Find by how much per cent. our crop is worse than this.

The Norfolk farmers were receiving in 1910 seventeen shillings and sixpence per ton for their sugar beet. Owing to the scarcity produced by drought we have sold ours this year at thirty shillings per ton. By how much per cent. has the drought increased the value of our crop?

ON LEVELLING.

The figure *B* (see p. 418) is a ground plan of the field levelled.

Draw the plan to a suitable scale.

Find its area in square yards.

In figure *C* (see p. 418) *abcdef* is the vertical section perpendicular to the long sides, not to scale. *afce* shows the original section, and *abcd* the levelled section.

Draw this section to same scale.

Find area of *bfc*.

Find *an*, *bn*. Hence find area of *abc*. (See note below.)

Find volume in cubic yards of the earth moved.

Figure *A* is a dimensioned drawing of a truck. Find the cubic yards it will carry, allowing for $\frac{1}{3}$ overloading.

It was found that 15 truck loads could be dealt with in $1\frac{1}{2}$ hours by 12 workers.

Find the number of workman-hours required. If the wage is 8d. per hour, find the cost of the levelling.

NOTE.—As a first approximation the area *afb* may be neglected.

FIG. A.

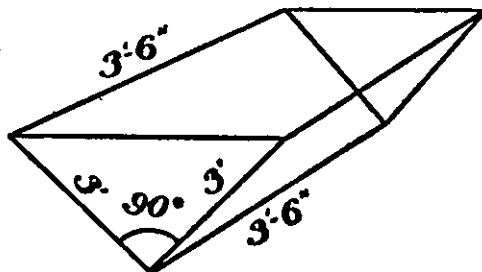


FIG. B.

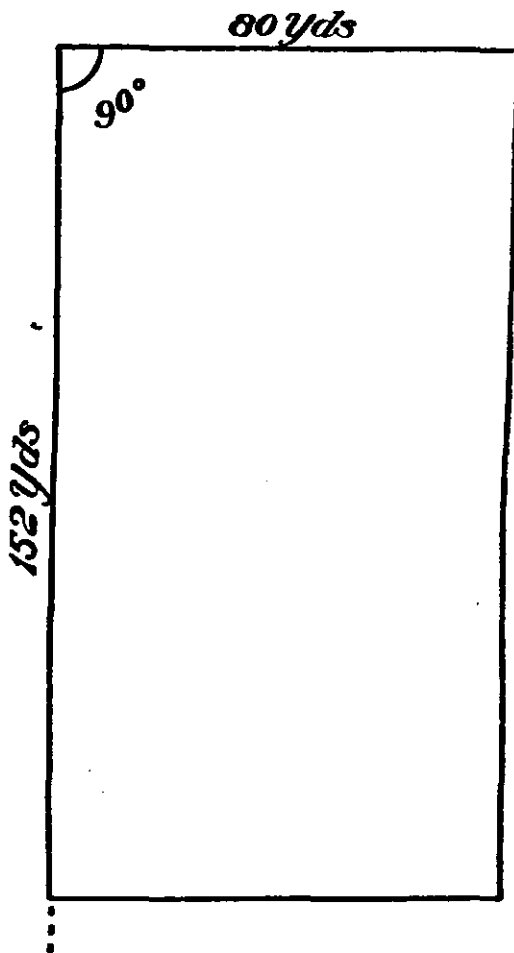
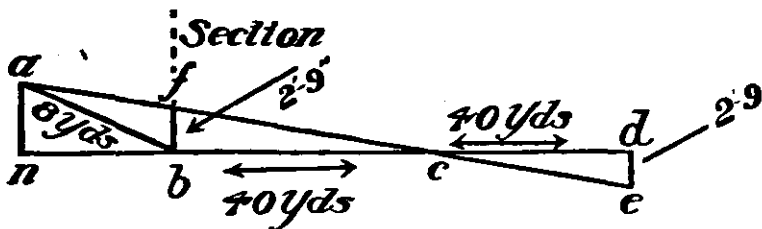


FIG. C.



SETTING OUT ARCS OF CIRCLES.

The figure A (see p. 419) explains itself.

- Prove (1) $\angle POQ = 2 \angle PP'Q$.
 (2) $\angle PQP' = 90^\circ$.
 (3) Δs $PNQ, P'NQ, PQP'$ are similar.
 (4) $\angle QPM = \angle PP'Q$.
 (5) $QN^2 = PN \cdot P'N$.

If $PM = y$
 $QM = x$
 Radius = a
 show that $y^2 = (2a - x)x$.

If $a = 100$ yards, find x for $y = 10, 20, 30, 40$ yards.

To the scale of 25 yards to the inch, draw the arc of the circle from the values of x and y .

Find its centre, and measure the radius.

In figure B , A and B show the positions of two gates, C the position of a door.

It is required to make a road as the arc of a circle passing through ACB .

Draw the triangle, and find the radius of the required circle.

It is proposed to set out the arc AC by three equidistant offsets.

Find from your drawing the lengths of the offsets.

Find the lengths of the offsets, using the equation $y^2 = (2a - x)x$.

FIG. A.

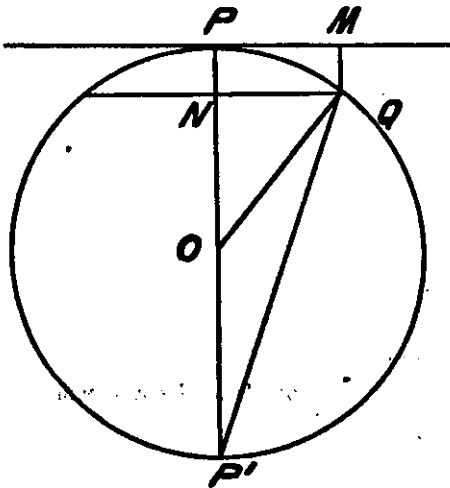
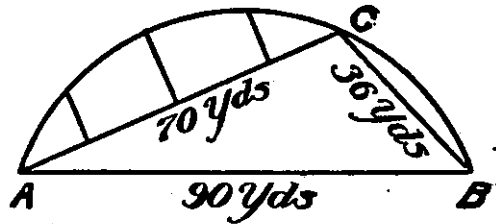


FIG. B.



STEAM ENGINE CRANK AND CONNECTING ROD.

Make a drawing of the crank, connecting rod, and piston of the given model. Describe the action.

BC is the connecting rod of an engine and is 4' long AB is the crank and is 1' 3" long.

$\angle BAN$ is 25° . AB is produced and CO is drawn perpendicular to AC to meet AB .

Draw the figure to a scale of either 1 cm. = 1 foot or 1 inch = 1 foot.

Draw the circle.

Using your tables find BN .

Using Pythagoras find AN .

Using Pythagoras find NC .

Hence find AC .

When the crank lies on AD find the distance of C from A .

Hence the travel of C (the piston) from the start to its present position.

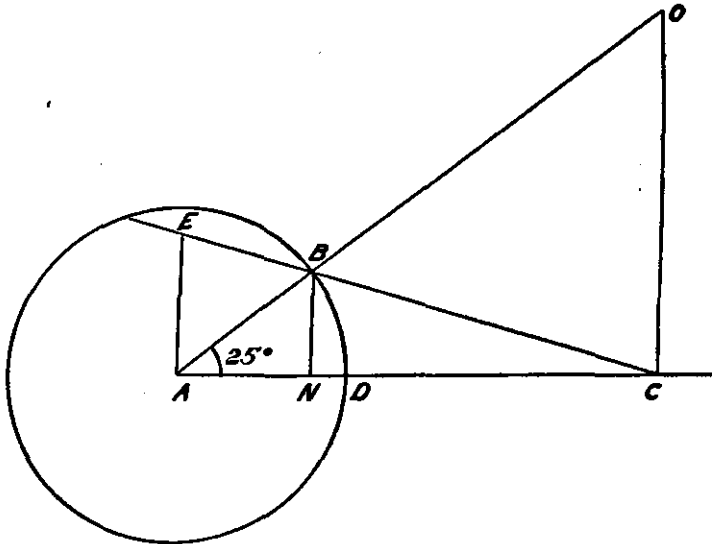
What fraction (decimals 2 places) of the stroke has the piston travelled?
 What fraction of 180° has the crank travelled?

Find OC ; OA ; OB ; and ratio of OC to OB .

Produce CB to meet the perpendicular to AC from A .

Find ratio of AE to AB .

Prove that $\frac{OC}{OB} = \frac{AE}{AB}$.



STEAM ENGINE.

It is required to find the weight of coal used per hour by a steam engine, the details of which will be explained to you.

1. Size of cylinder: stroke $1\frac{1}{2}$, diameter 8".

(Area = πr^2 .) Find the volume in cubic feet.

[Take into account a clearance space of 5%.]

2. Find the Density of the Steam—admission pressure is 82 lbs. per square inch. Use the steam tables.
3. Find Mass of Steam used in one hour.

Formula:—

$$M = V.D.N. 60. \quad (\text{Why?}) \quad N = 150.$$

4. Latent Heat at t° C. is given by

$$L = 806.5 - .7t.$$

Find from steam tables temperature at 82 lbs. pressure. Use squared paper.

Find L at this temperature.

5. Total Heat required is given by

$$H = M(t - 15) + ML.$$

Find the heat.

6. Find the ft.-lbs. of energy of this heat by multiplying by 1400.
7. One lb. of coal will give 12 million foot-pounds.

Find weight of coal used in the hour—

- (1) If all the heat of combustion goes into the steam.
- (2) If only 60% goes into the steam.

From the tables:—

$t^{\circ} \text{C}$	p (lbs. per sq. inch).	D (lbs. per cub. foot).
155° C.	79.03	0.1846
160° C.	89.86	0.2084

The interpolations required in (2) and (4) should be done arithmetically—and checked by a small graph.

8. In 4. Find the value of t at which $L = 0$.
What does this mean?

THE STEAM GAUGE.

1. The following are readings of the steam gauge:—

Pressure.	Gauge Reading.
10 lb.	10.5
15 „	15.8

Find the correct pressure when the dial shows 12.5, 13.5.

Do this by Arithmetic.

Do this by a small graph.

THE MOTOR.

2. The horse-power of the electric motor is given by the formula

$$H = \frac{(W_1 - W_2)C.n}{33000}$$

$$W_1 = 56 \text{ lb.}$$

$$W_2 = 4 \text{ „}$$

Circumference is 18.6 inches.

$$C = \text{circumference in feet.}$$

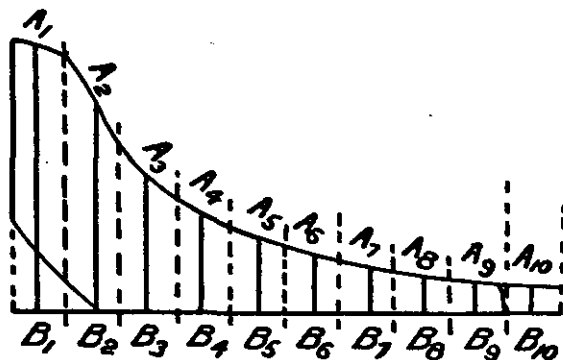
$$n = 1120$$

Find H .

3. If the horse-power is 3, and $W_1 = 56$, $W_2 = 4$, $C = 18.6$, find the number of revolutions per minute.

TEST OF THE DOUBLE ACTING STEAM ENGINE IN THE LABORATORY.

1. To obtain the average effective pressure (P) acting on the piston it is necessary only to find the average breadth of the indicator diagram.



Scale, 1 inch represents a pressure of 40 lbs. per sq. inch.

$$\text{The average breadth} = \frac{A_1 B_1 + A_2 B_2 + \dots + A_{10} B_{10}}{10}$$

The average force (measured in lbs.) driving the piston forward = average pressure (lbs. per sq. inch) \times area of piston (in sq. inches) A . This multiplied by the length of stroke (L) in feet gives the work done in foot-lbs. per stroke. Multiply by the number of strokes per minute and you get the work done in foot-lbs. per minute. Divide by 33,000 and you get the H.P. (horse power) "indicated." (This is generally known as the Indicated Horse Power—I.H.P.) The rule is easily remembered in the form

$$\text{I.H.P.} = \frac{\text{PLAN}}{33000}$$

2. Dimensions of engine: Stroke = 1 foot; Diameter of piston = 5 inches; Speed of engine revolution per minute (R.P.M.) = 220. (The number of strokes per minute, N = twice the R.P.M.)
3. The work which the engine did against a brake in this test may be obtained from the following information:—

Circumference of brake-pulley = 13.1 feet.

$W_1 = 58$ lbs.; $W_2 = 3$ lbs.

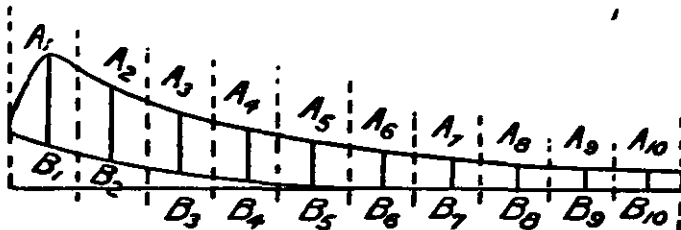
Formula—

$$\text{Brake H.P.} = \text{B.H.P.} = \frac{(W_1 - W_2) \times C \times N}{33000}$$

where N = R.P.M. in this formula.

TEST OF THE GAS ENGINE IN THE ENGINEERING LABORATORY.

The following indicator diagram is the mean of a series of diagrams taken on the same indicator card with a constant load on the engine:—



Scale, 1 inch vertically = 240 lbs./sq. inch.

To obtain the average effective pressure (P) acting on the piston it is necessary only to find the average breadth of the indicator diagram.

$$\text{The average breadth} = \frac{A_1 B_1 + A_2 B_2 + \dots + A_{10} B_{10}}{10}$$

The average force (measured in lbs.) driving the piston forward after an explosion is equal to the average pressure (lbs. per sq. inch) multiplied by the area of piston (A) in sq. inches. This multiplied by the length of stroke (L) in feet gives the work done in foot-lbs. per explosion stroke. Multiply again by the number of explosions per minute and you get the work done in foot-lbs. per minute. Divide by 33,000 and you get the "indicated horse power" (I.H.P.). The rule is easily remembered in the form—

$$\text{I.H.P.} = \frac{\text{PLAN}}{33000}$$

where N = the number of explosions per minute, which in this experiment = 65.

Dimensions of engine.—Stroke of engine, 1.83 feet; diameter of piston, 9.5 inches.

The gas engine was driving a dynamo through a flexible coupling. The dynamo in this test was supplying a current of 60 ampères to a number of electric lamps at a pressure of 106 volts. The horse-power developed by the dynamo, which for convenience we call electrical horse-power (E.H.P.), may be easily obtained from the well-known formula—

$$\frac{V \times A}{746}$$

Assuming the efficiency of the dynamo to be 90 per cent., the H.P. supplied to the dynamo by the engine = $\frac{\text{E.H.P.}}{0.9}$. The “mechanical

efficiency” of the gas engine is then = $\frac{\text{E.H.P.}}{\text{I.H.P.}}$

The horse-power lost in overcoming engine friction = $\text{I.H.P.} - \frac{\text{E.H.P.}}{0.9}$; and the proportion of the indicated horse-power lost in this way =

$$\frac{\text{I.H.P.} - \frac{\text{E.H.P.}}{0.9}}{\text{I.H.P.}}$$

TESTS ON ELECTRIC LAMPS.

The other day I found in the Engineering Laboratory that 10 carbon filament lamps working on a 100-volt circuit took a current of 3.4 ampères. Each of these lamps gave a light of the same intensity as 8 standard candles all lighted together. How many watts per c.p. (candle-power) were these lamps using? [Note.—1 ampère \times 1 volt = 1 watt.]

1. What current does a 32-c.p. carbon filament lamp take when working on a 220-volt circuit.

According to some of the most recently published tests a Tungsten lamp takes 1.2 watts per c.p. and a Tantalum lamp 1.9 watts per c.p. [These are two of the most successful metallic filament lamps.]

2. What current does (i) a 32-c.p. Tungsten lamp, and (ii) a 32-c.p. Tantalum lamp take on a 220-volt circuit?

[Note that it is much more economical to use metallic filament lamps than carbon filament lamps.]

3. Find the candle-power of (i) a Tungsten lamp, (ii) a Tantalum lamp, and (iii) a carbon filament lamp, if each lamp takes 0.32 ampères on a 110-volt supply.

EXAMINATION PAPER.

- A. A straight line BPQ through an external point B cuts a circle centre A, in P and Q, and you are given $AP = 2''$, $PB = 4''$, $\angle PAB = 35^\circ$

Construct the figure as accurately as you can.

Draw PN perpendicular to AB. Measure PN.

By means of the sine of 35° calculate PN.

Measure the angle B. From your calculation of PN find $\frac{PN}{PB}$ and hence calculate angle B.

Produce BP to cut the circle again at Q. Measure BQ, BA.
From these measurements work out $BP \times BQ$ and $BA^2 - AP^2$.
They should come to the same.

B. Draw any quadrilateral. Measure all its interior angles and add them together. How many right angles are there in the sum?

Draw any pentagon (5 sides) and do the same. Also do the same with a hexagon (6 sides).

Prepare a table as follows:—

—	N.	S.	$2N - S.$
Triangle - - - - -	3	2	4
Quadrilateral - - - - -	4	4	
Pentagon - - - - -			
Hexagon - - - - -			

Where N = number of sides

S = the sum of all the angles expressed in right angles.

What do you find about $2N - S$.

Hence give a formula for the sum of the angles of any N-sided figure.

C. Draw a large triangle ABC and take any point O inside it.

Produce AO, BO, CO to meet the opposite sides at X, Y, Z.

Measure BX, XC, &c. carefully and work out the values of

$$AZ \times BX \times CY$$

$$\text{and } AY \times CX \times BZ.$$

These results ought to come nearly equal.

Repeat with a new figure and state the result you find clearly.

If you had the measurements

$$BX = 3'', \quad CY = 2.5'', \quad AZ = 4''$$

$$XC = 5'', \quad YA = 2'', \quad ZB = 3''$$

ought AX, BY, and CZ to meet in a point?

D. The time of swing of a pendulum is $2\pi\sqrt{\frac{x}{g}}$ secs.

$$\text{where } \pi = 3.142,$$

$$x = \text{length of pendulum in feet,}$$

$$g = 32.2.$$

Calculate the times of swing for pendulums of length 4, 7, 10, 16 feet.

Draw a graph showing the time in terms of the length, making a smooth curve, and from it find the length so that the time of swing may be $3\frac{1}{2}$ seconds.

Check this by finding x from the equation

$$3\frac{1}{2} = 2\pi\sqrt{\frac{x}{g}}$$

E. x and t are the distance in miles and the time in hours of a train from the railway station.

Plot on square paper. Describe why it is that the slope of the curve shows the speed. Where is the speed greatest and least, and what are those speeds (m.p.h.)?

a	0	1.5	6.0	14.0	19.0	21.0	21.5	21.8	23.0	24.7	26.8
t	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0

F. The distance a of various planets from the sun is given in the table below. [Unit is earth's distance.]

The time of one revolution about the sun is T years.

	Venus.	Earth.	Mars.	Jupiter.
a	.7233	1	1.5237	5.203
T	.6152	1	1.8808	11.862

Kepler's Third Law asserts that $\frac{a^3}{T^3}$ is the same for each.

Calculate $\frac{a^3}{T^3}$ for each by logarithms to verify the law and find T for Saturn for which planet $a = 9.539$.

G. The range of a gun is given in feet by the formula $\frac{u^2 \sin(2\theta)}{g}$ where

u = velocity of shot.
 θ = angle of elevation.
 $g = 32.2$.

Calculate the range when $u = 432$ feet per second for the values $\theta = 10^\circ, 20^\circ, 30^\circ, 40^\circ$.

For what value of θ is the range half a mile?

H. A belt passes round an engine fly-wheel 10 feet in diameter with centre O and a dynamo pulley 2 feet in diameter with centre M , the centres being 12 feet apart.

Draw an accurate diagram, construct the straight parts of the belts as common tangents PH, QL , and find angles POQ, HML . Hence find the total length of the belt.

If the dynamo revolves 1,200 times per minute, find the number of revolutions of the engine and the speed of the belt in feet per second.

J. The horse-power of an engine is given by the formula

PLAN P = pressure in lbs. per sq. in.
 33000 L = stroke in feet.
 A = area of piston in sq. in.
 N = number of strokes per minute.

If the piston is a circle of diameter 10" and $P = 122, L = 1.4$ feet, $N = 150$, find the horse-power.

K. Explain all the ways you know of solving a quadratic equation, and prove your rules and give simple examples.

Solve to three significant figures

$$3.25x^2 - 75.1x - 115.3 = 0.$$

L. Draw a triangle ABC having

$AB = 3.5''$,
 $AC = 4.2''$, and $\angle BAC = 43^\circ$.

Calculate and measure BN ,
 the perpendicular from B to AC .

Find the area.

Also find AN, NC.

By Pythagoras calculate BC.

Prove that the area of any such triangle is $\frac{1}{2}AB \cdot AC \cdot \sin A$.

M. Simplify the expression

$$2(x - 3) + 5(4 - 2x) - 3(2x - 1)$$

and find what value must be given to x in order to make it equal to 0.

N. Draw graphs on the same paper of $y^2 = 6(x + 2)$ and $2y + 3x = 20$ so as to show where the graphs cut. (Find a suitable scale by trial.)

Calculate the co-ordinates of the points of intersection by solving the equations and verify by the figure.

P. CD is a ladder 30 feet long with one end C on the ground and resting on a wall AB 8 feet high. If $CB = x$ find a formula for the height (DN) of the end D in terms of x .

Calculate DN when $x = 8, 10, 12$, and find x when DN = 14 feet.

Sketch the path of D as C slides along the ground.

Q. In a triangle ABC the side $AB = 3''$, $AC = 4''$ and the angle BAC is 55° .

Measure the third side BC.

Draw BN perpendicular to AC.

Measure BN and find the area of the triangle by the rule $\frac{1}{2}$ base \times height.

Check by drawing another perpendicular from C and using the same rule.

PAPERS.

1. M. I., M. II., C. II., C. III.

(Do two questions.)

1. D.

2. E.

3. H.

3. M. V., M. VI.

(Do two questions.)

1. J.

2. K.

3. B.

5. C. V. A and B.

(Do two questions.)

1. L.

2. F.

3. C.

7. M. VII., C. VII.

(Do three questions.)

1. J.

2. M.

3. B.

4. Q.

2. M. III., M. IV.

(Do two questions.)

1. A.

2. F.

3. G.

4. C. IV.

(Do two questions.)

1. P.

2. N.

3. F.

6. C. VI. A and B.

1. M.

2. B.

3. C.

[In the examination rough diagrams ready drawn were supplied for questions A, H, I, and P.]

PRACTICAL MATHEMATICS AT WINCHESTER COLLEGE.

Fittings.

1. Five tables, 6 feet by 3 feet (about 3 feet high), each with four numbered drawers, and with tap and sink at one end.
2. A demonstration table, of the same type but larger.
3. A large sink and drying table.
4. Gas is laid on at all tables, but this is a luxury. It would be sufficient to have it at the demonstration table.
5. Store cupboards, black-board, &c.

Apparatus.—*In each drawer* the apparatus in general use, *i.e.*, graduated rules, scissors, callipers, paper, cotton, mercury tray, wooden cuboids, cylinder, &c.

In the Store Cupboards.—Balances and weights, spring balances, measuring cylinders, &c., &c.

The Syllabus.

Part I.—Measurements of length, area, and volume. Determination of π , comparison of litre and pint, &c.

 Weighing; comparison of areas by weighing.

 Weight of 1 cc. of water; specific gravity by direct weighing; by specific gravity bottle.

 Flotation; Archimedes' principle; equilibrium and pressure of fluids; barometer; cartesian diver; syphon; pendulum.

Part II.—Force, tension; levers, moments, parallel forces. Centre of Gravity. Pulleys. Parallelogram of forces.

Part III.—Vernier; micrometer screw-gauge; spherometer. Miscellaneous experiments.

Every boy has his own text-book. In Parts I. and II. full instructions are given for each experiment, and spaces are left for the boy to fill in measurements taken and calculate results. In Part III. general instructions are given, and the boy has to set out his work in a systematic way on paper or in note-book.

The boys in the *middle block* of the school have one hour a week in the laboratory. They are taken in their mathematical sets by the mathematical master in charge of that set (18–22 boys). They work independently or in pairs. All of these boys finish Part I., most of them finish Part II., and a few reach Part III.

In the *senior block* of the school, specialists (*i.e.*, mathematicians, army boys, engineers, &c.) do more work in the laboratory. They finish the text-book, and do further experiments of the same type. Some few boys go on still further to experiments in rigid dynamics, &c.

Expense.—The cost of upkeep is very slight—not more than 10*l.* a year, and this includes the wages of a boy who lays out the text-books in preparation for a class, gets out the apparatus, tidies up, and cleans at the end of the hour.

Objects aimed at, and Results obtained.—This work is the boy's first experience of experimental science, and he enjoys it. He learns scientific method, observation followed by deduction. He finds by bitter experience that "answers do not come out," and he sees the advantages of approximate methods and the uses of logarithms. He appreciates, at any rate, the use of Mathematics as a handmaid of science. He makes a good deal of noise over it, and a master with nerves—or a classical division in the room below—may be worried by the dropping of weights and water; but the boy is keen on the work, and there is no difficulty about discipline.

From Part II. a boy gets a working knowledge of forces, moments, &c., and his mind is prepared for the more mathematical treatment of statics and dynamics later.

Results, as tested by an examination, may be disappointing. It is hard to devise an examination which is a fair test, but this does not condemn the work. The boy *understands* what he is doing, as is shown by the fact that he can make use of previous experiments, and make correct deductions from them; but it needs a much higher standard of training before he can show an examiner that he understands. For example, a boy understands and can make use of Archimedes' principle long before he can, of his own free will, state it precisely and correctly.

It is, perhaps, worth while giving a little time in the classroom to help the boys to state clearly the results of their deductions from experiments in the laboratory. A few hours of this makes all the difference in an examination, but it is rather an artificial precision of statement and thought which is produced, and too much time should not be given to it at this early stage of experimental work.

G. M. BELL.

METHODS OF INTUITION AND EXPERIMENT IN SECONDARY SCHOOLS.

One of the questions proposed for discussion at the International Congress of Mathematicians (Cambridge, 1912) is indicated at the head of the present report. In order to ascertain the prevalence of such methods in England, a questionnaire was sent out to 551 schools; namely to 112 schools represented on the Headmasters' Conference, and to 439 schools represented on the Incorporated Association of Headmasters. Replies were received from 74 and 295 schools in the two groups respectively, which give percentages of 66 and 67. The result of the enquiry should therefore be very fairly representative of the work done in these schools in England and Wales, but throws no light on the conditions in Scotland and Ireland as few replies were received from these parts.

The constitution of the Headmasters' Conference provides that "as a general rule there should be 100 boys at least in any school represented at the Conference, and about 10 resident undergraduates at the Universities who have gone direct from the school." It is therefore fairly accurate to describe Conference Schools as large schools that prepare boys for the Universities—mainly, of course, the ancient Universities of Oxford and Cambridge.

The school life is usually 13 to 18 or 19. These schools are described loosely as "public schools."

The schools whose Headmasters have joined the Incorporated Association of Headmasters are secondary schools, varying greatly in size and character, and including both grammar schools of ancient foundation and new municipal secondary schools. The leaving age is as a rule lower than in the public schools; perhaps about 16 is practically the upper limit in the majority of cases. The age of entry varies greatly; the lower limit is eight or nine, but many boys enter from elementary schools up to the age of 14 or so. Generally speaking these schools do not prepare boys for Oxford and Cambridge; on the other hand they supply a large proportion of students to the numerous modern Universities.

The following questionnaire was sent out:—

A. Name of School.

B. Does the Headmaster object to the name of your school appearing in the report?

- C. *To what extent and in what classes (upper, middle or lower school, or Army class) do you introduce the following?*
1. Practical fieldwork.
Are measurements made and numerical results deduced?
Do you use theodolite (or equivalent instrument)?
plane-table? other instruments?
 2. Practical astronomy.
 3. Descriptive geometry (plan and elevation).
 4. Perspective drawing.
 5. Is the teaching of descriptive geometry and perspective associated with that of solid geometry?
 6. Is descriptive geometry and perspective taught by a mathematical or an art master?
 7. Accurate drawing work in connection with projective geometry: construction of conics by anharmonic ratio properties, &c.
 8. Graphical study of statistics, &c.
 9. Graphical representation of functions.
 10. Graphical treatment of vectors.
 11. Graphical statics.
 12. Formal methods of contracted multiplication and division.
 13. Use of slide rule (a) front scales, (b) trigonometrical scales.
 14. Tables of squares.
 15. Tables of square roots.
 16. Tables of cube roots.
 17. Tables of logarithms.
 18. Tables of trigonometrical functions.
 19. To how many places are the tables you use calculated?
 20. Graphical work in solution of transcendental equations
(e.g. $\log x = 1 + \frac{1}{x}$), in conjunction with Newton's rule, processes of trial and error, &c.
 21. Estimation of area by means of (a) squared paper, (b) planimeter.

Signature of informant.

In public schools the ages of lower classes may be taken as 12 to 15; of middle classes, 14 to 17; of upper classes, 16 to 18 or 19. But of course it may happen that a clever boy of 14 is in the highest mathematical class, and corresponding anomalies of age may be found in the other classes.

Public schools are usually reclassified for Mathematics: thus, "upper classes" means "upper mathematical classes" and so forth. The boys in the highest mathematical classes are usually working for mathematical scholarships at the Universities. In the larger schools there are also, as a rule, a number of boys preparing for the competitive examination that admits to the Army colleges, and these boys are often grouped as a

separate class. If numbers permit, there may be a separate class for future engineers or surveyors. These classes—Army, engineers, surveyors, are grouped together in the present report as “specialists.”

It is difficult to estimate the ages corresponding to “lower,” “middle,” “upper” classes in other secondary schools, on account of the great variety in the character of these schools. Perhaps in the majority of the large secondary day schools the ages are—lower, 8 to 13; middle, 12 to 15; upper, 14 to 16. Army classes do not usually exist in these schools. Reclassification for Mathematics is by no means universal. The number of boys working for scholarships is in most cases too small to affect the general style of the work.

In all schools the teaching of specialists (as defined above) tends to be of a more intuitional and experimental character than that of other boys, a result ultimately connected with the requirements of their chosen professions, and in the case of Army boys caused more directly by the style of the papers set in the entrance examination. The Army requirements during the last 10 years have no doubt had a powerful reaction on the character of the teaching throughout English public schools.

The following is an analysis of the answers received. Percentages for each class of school have been calculated on the number of replies received from schools of that class.

1. Practical Fieldwork, in which Measurements are made and Numerical Results deduced:

Public Schools.—Not done, 58 per cent. In the remaining 42 per cent. such work is done, but as a rule only by special classes of students (officers' training corps, surveyors, engineers, &c.). In only two schools is this stated to be part of Army class work; in two others it is classed with geography. On the whole it is more frequently done in upper classes than in middle and lower. As to instruments, theodolite is specified in 14 per cent., simplified angle-instruments in 5 per cent., sextant in 7 per cent., prismatic compass in 1 per cent., plane table in 9 per cent., level in 4 per cent.

Other Secondary Schools.—Not done, 44 per cent. In the remaining 56 per cent., upper, middle and lower classes are mentioned with equal frequency. Apparently more generally done than in public schools, and less confined to specialists. Classed with geography in 14 per cent. Theodolite 23 per cent., simplified angle-instruments 9 per cent., sextant 10 per cent., compass 3 per cent., plane table 25 per cent., level 3 per cent.

2. Practical Astronomy.—Very rarely attempted, viz., in 15 per cent. of public schools, 8 per cent. of other secondary schools. Perhaps the climate is not very suitable. Four schools in all are provided with observatories. Where the work is done, it is concerned with determination of latitude, direction, and time.

3. *Descriptive Geometry, after Monge (Plan and Elevation, &c.)*—Schools are in a transitional state in the treatment of this subject. Formerly it was only taught to boys who needed it for special examinations, and then generally by the art master; this commonly meant working by rule with great attention to artistic ink-work. Two or three causes are at work to change this position: (1) The Army examiners now treat the subject as part of Mathematics, and do not require inking-in; mathematical masters are, therefore, not discouraged from teaching the subject by consciousness of their technical weakness on the artistic side; (2) Descriptive geometry is now required for an Honours degree in Mathematics at Cambridge; this circumstance will result in an increased output of mathematical masters interested in descriptive geometry; and, further, will have a direct reaction on schoolwork. (3) Boys intending to study engineering at the University are encouraged to master descriptive geometry and perspective at school.

The figures are:—

Public Schools.—Not done, 23 per cent.; done in upper classes, 38 per cent.; middle, 8 per cent.; lower, 3 per cent.; Army class, 38 per cent.; specialists other than Army class, 23 per cent. In some of the above schools the subject is being taught in more than one place, *e.g.*, both in upper classes and in the Army class. Where taught, it is associated with solid geometry in 66 per cent., taught by a mathematician in 42 per cent., by an art master in 58 per cent.

Other Secondary Schools.—Not done, 40 per cent.; done in upper classes, 14 per cent.; middle, 21 per cent.; lower, 18 per cent.; specialists, 3 per cent., the rest unclassified. A feature in this group of schools is the teaching of plan and elevation in connection with manual work, generally carpentry (18 per cent.); this may explain the fact that it is taught lower in the school than is the case in the public schools. Where taught, it is associated with solid geometry in 54 per cent., taught by a mathematician in 35 per cent., by an art master in 44 per cent., and by a manual instructor in 20 per cent.

4. *Perspective.*—This subject has not been annexed by the mathematician to the same extent as descriptive geometry; it is still mainly an art subject. The correlation with Mathematics probably extended no further, in most cases, than an explanation to the more advanced pupils of the mathematical principles of perspective.

Public Schools.—Not done, 38 per cent.; done in upper classes, 22 per cent.; middle, 14 per cent.; lower, 4 per cent.: Army class, 11 per cent.; other specialists, 22 per cent.; largely a specialist subject, it appears. Where taught, associated with solid geometry in 50 per cent., taught by a mathematician in 23 per cent., by an art master in 77 per cent.

Other Secondary Schools.—Not done, 64 per cent.; done in upper classes, 9 per cent.; middle classes, 14 per cent.; lower classes, 7 per cent., the rest unclassified. Where taught, associated with solid geometry in 29 per cent., taught by a mathematician in 17 per cent., by an art master in 83 per cent. Mainly an art subject in these schools.

5. *Accurate Drawing Work in connection with Projective Geometry; Construction of Conics by Anharmonic Ratio Properties, &c.*—This is done to some extent in 28 per cent. of public schools, and confined to the few best boys who reach this comparatively advanced work. It hardly enters into the scheme of other secondary schools; 11 per cent. gave an affirmative answer of some sort, but it seemed possible that the term “projective geometry”—the geometry of Chasles—was not always understood in this sense.

6. *Graphical Study of Statistics:*

Public Schools.—91 per cent. do this work, beginning it generally in the lower or middle classes; it is confined to upper classes in only 12 per cent., and to Army class only in 4 per cent.

Other Secondary Schools.—An affirmative answer came from 85 per cent., here again mainly in lower and middle classes, 11 per cent. only confining it to upper classes. In 6 per cent. it is said to be correlated with geography teaching.

This, then, is a very usual feature in English secondary schools. As a rule it does not amount to much more than plotting a few tables of statistics and deducing information by interpolation. Probably even higher percentages really do this; in some cases there was reason to believe that informants were misled by the rather ambitious phrase “study of statistics,” taking it to imply excursions into economic science.

Mr. A. N. Whitehead has advocated a great extension of this type of work. “There are to our hands statistics of trade, both external and internal, statistics of railway traffic, statistics of crime, of income tax returns, of national expenditure, of weather, of pauperism, and of times of sunrise and sunset throughout the year. The reduction of these to graphs, the careful study of the peculiarity of these graphs, the search for correlations among them, and the study of the public events which corresponded in time to peculiarities of graphical form, would teach more mathematics and more knowledge of modern social forces than all our present methods put together.”* In his paper

* Journal of the Assoc. of Teachers of Maths. for S.E. England, Dec. 1911 (copies from the Hon. Sec., the School, Tonbridge).

on "Higher Mathematics for the Classical Sixth Form,"* Mr. W. Newbold has described his experiments on these lines with the senior non-mathematical boys at Tonbridge School; but this is pioneer work, and not attempted generally.

7. *Graphical Representation of Functions :*

Public Schools.—There were only two answers in the negative, and I was doubtful as to the informant's meaning in each case. Practically this work is done in all public schools, and indeed is required for all the ordinary external examinations. In 17 per cent. it is begun in the upper classes, 55 per cent. in middle, 25 per cent. in lower, the rest unclassified.

Other Secondary Schools.—Here again this work is very general, being done in 97 per cent.; 21 per cent. beginning it in upper classes, 54 per cent. in middle, 17 per cent. in lower, the rest unclassified.

Probably in most cases this work resolves itself into plotting the functions, and using the graphs as an aid to the solution of equations. The somewhat subtle ideas suggested by the word "functionality" are perhaps not so generally appreciated. But at any rate the graph has a firm place in lower and middle teaching, and gives a base from which further advances in teaching may be made.

8. *Graphical Treatment of Vectors* is taught in 80 per cent. of public schools and 41 per cent. of other secondary schools, mainly in the upper classes. There was nothing in the answers to indicate how far graphical treatment of vectors is associated with the study of complex numbers, but it is known to be thus associated in some schools. As a rule, it is probably taken in connection with mechanics (velocity, acceleration, force), which is usually in England a part of the mathematical course. The question might with advantage have been expressed more clearly.

9. *Graphical Statics :*

Public Schools.—95 per cent. do this subject, viz., in upper classes, 69 per cent.; middle, 12 per cent.; lower, 1 per cent.; specialists only, 13 per cent.

Other Secondary Schools.—63 per cent., including 38 per cent. which take the subject in upper classes, 14 per cent. in middle; specialists only, 9 per cent.; the rest unclassified.

This teaching is therefore practically universal in public schools, and common in other secondary schools.

* See p. 381 above.

10. *Formal Methods of Contracted Multiplication and Division:*

Public Schools.—In 84 per cent., viz., in upper classes 19 per cent., middle 38 per cent., lower 27 per cent.

Other Secondary Schools.—In 93 per cent., viz., in upper classes, 24 per cent.; middle, 49 per cent.; lower, 16 per cent.; rest unclassified.

There was some reason to think that the force of the word "formal" was not appreciated in all cases. There are indications that the use of formal rules is diminishing. On the one hand, the omission of useless figures and the presentation of results to a given number of significant figures must become increasingly general as practical and laboratory work in the lower classes gains ground; on the other hand, informants say that logarithms prove more useful than contracted methods; that contracted rules are learnt, but seldom used afterwards; that they have given up the use of such rules after trying them for many years; and so forth.

11. *Use of Slide Rule—(a) Front Scales; (b) Trigonometrical Scales:*

The replies did not discriminate between (a) and (b).

Public Schools.—Slide rule used in 70 per cent., but mainly by specialists (55 per cent.). In the ordinary course for upper classes, 10 per cent.; for middle, 3 per cent.; for lower, 1 per cent.

Other Secondary Schools.—Used in 33 per cent., including upper classes 15 per cent., specialists 11 per cent.

On the whole, the slide rule is used by mathematical teachers only under compulsion.

12.—(a) *Tables of Squares:*

Public Schools.—Used in 61 per cent., viz., upper classes, 23 per cent.; middle, 15 per cent.; lower 3 per cent.; specialists only, 11 per cent.; not classified, 9 per cent.

Other Secondary Schools.—Used in 33 per cent., viz., in upper classes, 18 per cent.; middle, 8 per cent.; lower, 1 per cent.; not classified, 6 per cent.

12.—(b) *Tables of Square Roots.*—Practically the same results as in 12 (a).

12.—(c) *Tables of Cube Roots:*

Public Schools.—Used in 45 per cent., viz., in upper classes, 15 per cent.; middle, 9 per cent.; lower, 3 per cent.; specialists only, 11 per cent.; not classified, 7 per cent.

Other Secondary Schools.—Used in 30 per cent., viz., in upper classes, 17 per cent.; middle, 6 per cent.; lower, 1 per cent.; not classified, 6 per cent.

12.—(d) *Tables of Logarithms :*

Public Schools.—Used in 100 per cent., viz., in upper classes, 49 per cent. ; middle, 37 per cent. ; lower, 4 per cent. ; specialists only, 3 per cent. ; not classified, 8 per cent.

Other Secondary Schools.—Used in 97 per cent., viz., in upper classes, 58 per cent. ; middle, 30 per cent. ; lower, 2 per cent. ; not classified, 7 per cent.

12.—(e) *Tables of Trigonometrical Functions :*

Public Schools.—Used in 100 per cent., viz., in upper classes, 60 per cent. ; middle classes, 25 per cent. ; lower, 4 per cent. ; specialists only, 3 per cent. ; not classified, 8 per cent.

Other Secondary Schools.—Used in 88 per cent., viz., in upper classes, 63 per cent. ; middle classes, 17 per cent. ; lower classes, 1 per cent. ; not classified, 7 per cent.

12. - (f) *Number of Figures to which Tables are calculated :*

Various combinations of 4, 5, 6, 7 occurred, of which the following were the most general.

Public Schools.—4-figures, 65 per cent., 4-figures supplemented by 7-figures in upper classes 16 per cent.

Other Secondary Schools.—Reckoning percentages on the total number of schools that use tables, the result is : 4-figures 69 per cent., 4-figures supplemented by 7-figures in upper classes 14 per cent.

12.—(g) *Résumé of Returns dealing with Tables :*

Different tables are used in the following descending order of frequency—logarithms, trigonometrical functions, squares and square roots, cube roots. Many schools are of opinion that the use of logarithms obviates the necessity of tables of squares and square roots, but there is no doubt that these latter tables have their place in schoolwork. The extent to which the everyday use of tables has established itself is shown by the prevalent use of 4-figure tables ; 7-figure tables are too cumbrous for most school purposes. 4-figure tables used to be used mainly by science masters, but have now been largely adopted by mathematicians as well.

13. *Graphical Work in Solution of Transcendental Equations (e.g., $\log x = 1 + \frac{1}{x}$), in conjunction with Newton's Rule, Processes of Trial and Error.*

Public Schools.—Done in 47 per cent. of schools, viz., in upper classes, 29 per cent. ; middle, 1 per cent. ; specialists only, 17 per cent.

Other Secondary Schools.—In 16 per cent., practically always in upper classes.

This work, then, is not very usual, and seldom goes beyond a few illustrative examples.

14.—(a) *Estimation of Area by means of Squared Paper.*

Public Schools.—Done in 93 per cent., viz., in upper classes, 3 per cent. ; middle, 35 per cent. ; lower, 46 per cent. ; specialists only, 9 per cent.

Other Secondary Schools.—Done in 93 per cent., viz., in upper classes, 3 per cent. ; middle, 23 per cent. ; lower, 55 per cent. ; not classified, 12 per cent.

This work appears to be a standard part of the curriculum for the middle and lower classes. A rather curious feature is that it seems to be done later in public schools than in other secondary schools. There is some evidence that this is connected with the greater prevalence of geography teaching in other secondary schools, determination of areas by squared paper often being associated with geography.

14.—(b) *Estimation of Area by means of the Planimeter.*—Taught in only 15 per cent. of public schools and 6 per cent. of other secondary schools, and in these cases only to a few boys in the highest class.

15. *General Remarks.*—In framing the questionnaire perhaps I did not keep very closely to my brief, as some of the questions have no particular bearing on methods of intuition and experiment. The questionnaire followed the lines of a circular issued by the Central Committee of the International Commission on Mathematical Education, and in spite of the criticism just made it may fairly be said that all the topics referred to have a somewhat modern flavour, while it is equally true that intuition and experiment is the keynote of modern mathematical education. In a broad sense, then, the questionnaire is appropriate to its purpose.

The object of this report is to state facts, not views. If I speak of the spread of intuitional and experimental methods as an advance, I do so merely to save words, and with all possible respect to the views of those who would prefer another description. With this proviso, it is clear that there has been a marked advance in England during the present century.

The use of graphical methods in elementary algebra teaching is universal and entirely a 20th-century development. Other aspects of the same movement are the adoption of descriptive geometry by the mathematicians, the use of handy 4-figure tables, and of graphical methods in statics, and, though, in these cases, the victory is less complete than that of the "graph," it is remarkable and equally modern. The absence of practical astronomy from the ordinary curriculum is perhaps not surprising, in view of the difficulty of organising a systematic

course of observations ; but a good deal of interesting work is done in a few schools with sun and shadow, and in determination of latitude ; here there is room for further advance, where time permits. Practical fieldwork has not the place that it deserves in English schools.

A close connection will be noticed between the activities of the schools and the demands of examining bodies ; these bodies have to bear the chief responsibility for what goes on in the schools. To a great extent, they have considered favourably the suggestions made by representative bodies of teaching ; and to this is due the wide adoption of some of the subjects referred to in this report. In other subjects, such as practical fieldwork, which are somewhat unsuited for examination tests, it is easier to measure the advance that teachers have made spontaneously.

C. GODFREY.

EXAMINATIONS FROM THE SCHOOL POINT OF VIEW.

In the large public schools of England, and in many of our other schools of established reputation, it has not, as a rule, been considered essential for all boys who are drawing near the end of their school career to pass any examinations in addition to those which are regularly held by the members of the school staff. In such schools the competition amongst the pupils is so keen, and the standard reached by the various forms so well known, that the fact that a pupil has reached a certain position in the school is a sufficient hall-mark for ordinary purposes. If, however, a boy wishes to complete his education at one of the Universities, or to enter one of the learned professions, he is required to pass some preliminary examination as a test of his general education.

In all these preliminary examinations a certain amount of mathematical knowledge is obligatory. In some of them "More advanced Mathematics" is one of the special subjects from which the candidate for admission is required to select a given number.

When we come to examine in detail the syllabuses of the several preliminary examinations for the different Universities, and for those professions in which a knowledge of Mathematics is not directly useful, it is rather difficult to determine from them what amount of Mathematics is generally-considered to be necessary and sufficient for a liberal education. A certain amount of Arithmetic is required in all cases. A little Algebra and Geometry is also necessary for all, with the remarkable exception of Oxford University, where the candidate must *take his choice between* Algebra (to simple equations) and Elementary Geometry (substance of parts of Euclid, Books I., II., III.) The standard required for the Law is fixed by simple equations and four books of Euclid; for the medical profession quadratics and three books of Euclid; the requirements for the Army are frequently changing, and the examination is of a special type, which must be considered separately.

Of the Universities, the amount of Mathematics required for entrance at Oxford is practically fixed by the standard of Responsions as given above; that for Cambridge by the Previous Examination, in which the Algebra includes progressions, graphs, and logarithms, and the Geometry some knowledge of similarity. Corresponding to these two examinations, separate Matriculation Examinations are held by the Universities of London, Birmingham, and Durham, by the Joint Board of the Manchester Liverpool, Leeds, and Sheffield Universities, by the Welsh

University, by the various Scotch and Irish Universities, and by certain University Colleges and Technical Colleges. Some idea of the number of different Matriculation Examinations may be gathered from the fact that no less than sixteen such examinations are specified by the regulations of the Institute of Civil Engineers as exempting, under stated conditions, from the examination for admission as a student of that Institution. This number does not include several Colonial University Examinations, nor the local and other similar certificates which also appear in the list of exemptions.

The general standard of the Elementary Mathematics demanded in these examinations is roughly the same as that of the Cambridge Previous Examination given above. There are various differences in detail, such as the inclusion or omission of the theory of indices, surds or proportion in Algebra, and similarity in Geometry. The questions set by the various examining bodies, however, differ so widely in character that this equality of standard is hardly more than nominal.

In addition to the educational work carried on by the various Universities as teaching centres for higher education, they all give practical recognition of their duty to the sources from which they derive their students by holding frequent "local examinations" at various centres, and by examining and inspecting those schools in their various districts which desire such assistance. At these examinations certificates are given to the successful candidates, by means of which they can obtain exemption from most of the preliminary or matriculation examinations, provided that the range of subjects in which they have passed and the standard reached in them are satisfactory.

Of late years the number of schools which avail themselves of the services of the University Boards to examine the upper forms has considerably increased. In some cases the lower forms are also examined, and junior or preliminary certificates awarded to the pupils who reach a high enough standard.

Passing over these examinations for lower forms, we may roughly divide the jumble of pass examinations open to those who have just left or are just leaving school into four classes, according to the nominal standard required for a pass in Elementary Mathematics.

1. The Oxford Higher Locals, the Cambridge Higher Locals—candidates for which must have reached the age of 17, or have passed certain of the examinations in Class 2—and the Preliminary Examination of the Institute of Civil Engineers.

2. The Oxford Senior Locals, the Cambridge Senior Locals, the Cambridge Previous, about sixteen different Matriculation Examinations, some of the harder Preliminary Examinations, the College of Preceptors 1st Class, and the examinations for School Certificates, Higher School Certificates, Senior School Certificates, or School Leaving Certificates held by the Scotch Education Department, the Intermediate Education Board of Ireland,

the Oxford and Cambridge Schools Examination Board, the Delegates of the Oxford Locals, the Cambridge Local Syndicate, and the Matriculation Boards of most of the other Universities and University Colleges.

3. The Junior Locals, Junior School Certificates, College of Preceptors 2nd Class, and the easier Preliminary Examinations.

4. Oxford Responsions.

In many of the above Certificate Examinations Algebra and Geometry are not compulsory subjects, but a certificate is given to any candidate who satisfies the examiners in a stated number of subjects. The certificate given states the subjects in which the candidate has passed, and the subjects, if any, in which distinction has been gained. Such certificate exempts the successful candidate from the various Matriculation and Preliminary Examinations, providing that it shows a sufficiently high standard in the subjects required for the particular examination which it is desired to escape. Certificates are also granted which exempt from various examinations candidates who have not obtained a full certificate for the examination which they have taken, *e.g.* in the examination held in July 1910 for the Higher Certificate of the Oxford and Cambridge Schools Examination Board, there were 1,706 candidates for full certificates, of whom 911 were successful, and 294 of these obtained distinction in one or more subjects; 440 candidates, who had already obtained certificates in a former examination, were now trying to gain "distinction" in one or more subjects; 158 were successful in this. Of those who failed to gain a full certificate 24 obtained certificates to exempt them from Responsions, and 103 were exempted from a part or parts of the Previous Examination.

Similarly in the examination for School Certificates held by the same body in July 1910 we find that, besides those who gained full certificates of types which would exempt from Responsions or the Previous Examination, there were 14 candidates who obtained certificates to exempt them from Responsions, and 15 who secured exemption from parts of the Previous Examination. Thirty-three gained School Certificates qualifying for the Army, and 60 obtained Leaving Certificates for Army purposes.

So far we have considered the various examinations from the point of view of the student who must pass some examination for the definite object of entering a profession or one of the centres of higher or specialized education. We have found that he has in general a great variety of examinations to choose from which are nominally of the same standard. But these examinations differ in certain details, so that a complicated system of requirements is insisted upon in order that any one of these examinations may exempt the successful candidate from passing any other of the same class. (There are six pages of about 500 words each in the Calendar of the University of London devoted

to Exemptions from the Matriculation Examination, and Privileges and Exemptions granted by other bodies to those who have passed the London Matriculation.)

There is, however, another point of view from which most of the examinations already considered must be regarded. As I have already stated, there is no necessity for anyone to possess a certificate relating to the thoroughness or success of his education, unless he requires it for one of the purposes previously mentioned. The large majority of those who leave school for business of any kind are glad enough to escape any direct test of their literary or scientific attainments; but there are always a large number who themselves desire, or their parents or guardians desire for them, something in the nature of a certificate to show what their education has been and how they have profited by it. This desire is especially strong amongst those who are conscious of good abilities, but have not had the opportunity of going to school and comparing themselves with others of the same age and class. This leads many students, who do not intend to continue their education at the University or to enter any profession, to pass one of the Matriculation Examinations, or to take as good a certificate as they can at one of the Local Examinations or at a school examination held by one of the recognized examining bodies. Many schools also find it an advantage to pass as many candidates as possible at these examinations, and there is considerable competition amongst schools of a certain type to make the best show in them.

To meet the requirements of such candidates, for whom the several examinations are not merely stepping stones to enable them to reach some desired goal but represent the goal itself to be reached, it is necessary to have some system of classification by which the proficiency of the successful candidates may be gauged. This is done either by dividing those who have passed into classes, or by awarding distinction in the subjects in which especial proficiency has been shown and recording the fact upon the certificate, or by a combination of these two methods. It is impossible to deal with all the alternatives, but the following may be considered as typical:—

In the London Matriculation Examination the names of the successful candidates are published in two divisions, each division in alphabetical order. Subjects taken are entered upon the certificate, but no credit is given for distinction in any subject. Elementary Mathematics is a compulsory subject. "More advanced Mathematics" is one of the optional subjects of which two must be taken. It includes the binomial theorem (positive index), Trigonometry to solution of triangles, and the straight line and circle in Analytical Geometry. Mechanics stands by itself as one of the optional subjects.

In the corresponding examination held by the Joint Matriculation Board of the four younger Universities the names of successful candidates are published in alphabetical order in two

divisions, but supplementary lists are published of those who gain distinction in certain subjects. In order to gain distinction higher alternative papers must be taken. These higher papers in Mathematics, which may be taken instead of the compulsory pass papers, include questions on the binomial theorem, Trigonometry to solution of triangles, Analytical Geometry to the end of the ellipse, and Pure Solid Geometry of the straight line and plane. Mechanics or Physics ranks as one of the optional subjects of which two must be taken, and the corresponding higher alternative paper includes both Mechanics and Physics.

Those who pass the Cambridge Previous Examination are also arranged in classes: but this examination is only open to those who have arranged to come into residence as members of the University, and is not available for the purpose now under consideration.

To gain a first-class certificate from the College of Preceptors a candidate must pass in six subjects, of which Arithmetic and English are compulsory, and may offer twelve. Honours are gained by reaching a certain aggregate of marks, and distinction may be gained in any subject by getting three-quarters of the maximum for the paper in that subject. Algebra, Geometry, Trigonometry, and Mechanics are separate subjects. The Algebra includes progressions; Geometry the standard of Euclid VI. 1-20; Trigonometry solution of triangles.

In the Oxford Senior Locals the subjects in which a candidate may be examined are divided into twenty-three sections, and a certificate is granted to any candidate who passes in one examination in five sections. No section is compulsory, but there are certain restrictions as to choice. The title of Associate of Arts is conferred upon successful candidates who are under 19 years of age. "Over age" certificates are also awarded. Subject to the above limit of age the successful candidates are arranged in three honour classes and a pass list. Those in the first class are in order of merit; the second class is arranged in not more than five divisions; the pass list is in two divisions. Special lists in order of merit give the names of those who have gained distinction in any of the sections. The holder of a certificate can enter for any sections at a subsequent examination and gain supplementary certificates, which may include distinction if the candidate is still under 19. Of the above twenty-three sections the three which interest us are Arithmetic, Mathematics, and Higher Mathematics. In Arithmetic distinction is not awarded. In Mathematics Algebra to the binomial theorem (positive integral index) and Geometry to the end of the circle, suffice for a pass, but Trigonometry to the solution of triangles must also be successfully offered in order to gain distinction. In Higher Mathematics there are three divisions: (a) Higher Algebra and Plane Trigonometry (including some differential), (b) Higher Geometry including analytical conics

and some tracing of curves of a higher degree, (c) Applied Mathematics. A pass may be secured by doing sufficiently well in one division, a pass in any division by doing well enough in either of the subjects included in the paper. For distinction it is necessary to pass in two of the three divisions at least.

In the Cambridge Senior Locals there are seventeen sections, not counting Arithmetic, which is compulsory for all. A pass may be obtained by passing in three sections. A candidate may offer seven sections and one subject from an eighth with some restrictions as to choice. There are three honour classes and a pass list, each of which is arranged in alphabetical order. Those over 19 are placed in a pass list by themselves. The honour classes are determined solely by the total number of marks gained by the student. Distinction in any subject is noted on the certificate given, and separate lists of those who gain distinction in particular parts of the examination are published. These lists are in order of merit. Honours and distinction can only be gained by those under 19. Supplementary certificates can be gained by those who already hold certificates. In this examination Mathematics is one of the sections, and the subjects included are Geometry, Algebra, Trigonometry, Differential, Analytical Geometry, and Applied Mathematics. To pass in the section it is only necessary to pass in Geometry and Algebra. For a pass in Geometry the candidate must satisfy the examiners in Plane Geometry including similar triangles, but questions are set on the elements of solid for those who aspire to distinction. Similarly in Algebra the limit necessary for a pass is the binomial theorem (positive integral exponent), and in Trigonometry solution of triangles; but higher questions are set in both these papers for the benefit of aspirants to honours.

In the Oxford Higher Locals and Cambridge Higher Locals the number of sections is less and the range of each section or group of subjects is wider. The successful candidates are arranged in three classes, of which the third denotes a pass without honours, the first and second are honour classes. To gain the Oxford Certificate one of the groups taken successfully must be either Languages, or Mathematics, or Natural Science. For the Cambridge Certificate a pass in Languages or Mathematics is essential. For either Certificate Arithmetic is a compulsory subject, and a candidate must also pass in three sections. For honours either a pass in three sections with a first or second class in two of the three, or a pass in four sections with a first class in one of the four must be gained. (There is some relaxation of the requirements in favour of candidates who have gained honours in the Senior Locals or distinction in the Higher Certificates of the Oxford and Cambridge Schools Examination Board.)

To gain a pass in Mathematics in the Oxford Higher Locals candidates must pass in Algebra to the binomial theorem (positive

integral index) and in Geometry, including some analytical conics and curve tracing. For honours they must also offer Higher Algebra, the subject matter of Euclid XI. 1-21, Plane Trigonometry, including De Moivre's theorem and its applications, and *either* Mechanics *or* Conics treated both geometrically and analytically (if they wish they may offer both).

To gain a pass in Mathematics in the Cambridge Higher Locals, Geometry to similar triangles and Algebra to the binomial theorem with positive integral indices is required. For Honours some knowledge is required of the Geometry of the plane and the sphere, and of more advanced Algebra, as well as of some of the following subjects: Trigonometry, Geometrical and Analytical Conics, Mechanics, Hydrostatics, Astronomy, and Differential and Integral Calculus.

For the Higher Certificate of the Oxford and Cambridge Schools Examination Board the school subjects are divided into four groups: Group I., Languages; Group II., Mathematics; Group III., English Subjects; Group IV., Science. For a pass the candidate must pass in at least four subjects taken from three or more Groups, or in one subject in Group II. and three in Group I. (There are various special cases which we need not consider.) Group II. comprises (a) Elementary Mathematics, (b) Additional Mathematics; (a) includes Arithmetic, Algebra to quadratics, and Geometry to the end of the circle. For a pass in (b) a candidate must satisfy the examiners in each of the subjects Arithmetic, Algebra to progressions, Geometry including similar triangles, Trigonometry to solution of triangles, and *either* Elementary Statics *or* Dynamics (as a further alternative a compound paper in Statics *and* Dynamics may be taken). A candidate for distinction must also take "special distinction papers" in Algebra, Pure Geometry, Analytical Geometry, and elementary Differential and Integral Calculus. These include some higher algebra, the earlier part of theory of equations, and a considerable amount of higher geometry and geometrical conics. He must also take both Statics and Dynamics. (Seven pages of the Regulations are devoted to "Exemptions by means of the Higher Certificate Examinations.")

The various School Certificates are as a rule awarded subject to the condition that the student should have been for so many years at a school or schools periodically inspected and approved by the examining body. Certificates are granted to exempt from the chief preliminary examinations. The examinations for these certificates are as a rule pass examinations, no effort being made to pick out the candidates worthy of distinction. Many schools are examined by the various Boards, and a School Certificate Examination is regarded as a good test of general education for candidates of about 17 years of age.

This necessarily incomplete sketch of the complicated and confusing regulations of a few of the examinations provided for testing the proficiency of those who choose to apply the test will

show that there is no lack of opportunity for doing so. A sufficiently wide range is covered by many of the above schedules to make it certain that a candidate who has gained honours in any subject has been well grounded and has shown considerable ability. The certificates given are so explicit that no mistake should be possible as to the degree of proficiency indicated by them. Unfortunately it appears that very few people take the trouble to form a just estimate of the range required for different examinations, with the result that quite undue importance is attached to success in some examinations in which the range of knowledge required is very limited.

The most glaring instance of this is the London University Matriculation. This examination is rated so highly in the public estimation that I find that out of 500 schoolmasters who retail their various qualifications in the *Schoolmaster's Year Book*, no less than 108 mention the London Matriculation and seven mention various other Matriculations, while only one refers to his success in the Oxford Locals. These 500 names were taken at random, in batches of 100 each, from the Directory. A good many graduates of London do not mention the Matriculation and these I have not counted. Of the 108 who mention the Matriculation 55 have graduated at London University; 22 have passed the Intermediate in Arts or Science, of whom 2 graduated at Oxford, 1 at Cambridge, 1 at Manchester, and 1 at Durham, while 17 have taken no higher University qualification; 31 of the 108 have proceeded no further at London, of whom 11 have graduated at other Universities, but 20 can claim no higher University qualification than the London Matriculation. This frequent mention of the examination and the fact that it is mentioned by men who have graduated with high honours, show that it is regarded by the general public as a qualification of considerable importance. This fact is still further impressed upon schoolmasters by the frequency with which parents desire them to prepare for this examination boys who have apparently no particular reason for passing it.

A few figures taken from the statistics published in the *Calendar of the University of London (1910-11)* give some idea of the large number of students who take the Matriculation Examination without proceeding further with the London University course. The figures given for the seven years 1903-1909 inclusive show that during that period 44,065 candidates presented themselves for the Matriculation Examination, of whom 18,979 (or 43.1 per cent.) passed. During the same period 16,833 candidates presented themselves for the various Intermediate Examinations, of whom 9,330 (or 55.4 per cent.) passed, and 8,067 for the various Bachelors' Degrees, of whom 4,902 (or 67.7 per cent.) passed. The number who present themselves for the Intermediates (which number must include a good proportion of duplicate and triplicate entries of the same individual in an examination which rejects 44.6 per cent. of the

candidates) is 88·7 per cent. of the number who have qualified themselves to do so by passing the Matriculation Examination. This proportion would not appear to imply any very marked leakage, but examination of the list of undergraduates in the same Calendar discloses the fact that there are roughly 120 columns of names of those who have passed the Matriculation Examination, and 68 columns of those who have obtained exemption by means of school certificates or by passing other recognised examinations. As the large majority of these must have matriculated with a view to proceeding further with the University course, it seems safe to assume that at least one-third of those who present themselves for the Intermediate are not included in the number who have passed the Matriculation. It seems certain, therefore, that at least 30 per cent. of those who submit themselves to the London Matriculation Examination do so solely for the sake of the *kudos* to be gained by passing it.

Now, if we examine in detail the particulars of the Mathematics required for the London Matriculation, we find that the range is very limited.

The Elementary Mathematics, which is a compulsory subject, is tested by two papers—one in Arithmetic and Algebra, the other in Geometry. The Algebra includes the three progressions, but excludes surds and indices, ratio, proportion, and variation. The Geometry, the subjects of Euclid I.-IV., with simple deductions, easy loci, areas of triangles and parallelograms, of which the bases and altitudes are given commensurable lengths. (Euclid's proofs are not insisted upon.)

Mathematics (more advanced) is one of a group of eighteen optional subjects, of which two must be taken. (One of these must be a language, unless Latin has been selected from the previous group of subjects.) It is tested by one paper, which includes Algebra, indices, logarithms, and binomial (positive index). Geometry, similar figures, mensuration of the circle, and elementary co-ordinate geometry of the straight line and circle. Trigonometry, including solution of triangles. In the three examinations, June and September 1909 and January 1910, this subject was selected by 8·4 per cent. of the whole number of candidates, and of these 47·5 per cent. were rejected.

It appears, then, that about 95 per cent. of the London Matriculation certificates can bear no higher evidence of mathematical reading than the range covered by the two papers in Elementary Mathematics—a range which is obviously insufficient to rank highly as a qualification for the teaching profession, or for any business or profession in which knowledge of Mathematics is important.

It seems, then, fair to take the London Matriculation Examination as an example of an examination to which exaggerated importance is attached as a test of knowledge and ability. Nor is the reason for this far to seek. As stated above, the number who have passed this examination during the last seven years

recorded is only 43·1 per cent. of the total number of candidates. The total number of successful candidates from 1838 to 1909 inclusive is 49·1 per cent. of those entered. The examination is obviously a difficult one to pass, and it has always had that reputation. The majority of the general public who have not the inclination, or perhaps lack the ability, to see for themselves what ground is covered by the syllabus of the examination, are quite prepared to accept it as a good test, because they know that so many have failed to pass it. Now, it is clearly objectionable that an examination, of which the syllabus only covers the minimum range of knowledge required from boys of the most moderate ability, should be generally recognised as a searching test of proficiency. This is doubly objectionable if that reputation is gained by setting difficult and catchy questions of a puzzling or confusing type. Every schoolmaster of experience knows that such questions are quite unnecessary for the purpose of weeding out the stupid or ignorant, and that their use is always liable to produce a false and fluky order in which occasional very moderate boys come out far too high, while occasional boys of quite good ability come to unexpected grief. Experience shows that this is precisely what does happen in this particular examination. You can never guarantee that a boy of good ability will get through, but you need never feel surprised if a boy of very moderate power appears in the First Division.

But the great objection to this style of examination from the schoolmaster's point of view lies in the fact that it is almost hopeless to send in a candidate without special preparation. Even a quite good boy requires a certain amount of practice in the papers which have been set, or he is almost certain to fail to do himself justice. The average boy requires careful coaching in the art of doing the papers. He may have to waste months, possibly more than a year, just at a time when he is beginning to go ahead with his Mathematics, in learning nothing new except how to cope with a particular kind of examination paper. The work on which he is to be examined in Mathematics is 3rd Set or 4th Set work, and if he is going in for the London Matriculation it is useless as a rule to take him up into the 2nd Set until he has passed the examination. He may be quite fit to come up and get on, but experience shows that it is wisest to let him stop where he is and grind away at Matriculation papers until he is through. The moral effect upon the boy's work is usually disastrous; he is checked back just at the time that it is important for him to be pressing forward, and he is quite likely never to get on the move forward again with any heart in his work.

It seems clear that an examination which is to pick out the first 50 per cent. of the candidates should be arranged on lines which include the work which the first 50 per cent. may be supposed to have done, not set solely upon the work of the last 50. Any attempt to select the first 50 per cent. by setting

questions confined to the work which may reasonably be expected of the lowest 50, but making these questions hard enough to be beyond the power of the lowest 50, must be unsatisfactory as a test of knowledge, and must lead to unnecessary dislocation of work for the candidates who are preparing for the examination. An examination which only professes to test the general education of the candidates examined should be a purely pass examination presenting no special difficulty to a boy of ordinary ability whose education has covered the range required. If it is desired to pick out candidates of special ability, this should be done by means of special alternative questions, or preferably by special papers suited to the purpose. We have already seen that one or other of these plans is adopted in most of the examinations in which honours or distinction are awarded.

What has been said about the London Matriculation applies, though in a less degree, to most of the pass examinations in the second group—the questions set are of a harder type than is necessary or desirable for the purpose for which the examination is intended. Examiners, whose business it is to weed out the inefficient, seem unable to realise that this should be the sole object of the papers set. They will try at the same time to separate the extra-efficient. The two things cannot be done satisfactorily by means of a single paper in which only very elementary portions of the subject tested are to be included. If the questions set are such as test the ability of the best candidates, they are too hard for those of moderate ability whose knowledge and efficiency should entitle them to a pass. Consequently either a large number of the latter class must be rejected, or the standard to be reached on the paper must be fixed so low that a pass is very little criterion of the required proficiency. In either case the element of luck, which cannot be altogether eliminated from examinations, is far too large a factor in the success or failure of those who are near the line.

Another fact which is worthy of careful consideration presents itself at this point. A number of the examinations under consideration reject about 50 per cent. of the candidates examined. Suppose the average of passes to be 50 per cent. This implies that all the candidates whose performance is below the mean fail. Now, if an examination which professes to be instituted merely for the sake of weeding out those who are inadequately educated is so arranged that the mean boy can only just scrape through, either there must be something very wrong with our system of education, or the standard required must be beyond that which is reasonable for the purpose professed.

Take for example, the School Certificate Examination of the Oxford and Cambridge Schools Examination Board—a body whose authority on the standard of proficiency which may reasonably be expected ought to be beyond reproach. First let us see what they profess. In the Regulations we find—

"The School Certificate Examination is intended for Fifth Form candidates of about 17. It is a Pass Examination intended as a test of general education." In the report for 1910, "The results of the School Certificate Examination confirm the opinion of the Board that this examination is a good test of general education for candidates of about 17 years of age." I take it that a *good* test of general education means a test of the standard which may reasonably be expected from candidates who have been properly educated. Every candidate for a school certificate must be a member of a school which has been inspected and approved by "this Board," or inspected by the Board of Education and approved by "this Board." Consequently we may assume that in the opinion of the Oxford and Cambridge Schools Examination Board they have all had the opportunity of being properly educated. Now let us examine the results of this good test. In the four examinations of which statistics are given in the report for the year ending October 31st, 1910, 1,300 candidates for School Certificates were examined; 537 certificates were awarded; 41·3 per cent. of the candidates were successful, and 58·7 were rejected. Of course, a good number of the candidates may not have fulfilled the condition of being "about 17 years of age"; but, presumably, they would be all, or nearly all, up to Fifth Form standard, and it is not as a rule the youngest members of a high form who are most likely to fail in examination. The conclusion is almost irresistible that the requirements of the examiners are beyond what may reasonably be expected for a properly educated Fifth Form candidate. If we examine in detail the number who passed in each subject, we find that the total number of candidates includes 266 who were not candidates for School Certificates, in addition to the 1,300 candidates already mentioned. These were either taking one or more subjects to complete certificates already obtained, or candidates for a certificate exempting from some other examination. Of the total number examined in Latin 60·9 per cent. were successful, in Greek 58·6, in French 64·3, in German 66·7; of the total number examined in Mathematics 38·2 per cent. were successful in Elementary Mathematics, and 50·3 in Additional Mathematics, but we are not told how many of these took both subjects. In Scripture Knowledge 54·6 per cent. passed, in English 74·8, in English History 66·9, in Geography 63·1, in History and Geography 49·9, in Mechanics 57·7, in Physics and Chemistry 67·3, but here again there may be some overlap in the three subjects Physics, Chemistry, and Physics-and-Chemistry, which I have found it necessary to group together.

I consider, for reasons which I shall endeavour to substantiate presently, that an average of about 75 per cent. might be reasonably expected to pass an examination properly designed to weed out the inefficient, and allow those who have made good use of their educational advantages to pass, provided that their

education has been on the right lines, and has covered the range which the examination tests. The above figures show that this ideal has been actually reached in English, and has been approached within fairly reasonable limits in several of the other subjects tested. The question naturally suggests itself, "Why then does the percentage of passes for the certificate fall so very far below the ideal?" It appears to be pretty certain that failure is largely due to the difficulty which the ordinary candidate finds in getting up to the required mark in so many as five subjects at once.

This view is supported by the facts disclosed by a study of the statistics given for the Higher Certificate Examinations held by the same Board in 1909 and 1910. Comparing the percentage who passed in the different subjects we find that in only two subjects, Greek and Geography, is this percentage higher for the Higher Certificate than for the School Certificate. In the remaining subjects, which can be compared, the percentage who passed in the higher examination is less by an amount which varies from about 1 to 12 or 13. Yet out of 3,540 candidates for the Higher Certificate, 1,879, or 53 per cent., were successful, as against 41.4 per cent. of the 1,298 candidates for a School Certificate. Here we have two examinations conducted by the same Board. There are nearly three times as many candidates for the higher as there are for the lower of the two certificates. The difficulty of passing in the several subjects of the higher is by no means less. Yet the percentage of successful candidates in the lower examination is only about three-quarters of the percentage who succeed in the higher. This may be accounted for by the difference in the number of subjects in which a pass must be obtained in order to secure a certificate, which is four for the Higher Certificate and five for the School Certificate. It should be noted that a candidate for the Higher Certificate may offer six subjects and fail in two of them, and yet get the Certificate. A candidate for the School Certificate may offer any number of subjects which do not clash with each other in the time-table. This may account, to some extent, for the percentage of passes in the several subjects being so much higher than the percentage of certificates granted would lead one to expect; for many candidates of more than average ability will have no difficulty in passing in six or more subjects. It is the boy or girl of moderate, or average ability, who finds a difficulty in reaching the required standard in five subjects. The low percentage of certificates granted to candidates whose course of education has been inspected and approved, is a sufficient proof of this. If inspection of a school course declares it worthy of approval, and examination of the results pronounces that only 40 or 50 per cent. have been properly educated, it would seem that either the inspection or the examination must be faulty. I think we may safely assume that schools which are regularly inspected and approved are doing their work well,

and are giving a reasonably good education. Assuming for the present that about 75 per cent. of the pupils so educated should be entitled to a pass, it follows that any School Leaving Examination which only allows a pass to 40 or 50 per cent. of the candidates, is not fulfilling its legitimate purpose—the purpose, namely, of weeding out those pupils only who have shown themselves incapable of profiting duly by the educational advantages offered to them, and cannot, therefore, be expected to take due advantage of any higher educational course.

It may fairly be argued, of course, that the question of fitness to enter any University as a student implies something more than the mere capability of profiting by the special course which the student intends to take at the University, that the University has a right to stipulate that every student shall have received a good general education before entering upon such special course, and that the requirements of the Matriculation or equivalent examination represent the minimum which can reasonably be offered as evidence of a good general education. In support of this it may be argued that some knowledge of five subjects such as candidates may select—say Latin, Greek, Elementary Mathematics, English, and English History; or French, Elementary Mathematics, English, Physics, and Chemistry—is by no means too much to expect from an ordinary well-educated boy or girl. It may further be argued that a Matriculation Examination or its equivalent is the only safe means of ensuring that a student's general proficiency is not being sacrificed to the desire for early specialisation, and that if the number of subjects demanded were reduced, general proficiency would be certain to suffer.

This is reasonable enough, but the fact remains that most of the examinations are so conducted that many more candidates are rejected than is necessary for the purpose of the examination. A false standard of proficiency is exacted, such that the natural ability of the candidate counts too much, and the result is that those who are not distinctly above the average in this respect find considerable difficulty in passing. This policy defeats its own object, as the inevitable result is that a candidate who has failed in one or other of his subjects must specialise more or less in his weak subject to the neglect of those in which he is making good progress, and this is surely specialisation of the worst type. Examining Boards should realise that education has its limits. You can educate your pupils up to a certain point; but with most of them you soon reach a clearly-defined limit beyond which the mind does not work. You can take your pupil across this limit and persuade him to acquiesce in the things which you tell him that you see there; you can teach him to answer questions on those things, and to make quite a good show when examined upon them, if he has plenty of time to get his subject up; but the whole thing is hollow, there is no conviction, only acquiescence; he takes what he is told and accepts it without appreciation, and his mind receives no training

as the result of your earnest efforts except in the virtue of obedience. For example, you may teach such a pupil that he is not to say that $a^{\frac{p}{q}}$ is a multiplied by itself $\frac{p}{q}$ times, and illustrate the futility of his definition by particular examples; he regards this as a foolish fad of yours, but he acquiesces. You then teach him to say that $a^{\frac{p}{q}} = \sqrt[q]{a^p}$ and you show him why that is so, and illustrate your proof by a number of particular cases. He easily learns to write out your proof, and even to answer all ordinary questions upon it in appropriate jargon; but the chances are that he fails to have the slightest appreciation of the thing which he has learnt, and that he now secretly cherishes in the back of his mind the conviction that $\sqrt[q]{a^p}$ means that a has to be multiplied by itself $\frac{p}{q}$ times.

To such a mind as the one I have selected, it is useless for the present to appeal by other than purely arithmetical ideas and processes of reasoning. His imagination may gradually be developed up to the point of appreciating the proof that he has learnt, but it will take time. In the meanwhile he may be pushed on upon the lines along which he is capable of moving, and learn much that will be useful to him, not only as a mental asset, but also as mental training. By adopting this course with pupils of this type we can turn them out well fitted to cope with the ordinary problems which have to be faced by ordinary men, and well qualified to hold their own in any business or profession for which their natural mental tendencies have marked them out as suitable. In other words, they will be properly educated. By devoting more time to the weak side of their nature, we may work them up to the point of being able to reproduce things which they have heard, but not appreciated, in a sufficiently faultless form to answer such questions as are likely to be set them in examinations, but the effect upon the mind is only skin-deep. They are better qualified to pass certain examinations, but less well fitted to hold their own in the world than they would have been if allowed to develop on the lines natural to them. They are less well educated.

It is the same in every subject which the pupil takes up. Unless he is exceptionally endowed by nature, there is a well-defined line which separates the things for which his mind is adapted from the things to whose inner meaning his mind is blind. These latter things may be scratched upon the surface of his mind by constant reiteration, but they can never be burnt in so as to influence seriously his mental development. He should be allowed to know something of these things; but he should not be kept back in his knowledge of things more suitable to his type of mind, in order to try to master thoroughly the things which lie beyond the sphere within which his mind is free to move.

Now, this mental limitation, of whatever kind it may be—want of geometrical feeling, want of imagination, want of taste, or what not—will in general make itself felt in a different degree in different subjects, so that a much higher degree of proficiency may be obtained in two or three selected subjects than in any other, and these selected subjects are the most suitable to form the main stand-by of the pupil's mental training. Other subjects should not be neglected, but no high degree of proficiency should be expected in them. A pass examination in general education should take count of them to the extent of ensuring that due attention has been given to them as secondary subjects, but should not exact so high a degree of proficiency that some months of special cramming in them is required to ensure a pass—cramming which involves neglect of the more important subjects in which the pupil is making good progress.

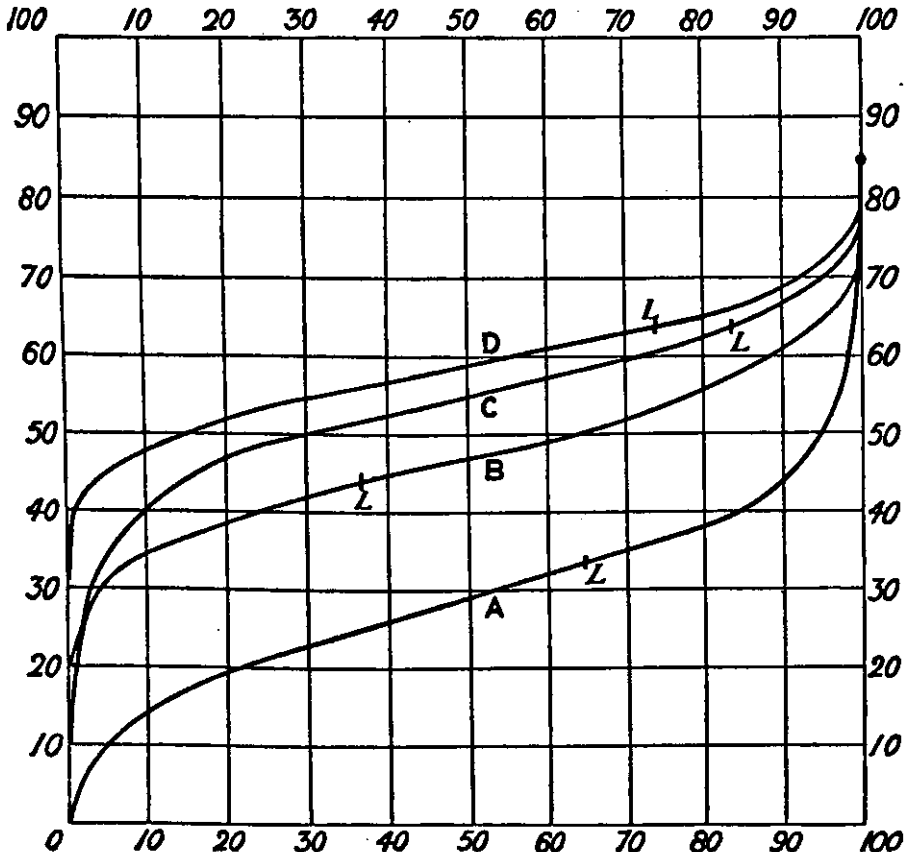
Before summing up the conclusions which may be drawn from the various statistics and general considerations which I have put forward, I will state briefly my grounds for objecting to the line in a pass examination being drawn at or near the mark which excludes 50 per cent. of the candidates. I will at the same time try to justify my preference for a line at about 25 per cent. or 75 per cent., according to the purpose for which the examination is intended.

I suppose that the general preference for the 50 per cent. line is grounded on the idea that all pupils ought to be able to reach an average degree of proficiency, and that 50 per cent. represents about the average. In other words, the average boy ought to pass; those below the average ought not. The objection to this plausible contention is that, as every student of statistics knows, "mediocrity is always the commonest condition" (Sir Francis Galton, "Natural Inheritance," p. 41). If a large number of individuals are compared with respect to any measurable quality, the rule is invariable that they cluster about the mean in such a manner that differences are very slight between the first 25 per cent. below the mean and the first 25 per cent. above, compared with the differences outside these limits. Consequently, if you draw your line at 50 per cent. you are "taking a shot into the brown," and it becomes a pure matter of chance which of the candidates who are just about the line do well enough to get through and which of them fail.

The normal curve of distribution is very nearly straight between the 25 per cent. mark and the 75 per cent. mark. Near these points it turns almost abruptly down or up and the gradient becomes rapidly more marked. This is the general shape of the curve when any physical attribute is taken as the subject of the scheme and it seems reasonable to suppose that if we could measure mental attributes precisely the same general law would hold good—viz. 40 or 50 per cent. of those examined would not differ markedly from the mean. 20 per cent. would pick themselves out at each end of the curve, the

difference of capacity from the general mean becoming rapidly more and more marked, as the individuals approach each end of the scale. Consequently I hold that in a pass examination, in which the range of subjects has been covered by all the candidates, the line should come on the average somewhere about the 25 per cent. mark if those only whose capacity is distinctly inferior are to be rejected. Of course a certain number of those who would naturally be fitted to pass would fall below the line owing to careless or insufficient work, but I take it that about as many would be able to raise themselves above it by unusual diligence and determination.

To illustrate this point of view I have constructed the following curves of distribution of the candidates for various public examinations. The figures in the vertical columns represent percentages of the maximum number of marks obtainable; those in the horizontal line represent percentages of the whole number of candidates. Each point on one of the curves indicates the percentage of the candidates in the particular examination or examinations to which that curve belongs, who failed to obtain the percentage of the maximum which corresponds to that point in the vertical column, *e.g.*, the point *L* on curve *A* corresponds



to 64·5 on the horizontal scale and 33 on the vertical. This indicates that 64·5 per cent. of the candidates failed to get 33 per cent. of the maximum in the examination represented by curve *A*, and 35·5 per cent. got 33 per cent. of the marks or more. If we follow the curve up we see that 14 per cent. of the candidates got more than 40 per cent. of the maximum, between 4 and 5 per cent. got more than 50 per cent. of the marks and so on. In the above diagram—

A is a curve of distribution by marks of the candidates for First Division Clerkships in the Civil Service in 1905 and 1907 (412 candidates).

D is a similar curve for the competition for Second Division Clerkships in October 1892 (712 candidates).

C co-ordinates the candidates for intermediate clerkships in the Civil Service, who competed for junior appointments in various Government Departments in July and December 1908 and July 1909 (564 candidates).

B is compiled from the four competitive examinations for the Royal Military College held in June and November 1908 and 1910 (1,008 candidates).

The various dates were not in any way selected. These were the first reports of examinations of the various types which came to hand. The observations were collated to each 5 per cent. of the maximum number of marks and the curves are figured, without any smoothing, exactly as they worked out. The letter *L* in each curve denotes the point at which the line was drawn between the successful and unsuccessful candidates. The four curves get mixed at the upper end and *B* and *C* cross at the lower end.

As the marks for purely pass examinations are not published I was unable to construct a curve of distribution founded upon any pass examination; but I was fortunate in discovering the record of the examination which gives us the curve *D*. This has all the characteristics which a purely pass examination ought to give, as the subjects tested were well within the scope of all the candidates. From the position in which the point *L* is placed it will be seen that the examination has answered its purpose well in picking out the best candidates from those who offered themselves for the vacant posts, as the rapid increase of gradient of the curve above the point *L* suggests that the candidates whose marks are represented upon this portion of the curve are for the most part considerably superior to the mass of those who come below. If the number of posts offered had made it necessary to draw the line somewhere near the 50 per cent. mark the best candidates would still have been included; but there would have been little reason to suppose that the selected candidates near the line of division were at all superior to the best of those rejected. On turning to the actual marks I find that the curve hardly shows this so clearly as it ought, for the marks of 84 candidates, or 11 per cent. of the whole number,

differ from the median mark by less than 1 per cent. of the maximum, a difference of standard which no examination can define with certainty. As the line stands, the examination may be regarded as successful from the point of view of the examining body, who may feel sure that they have secured the best men and that the lowest they have been obliged to take are well up to their work. But it is not quite so satisfactory from the point of view of the candidates who have just failed to secure an appointment; for there are still 62 candidates, or more than 8 per cent. of the whole number, whose marks differ from that of the last successful candidate by less than 1 per cent. of the maximum, and whose place is consequently largely a matter of luck. This suggests that the range of subjects ought to be enlarged if it is desirable to make selections for these posts fairly by open competition—a plan which has since been adopted.

If we take this curve *D* as being a typical curve to represent fairly the standard of efficiency of a typical group of candidates for a pass examination, its form seems to indicate that the line for a pass should be drawn between the 20 and 30 per cent. marks on the horizontal scale. If distinction is to be awarded first-class honours might be given to the top 6 or 7 per cent. and second-class honours, indicating extra efficiency, to those within the top 20 per cent.

The curve *A* is typical of the high-class competitive examination results. By a process of natural selection the candidates for First Division Clerkships are all or nearly all picked men. There are few, if any, of the candidates whose mental calibre would be insufficient to gain them a place in the top 10 per cent. of the competitors co-ordinated in curve *B*; yet we see that the curve has the same general characteristics as the other curves. The gradient is higher throughout the general course of the curve, but we have the same rapid increase of gradient in the top 15 per cent. going upwards and the bottom 20 per cent. going downwards from the median position on the curve. There can be no doubt that the most capable men have been selected, if any examination can do it.

Curves *C* and *B* represent competitive examinations of a less searching character. The position of the point *L* is an index of the keenness of the competition. This is sufficient to ensure a high standard of efficiency in the successful candidates for junior appointments in Government Departments, where the pay is adequate. If we assume that the competitive examination for the Royal Military College can be taken as a true test of the efficiency of those candidates who have survived the preliminary sifting of the qualifying examination, then the position of the point *L* in curve *B* appears to indicate that the competition for the honour of serving in the Army amongst men who can afford to do so is only sufficiently keen to eliminate those who are relatively inefficient.

It cannot be too strongly insisted upon that it is this question of keenness of competition which must settle the intellectual status of men selected by competition. All that a competitive examination can hope to do is to pick out the best from amongst the men who offer themselves. It is impossible to secure a better stamp of man by increasing the difficulty of the examination. As a rule (there are of course exceptions) men of ability can only be secured by offering them attractive terms. The English Army is at present no place for a young man who has no private means. It is this fact, and not want of patriotism, which limits the competition, with the result shown by the curve *B*. The style of the examination may, however, deter a certain number of desirable candidates from offering themselves, as we shall see later.

We are now in a position to sum up the conclusions to which we are led by what has gone before, and we see that:—

Examinations may be roughly divided into three classes:—

- I. Competitive examinations to pick out the best men for certain definite posts.
- II. Pass examinations to pick out those who are worthy of distinction, or distinctly above the average in any subject or subjects.
- III. Pass examinations to certify that the successful candidates have been reasonably well educated, and to weed out those whose lack of enterprise or want of ability have resulted in inefficiency.

We see further that examinations of each type must influence, more or less, the work of the candidates for some little time before they offer themselves for the approval of the examining body. For this influence to be beneficial it must be so exerted as to encourage the candidates to press on to higher work, without interfering unduly with a reasonably conceived and efficiently worked curriculum.

In our present system of examinations prejudicial interference with class work is the inevitable result of the extraordinary diversity and complexity of the requirements of different examining bodies, which all profess to have the same object in view. Further dislocation is caused by the almost invariable practice of examinations of Class III usurping the functions of Class II, and trying to pick out the best candidates, when they ought to devote their whole attention to the termination of the worst.

The interests of education call for the abolition of both these disturbing causes. To do away with the first, some general standard of requirements should be agreed upon, which could be comfortably covered by any approved curriculum, so that no specialisation would be required for examinations which do not profess to be of a special type.

To get rid of the second it is essential that the legitimate object of pass examinations in general education should be kept

in mind by the examiners, and that they should clearly realise that they are required to differentiate at the bottom and not at the top of the scale.

They should bear in mind that the receptive capability of the average boy—in which class I would include the whole of the 50 or 60 per cent. who cluster about the intellectual mean—is very limited compared with that of the top 5 or 6 per cent.; but that this limitation does not so much affect the amount that the average boy can take in, as the quality of the conceptions which he can master, and the complexity of the operations which he can manipulate. To differentiate the average boy from the inefficient it is useless to set questions which are beyond the scope of the average boy. This can easily be done in questions of the most elementary range, and constant care is necessary in order to avoid doing it.

Examining boards should so frame their schedules for testing the general education of the boy who is leaving school that the great mass (70 or 80 per cent.) of those who have been through an approved course could obtain a certificate without having to specialise in their weak subjects to the detriment of their stronger ones. The candidates should not require to do more than devote a certain amount of time out of school hours to rubbing up their weak subjects. This could be secured by allowing the candidate to select one or two subjects in which he is prepared to show a higher degree of proficiency than he proffers in the rest. These would naturally be, in most cases, the subjects on which most stress is laid in the course of study which he is taking at school. Obviously a Modern Side boy ought to be prepared to take up a wider range in mathematics or science than a Classical Side boy, and the extra efficiency ought to be shown by offering a wider range, it ought not to be tested by setting questions on the same range of reading, but of a more subtle or complicated nature. The average boy ought to be encouraged, I say again, to push on according to his power and not kept back to struggle with abstruse conceptions and delicate niceties of manipulation for which his mental qualifications are inadequate. I suggest, therefore, that a fairly wide range should be demanded in the one or two subjects which form the mainstay of the candidate's education, and that he should only be required to show efficient knowledge of a much narrower range in the remaining subjects which he takes up. All questions should be of a simple and straightforward type, and a higher percentage of the maximum should be exacted than is possible with questions of the type now generally set.

The papers set for picking out the candidates who rank distinctly above the average boy ought to be quite distinct from those already discussed. They should contain a sufficient number of straightforward questions dealing with a higher range of subject-matter to enable the hard-headed candidate to

show his efficiency, together with some carefully graduated questions of a more scholarly type to give the brilliant a chance of showing their calibre. In this way a class list should be issued which would really carry some weight as providing an index of higher proficiency.

In the higher certificates granted on the suggested lines specialisation ought to be allowed to those who have already gained pass certificates. This is practically done already in the higher certificate examinations to those who have already gained a certificate of the same class. The mathematical schedule should be so arranged as to encourage extension of reading upon lines likely to be practically useful to the searcher after higher knowledge; but some variation should be allowed to the man who has fixed upon his future calling as that of an architect, engineer, chemist, electrician, actuary, or a member of any other business or profession in which Mathematics of a highly specialised type is necessary to success. Artificial limitations to progress should be relentlessly lopped off, such as the division of conic sections into geometrical and analytical and the rejection of the methods of the calculus for any purpose for which those methods are the most suitable. The advanced students should be encouraged to make use of the calculus at an earlier stage than it is at present allowed, and by no means encouraged to waste valuable time struggling with the calculus in disguise, applied to avoid the use of the calculus in forbidden areas. A sensible straightforward method of pushing on to higher knowledge is of far greater educational value than any amount of dallying by the way to acquire methods of work, which will be abandoned later, for the sake of the training which these methods are supposed to give.

The examinations which we have already considered are all conducted by bodies which are closely in touch with the teaching profession. With most of them there is a certain amount of friendly competition and desire to attract the rising talent. Consequently there is generally a tendency amongst them to meet the wishes of teachers when these find authoritative expression, and, notwithstanding the harassing diversity in detail which is the natural result of their independent origin, there is a general agreement in the scope of the examinations, and it would require no revolutionary measures to be adopted to bring them accurately into line. As regards the two premier Universities, if I may call them so without offence, Oxford has never done more than tolerate a certain amount of knowledge of Elementary Mathematics in the candidates for Responsions. Consequently a Classical Side boy who has fixed upon Oxford as his future destination is free, as far as that University is concerned, to abandon the study of Mathematics at an early age, or only devote an hour or so a week to a half-hearted keeping up to "the Oxford Standard." Cambridge, the leading authority upon mathematical teaching,

has kept well in touch with the body of schoolmasters, and any boy who has reached a respectable standard at school can be sent up for the Previous Part II. without any reasonable doubt that he will succeed in passing it. If it is necessary in special cases to work specially for this examination, the special work consists in pressing a backward boy forward along the lines in which he is already working, and produces no dislocation of his school work. All that is required to bring the Cambridge Previous Examination into line with the school of to-day is the abolition of compulsory Greek and the recognition of the Modern Side as the predominant partner in the existing scholastic firm. The London Matriculation has already been fully dealt with. The want of sympathy which this University shows towards the teaching profession is a survival from the days when London University considered teaching as quite outside its sphere. Now that it has realised that this is too narrow a view to take of its duty to the nation, we may confidently hope that it will soon give more weight to the opinions and desires of the teacher. The younger Universities appear to have felt the influence of London very strongly when preparing the schemes of their Matriculation Examinations, but these are all essentially teaching centres, and are naturally more in line with the schools from which they chiefly draw their material.

We must now consider the competitive examinations for appointments in the Army and Civil Service. These examinations are conducted by the Civil Service Commission—a body which, with the best intentions, has never succeeded in getting into sympathy with the teaching profession in its method of selecting the best men for the Army. The Commission has never realised the fundamental truth already fully dealt with, that education has its limits—that you cannot, as the schoolmaster phrases it, put a quart into a pint pot. Consequently the authorities are constantly raising a plaintive cry for “better educated” men, when what they really desire is better men. I have already pointed out that better men can only be secured by offering more attractive terms, and as long as the competition for the Line is limited by the impossibility of a subaltern living upon his pay in the style required of him, so long and no longer there will be a difficulty in getting “better educated” men to compete for the honour and distinction attached to a Commission in the Line and the more solid satisfaction of holding a position in which a young and patriotic man can feel that he is serving his country.

This fundamental error with regard to the creative power of education, coupled with a fallacious notion that the education of candidates can be improved by tinkering with the examination, has led to a perpetual series of changes in the regulations for the Woolwich and Sandhurst examinations, and in the type of questions set. This process is going on while I write, and the new regulations, which are being evolved, appear to be designed

upon lines which will materially increase the difficulty of securing good men. The complaint is constant that the candidates are not sufficiently well educated, and a suggestion has been made (not official) that the Army will have to adopt the plan of the Navy, and educate their officers from an early age in Military Colleges. The suggestion is not a practical one. It indicates a strange want of appreciation of the real difficulty. The desire to go to sea is strong in early youth, and is encouraged by parents, who are glad to see their boys provided at an early age with a career which will certainly support them, and may bring them distinction. Consequently there is considerable competition for cadetships in the Navy, and a sufficiently large surplus can be selected to provide for the considerable leakage due to want of suitability for the profession, which takes place as time goes on. Desire to enter the Army is not so usual in young boys, unless they are born into the Army. The Army as a future career for their sons is not attractive to the majority of parents, because they cannot tell whether they will be able to provide the sum necessary for the maintenance of one or more subalterns some years hence. The competition for cadetships would almost certainly be insufficient to provide the number of officers required.

The function of a competitive examination is to pick out the best of the competitors. No examination can be made so attractive as to improve materially the intellectual status of the competitors; but an examination can easily be so arranged as to check competition. This is precisely what the Army examiners have always succeeded in doing; for there has been one invariable characteristic of the examination which has persisted throughout its many transformations. It has always been more or less different from any other examination, and has always demanded a type of work peculiar to itself. The result of this individualism was in years gone by to create a race of Army crammers, who almost held the monopoly of teaching our prospective officers. Their mantle has descended upon the Army Class tutors in our Public Schools, who now possess the largest share in the monopoly. It is hardly possible for any but the best boys to get into the Army without a year in the Army Class. It is hopeless for the average boy to attempt it without two or three years. Now many Army candidates are, so to speak, born so. Their family traditions all point to the Army as their natural destination. But the desire to enter the Army does not, as a rule, affect other boys strongly until they are 16 or 17. When it does come upon a boy, it is not, as a rule, a desire which commends itself to his parents, unless he is fairly certain to get into Woolwich or the Indian Army. So it comes about that parental sanction is often sought too late, and still more often obtained too late, to give the would-be soldier any chance of successful preparation. The idea is consequently dropped, and a promising candidate is lost.

To strengthen competition the lower limit of age should be kept up, so that a boy who makes up his own or his mother's mind late may not find that his chance of success in the profession would be seriously affected by want of seniority at entrance. More important still, the examiners should cease to affect a style of question which differs so much from the ordinary type that the ordinary boy requires special preparation to give him the necessary facility to cope with it. The number of subjects to be offered in the competitive examination should be limited to four at most, and the subjects from which these are to be selected should not be mutually exclusive. Special cramming of uncongenial subjects would not then be necessary for the average boy; whilst a good boy who makes up his mind to enter late would be able to compete with some chances of success by taking up his four best subjects.

This question of mutually exclusive subjects is an important one. By fixing the compulsory subjects, and by making Latin and Greek mutually exclusive, the Army Council say practically: "No Classical Side boy need apply. You must be educated on the lines of the Army Class, or you will not do for the Army."

It cannot be necessary for an officer in the Army to be educated on these particular lines. It is more than doubtful whether it is beneficial for him. In any case it not only shuts out all average boys who have not made up their minds at a comparatively early age to adopt the Army as a profession; but it shuts out relentlessly many of the best class of boys who have sufficient ability to get in with a minimum of special preparation, if the course of study, which has made them what they are, were more generously treated. I have said that the Modern Side is the predominant partner in the scholastic firm. Numerically this is becoming more the case every year; but it is still the Classical Side which gets the first choice of material. It is on Classical Side lines that a young boy's education first begins. If he has more than his share of grit or ability it is in his classical work that he gets the first chance of showing it, and it is naturally the Classical Side which is fed with most of the superior material. A glance at our school list for this term discloses the fact that there are 299 boys on the Modern Side, 168 boys on the Classical, 39 in the Army Class. Thus one third of the school are on the Classical Side. Now let us examine the list of School Prefects and House Prefects—the boys who have been selected for authority because they have shown by what they have done and what they are doing that they are trustworthy, and capable of exerting a good influence and keeping order. On the Modern Side we have 26, or 8·7 per cent. of the number of boys on the Modern, on the Classical there are also 26 (15·5 per cent.), in the Army Class there is 1 (2·5 per cent.). The proportion of prefects in the Army Class is unusually low this term, but that does not affect the point I

wish to make—the point, namely, that if the Commission deliberately reject all those who have received a classical education, they are excluding about half the boys of the very type they should wish to attract, the boys whose ability and character have marked them out as pre-eminently suited to hold command. It is *character* which is wanted in an officer more than anything which can be directly tested by examination, and it cannot be imagined that years of working to a syllabus, years of neglecting all knowledge which is not likely to pay, years of regarding everything that has to be learnt only from the point of view of the mark-grubber can be the best training in unselfish devotion to duty, or high-toned *esprit de corps*; nor does experience suggest that the moral effect of such a course is more salutary than theory would lead one to expect. It appears certain that both morally and intellectually the general standard reached by the competitors for the Army Entrance Examination must be lowered to some extent by the exclusive nature of its syllabus and the unconformability of its requirements.

The Mathematics which may be offered for the Army competition is divided into two groups, Mathematics I. and Mathematics II. For Woolwich, Mathematics I. is rightly one of three compulsory subjects which must be taken up, Mathematics II. is one of five optional subjects of which two may be taken. For Sandhurst, Mathematics I. and Mathematics II. appear in a list of six optional subjects, of which two may be selected. They are not mutually exclusive; but a candidate of sufficient mathematical ability to offer both with any chance of doing well in Mathematics II. is almost always a Woolwich candidate.

The following is the syllabus for Mathematics I. :—

Arithmetic.—The ordinary rules, with applications more especially to the mensuration of plane figures and solids. Use of four-figure logarithms and the slide-rule. The metric system and the use of decimals in approximate calculation will be insisted upon. The use and theory of recurring decimals are not required.

Neatness and accuracy of working are expected, and the methods of solution employed must be clearly indicated. Results may be obtained by the slide-rule in cases in which it allows of sufficient accuracy. There will be no objection to the intelligent use of algebraic formulas and symbols.

Geometry.—Geometrical drawing and practical geometry of plane figures, including the use of Marquois and other scales. The substance of Euclid, Books I. to VI. The special treatment of incommensurables will not be required. Proportion may be treated algebraically, and the complications of Euclid's definitions and nomenclature avoided.

Algebra.—Including indices and the simpler properties of surds; graphs of the simpler algebraic functions; quadratic equations; use of graphs in solving equations, and in illustrating

and solving practical questions; rate of variation of a function and gradient of a graph; graphic interpolation.

Grasp of elementary principles and readiness in practical application will be looked for. Numerical results should be worked out to a few significant figures, and candidates should be accustomed to checking the accuracy of their results. Skill in elaborate analysis, such as the simplification of complicated fractions, will not be looked for.

Trigonometry.—Up to and including solution of plane triangles; graphs of trigonometrical functions; use of four-figure tables.

Readiness in straightforward practical applications, but no great analytical skill, will be demanded.

Dynamics.—Graphical proofs of formulas for uniformly accelerated motion; impact, work, energy, circular motion.

Statics.—Composition and resolution of forces; parallel forces; centre of gravity; three-force problems; friction; mechanical powers, e.g., lever, wedge, pulleys, etc.

Practical Work.—Measurement of length—vernier, calipers, micrometer screw-gauge, spherometer. Measurement of angles—use of protractor, etc. Measurement of areas—by dimensions, by squared paper, by weighing; area of cross section of a tube, calibration of a tube. Measurement of volume—by dimensions, by graduated jar or burette. Measurement of weight—use of balance. Principle of Archimedes—measurement of volume by weighing. Specific gravity. Experimental verification of the above theoretical work, such as the measurement of velocity, impact, work, energy, etc. Construction of the mechanical powers.

This syllabus leaves little to be desired, and the requirements are obviously not excessive for candidates for the Engineers or Artillery; any boy of good ability who has been reasonably well taught in Mathematics should be able to do well on it without special preparation, if the examination were to aim at testing the ability of candidates who have been trained upon ordinary lines.

It does not, however, appear that the examiners attempt to discover whether the candidates have any thorough insight into the Mathematics required as indicated by the range of the syllabus. There are only two papers set upon the whole range of Mathematics I., and each paper consists of six questions, each question being, as a rule, sub-divided into three or four either very similar in character, or forming a chain of which each link depends upon the preceding link. Obviously this must be a bad plan for testing a boy's knowledge of six subjects ranging from Arithmetic to circular motion. Nor are the questions set such as to make the most of the plan adopted. The questions on Dynamics and Statics, as a rule, are good, and should test both the knowledge and ability of the candidates. The questions in Arithmetic have a great want of variety, and generally

involve a wearisome reiteration of the same process, which must test the patience of the candidates if it tests nothing else. Knowledge of the other subjects included in the syllabus can hardly be considered as tested at all in most of the papers set. Most of the questions which should be devoted to Geometry, Algebra, and Trigonometry, are given up to graphical work, and the drawing of graph after graph. For constructing these the details given are often so explanatory as to remind one of the directions for turning the heel of a stocking, so that success with the question can hardly be considered as better evidence of mathematical ability than that afforded by the successful negotiation of the stocking heel. Some of the graphs set are very good questions, and questions which a mathematical master would be sorry to see excluded from the paper; but many of them are no test of any quality except neatness and patience, and not all are capable of accomplishing the task which they are expected to perform. Anyway, there is far too much graphical work of various kinds to leave room in the papers for questions of the stamp which the schoolmaster would desire to see included—questions of sufficient variety to test the candidate's knowledge throughout the whole range of Mathematics covered by the syllabus, and his appreciation of the mathematical principles upon which his knowledge is built up. The average boy must be specially prepared or he is all at sea with the papers of Mathematics I., and the preparation must be of a pitifully one-sided nature. This special preparation involves the neglect of a great deal which is far more valuable as a training of the mind than any lessons to be learnt by the acquisition of a finicking nicety in the use of squared paper, more valuable even than the intelligent appreciation of the rough results, which may be read from an elaborate graph—results of which the meaning is liable to be very obscure to a mind trained chiefly on a graphical basis.

The syllabus of Mathematics II. includes that for Mathematics I., together with:—

Geometry.—The substance of Euclid, Book XI., with application to mensuration of solids. Elements of solid geometrical drawing.

Algebra.—Elementary knowledge of the use of indeterminate coefficients, especially with partial fractions. A working knowledge (without rigorous fundamental demonstrations) of the elementary infinite series for $(1 + x)^n$, e^x , $\log(1 + x)$, $\sin x$, $\cos x$, $\tan^{-1}x$, and their use in approximate calculations, especially in finding the slope at a given point of the graph of a function.

Differential and Integral Calculus.—A working knowledge of the notation and fundamental principles in so far as they can be illustrated graphically, with simple applications to the properties of curves, to turning values, and to easy mechanical and physical problems. Further discussion of geometrical applications to be treated under co-ordinate geometry.

Co-ordinate Geometry.—The elementary principles and methods, with straightforward applications to the straight line, circle, ellipse, parabola, hyperbola, cycloid, catenary, logarithmic spiral, and other curves of common occurrence; also to the straight line and plane in space of three dimensions. A systematic knowledge of conics (either geometrical or analytical) is not required: thus, for example, the general theory of the classification of curves of the second degree would not be asked for. The methods of the calculus may be used freely.

Mechanics.—Elementary statics of liquids and gases.

Practical Work.—The practical syllabus of Mathematics I., treated with greater fullness and more reference to exact measurement, and, in addition, centre of gravity and centre of pressure; barometer; Boyle's law; laws of pendulum; determination of g ; impact; coefficient of restitution; compressibility, rigidity; Young's modulus, by stretching and flexure.

This syllabus is a little vague. It seems a pity that graphical illustration should take the place of rigorous demonstration to so great an extent as the syllabus suggests. Many of the boys who are capable of covering the ground of the syllabus should also be capable of doing it thoroughly, and such boys should be given the chance of asserting their superiority to the boys who can only scratch the surface of the subject. Still, the syllabus is practical and progressive, and the fault, if fault it be, is on the right side.

The papers set in Mathematics II. are a good test of knowledge and ability. It would be difficult to improve upon them; but the most liberal interpretation of the syllabus would hardly manage to include some of the questions set.

To conclude with a few general remarks about the mathematical papers which I have studied for the purpose of this paper.

The first thing which suggests itself is that for all important examinations the papers set by any examiner should be submitted to a second examiner and worked through by him from the proofs. This should render it impossible for a question such as the following to appear:—

“A bankrupt has $m + n$ creditors. To m of these he owes $\pounds p$ each, to the remaining n $\pounds q$ each. If all his creditors had an equal claim they would each receive r shillings in the pound. How much does each of the m , and how much each of the n , actually receive?” (*London Matriculation, June 1904.*)

Examiners should study the syllabus before setting a paper and be careful to keep within it. It is very important that papers should not be too long. Long papers encourage loose and inaccurate work, and penalise the painstaking candidate who tries to be thorough. A question which does not conform to the syllabus is at best a question wasted. It may also be unfair to the majority of the candidates, by giving an advantage to a few which is not contemplated by the syllabus, and it may

puzzle and dishearten the painstaking candidate who fails to recognise its nature. Examples of this kind are far too common, e.g. "Define *parallel lines*, and assuming only your definition and either Euclid's twelfth axiom or Playfair's axiom, prove that when a straight line cuts two other straight lines, if a pair of alternate angles are equal, the two straight lines are parallel. Also state and prove the converse." (*College of Preceptors, Certificate Exam. Midsummer 1909.*)

The syllabus of requirements in Geometry for this examination states that "any proof of a proposition will be accepted which appears to the Examiners to form part of a systematic treatment of the subject." To insist upon Euclid's or Playfair's axiom is clearly contrary to the spirit of this regulation. There are other and still more serious faults or blunders in the question as set, which are most unfair to the painstaking candidate.

Questions should be as far as possible independent. There is a great temptation, as I know from experience, to string questions together and set a second and perhaps a third depending upon what has already been done. Experience shows that such questions do not work well. The penalty for error in the first answer is too great when compared with that for a corresponding error in the last. Consequently it is better to avoid such questions though there is much to be said in their favour. Theoretically they are good, but in practice they generally work out badly.

Questions should be as simple and direct as possible, and pitfalls should be avoided—the majority of candidates will dig plenty for themselves on level ground. Chatty questions, anecdotal questions and questions in which some scientific principle is enunciated for no apparent object, only waste the time and patience of the candidate and serve no useful purpose for the examiner. In a recent examination for junior appointments in the Admiralty, etc., a question was set which began with an elaborate description of a soap bubble experiment, illustrated by diagrams, and ended up by asking for the diameter and area of a circle of which the circumference was a given length. A candidate who worked through the paper for me, waded through the details of the experiment, carefully examined the pictures to which he was referred, and ultimately came across the question. He felt, of course, that it was hardly worth while to have gone through so much to find so little, found the diameter of the circle and passed on without discovering that he had also to find the area. It was foolish of him, but natural, considering the way in which the question was buried, and the penalty in lost marks would be out of all proportion to the error committed. The whole of the soap bubble story with its pretty pictures was an excrescence and could serve no useful purpose.

Questions should not be set which penalise knowledge, e.g., "An upright pole 10 feet high casts a shadow 12 feet 6 inches

long at midday on a certain day. Another upright pole of the same height 100 miles further north casts a shadow 13 feet 2 inches long at the same time. Deduce the earth's perimeter, supposing the earth a sphere." (*Woolwich Entrance, November 1903.*)

The answer to the above works out to 28,000 miles approximately, which is, perhaps, as near as could be expected from the roughness of the method used to determine the measurement required. But it is sufficiently far off the correct result to rouse doubts of the accuracy of his work in a boy who knew what that ought to be, and this puts him at a disadvantage in competing with an ignorant boy to whom almost any value would be equally probable. It would be easy to add a short question on percentage error so worded as to remove this disadvantage.

There are many other types of questions which are open to criticism as being catchy, or as requiring dodges for their solution which are not of use in questions which are likely to turn up in the course of practical work; but all such doubtful questions will be avoided if examiners will aim at clearness of expression, and simplicity in setting problems and examples, and will endeavour so to frame their questions that every candidate who has the knowledge and ability required for the actual question asked, will find nothing in the setting of the question to puzzle him and prevent his showing the knowledge and ability which it is desired to test.

Examples of papers which have been set in the more important examinations which I have discussed are appended.

CECIL HAWKINS.

APPENDIX.

I have to thank the following authorities for permission to reproduce the appended papers recently set. In some cases the corresponding papers of 1900 have been included for comparison.

The Syndics of the Cambridge University Press.
 The University of Cambridge Local Examinations and Lectures Syndicate.
 The Delegates of Local Examinations, University of Oxford.
 The Oxford and Cambridge Schools Examination Board.
 The Senate of the University of London.
 The Joint Matriculation Board of the Universities of Manchester, Liverpool, Leeds, and Sheffield.
 The Controller of His Majesty's Stationery Office.

UNIVERSITY OF CAMBRIDGE.
 PREVIOUS EXAMINATION, PART II.

December, 1910.

ARITHMETIC AND ALGEBRA.

(Three hours.)

1. Simplify

$$\frac{14\frac{1}{2} - \frac{5}{8}}{11\frac{3}{4} + 19\frac{1}{4}} - \frac{3\frac{1}{2} - 2\frac{3}{4}}{6 - \frac{1}{4}}$$

2. Taking 1 lb. to be equal to 454 kilogrammes, and 1 cubic yard to be equal to 764 cubic metres, find in lbs., correct to two places of decimals, the weight of a cubic foot of a substance of which a cubic centimetre weighs 8 grammes.

3. Find to the nearest foot the length of the side of a square field of which the area is $8\frac{1}{2}$ acres.

4. By selling articles at 33s. a dozen a man makes a profit of 10 per cent on his outlay. What profit per cent. does he make if he sells them at £13 a hundred?

5. A man invests £544 10s. in 3 per cent. stock at $82\frac{1}{2}$, and £409 10s. in $3\frac{1}{2}$ per cent. stock at $94\frac{1}{2}$; find his total income. If he sells out the latter stock, when the price rises to $97\frac{1}{2}$, and invests the proceeds in the former stock at its original price, what is the change in his income?

6. Multiply $a^2(b+c) - b^2(c+a) + c^2(a-b)$ by $a+b-c$.

Simplify

$$\frac{2a-b}{a(2a+b)} - \frac{2a+b}{b(2a-b)} - \frac{2(3b-2a)}{b^2-4a^2}$$

7. Solve the equations:

$$\begin{aligned} & \text{(i) } \frac{1}{2}(7x-1) - \frac{1}{3}(5x-3) = 3x+1. \\ & \text{(ii) } (b+c)(x-a) + (a-c)(x-b) = (a-b)(x-c). \\ & \text{(iii) } \left. \begin{aligned} 5x+6y &= 2 \\ 3x+4y &= 0 \end{aligned} \right\} \end{aligned}$$

8. A man can walk a certain distance by taking 120 steps of equal length. If he takes steps 3 in. longer for one half of the distance and 3 in. shorter for the other half, he will take altogether 121 steps. Find the distance, and the length of his steps.

9. Define ratio. What is a fourth proportional?

Prove that, if $x - s : y - s = x^2 : y^2$, then either $x = y$ or $\frac{1}{x} = \frac{1}{y} + \frac{1}{y}$.

10. Find the sum of n terms of a geometrical progression of which the first term is a and the common ratio r .

Sum to seven terms the progressions

$$(i) 48 + 60 + 75 + \dots$$

$$(ii) 3\frac{1}{2} - 2\frac{1}{2} + 2 - \dots$$

11. Prove that $\log_{10} x^m = m \log_{10} x$.

Find, by using the tables of logarithms, the values of

$$(i) \frac{30.76 \times 0.347}{0.092},$$

$$(ii) \sqrt[3]{5439}.$$

12. Draw graphs of

$$(i) y = \frac{1}{2}(7 - 3x),$$

$$(ii) y = \frac{5 + 4x}{7 + 2x},$$

from $x = -2$ to $x = 4$; and in both the graphs find the value of x when $y = -\frac{1}{2}$.

GEOMETRY.

(Two hours and a half.)

[In all questions of Practical Geometry the methods of construction must be clearly indicated.]

1. Construct a triangle ABC such that AB is 4.2 cm., BC is 3.4 cm., and CA is 5 cm. Describe a circle touching AB and AC produced, and BC . Measure its radius.

2. Prove that if a straight line cut two parallel straight lines, it makes the alternate angles equal, and also the corresponding angles equal.

3. Prove that if, in a triangle ABC , the angle BAC is greater than the angle ABC , then the side BC is greater than the side AC .

Prove that if the angle BAD in a parallelogram $ABCD$ is obtuse, then the diagonal BD is greater than the diagonal AC .

4. Prove that if three parallel straight lines cut one straight line in A , B , C and another in D , E , F , and if AB and BC are equal, then DE and EF are equal.

5. State, and prove, a proposition giving the difference between the square on the side opposite an obtuse angle in a triangle, and the sum of the squares on the sides containing the obtuse angle.

Prove that if P is any point, and if C is the middle point of AB , where A and B are fixed points, then the difference between twice the square on CP and the sum of the squares on AP and BP is the same for all positions of P .

6. Show how to construct a square equal in area to a given rectangle, and prove the correctness of your method.

Construct a square $ABCD$ with each side 7.5 cm. Join P , Q the middle points of AB , AD . Then construct a square equal in area to the figure $BCDQP$, and measure a side of this square.

7. Draw two circles whose radii are 2.5 cm. and 4 cm., and their centres 9 cm. apart. Construct two common tangents cutting each other between the circles, and measure the angle between them.

8. Prove that if one circle touches another internally, the straight line through their centres passes through the point of contact.

9. Prove that if a straight line touch a circle and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

10. Prove that if two triangles have one angle of the one equal to one angle of the other, and the sides about these angles proportional, the triangles are similar.

Prove that if the two triangles ABC , DEF have their corresponding sides parallel then the straight lines AD , BE , CF meet in a point.

PREVIOUS EXAMINATION. ADDITIONAL SUBJECTS.

MECHANICS I.

(Two hours and a half.)

1. Prove
- (1) $\sec^2 A - 1 = \tan^2 A$.
- (2) $\frac{1}{1 + \tan A} + \frac{1}{1 + \cot A} = 1$.

2. If a , b , c be the sides of a triangle ABC , opposite the angles A , B , C respectively, find a formula connecting a , b , c $\cos A$ and $\cos C$. If $a \cos C = 5$ centimetres, $b = 14$ centimetres, $c = 15$ centimetres, find $\cos A$; and from the Tables find A .

3. If three forces acting at a point are in equilibrium, prove that each is proportional to the sine of the angle between the directions of the other two.

If the three forces be P , Q , R ; and if the angle between P and Q be 110° , and that between Q and R be 130° , find Q when $R = 342$ lbs.

4. Find in magnitude and position the resultant of two like parallel forces.

Two men carry a weight of 90 lbs. hanging from a uniform straight pole 12 ft. long, whose ends rest on their shoulders. The weight is so placed that the pressure due to it on one man's shoulder is a half more than the pressure due to it on the other man's shoulder. Each man now moves one foot nearer the centre of the pole, without altering the position of the weight on the pole; which of the two men profits by the change, and by how much?

5. If two forces act at a point, prove that their moments about any point in the line of action of their resultant are equal and opposite.

If a straight line be drawn parallel to their resultant and in their own plane, show that the difference of the moments of the two given forces about any point in this line is always constant.

6. Weights of 2 lbs. each are placed at two points A , B , and another weight of 4 lbs. is placed at a third point C , not on the straight line through A and B ; find the centre of gravity of the three weights.

A uniform straight bar AB , 3 ft. long, is bent at right angles at C so that AC is twice BC ; the end A rests on a smooth peg, and the end B is supported by a vertical string so that AC is horizontal. Find the pressure on the peg and the tension of the string, the whole bar weighing 6 lbs.

7. In the system of pulleys in which one string passes round all the pulleys, find the relation between the power, the weight and the number of the pulleys.

In such a system containing 4 pulleys in each block it was found that a weight of 7 lbs. could support a weight of 51 lbs.; find the weight of the lower block, and the weight required to support a weight of 83 lbs.

8. A weight W rests on a smooth inclined plane making an angle α with the horizontal, being held up by a force acting along the plane. Find the magnitude of R , the pressure on the plane; and if the force along the plane be P , show that

$$P^2 + R^2 = W^2.$$

MECHANICS II.

(Two hours and a half.)

1. Find, without using the tables, the values of

$$\cos 30^\circ, \sin 90^\circ, \cot 120^\circ.$$

Find to the nearest minute the value of θ less than 180° which satisfies the equation

$$2(1 + \cos \theta) = 3 \sin^2 \theta.$$

2. Prove that in any triangle ABC

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

In a triangle ABC , $B = 50^\circ$, $C = 75^\circ$ and $a = 10$ cms. Find the lengths of the sides b and c .

3. Assuming the truth of the proposition known as the Parallelogram of Velocities, show how to find the magnitude and direction of the resultant velocity of a point whose component velocities are u , v in two directions inclined at an angle θ .

From a ship travelling at 30 miles per hour a shell is fired with a velocity of 2,500 ft. per second from a gun which is trained at right angles to the ship's course. If the object aimed at be a point on a sea wall which is parallel to the ship's course and distant from it 1 mile, show that the shell will strike the wall nearly 93 ft. in front of the point aimed at.

4. Define uniform acceleration, and prove the formula $v = u + at$.

A coach slipped from a train is uniformly retarded by the brakes and comes to rest in 2 minutes after travelling 1 mile. Find the rate at which the train was travelling when the coach was slipped, and assuming that the train maintains this speed constant, show that it will be 1 mile distant from the coach when the latter comes to rest.

5. Explain what is meant by a pound, a poundal, a pound's weight.

A train travelling at 50 miles per hour is brought to rest in 220 yards by a uniform force exerted by the brakes. Show that this force is approximately equal to the weight of 2.5 cwt. per ton of the train.

6. Two masses M and m are connected by a fine string passing over a smooth pulley. Find the distance travelled from rest by either mass in t seconds.

In the above system, if $M = 22$ grams and $m = 18$ grams, and if M be initially moving upwards with velocity 8 ft. per second, show that it will come to rest after moving 10 ft.

7. A particle is projected with velocity 241 ft. per second directly up a smooth plane inclined to the horizon at an angle $28^\circ 49'$. Find the distance travelled by the particle when it has reached its greatest height.

8. A body is projected with velocity V at an inclination θ to the horizontal. Find its range on the horizontal plane through the point of projection.

A shot is fired horizontally from the top of a tower 80 ft. high with a velocity 1,200 ft. per second; find how far from the foot of the tower the shot will strike the ground.

UNIVERSITY OF CAMBRIDGE. HIGHER LOCAL EXAMINATIONS.

June 1900.

EUCLID.

(Two hours.)

1. Prove that if one side of a triangle is greater than another side the angle opposite to the first will be greater than the angle opposite to the second

ABC is a triangle, D the middle point of BC . Show that if BAC is an acute angle, then AD will be greater than half BC .

2. Prove that the interior angles of any rectilinear figure together with four right angles are equal to twice as many right angles as the figure has sides.

Find the number of sides of a regular polygon, each angle of which is equal to the sum of an angle of an equilateral triangle and an angle of a regular pentagon.

3. Prove that the opposite sides and angles of a parallelogram are equal to one another.

Find two points in two given straight lines which are not parallel, such that the line joining them may be bisected in a given point.

4. Prove that, if a straight line is divided into two equal parts and also into two unequal parts, then the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Show that, if two adjacent sides of a rectangle are together equal to a given straight line, the rectangle will be greatest when it is also a square.

5. Show how to draw a straight line to touch a given circle from a given point without it.

The point A is fixed, and B, C are two points on fixed straight lines passing through A . Show that if the perimeter of the triangle ABC is constant then BC touches a fixed circle.

6. Prove that, if from a point without a circle two straight lines are drawn, one of which cuts the circle while the other touches it, then the rectangle contained by the whole line which cuts the circle and the part of it without the circle will be equal to the square on the straight line touching the circle.

If tangents drawn to two circles from a point T are equal to one another, and the circles intersect in P and Q , show that P, Q , and T lie in the same straight line.

7. Show how to inscribe a circle in a given triangle.

Describe a circle which shall touch two given straight lines and pass through a given point.

8. Prove that, if a straight line is drawn parallel to one of the sides of a triangle, it will cut the other sides or those sides produced proportionally.

Two parallel lines ABC, DEF are divided so that

$$AB : BC :: DE : EF;$$

prove that AD, BE, CF will meet in a point or be parallel.

9. Prove that similar triangles are to one another in the duplicate ratio of their homologous sides.

If one diagonal of a quadrilateral bisects the angle between two of the sides and is a mean proportional between them, show that the segments of the other diagonal will be in the duplicate ratio of the other sides.

10. Prove that if two straight lines are parallel, and one of them is at right angles to a plane, then the other will also be at right angles to the same plane.

Show how to draw a straight line perpendicular to two given straight lines which do not intersect.

ALGEBRA.

(Two hours.)

1. Prove that if an expression $a_0x^n + a_1x^{n-1} + \dots + a_n$ vanishes when $x = b$, then $x - b$ is a factor of the expression.

Factorize completely $x^2 + 3x^2 - 4$.

2. Solve the equations

$$(i) \frac{4x-1}{2x+1} + \frac{3x-2}{3x+2} = \frac{12x-1}{4x+3}$$

$$(ii) \begin{aligned} 4x + 3y - 3z &= 2 \\ 3x - y + z &= 8 \\ x - 4y + 2z &= 0. \end{aligned}$$

3. The number of inhabitants per square mile is three times as great in Surrey as in Norfolk; the area of the two counties together may be taken to be 2,700 square miles, and their joint population 1,025,000. If Surrey were only as densely populated as Norfolk actually is, and if Norfolk were as densely populated as Surrey actually is, the population of Norfolk would exceed eight times that of Surrey by 100,000. Find the number of inhabitants per square mile in Surrey.

4. Find the conditions that the quadratic equation

$$ax^2 + bx + c = 0$$

should have its roots real and positive.

5. Solve the equations

$$(i) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 2,$$

$$(ii) \begin{aligned} x^2 + y^2 &= 3x + 1 \\ x^3 + y^3 &= 3y + 7. \end{aligned}$$

6. Define an *index*, and prove that, if m and n are positive integers,

$$a^m a^n = a^{m+n}.$$

Simplify

$$(a^2 - ab^2 + a^2b - b^3)(a + a^2b^2 + b).$$

7. Establish the formula

$$s = \frac{a(1-r^n)}{1-r}$$

for the sum of n terms of the series a, ar, ar^2, \dots

The sum of the first 4 terms of a geometrical progression is 15, and the ratio of the 4th term to the sum of the 2nd and 3rd terms is $-\frac{1}{4}$. Find the series.

8. Find the number of permutations of n things taken all together, when p of them are alike and of one kind, q of them alike and of another kind, and so on.

Find the number of permutations, taken 3 together, of the letters in the word *sequences*.

9. Prove the binomial theorem for a positive integral index

Show that

$$\begin{aligned} 2^n &= 1 + (2n+1) + \frac{(2n+1)2n}{2!} + \dots \\ &\quad + \frac{(2n+1)2n(2n-1)\dots(n+2)}{n!} \end{aligned}$$

10. Define a *logarithm*, and prove that

$$\log_b b \times \log_a a = 1.$$

Find
given that

$$\begin{aligned} \log_{10} 0.025, \\ \log_{10} 2 = 0.30103. \end{aligned}$$

ALGEBRA (ADVANCED) AND TRIGONOMETRY.

(Two Hours.)

1. Explain the meaning of *convergent* as applied to an endless series, and find for what values of p and x the series

$$(i) \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

$$(ii) \frac{x}{1^p} + \frac{x^2}{2^p} + \frac{x^3}{3^p} + \frac{x^4}{4^p} + \dots$$

are respectively convergent.

2. Prove that if x is numerically less than unity

$$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$$

Given that $\log_e 9 = 2.197225$, find $\log_e 11$ to five places of decimals.

3. Prove that if p_n, q_n be the numerator and denominator of the n th convergent to the continued fraction

$$a^1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}$$

then

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Express $\frac{103993}{33102}$ as a continued fraction of the above form.

4. Prove that $x^{p-1} - 1$ is a multiple of p , if p is any prime number and x prime to it.

Prove that $6^p - 5^p - 1$ is a multiple of $30p$, if p is a prime number greater than 5.

5. Define the *sine* and *cosine* of an angle; and prove that if A is an acute angle $\cos(180^\circ - A) = -\cos A$.

Find all the values of θ , between 0° and 360° , which satisfy the equation

$$\sin \theta = \cos 4\theta$$

6. Prove that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

and that

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}.$$

Prove that, if $A+B+C = 90^\circ$, then

$$4 \cos \frac{B+C}{2} \cos \frac{C+A}{2} \cos \frac{A+B}{2} = \cos A + \cos B + \cos C.$$

7. When the altitude of the sun above the horizon is A , a man's shadow is a feet longer than when the altitude is B . Find a formula for the height of the man; and find his height to the nearest inch if

$$A = 30^\circ, B = 60^\circ, \text{ and } a = 7.$$

8. Find an expression for the area of a triangle ABC in terms of its sides, a, b, c .

$$\text{Express } a \sin A + b \sin B + c \sin C$$

in terms of a, b, c .

9. Sum the series

$$(i) \sin x + \sin(x+y) + \sin(x+2y) + \dots + \sin(x+2ny);$$

$$(ii) \sin^2 x - \sin^2(x+y) + \sin^2(x+2y) - \sin^2(x+3y) + \dots + \sin^2(x+2ny).$$

10. Prove that the limit of $\sin \theta/\theta$, when θ is indefinitely diminished, is 1.

Find the limit under the same circumstances of

$$\sin^2 3\theta/(1 - \cos 4\theta).$$

DIFFERENTIAL AND INTEGRAL CALCULUS.

(Two hours.)

1. Define a *differential coefficient*, and give a geometrical illustration. Obtain, directly from the definition, the differential coefficients with respect to x of $\sqrt{(x^2 + a^2)}$, $\tan \frac{x}{a}$, a^x .

2. Find the first differential coefficient of $\frac{f(x)}{\phi(x)}$ with respect to x , and also the second differential coefficient.

If $y = \cot^{-1} x$, prove that

$$\frac{d^n y}{dx^n} = (-1)^n (n-1) \sin^n y \sin ny.$$

3. Assuming that $f(x+h)$ can be expanded in a convergent series of the form $a + bx + cx^2 + \dots$, find the values of a , b , c , &c.

Find by any method the first four terms in the expansion of $\sin \{\log(1+x)\}$.

4. Show that, if both $\phi(x)$ and $\psi(x)$ vanish when $x = a$, the limit of the fraction $\frac{\phi(x)}{\psi(x)}$ is $\frac{\phi'(a)}{\psi'(a)}$.

Find the limiting values of $x^n e^x$ and $x^{1/x}$ when $x = 0$.

5. Give a method of finding the rectilinear asymptotes of a curve, pointing out the difficulties which arise in the case of parallel asymptotes.

Find the asymptotes of the curve whose equation is

$$(x-y)(x^2 - y^2 + 2x + 3y) = 1.$$

6. Define a *point of inflexion*, and find the condition that a point (x', y') on the curve $y = f(x)$ should be a point of inflexion.

By tracing a portion of the curve $r^3 = a^3(\theta + \sin \theta)$, (or otherwise,) show that it has a point of inflexion at the origin.

7. Find the values of the integrals

$$\int \frac{x+a}{x+b} dx, \int x^2 \log x dx, \int \sin^2 x \cos^3 x dx, \int \frac{dx}{x^2 + 2x + 3}.$$

8. Show that the length of a curve whose polar equation is given is the integral $\int \left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} d\theta$ taken between suitable limits.

9. Find the area bounded by the lines

$$y = x, \quad y = \sqrt{3}x, \quad r = c \tan \theta.$$

CONIC SECTIONS.

(Two hours.)

1. Prove that the tangents at the extremities of a focal chord of any conic meet on the directrix.

The tangent at an extremity of the latus rectum intersects any ordinate NP produced in R ; show that $NR = SP$.

2. Prove that the rectangles contained by the segments of any two intersecting chords of a parabola are as the parameters of the diameters which bisect them.

3. Prove that the tangent at any point P of an ellipse makes equal angles with the focal distances SP and $S'P$.

If the normal at P cuts the minor axis in g , show that a circle can be described passing through S, P, S', g .

4. If AB, CD are two chords of an ellipse making equal angles with the major axis, but not parallel, show that AD, BC and also AC, BD make equal angles with the axis.

5. Prove that the triangle contained by any tangent to a hyperbola and its asymptotes is of constant area.

Having given the asymptotes of a hyperbola and a point on the curve, show how to determine the foci, directrices, and vertices.

6. Find the condition that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

may represent two straight lines.

Find also the conditions that the straight lines may be (1) parallel, (2) coincident, (3) at right angles.

7. Find the equation of the common chord of the circles whose equations are $x^2 + y^2 = a^2$ and $x^2 + y^2 + 2gx + 2fy = 0$, and show that the length of this common chord is

$$a \left(4 - \frac{a^2}{f^2 + g^2} \right)^{\frac{1}{2}}.$$

8. Find the equation of a parabola referred to a diameter and the tangent at its extremity as axes.

PN is an ordinate of the diameter ON , and Q divides PN in a given ratio. Prove that the tangent at P intersects the corresponding tangent to the locus of Q in NO produced.

9. Find the equation of the normal at any point of the ellipse whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, in terms of the eccentric angle of the point.

Show that the angle between a diameter and the normal at its extremity is greatest when the diameter is one of the equi-conjugates.

10. Find the condition that the straight line whose equation is $lx + my = 1$ should touch the conic whose equation is $ax^2 + by^2 = 1$, and determine the co-ordinates of the point of contact.

STATICS AND DYNAMICS.

(Two hours.)

1. Prove that if three forces acting on a particle are in equilibrium, each is proportional to the sine of the angle between the other two.

The ends of a string, which may be regarded as weightless, are fastened at two points A, B , the distance between which is $\frac{1}{\sqrt{2}}$ times the length of

the string. A weight W is fastened at the middle point C of the string. If AB makes an angle 15° with the horizontal, find the tensions of AC, BC when there is equilibrium.

2. Prove that the sum of the moments of two intersecting forces about any point in their plane is equal to the moment of their resultant.

A body is in equilibrium under the action of forces along the sides of a rectangle. Prove that each force is proportional to the length of the side along which it acts.

3. Prove that if forces acting at a point are represented in magnitude and direction by $m_1OA, m_2OB, m_3OC, \dots$, their resultant is represented by $(m_1 + m_2 + m_3 + \dots)OG$, where G is the centre of inertia of masses m_1, m_2, m_3, \dots , at A, B, C, \dots , respectively.

If I is the centre of the inscribed circle of a triangle ABC , and D any other point, show that a force represented by ID may be replaced by forces represented by $\frac{a}{2s}AD$, $\frac{b}{2s}BD$, $\frac{c}{2s}CD$, where a, b, c are the sides of the triangle and $2s$ its perimeter.

4. Define the coefficient of friction, and show that a heavy body (incapable of rolling) can rest on a rough inclined plane if the tangent of its inclination does not exceed the coefficient of friction.

A heavy particle can just rest on an inclined plane; show that if it is acted on in addition by a force equal to its weight, at right angles to the plane and towards it, the particle can rest if the inclination of the plane is not more than twice as great as before.

5. Enunciate and prove the principle of virtual work for a particle.

Apply the principle to show that if the lines of action of three forces P, Q, E acting on a particle O meet any straight line in K, L, M respectively, and if their resultant S meets it in N , then

$$S/ON = P/OK + Q/OL + E/OM.$$

6. Define *uniform velocity*, and explain carefully what is meant by the velocity of a point when it is not uniform.

A body is moving in a straight line with uniform acceleration; it is observed to pass through 72 feet during one second, and through 86 feet during the following two seconds. Find the acceleration, and find when the body will be at rest.

7. A string attached to a weight W passes over a fixed pulley and is attached at the other end to a pulley of weight w . A second string passes over the latter pulley and carries at its ends weights W_1, W_2 . Find the acceleration of W and the tension of the second string.

8. Define *kinetic energy, work, power*.

A train running at a full speed of 44 ft. per second, slips two carriages each of mass 10 tons, and the full speed gradually rises to 55 ft. per second; find the horse-power of the engine and the mass of the train, the resistance due to friction being 12.5 lb. per ton.

[One horse-power is 550 foot-pounds per second.]

9. Find the acceleration of a point moving with uniform speed v in a circle of radius a .

A particle is projected along the inner surface of a smooth vertical circle of radius a , starting at the lowest point with velocity u . Show that if $u^2 = \frac{1}{2}ga$, the particle will leave the circle at an angular distance 60° from the top and will then describe part of a parabola of latus rectum $\frac{1}{2}a$.

HYDROSTATICS AND ASTRONOMY.

(Two hours.)

1. Show that if a uniform liquid is at rest under the action of gravity the difference of the pressures at any two points is proportional to the difference of their depths.

A wedge-shaped box the length of which is 50 centimetres, and any section of which parallel to the ends is an equilateral triangle with sides 10 cm. long, rests with its ends vertical and one of its sides horizontal. Find the resultant pressure (1) on the base, (2) on one of the oblique sides, when it is just filled with water.

[The mass of a cubic centimetre of water is a gramme.]

2. Define the *centre of pressure* of a plane surface immersed in any fluid, and find its position for a rectangle placed in water with one edge in the surface.

One side of a cubical box, resting on a table, is free to turn about a horizontal axis in its own plane, the distance of which from the lower edge

is $\frac{1}{2}$ the height of the box. Find how much water there must be in the box in order that none may escape.

3. Prove that when a body is immersed in a heavy liquid, the resultant pressure on it is equal and opposite to the weight of the liquid displaced.

A vessel contains water, above which is a layer of oil of specific gravity .9. A sphere floats so that its lower half is in the water and its upper half in the oil. Find its specific gravity.

4. State Boyle's law connecting the pressure and density of a gas, and show how it can be verified experimentally.

Find the depth below the surface of a river of the top of a cylindrical diving bell 10 feet high, when it is half full of water; the height of the barometer being 30 inches, and the specific gravity of mercury being 13.6.

5. Describe the common hydrometer, and explain how it may be graduated.

Show that if a small body the weight of which is $\frac{1}{n}$ th that of the hydrometer is attached to the top of the stem, a liquid of specific gravity s will appear to be of specific gravity $\frac{n}{n+1}s$.

6. Describe the apparent daily motions of the stars, and show how they can be accounted for by the rotation of the earth.

At midnight on June 29 the star Vega is seen by an observer in London (lat. 51°) to be on the meridian 12° south of the zenith. Whereabouts in the sky should the same observer look for it at 9 p.m. on November 29?

7. Explain some of the reasons for believing that the earth goes round the sun rather than the sun round the earth.

8. Describe and explain the phases of the moon.

Why is there on the whole more moonlight in winter than in summer?

9. State and briefly explain Kepler's laws of planetary motion.

If the mean distance of a planet from the sun is $2\frac{1}{2}$ times that of the earth from the sun, find the average interval between successive oppositions.

10. How can the latitude of a ship be determined by observation of the sun at noon?

At noon on November 23 in longitude 90° W. the altitude of the sun above the S. point of the horizon is observed to be $80^\circ 32'$; the correction for dip and refraction together is $5'$; the sun's declination at Greenwich noon on November 23 is $20^\circ 15'$ S. and on the following noon is $20^\circ 37'$ S. Find the latitude.

June 1910.

GEOMETRY.

(Two hours.)

Attention is called to the alternative questions A, B, C.

In all questions of Practical Geometry the methods of construction should be clearly indicated.

1. Construct a triangle whose sides are respectively 4, 7, and 9 centimetres in length. Inscribe a circle in the triangle, and measure its radius.

2. Prove that if a straight line cuts two other straight lines so as to make the alternate angles equal, then the two straight lines are parallel.

Describe a trapezium of which the two parallel sides are respectively 10 and 4 centimetres in length and the other sides respectively 7 and 8 centimetres in length.

3. Show that if a parallelogram has all its sides equal, its diagonals bisect one another at right angles.

The area of such a parallelogram is 120 sq. inches, and the length of one diagonal is 10 inches; find the lengths of the other diagonal and of a side.

4. Give geometrical illustrations of the following identities :—

(i) $(a + b)^2 = a^2 + b^2 + 2ab$,

(ii) $(a + b)^2 - (a - b)^2 = 4ab$,

(iii) $(2a - b)^2 + b^2 = 2a^2 + 2(a - b)^2$.

5. Prove that if a straight line touch a circle and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

Two circles touch internally at A and a chord QB of the larger circle touches the smaller at P . Prove that AP bisects the angle QAB .

6. Prove that if two chords of a circle intersect inside the circle, the rectangle contained by the parts of the one is equal to that contained by the parts of the other.

Hence prove that of all rectangles of the same area the square has the smallest perimeter.

7. Show that a straight line drawn parallel to one side of a triangle cuts the other sides proportionally.

A straight line DEF cuts the sides BC , CA , AB of a triangle ABC or those sides produced, and is equally inclined to AB and AC . Prove that

$$\frac{BD}{DC} = \frac{BF}{FC}$$

What does this theorem become when E and F move up to and coincide with A ?

8. Prove that the shortest distance between two given non-intersecting straight lines is perpendicular to both, and give a construction for finding it.

9. Points D , E , F are on the sides BC , CA , AB of the triangle ABC or on the sides produced; find the condition that DEF shall be a straight line.

10. Show that the feet of the perpendiculars on the sides of a triangle ABC from any point P on its circum-circle are collinear.

Show also that this line of collinearity bisects the line joining P to the orthocentre of the triangle.

N.B.—Any of the following questions may be taken instead of any of the questions 8, 9, 10, but considerably lower marks will be assigned to them.

A. Prove that parallelograms on the same or equal bases and of the same altitude are equal in area.

B. Find the locus of a point which is equidistant from two given intersecting straight lines.

Describe a circle of given radius to touch two given intersecting straight lines.

C. Prove that in any right-angled triangle the square on the side subtending the right angle is equal to the sum of the squares on the sides containing it.

ABC is a right-angled isosceles triangle, C being the right angle. AD bisects the angle A and meets BC in D . Prove that $BD^2 = 2DC^2$.

ALGEBRA.

(Two hours and a half.)

Tables of logarithms and squared paper can be obtained from the Presiding Examiner.

[N.B.—Attention is called to the alternative questions A, B, C, D at the end of the paper.]

1. Find the factors of the following :

(i) $4x^2 - 3x - 27$,

(ii) $a^4 + a^2b^2 + b^4$,

(iii) $by(2ax - by) + c^2y^2 - a^2x^2$.

2. Solve the equations:

(i) $\frac{3x+2}{4x+3} + \frac{4x+3}{3x+2} = 2\sqrt{6}$;

(ii) $3x^2 + xy = 2, y^2 + 2xy = 5$;

(iii) $\frac{y+z}{1-x} = \frac{x+z}{2-y} = \frac{x+y}{3-z} = \frac{6}{x+y+z}$.

3. Prove the validity of the ordinary method for finding the Highest Common Factor of any two numbers.

4. Prove that the ratio of any two positive numbers is made nearer to unity by the addition of any positive quantity to each of the numbers.

Find the ratio of a to b if

$$3ab + 2b^2 : 2a^2 - ab = 9 : 5.$$

5. What is a "rational integral function" of an algebraical quantity? Show that if any rational integral function of x vanishes when $x = c$, then $x - c$ is a factor of the function.

Prove that $a + b + c$ is a factor of

$$(b+c)^3 + (c+a)^3 + (a+b)^3 + 3abc.$$

Show also that the other factor is positive for all real values of a, b , and c .

6. Find (i) the product of the roots, and (ii) the sum of the squares of the roots, of the equation

$$ax^2 + bx + c = 0.$$

Find λ so that the sum of the squares of the roots of this equation and of

$$ax^2 + 3bx + \lambda = 0$$

may be equal.

7. Find the sum of n terms of a Geometrical Progression.

Find the sum of n terms of the series $1^2 + 2^2 + 3^2 + \dots$, and of the series of which the r th term is $1 + 2^r + 3r^2$.

8. Find the number of permutations which can be made of n things taken r at a time.

Find in how many ways 52 cards can be equally divided between four persons.

9. Write down the expansion of $(1-x)^n$, and show that the sum of the coefficients of the first $p+1$ terms is

$$(-1)^p \frac{(n-1)!}{(n-p-1)!p!},$$

where n is a positive integer.

Find the coefficient of x^r in $(1-x)^{-n}$, and evaluate the series

$$1 + 3 \cdot \frac{1}{10^2} + \frac{3 \cdot 5}{1 \cdot 2} \cdot \frac{1}{10^4} + \frac{3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3} \cdot \frac{1}{10^6} + \dots$$

10. Prove that

$$\log_e x^r = r \log_e x,$$

and find s where $s^2 = 47 \cdot 83$.

Evaluate

$$\left(\frac{59 \cdot 7 \times \cdot 039}{\cdot 00732} \right)^{\frac{1}{2}}.$$

11. Express in partial fractions

$$\frac{x^2 + 5}{(3x-2)^2(x+4)},$$

and find the coefficient of x^r when it is expanded in powers of x .

[N.B. Any of the following four questions may be taken instead of any of the questions 8, 9, 10, 11, but considerably lower marks will be assigned to them.]

A. Simplify:

$$(i) \left(\frac{a^2 - ab + b^2}{a - b} + \frac{a^2 + ab + b^2}{a + b} \right) \div \frac{b^2}{a^2 - b^2}$$

$$(ii) \frac{a^2 - b^2}{a^2b + ab^2} \times \left(\frac{1}{a - b} - \frac{1}{a + b} \right) \div \frac{2}{a^2 + ab}$$

B. If the speed of a train is increased by two miles per hour a saving of 12 minutes is effected in a journey of 240 miles. Find the original speed of the train.

C. Draw graphs of

$$y = 2x \text{ and } y = 1 + \frac{4}{x}$$

for values of x between -3 and $+3$, and hence find the values of x for which

$$2x = 1 + \frac{4}{x}$$

D. Reduce to simpler surds the expressions

$$(\sqrt{2} - \sqrt{3} + \sqrt{6})(\sqrt{2} - \sqrt{3} - \sqrt{6}),$$

and

$$\frac{\sqrt{5} + \sqrt{3}}{11 - 2\sqrt{15}}$$

TRIGONOMETRY.

(Two hours.)

Tables of logarithms &c. can be obtained from the Presiding Examiner.

[N.B.—Attention is called to the alternative questions A, B, C, D at the end of the paper.]

1. Show how to construct an angle whose sine is $\cdot 55$. Find the tangent of the angle. Verify the result by use of the Tables.

2. Prove the formulæ:

$$(i) \cos(A + B) = \cos A \cos B - \sin A \sin B;$$

$$(ii) \cos A - \cos B = -2 \sin \frac{A - B}{2} \sin \frac{A + B}{2};$$

$$(iii) (\sin A - \cos A)^2 + (\sin A + \cos A)^2 = 2 - \cos 4A.$$

3. Find a formula for all angles which have the same sine as the angle A .

Show by a figure how many values $\cos \frac{A}{2}$ may have when the value of $\sin A$ is given.

4 Show that if P be a point in the side AB of a triangle ABC , and if the angle $BPC = \theta$, and $AP : PB = x : y$, then

$$y \cot A - x \cot B = (x + y) \cot \theta.$$

5. Prove that, in a triangle ABC ,

$$(i) \tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2};$$

$$(ii) \sin 4A + \sin 4B + \sin 4C = -4 \sin 2A \sin 2B \sin 2C.$$

6. From a point P due South of a tower the altitude of the tower is a , from a point Q due East of P its altitude is β , and the distance of Q from P is c . Find the height of the tower.

Find the number of feet in the height if $a = 21^\circ$, $\beta = 13^\circ$, and $c = 150$ feet.

7. If B be the radius of the circumcircle of a triangle ABC , prove that

(i) $4BS = abc$,

(ii) $2B^2(a \cos A + b \cos B + c \cos C) = abc$,

S being the area of the triangle.

8. State and prove Demoiivre's Theorem for the case where the index is a positive fraction, assuming its truth when the index is a positive integer.

Find five different values of $(-1)^{\frac{1}{5}}$.

9. Sum each of the following two series to an infinite number of terms :

(i) $\sin \alpha + \alpha \sin(\alpha + \beta) + \alpha^2 \sin(\alpha + 2\beta) + \dots$,

where $\alpha^2 < 1$;

(ii) $\sin \theta \sin^2 \frac{\theta}{2} + 2 \sin \frac{\theta}{2} \sin^2 \frac{\theta}{4} + 4 \sin \frac{\theta}{4} \sin^2 \frac{\theta}{8} + \dots$

10. Express

$$\sin(\theta + \phi i) \text{ and } \log(x + yi)$$

in the form $A + Bi$.

If

$$\log \sin(\theta + \phi i) = A + Bi,$$

show that

$$\cos 2\phi - \cos 2\theta = 2e^{2A}.$$

[N.B. Any of the following four questions may be answered instead of any of the questions 7, 8, 9, 10, but considerably lower marks will be assigned to them.]

A. Find the values of $\sin 18^\circ$, $\cos 216^\circ$, without using the Tables.

B. Prove that

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A};$$

and that, in any triangle ABC ,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

C. Prove that

$$\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A};$$

and deduce that

$$\tan 37\frac{1}{2}^\circ = \sqrt{6} - \sqrt{2} + \sqrt{3} - 2.$$

D. Show that, if $x + y + z = 2\pi$,

$$1 - \cos^2 x - \cos^2 y - \cos^2 z + 2 \cos x \cos y \cos z = 0$$

DIFFERENTIAL AND INTEGRAL CALCULUS.

(Two hours.)

I. Determine from first principles the differential coefficients of $\log x$ and $\tan x$.

Prove that

$$\frac{d^2}{dx^2}(\tan x) - 2 \frac{d}{dx}(\tan x) = 2 \frac{d}{dx}(\tan^2 x).$$

2. Explain the rule for finding the differential coefficient of the product of two functions of a variable.

If $y = e^x \tan^{-1} x$, prove that

$$(1 + x^2) \frac{d^2y}{dx^2} - 2(1 - x + x^2) \frac{dy}{dx} + (1 - x)^2 y = 0.$$

Deduce the expansion of y in ascending powers of x as far as the term involving x^4 .

3. A curve is defined by its polar equation; show that the angle between the radius vector and the tangent is

$$\tan^{-1} \left(r \frac{d\theta}{dr} \right).$$

Find this angle in the case of the cardioid $r = a(1 + \cos \theta)$, and prove that if p is the perpendicular from the origin on the tangent at the point whose radius vector is r , then $r^2 = 2ap^2$.

4. Trace the curve

$$(y^2 - 1)y = (x^2 - 4)x.$$

5. Investigate a method for finding the maximum and minimum values of a function of a single variable.

From a given circular sheet of metal it is required to cut out a sector so that the remainder can be formed into a conical vessel of maximum volume. Show that the angle of the sector removed must be about 66° .

6. Prove the formula

$$\rho = \frac{\left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Find the radius and centre of curvature at any point of the catenary

$$y = \frac{a}{2} (e^{\frac{x}{a}} + e^{-\frac{x}{a}})$$

7. Find the values of

$$\int \frac{dx}{\sqrt{a^2 - x^2}} \quad \int \frac{dx}{\sqrt{x^2 - a^2}} \quad \int \frac{dx}{\cos x} \quad \int \frac{dx}{(x-1)^2(x-2)}$$

8. Obtain a formula of reduction for $\int \tan^n \theta d\theta$, where n is a positive

integer; and evaluate $\int_0^{\frac{\pi}{4}} \tan^7 \theta d\theta$.

9. Find:—(i) the length of the perimeter of the cardioid

$$r = a(1 + \cos \theta),$$

(ii) the area included between the curves

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and

$$y^2 = 2bx + b^2.$$

CONIC SECTIONS.

(Two hours.)

The questions may be treated either geometrically or analytically.

1. Given the focus, directrix, and eccentricity of a conic, show how to obtain points on the curve.

The directrix and two points on the conic are given; find the locus of the focus.

2. Prove that tangents at the ends of any focal chord of a parabola meet at right angles on the directrix.

PQ is a focal chord, B any point on the parabola. BP, BQ meet the directrix in Z, Z' respectively. Prove that $ZS, Z'S$ are at right angles.

3. Show that the tangent at any point on an ellipse is equally inclined to the focal distances of the point.

C is the centre of the ellipse, SY, HZ are the perpendiculars from the foci on any tangent. $CYZQ$ is a parallelogram. Prove that the locus of Q is a circle.

4. If from any point P on a hyperbola, PL, PM are drawn parallel to the asymptotes to meet them in L and M , prove that $PL \cdot PM$ is constant.

The tangent at P meets the asymptotes CL, CM in T and T' . $T'N$ is drawn parallel to CT to meet LP produced in N . Prove that $T'N$ is bisected by the hyperbola.

5. Find the length of the perpendicular from the point $(x'y')$ on the line $Ax + By + C = 0$, and prove that the product of the perpendiculars from $(x'y')$ on the two lines given by

$$ax^2 + 2hxy + by^2 = 0$$

is

$$\frac{ax'^2 + 2hx'y' + by'^2}{\sqrt{(a-b)^2 + 4h^2}}$$

6. Show that the locus of a point, such that the lengths of the tangents from it to two given circles are in a constant ratio, is either a circle or a straight line.

7. Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ and give the geometrical meaning of t .

PQ is a chord of a parabola subtending a right angle at the vertex and cutting the curve normally at P . Find the abscissa of P .

8. Find the equation of the diameter of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is conjugate to the diameter $y = mx$.

If the tangents at two points whose eccentric angles are θ and ϕ meet on the director circle of the ellipse, prove that

$$\tan \theta \tan \phi = -\frac{b^2}{a^2}$$

9. Obtain equations for determining the co-ordinates of the points of intersection of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

and the rectangular hyperbola $xy = k^2$, and prove that the sum of the squares of the distances of these points from the centre of the hyperbola is equal to the square on the diameter of the circle.

STATICS AND DYNAMICS.

(Two hours.)

1. If three forces acting at a point are in equilibrium, show that each is proportional to the sine of the angle between the other two.

If the forces are of magnitude $P, 2P, \sqrt{3}P$, find the angles between them.

2. Show that if two forces act at a point their moments about any point in the line of action of their resultant are equal and of opposite sign.

Three forces P, Q, R act along three sides BC, CA, AB of a triangle, and their resultant acts along the line joining the orthocentre to the centre of gravity of the triangle. Show that

$$P : Q : R = \sin 2A \sin (B - C) : \sin 2B \sin (C - A) : \sin 2C \sin (A - B).$$

3. Show that the line of action of the resultant of the weights of any rigidly connected system of particles passes through a point fixed relatively to the particles:

Each edge of a regular hexagonal lamina is of length 6 inches. A triangle formed by joining two adjacent angles to the centre is cut out. Find the distance from the centre of the centre of gravity of the remainder.

4. Define the cone of friction.

Show how the coefficient of friction between two surfaces can be practically found.

5. Show that, if three forces acting at a point are in equilibrium, the total virtual work done in any small displacement of the point is zero.

A frame $ABCD$ consists of four equal rods, each of weight W , freely jointed together. It is hung from the point A , and C is joined to D by a string so that the rods form a square. Find the tension of the string.

6. What is the relation between the units of force, mass, and acceleration in order to satisfy the equation $P = Mf$? What are these units in the ft.-lb.-sec. system and in the c.g.s. system?

A constant force acting on a body initially at rest moves it through 36 ft. in 3 secs. Find the ratio of the force to the weight of the body.

7. A mass M lb. resting on a smooth table is attached to a string which, passing at right angles over the edge, has a mass m lb. at its other end. Find the motion when the system is set free.

If the mass of M lb. is also attached to another string lying in the opposite direction, which hangs over the parallel edge and has a mass m' lb. at its other end, find the motion and the tension of the two strings.

8. A particle is projected with horizontal velocity u and vertical velocity v . Find its greatest height and its range on a horizontal plane through the point of projection.

An elastic ball projected with these initial velocities from a point in a horizontal floor strikes a vertical wall at horizontal distance l , and after striking the floor once reaches the point of projection. Show that the coefficient of restitution is $gl/2uv$.

9. A weight is attached by a string of length l to a fixed point, and describes a horizontal circle such that the string makes a constant angle α with the vertical. Show that the number of revolutions which it makes per second is $\frac{1}{2\pi} \sqrt{\frac{g}{l \cos \alpha}}$.

A mass of 5 lb. is attached by a string of length 3 ft. to a fixed point in a smooth horizontal table. If it is projected so as to describe a circle in half-a-second, what is the tension of the string in lbs. weight?

HYDROSTATICS AND ASTRONOMY.

(Two hours.)

1. Prove that the pressure at a depth s below the surface of a liquid is given by the formula

$$p = ws + \pi,$$

stating in what units all the quantities involved are to be measured.

If the pressures at two points in a water pipe are 15 lb. per sq. inch and 40 lb. per sq. inch, find, in feet, the difference between their levels, taking 1,000 oz. as the weight of a cubic foot of water.

2. Find the specific gravity of a mixture of two substances in terms of the specific gravities and the weights of the parts.

Show that a mixture of equal volumes of two different substances has greater specific gravity than a mixture of equal weights of the same substances.

3. Find the resultant pressure of a fluid upon any body wholly immersed in it.

A uniform rod of weight 3 lb. rests in an oblique position with two-thirds of its length immersed in water and its upper end supported by a string. Find the tension of the string, and the specific gravity of the rod.

4. Describe the mercurial barometer, and show how to find by means of it a measure of the atmospheric pressure in lbs. per sq. inch.

If the specific gravity of mercury is 13.6 and that of the air is .0013, find approximately to what height a barometer must be carried for the reading to fall from 30 inches to 29 inches.

5. Define the following terms: First Point of Aries, Right Ascension, Celestial Longitude, Azimuth.

The Right Ascension of Aldebaran is 4 h. 30 m. At what time of the year does it transit at 10 p.m.?

6. Give a short description of a transit circle.

How can the declination of a star be found by the transit circle in combination with observation of its reflexion in a mercurial trough?

7. Examine at what part of the year the part of the Equation of time due to the eccentricity of the Earth's orbit is positive.

What is the clock time when the time by a sundial is 3 p.m. on a day on which the Sun rises at 7.21 a.m. and sets at 4.9 p.m.?

8. Describe the effect of refraction upon the apparent position of a star. Does it make a star appear to rise and set earlier or later.

9. Show how to deduce the synodical period of a superior planet from its period of revolution.

Assuming that the radius of the orbit of Mars is to that of the Earth as 3 : 2, show that the period of the revolution of Mars is about $1\frac{1}{2}$ years; and find approximately the interval between successive oppositions.

UNIVERSITY OF CAMBRIDGE.

LOCAL EXAMINATIONS.

SENIOR STUDENTS.

December, 1900.

EUCLID.

(Two hours.)

Any generally recognised abbreviations may be used, but the proof must be strictly geometrical.

1. Prove that if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, the triangles are equal in all respects.

2. Prove that if one side of a triangle be produced, the exterior angle is equal to the sum of the two interior and opposite angles.

3. From the angular points B, C of a triangle ABC , perpendiculars BQ, CB are drawn to AC, AB , intersecting in P . Prove that QB is at right angles to the straight line joining the middle points of AP and BC .

4. Prove that the square on a side subtending an acute angle of a triangle is less than the squares on the sides containing the angle by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular let fall upon it from the opposite angle and the acute angle.

D is a point in the side BC of a right-angled triangle ABC , of which C is the right angle. Find a point E in AB such that the rectangle BE, EA shall be equal to the rectangle BD, DC .

5. Prove that a straight line drawn through the centre of a circle perpendicular to a chord of the circle bisects that chord.

From the fixed points A, B of a circle parallel straight lines AP, BQ are drawn to cut the circle again in P and Q . Prove that the middle point of PQ lies on a concentric circle.

6. Prove that if a straight line be drawn from the point A of a circle at right angles to the diameter through A , it does not meet the circle in any other point than A ; and prove that if a straight line drawn through the point A of a circle do not meet the circle in any other point than A , it is at right angles to the diameter through A .

7. If points X, Y, Z be taken on the sides BC, CA, AB respectively of a triangle ABC , prove that the circles circumscribed about the triangles AYZ, BZX, CXY meet in a point.

8. Prove that if the angles of one triangle are severally equal to the angles of another triangle, the sides of the former are proportional to the sides of the latter.

A quadrilateral is inscribed in a parallelogram so as to have two opposite sides parallel to a diagonal of the parallelogram. Prove that its area is greatest when its diagonals are parallel to the sides of the parallelogram.

9. Prove that if two intersecting planes are both at right angles to a third plane, their common section is at right angles to the same plane.

If A, B, C, D be four points not in one plane, and planes be drawn through D respectively at right angles to BC, CA, AB , prove that these planes intersect in one straight line.

ALGEBRA.

(Three hours.)

1. Resolve into factors

$$(i) 24a^2 + 5ab - 36b^2, \quad (ii) x^4 - x^2y^2 + 16y^4.$$

Simplify

$$\frac{1}{x-a} - \frac{1}{x+a} - \frac{2x-a}{x^2-ax+a^2} + \frac{2x+a}{x^2+ax+a^2}.$$

2. Show that if $a + b + c + d = 0$, then

$$a^2 + b^2 + c^2 + d^2 = 3abcd \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right).$$

If

$$x + y + z = a,$$

$$x^2 + y^2 + z^2 = b,$$

$$x^3 + y^3 + z^3 = c,$$

find the product xyz in terms of a, b, c .

3. Solve the equations

$$\frac{a}{x+b} + \frac{b}{x+a} + 2 = 0;$$

$$(ii) \begin{cases} x(x-y) = 10, \\ y(x+y) = 24; \end{cases}$$

$$(iii) \frac{x}{2} = \frac{y}{3} = \frac{z}{6} = \frac{1}{2x+3y+6z}.$$

4. Find the sum of the roots of the equation

$$ax^2 + bx + c = 0.$$

If α, β are the roots of this equation, find an equation whose roots are

$$\frac{1+\alpha}{\beta} \text{ and } \frac{1+\beta}{\alpha}.$$

5. A sum of money is distributed among some children, each child receiving the same amount. If a shilling less had been given to each, 36 more children could have participated; and if a shilling more had been given to each, the number of children would have had to be reduced by 20. Find the sum distributed.

6. Prove that if a, b, x are all positive, and a less than b , then

$$\frac{a+x}{b+x} \text{ is greater than } \frac{a}{b}.$$

Find the least fraction which is greater than $\frac{1}{3}$ and has its denominator greater by 13 than its numerator.

7. Find the sum of all numbers between 400 and 1000 which are divisible by 17.

8. Find the sum of n terms of a Geometrical Progression whose first term is a , and common ratio is r .

Explain what is meant by finding the *sum to infinity* of this series.

Show that if all the terms are positive, the sum to infinity cannot be less than 4 times the second term of the series.

9. Of 10 things, 3 are alike and the rest unlike. Find (i) the number of permutations all together, and (ii) the number of combinations 5 together.

10. Write down the r th term in the expansion of $(a+x)^n$ in ascending powers of x .

Find what term in the expansion of $(3x+5)^{20}$ has the greatest coefficient.

11. Sum the series

(i) $\frac{1}{2 \cdot 4} - \frac{2}{3 \cdot 5} + \frac{3}{4 \cdot 6} - \frac{4}{5 \cdot 7} + \dots$ to infinity,

(ii) $1 \cdot 2 \cdot 4 + 2 \cdot 3 \cdot 5 + \dots$ to n terms.

Show that the sum to n terms of the series

$$n^2 + 2(n-1)^2 + 3(n-2)^2 + \dots \text{ is } \frac{n(n+1)^2(n+2)}{12}.$$

12. If $\frac{p}{q}, \frac{p'}{q'}$ are consecutive convergents to the continued fraction

$$a + \frac{1}{b + \frac{1}{c + \dots}}, \text{ show that } \frac{p}{q} - \frac{p'}{q'} = \pm \frac{1}{qq'}$$

Express the roots of $x^2 - 2x - 7 = 0$ as continued fractions of the above form, and find the first convergent to the positive root which differs from it

by less than $\frac{1}{100}$.

TRIGONOMETRY.

(One hour and a half.)

1. Define the unit of circular measure; and calculate it in degrees and minutes, taking π to be 3.1416.

What is the circular measure of $39^\circ 19'$ (to four places of decimals)?

2. Prove geometrically (where A and B are acute angles and their sum an obtuse angle) that

$$\cos B - \cos A = 2 \sin \frac{A-B}{2} \sin \frac{A+B}{2}.$$

Prove that, if

$$\cos x = \cos y \cos z + \sin y \sin z \cos \theta,$$

then

$$\tan \frac{\theta}{2} = \frac{\sin \frac{1}{2}(z+x-y) \sin \frac{1}{2}(z+y-x)}{\sin \frac{1}{2}(x+y+z) \sin \frac{1}{2}(y+z-x)}$$

3. Solve the equations

(i) $\sin x + \sin 3x = \cos 2x + \cos 4x,$

(ii) $\cot^{-1}(x-a) + \cot^{-1}(x-b) + \cot^{-1}(x-c) = 0.$

4. Show that in any triangle

$$(i) b^2 + c^2 - a^2 = 2bc \cos A,$$

$$(ii) \tan B \tan C + \tan C \tan A + \tan A \tan B \\ = 1 + \sec A \sec B \sec C.$$

Find the radius of the inscribed circle of a triangle in terms of one side and the angles.

5. Show how to find the angles of a triangle when the ratios of its sides are known.

The ratios of the sides are as 5 : 6 : 7; find the greatest angle in degrees and minutes, given that

$$\log 2 = \cdot 3010, \quad \log \tan 39^\circ 12' = \bar{1} \cdot 9115,$$

$$\log 3 = \cdot 4771, \quad \log \tan 39^\circ 18' = \bar{1} \cdot 9130.$$

6. An observer walks along a straight road and observes a tower whose foot is on the same level as the road. At the first observation the elevation of the top of the tower is α , and the line to the foot of the tower makes an angle θ with the direction of the road; after the observer has walked a distance c along the road the corresponding angles are β , ϕ . Show that

$$\cot \alpha \sin \theta = \cot \beta \sin \phi.$$

and find the height of the tower.

7. State De Moivre's Theorem, and use it to prove that

$$\sin n\theta = \cos^n \theta \left\{ n \tan \theta - \frac{n(n-1)(n-2)}{3!} \tan^3 \theta + \dots \right\},$$

the series containing only odd powers of $\tan \theta$, and the coefficients being those of the binomial series.

Use the above result and the corresponding result for $\cos n\theta$ to find $\tan 7\theta$ in terms of $\tan \theta$.

8. From the expressions for $\sin \theta$ and $\cos \theta$ as series in θ , show that $\tan \theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5$, if powers of θ higher than the sixth be rejected.

Prove that if θ be less than 3° , $\tan \theta = \theta$ accurately to four places of decimals.

CONIC SECTIONS.

(Two hours.)

Geometrical Conic Sections.

1. PN , PG are the ordinate and normal to a parabola at P and meet the axis in N , G respectively; show that $SP = SG$, and that NG is independent of the position of P on the curve.

If Q is any other point on the parabola, show that

$$QG^2 - PG^2$$

is equal to the square on the perpendicular from Q on PN .

2. Y , Y' are the feet of the perpendiculars from the foci S , S' on the tangent at P to an ellipse; show that Y , Y' lie on a circle concentric with the ellipse.

Prove that the ratio of SP to $S'P$ is equal to the ratio of the square on SY to the square on the semi-minor axis.

3. P , P' are two points on the asymptotes of an hyperbola, and PP' meets the curve in Q , Q' ; prove that $PQ = P'Q'$, and that the rectangle $PQ \cdot QP'$ is equal to the square on the semidiameter parallel to PP' .

4. T is any point on the tangent at P to a conic; TE , TN are the perpendiculars from T on SP and the directrix. Show that $SE = e \cdot TN$, where e is the eccentricity.

Prove that if two tangents are drawn to a conic from an external point, they subtend equal or supplementary angles at the focus S .

Analytical Geometry.

5. Find the area of the triangle ABC , where A, B, C are the points $(2, 3), (6, 4), (5, 7)$ respectively. Write down the equations to the lines AB, AC and show that the angle at A is

$$\tan^{-1} \left(\frac{13}{16} \right).$$

6. Find the equation to a circle of radius r with its centre at (a, b) ; and find the length of the tangent from the point (h, k) to this circle.

Show that the line $lx + my + n = 0$ will touch the circle if

$$(la + mb + n)^2 = r^2(l^2 + m^2).$$

7. Show that the locus of points $x = at^2, y = 2at$ is the parabola $y^2 = 4ax$, and that the chord joining two such points is $x - \frac{1}{2}(t + t')y + att' = 0$; find the normal to the parabola at the point $(at^2, 2at)$ and prove that it meets the parabola again in the point

$$x = a \left(t + \frac{2}{t} \right)^2 \quad y = -2a \left(t + \frac{2}{t} \right).$$

8. If r be the distance from the focus $(ae, 0)$ to the point $P(x, y)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that

$$r = a - ex.$$

If r' be the distance from the other focus show that $rr' = d^2$, $2d$ being the length of the diameter parallel to the tangent at P .

STATICS, DYNAMICS, AND HYDROSTATICS.

(Two hours.)

$[g = 32.2$ feet per second per second.]

1. Being provided with a wooden rod of known mass, turning on a knife edge, and also with weights and string, what experiments would you make in order to test the laws of parallel forces?

A uniform rod 2 ft. long, weighing 1 lb., has masses of 2 lb. and 3 lb. fixed to its ends. Find the point on the rod about which the system will balance.

2. State the principle known as the Parallelogram of Forces.

A picture weighing 10 lb. is supported by a string passing over a smooth nail. If the two parts of the string meet each other at the nail at an angle of 60° , find the tension of the string.

3. State Newton's Laws of Motion.

A lift or elevator is ascending with an acceleration of 2 ft. per second per second. Find the pressure on the floor of the lift due to a passenger whose mass is 100 lb.

Is the pressure experienced by the passenger equal to that exerted by him upon the floor of the lift?

4. When a train, whose mass is 400,000 lb., is moving with a velocity of 80 ft. per second, the brakes are suddenly applied and the train comes to rest in 1,600 ft. Supposing that the brakes give rise to a constant resistance, find (1) the retarding force, (2) the time taken by the train in stopping.

5. Explain the terms (1) Kinetic Energy, (2) Potential Energy.

Show that when a body falls in a vacuum the sum of its kinetic and potential energies is constant during the fall.

A mass of 5 lb. falls from rest. How much potential energy does it lose in the first second of time of its motion?

6. Define Momentum.

A mass of 10 kilogrammes moving northwards with a velocity of 20 centimetres per second meets a mass of 20 kilogrammes moving southwards with a velocity of 15 centimetres per second, and the two masses stick together. Find how the combined mass moves after the impact.

Is the total kinetic energy of the system changed by the impact?

7. Define the density of a substance.

A lump of glass weighs 300 grammes in air, 210 grammes in water, and 228 grammes in alcohol. Compare the densities of glass and alcohol with that of water.

8. Describe experiments by which the truth of Boyle's Law may be tested.

When the temperature is 0° C. and the barometric height is 76 centimetres, the mass of a cubic metre of air is 1.292 kilogrammes. What mass of air will enter a room whose volume is 100 cubic metres when the barometer rises 2 centimetres, the temperature remaining unchanged?

APPLIED MATHEMATICS.

(Two hours.)

1. A uniform rod is supported by two strings at its ends. Show that if the strings are at right angles, the inclination of the rod to the vertical is twice that of one of the strings.

2. Find the conditions of equilibrium of three forces in parallel lines.

An equilateral triangular frame ABC , formed of three rods freely hinged together, is in equilibrium under the action of three parallel forces at the angular points. Show that the stress along any rod is inversely proportional to the sine of the angle it makes with the direction of the parallel forces.

3. Find the centre of gravity of a uniform triangular lamina.

Points P, Q, R are taken in the sides BC, CA, AB of a triangle so that $BP:PC :: CQ:QA :: AR:RB$. Show that the centre of gravity of equal weights at P, Q, R coincides with the centre of gravity of the triangle.

4. Define the moment of a force about a point, and prove that when three forces acting at a point are in equilibrium the sum of their moments about any point in their plane is zero.

Find the relation between the power and the weight on a wheel and axle, when the friction of the axle in its bearing is considered.

5. Define the velocity and the acceleration of a moving point.

The velocity of a point changes from 20 miles an hour N.E. to 15 miles an hour due E. in 2 minutes under a constant acceleration. Determine the magnitude of this acceleration in foot-second units correct to the second decimal place.

6. Show that the speed at any point of the parabolic trajectory of a projectile is that due to a fall from the directrix.

If PT, TQ are the tangents at two points P, Q on a parabolic trajectory, prove that the speed at P and the speed at Q are in the ratio of PT and TQ .

7. Two weights connected by a string, length a , are held in contact. One is allowed to fall, and after a time $t (< \sqrt{2a/g})$ the other is let go. Show that after the second starts a time $\frac{2a - gt^2}{2gt}$ elapses before the string tightens;

and that the kinetic energy lost when the string tightens is

$$\frac{1}{2} \frac{ww'}{w+w} gt^2,$$

where w and w' are the weights.

8. Two weights are connected by a string. One rests on a smooth horizontal table and the other hangs vertically below the smooth edge of the table. When motion is permitted, the hanging weight descends with acceleration g/n . Show that the ratio of the weights is $n - 1 : 1$.

UNIVERSITY OF CAMBRIDGE.

LOCAL EXAMINATIONS.

SENIOR STUDENTS.

July 1910.

GEOMETRY.

(Two hours.)

The answers to questions marked A and B are to be arranged and sent up to the Examiner in two separate bundles.

[N.B. *All construction lines should be clearly shown. Attention is called to the alternative questions B ix, B x, B xi at the end of the paper.*]

A.

A 1. Prove that if a straight line cuts two parallel straight lines, the alternate angles are equal.

Show that the straight lines which bisect two opposite angles of a parallelogram are parallel or coincident.

A 2. Construct a quadrilateral $ABCD$ having the sides AB and DC parallel, AB of length 5.3 cm., DC of length 11 cm., and the diagonals AC and BD each of length 9.5 cm. Measure the angles ADC and BCD .

A 3. Prove that if there are three parallel straight lines and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

A 4. Prove that if, in a triangle ABC , AM is the perpendicular from A upon BC , the area of the triangle is half the rectangle $AM \cdot BC$.

Prove that, if D is the middle point of the side BC of a triangle ABC and O is any point in AD , the areas of the triangles AOB and AOC are equal.

B 5. Prove that, if A, B, C, D are four points in order on the circumference of a circle, then the angles ABD and ACD are equal.

AX and AY are two fixed straight lines. P, Q are points on AX, AY respectively such that PQ is of given length; and PO and QO are drawn perpendicular to AX and AY respectively. Prove that the straight line AO is of constant length.

B 6. Show how to construct a square equal in area to a given rectangle.

Describe a quadrilateral $ABCD$ with AB of length 3.5 cm., BC 10 cm., and CD 7 cm., the angle B 80° , and the angle C 45° . Construct a square equal in area to this quadrilateral, and measure its side.

B 7. Prove that if D, E are points in the sides AB, AC of a triangle ABC , such that DE is parallel to BC , then AD is to DB as AE is to EC .

Prove also that if F is any point in BC , and AF cuts DE in G , then DG is to GE as BF is to FC .

B 8. Prove that the section of a sphere by any plane is a circle.

Find the radius of a sphere circumscribing a cube of which each edge is of length 5 inches.

[N.B.—One or more of the following questions B ix, B x, B xi may be taken instead of an equal number of the questions B 6, B 7, B 8, but lower marks will be assigned to them.]

B ix. Construct a triangle with sides of length 4 inches, 5 inches, and 6 inches respectively. Without using the protractor bisect the largest angle, and measure the two parts into which the bisector divides the base.

B x. Enunciate the geometrical proposition corresponding to the identity

$$(a - b)^2 = a^2 - 2ab + b^2$$

and explain it by a figure.

B xi. Show that, if two circles touch one another externally, the straight line joining their centre passes through the point of contact.

Show how to describe a circle to touch a given straight line at a given point and to pass through another given point.

ALGEBRA.

(Three hours.)

Squared paper and tables of logarithms, &c., can be obtained from the Presiding Examiner.

N.B.—Attention is called to the alternative questions A, B, C, D at the end of the paper.

1. Prove the identity

$$2\{(b+c)(c-a) + (c+a)(a-b) + (a+b)(b-c)\} = (b-c)^2 + (c-a)^2 + (a-b)^2.$$

2. Simplify

$$\left(1 + \frac{2a}{2x+a}\right) \times \left(2 - \frac{a}{x+a}\right)^2 \times \left(x - a + \frac{2a}{a+3}\right)$$

3. Find the L.C.M. of

$$\begin{aligned} x^2 + x^2y - xy^2 - y^2, \\ x^2 + 2x^2y - xy^2 - 2y^2, \\ x^2 - 3xy^2 + 2y^2. \end{aligned}$$

and

4. Solve the equations

$$(i) \frac{2x+1}{x+3} - \frac{x-1}{x-3} = 1;$$

$$(ii) (x+a)^2 + x^2 = 3a^2;$$

$$(iii) 2x + 7y = 8,$$

$$6x^2 + 13xy - 28y^2 = 12 - 4x.$$

5. In a cricket match a player scored 6 in the first innings and 58 in the second. By the first innings his average score per innings for the season was reduced by one, but after the second innings his average score was one more than it was before the match. What was his average score before the match, and how many innings had he played?

6. Plot the graphs $y = \frac{1}{3}x(x+3)$ and $y = \frac{3}{2} + \frac{4}{x}$, from $x = -5$ to $x =$

2.5, using the same axes for both graphs. Hence obtain approximate solutions of the equation

$$2x^3 + 6x^2 - 9x - 24 = 0.$$

[Take ten small divisions of the squared paper as unit.]

7. If a, b, c are respectively the arithmetic, geometric, and harmonic means between the same two quantities, prove that $b^3 = ac$.

The sum of the first 4 terms of an A.P. is equal to the sum of the first 8 terms; and the sum of the first 3 terms is 27. Find the first term and the common difference.

8. Prove that $\frac{\log_a m}{\log_a b} = \log_b m$.

Evaluate $(.0079)^{\frac{1}{2}}$ and $\left(\frac{48.27 \times 12.36}{22.09}\right)^{\frac{1}{2}}$.

Find the least number of years in which £120 will amount to a sum exceeding £200 at 4% compound interest.

9. Find the number of permutations of n things, taking them all at a time, when a are alike of one kind, b are alike of another kind, and the rest are all different.

Find how many arrangements can be made of seven out of the eight letters of the word *football*.

10. Find limits to the value of x , if the r th term in the expansion of $(1+x)^n$ is the greatest.

Find the coefficient of x^r in the expansion of $\frac{(x+3x^2)^3}{(1-x)}$, where x is < 1 .

11. Express in terms of partial fractions

(i) $\frac{x+a}{(x+b)(x+c)}$,

(ii) $\frac{2x^2-11x+11}{(1-x)(x-2)^2}$.

Hence prove that if x is < 1 , $\frac{2x^2-11x+11}{(1-x)(x-2)^2}$ can be expressed in terms of

ascending powers of x as an infinite series whose $(n+1)$ th term is

$$\left\{ 2 + \frac{3(n+1)}{2^{n+2}} \right\} x^n.$$

12. Test the convergency of the infinite series

(i) $a + ar + ar^2 + \dots + ar^n + \dots$;

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots$;

(iii) $\frac{5^{\frac{1}{2}}}{2} + \frac{8^{\frac{1}{2}}}{15^{\frac{1}{2}}} + \dots + \frac{(n^2+4)^{\frac{1}{2}}}{(n^2+2n^2-1)^{\frac{1}{2}}} + \dots$

13. If $\frac{P_n}{Q_n}$ is the convergent to the continued fraction

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n + \dots}}}$$

corresponding to the partial quotient a_n , show that

$$P_n = a_n P_{n-1} + P_{n-2}.$$

If x and y stand for the infinite continued fractions

$$\frac{1}{3 + \frac{1}{4 + \frac{1}{3 + \frac{1}{4 + \dots}}}}$$

and

$$\frac{1}{4 + \frac{1}{3 + \frac{1}{4 + \frac{1}{3 + \dots}}}}$$

respectively, prove that $\frac{x}{y} = \frac{4}{3}$.

N.B.—Any of the following questions may be taken instead of any of the questions 10, 11, 12, 13, but considerably lower marks will be assigned to them.

A. Find the square root of

$$9x^6 + 6x^5 + 79x^4 - 22x^3 + 153x^2 - 208x + 64.$$

For what value of a is

$$ax^2 - 11xy - 21y^2 - 8x + 56y$$

divisible by $x - 7y$?

B. Solve the equations

$$(i) 2x^2 + (a + 5)x - a^2 - a + 2 = 0;$$

$$(ii) \begin{cases} x + 3y = 2, \\ x^2 + 9y^2 = 10. \end{cases}$$

C. A bill for £5 5s. is paid in florins and half-crowns. There are three more florins than half-crowns; how many are there of each?

$$D: \text{ If } \quad x + y : x - y = 7 : 3,$$

find the values of the ratios

$$\frac{x}{y} \text{ and } \frac{x^2 + xy}{4xy + 5y^2}.$$

TRIGONOMETRY.

(One hour and a half.)

Squared paper and tables of logarithms &c. can be obtained from the Presiding Examiner.

[N.B.—Attention is called to the alternative questions P, Q, at the end of the paper.]

1. Find the circular measure of the angles 10° , $\sin^{-1} \frac{1}{2}$.

Show that the sine of n' is approximately $29n/10^3$.

2. Define the tangent of an angle and find all the angles less than 360° whose tangent is $\frac{1}{2}$.

Two circles have radii 7 and 10 inches long, and centres 15 inches apart; find the angle between their common tangents.

3. Give the values of $\sin A + 3 \tan A$

$$\text{when } A \text{ is } 0^\circ, 10^\circ, 20^\circ, 30^\circ.$$

Make a graph to show the variation of this expression as A increases from 0° to 30° , and find the value of A for which its value is 1.7.

4. Show geometrically that, for cases in which A lies between 90° and 180° , and B between 0° and 90° ,

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}.$$

Prove that

$$(i) \sin(B+C) \sin A - \sin(C+A) \sin B = \sin(A-B) \sin C,$$

$$(ii) \tan^2 2A - \cot^2 A + \cot A \operatorname{cosec} A \sec^2 2A \cos 3A = 0.$$

5. Show that in any triangle

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

The sides of a triangle are 7, 8, and 11 cm. long; find the greatest angle, and the radius of the circum-circle.

6. AB is a horizontal road half a mile long running S.E. from A to B . At A a balloon is observed due east at an elevation of $61^\circ 10'$, and at B it is seen in a direction $25^\circ 15'$ east of north. Find the height of the balloon in feet.

7. Find all the values of $(-1)^{\frac{1}{2}}$.

If $\cos \theta + i \sin \theta$ represents one of these values, show that

$$z \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)$$

is wholly imaginary, and find its value.

8. Express $\cos n\theta$ in a series of descending powers of $\cos \theta$.
Prove that

$$\cos \frac{\pi}{12} \cos \frac{3\pi}{12} \cos \frac{5\pi}{12} \cos \frac{7\pi}{12} \cos \frac{9\pi}{12} \cos \frac{11\pi}{12} = -\frac{1}{32}$$

N.B.—Either of the following questions may be taken instead of either of the questions 7, 8, or both may be taken instead of the two questions 7, 8, but considerably lower marks will be assigned to them.

P. In a certain triangle

$$\tan \frac{A}{2} = \frac{1}{2}, \quad \tan \frac{B}{2} = \frac{2}{3},$$

find $\tan \frac{C}{2}$, and show that $a + b = 2c$.

Q. Find all the solutions of the equations

$$\begin{cases} \cos(3x + 5y) = \cos(x - 2y), \\ \tan(x + 3y) = \tan 5y. \end{cases}$$

ANALYTICAL GEOMETRY AND DIFFERENTIAL CALCULUS.

(Two hours.)

ANALYTICAL GEOMETRY.

1. Find the angle between the straight lines

$$ax + by + c = 0, \quad a'x + b'y + c' = 0.$$

Find the equations of the two lines through the intersection of the lines $x - 2y + 7 = 0$ and $2x - 3y - 3 = 0$ which make an angle of 45° with the line $y = 2x$.

2. Express the area of a triangle in terms of the coordinates of its vertices.

Show that the area of the triangle formed by the lines

$$ax^2 + 2bxy + cy^2 = 0, \quad y = mx + n$$

is

$$n^2(b^2 - ac)^{\frac{1}{2}} / (a + 2bm + cm^2).$$

3. Find the coordinates of the pole of the line $lx + my = 1$ with respect to the circle $x^2 + y^2 = a^2$.

Find the locus of the poles of chords of this circle which subtend a right angle at the point (h, k) .

4. Find the equation of the normal at the point $(am^2, 2am)$ of a parabola, and show that, in general, three normals can be drawn from a point to the curve.

Show that, if Q, R are the feet of the normals drawn from a point P of the curve itself, the line through P and the vertex meets QR on the directrix.

5. Show that the feet of the perpendiculars from a focus to the tangents of an ellipse lie on a circle.

Find, in terms of the eccentric angle of the point of contact, the length of the chord of this circle made by the tangent.

DIFFERENTIAL CALCULUS.

6. Define the term *differential coefficient* and give a geometrical interpretation.

Differentiate the following functions with respect to x :

$$\log \sin \sqrt{x}, \quad \tan^{-1} \frac{2x}{1+x^2}, \quad x^{1+x}.$$

7. Employ Maclaurin's Theorem to find the general term of the expansion of $\tan^{-1} x$ in powers of x .

8. Explain a method of determining the maxima of a function of one variable.

O is a fixed point on a circle, P any point on the circle, and OY the perpendicular from O on the tangent at P . Show that the greatest area of the triangle OPY is $3\sqrt{3}/8\pi$ of the area of the circle.

9. Find the equation of the tangent at (x, y) to the curve $x^3 + y^3 = a^3$. Show that the portion of the tangent intercepted between the coordinate axes is divided at the point of contact in the ratio $x^3 : y^3$.

APPLIED MATHEMATICS.

(Two hours.)

Squared paper, if required, can be obtained from the Presiding Examiner.

[The acceleration due to gravity may be taken as 32 foot-second² units.]

1. Enunciate and prove the *polygon of forces*.

ABC is an equilateral triangle and D the middle point of BC . Forces of 4, 2, 3, 7 pounds weight act, at a point O , parallel to AB, BC, CA, AD in directions indicated by the order of the letters. Find the magnitude of the resultant.

2. Show that if three forces in a plane keep a body at rest, they must concur or be parallel.

A uniform rod of mass M and length $2c$ is hung over a peg by a string of length $2a$ attached to the ends of the rod, and a mass m is affixed to one end of the rod. Show that in equilibrium the distances of the ends of the rod from the peg are $a\{1 \pm m/(M+m)\}$, provided m is not greater than $Mc/(a-c)$.

3. State the laws of friction, including those of limiting friction.

A solid sphere has a fine bore made through it along a chord subtending an angle 2α at the centre, and is free to move on a vertical rod fitting loosely into the bore. Show that the ball may rest in equilibrium with no further support if the angle of friction $> \alpha$.

4. Find the mechanical advantage of the system of pulleys in which one rope is used.

If there are six pulleys in all and a power of 7 lbs. supports a weight of 30 lbs., what weight will be supported by a power of 14 lbs.?

5. Define *relative velocity*.

Two points are simultaneously at A, B , and move with equal velocities towards C , where the angle CAB is 15° and ACB is a right angle; show that the least distance between them in the course of their motion is $\frac{1}{2}AB$.

6. Define *momentum*, and explain how it is affected by the action of a force.

Find the constant force which will bring to rest in 847 ft. a mass of a ton moving at the rate of $7\frac{1}{2}$ miles an hour.

7. Determine the motion of two weights suspended by a light string over a smooth peg.

ACB is a string, C is a smooth peg, and a weight of 5 lbs. is attached at A , one of 3 lbs. at B , and another of 3 lbs. between B and C . If the system

starts from rest with B 11 ft. above the ground, find the distance above B of the third weight in order that the latter may just reach the ground.

8. Explain the terms *kinetic energy* and *power*.

A fire-engine working at h horse-power is pumping x cubic feet of water per second from a pond at a depth d feet below the nozzle of the hose, which is inclined at an angle α to the horizon. Prove that the jet rises to a height

of $\left(\frac{44h}{5x} - d\right) \sin^2 \alpha$ feet. (Assume that 1 cubic foot of water weighs 1000 ounces.)

OXFORD HIGHER LOCAL EXAMINATION.

July, 1910.

ARITHMETIC.

(Two hours.)

* * * *The full working must be shown in all cases.*

1. How many people could receive £15 6s. 5d. each out of a sum of £1,072 9s. 2d.?

2. Simplify:

$$(1) \frac{2\frac{1}{2} + 1\frac{1}{2} - 3\frac{1}{2}}{1\frac{1}{2} + 2\frac{1}{2} - 2\frac{1}{2}} \text{ of } 1\frac{1}{4};$$

$$(2) 1\text{-}325 \text{ of } £3 \text{ 6s. 6d.} + 3\frac{1}{2} \text{ of } £2 \text{ 3s. 4d.}$$

3. Find the cost of 5 tons 6 cwt. 3 qr. 16 lb. at £9 6s. 8d. a ton.

4. At what rate per cent. per annum was a money-lender obtaining interest, if he received 1s. 3d. weekly on a loan of £6 10s.?

5. Find the length of the side of a square field of area $6\frac{2}{5}$ acres.

6. The average daily rainfall for one week was 0.025 in., and for the first three days of the next week it was 0.531 in. What was the average rainfall for the next four days, if the daily average for the whole fortnight was 0.184 in.?

7. A man spent 9 francs 55 centimes daily for three weeks. Find his total expenditure in English money, taking £1 = 25 francs 20 centimes.

8. (1) A tradesman allowed a customer 10 per cent. discount on an article marked at 20 per cent. above cost price. What was his profit per cent.?

(2) He bought two articles at a total cost of £60; one he sold at a gain of 10 per cent., the other at a loss of 10 per cent., and his total gain was £1 5s. 6d. Find the cost price of each article.

9. (1) Find the annual income obtained by investing £6,741 10s. in India 2½ per cent. stock at 69½. (Neglect brokerage.)

(2) Show that it is less profitable to invest in the above stock than in India 3 per cent. at 97.

ALGEBRA.

(Two hours.)

* * * *Candidates aiming at a First or a Second Class must omit questions marked with an asterisk.*

1. (1) Factorize:

$$4x^2 + 15xy - 4y^2; a^2 - 4(b - c)^2; x^4 + 10x^2 + 49.$$

(2) If $p + q = 1$, show that $p^2 + q^2 + 3pq - 1 = 0$.

2. Simplify:

$$(1) \left(p - \frac{2}{p-1}\right) \left(p - \frac{2}{p+1}\right) \left(q - \frac{2}{q-1}\right) \left(q - \frac{2}{q+1}\right);$$

$$(2) \frac{x^3 - 3x + 2}{x^3 - 7x^2 - x + 7}.$$

*3. Find the square root of

$$\left(a^2 - \frac{2}{a^2}\right)^2 - 4a^2(a-1) + \frac{8}{a}.$$

4. Solve the equations:

$$*(1) \frac{3}{4}x - \frac{1}{2}(4-x) = 2\frac{1}{2} - 3(x-2)$$

$$(2) x^2 - 3y^2 = 3x - 5y = 1;$$

$$(3) 2\sqrt{x} + \frac{1}{2} = \sqrt{5x+1}.$$

5. *(1) Square $2x^3 - x^{\frac{1}{2}} + x^{-\frac{1}{2}}$;

$$(2) \text{ If } \frac{x}{20} = \frac{1}{5 + \sqrt{5}}, \text{ show that } x + \frac{20}{x} = 10,$$

and find the value of $x^2 + \frac{400}{x^2}$.

6. If $f(x)$ denotes $x^2 + px + q$, and if $f(1) = 7$ and $f(2) = 16$, find the numerical values of p and q .

7. The cost of the annual dinner of a society was £8. Owing to the presence of two guests (who paid nothing), each member present had to pay four shillings more than he would have done if none of those present had been guests. How many were present at the dinner?

8. If a, b, c, d are four quantities in geometrical progression, show that

$$(b^3 + c^3)(a^2 - b^2 + c^2) = b^2(a^3 + d^3).$$

9. (1) Find the coefficient of x^6 in the expansion of $\left(1 + \frac{x}{3}\right)^8$.

(2) The population of a town of 80,000 inhabitants increased almost exactly at the rate of 5 per cent. per annum for 10 years. Then, after remaining stationary for 5 years, it decreased at the same annual rate for 10 years. Use the Binomial Theorem to show that the population at the end of this period was approximately 78,022.

10. If x and y represent positive rational quantities, show that $x^5 + y^5 \neq x^4y + xy^4$.

11. Show that every cube number is of the form $7m$ or $5m \pm 1$.

12. If p_n/q_n denotes the n^{th} convergent of a continued fraction, show that

$$p_n/q_n - p_{n-1}/q_{n-1} = (-1)^n / q_n q_{n-1}.$$

GEOMETRY.

(Two hours.)

* * Candidates aiming at a First or a Second Class must omit questions marked with an asterisk.

*1. Show that, if two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

*2. Show that, if the sides of a convex polygon are produced in order, the sum of the exterior angles thus formed is equal to four right angles.

One angle of a pentagon contains 140° . If the other angles are all equal, find the magnitude of one of them.

*3. (1) Show that any point which is equidistant from two points A and B lies on the perpendicular bisector of the line AB .

(2) Make an angle $POQ = 35^\circ$. In OP, OQ respectively take points A and B such that $OA = 3$ in., and $OB = 4.5$ in. Find accurately a point X which is equidistant from A and B , and is also equidistant from the lines OP, OQ . Measure OX in millimetres.

4. If ABC is a triangle, and D the middle point of BC , show that $AB + AC^2 = 2(AD^2 + BD^2)$.

Find the locus of a point which moves in such a way that the sum of the squares of its distances from two given points 4 in. apart is 26 sq. in.

5. Take four points A, B, C, D at random on a sheet of paper.

(1) Assuming that lines and angles can be accurately measured, describe two methods of determining (*without drawing a circle*) whether a circle can be drawn to pass through the four points.

(2) Prove the proposition on which one of your methods depends.

6. Two circles intersect at A and B , and the tangents to them at A meet the circles again at C and D . If the angle CAD is a right angle, show that CBD is a straight line.

7. Show that, if two triangles are equiangular, the sides about equal angles are proportional.

$ABCD$ is a quadrilateral: AX drawn parallel to BC meets BD in X , and BY drawn parallel to AD meets AC in Y . Show that XY is parallel to DC .

8. Show that the ratio of the areas of two similar triangles is equal to the ratio of the squares on corresponding sides.

9. Show that, if a straight line is perpendicular to each of two straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.

$OXYZ$ is a tetrahedron such that $OX = OY = OZ$, and OP is the perpendicular from O to the plane XYZ . Show that P is the centre of the circumcircle of the triangle XYZ .

PLANE TRIGONOMETRY.

(Two hours.)

Logarithm Tables can be obtained from the Superintending Examiner.

1. If the angle C of a triangle ABC is a right angle, find $\cos(A - B)$ and $\sin \frac{1}{2}(A - B)$ in terms of a and b .

2. Prove that:

(1) $\cos(A - B) = \cos A \cos B + \sin A \sin B$ when A and B are acute angles;

$$(2) \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{4} = \frac{\pi}{4};$$

(3) $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C$ if $A + B + C = 180^\circ$.

3. A man observed the angle of elevation of the top of a tower to be $9^\circ 50'$. After walking on a horizontal road for three minutes, at the rate of four miles an hour, directly towards the tower, he found the angle of elevation of the top to be 28° . Find the height of the tower to the nearest foot.

4. Solve the equations:

$$(1) \sqrt{3} \sin \theta + \cos \theta = 2;$$

$$(2) \sin 3x \cos 4x = \sin 2x \cos 5x.$$

5. In any triangle prove that:

$$(1) r = 4R \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C;$$

$$(2) a(\sin^2 \frac{1}{2} B + \sin^2 \frac{1}{2} C) + b(\sin^2 \frac{1}{2} C + \sin^2 \frac{1}{2} A) + c(\sin^2 \frac{1}{2} A + \sin^2 \frac{1}{2} B) = \frac{1}{2}(a + b + c).$$

6. Two sides of a triangle are in the ratio 11 : 17, and include an angle 38° . Find the remaining angles of the triangle.

7. Sum to n terms the series

$$\cos \alpha + \cos 2\alpha + \cos 3\alpha + \dots$$

8. Two of the escribed circles of the triangle ABC touch the sides BA and CA , produced beyond A , at D and E respectively. Prove that

$$DE = a \sqrt{(1 - \sin B \sin C)}.$$

9. Prove De Moivre's Theorem.

Find all the roots of the equation $x^5 + 1 = 0$.

MECHANICS.

(Two hours.)

1. If three forces acting in one plane maintain a particle in equilibrium, prove that each force is proportional to the sine of the angle between the other two.

2. The resultant of two forces X, Y acting at a point is $\frac{1}{2}Y$ perpendicular to the line of action of X . Find the angle at which the forces act, and the ratio of X to Y .

3. A rectangular column 15 ft. high and 3 ft. square, composed of material weighing 120 lb. per cubic foot, is resting on the ground. What horizontal force applied perpendicular to a vertical face at its middle point will cause the column to begin to tilt about the lower horizontal edge of the opposite face, the ground being so rough that the column does not slide?

4. A rod AB of weight W hanging by a string attached to it at A is drawn from the vertical by a force acting horizontally on B and equal to one-third of the weight of the rod. Find the tension of the string, and the tangent of the angle which it makes with the vertical when the rod is in the position of equilibrium.

5. $ABCD$ is a rectangular sheet of lead having its sides AB and BC 2 ft. long and 4 ft. long respectively. Through O , the centre of the rectangle, OL and OM are drawn parallel to AB and AD respectively, and the portion $OLCM$ is cut off. Show that the centre of gravity of the remainder is distant $\frac{1}{3}\sqrt{5}$ ft. from A .

6. Two particles of mass m, m' respectively are connected by a light inextensible string: the first particle is placed on a rough inclined plane (coefficient of friction = μ) and the string after passing over a pulley at the top of the plane supports the second which hangs freely. Show that the first particle will move upwards if

$$\frac{m'}{m} > \mu \cos \theta + \sin \theta$$

where θ is the inclination of the plane to the horizontal.

7. A particle is projected upwards at the same instant that another is let fall from a point vertically above it. The particles meet after 2 seconds and the one projected upwards is then moving twice as fast as the other. Find the original distance between the particles.

8. Two smooth rods are rigidly joined at A so as to include a right angle and fixed in a vertical plane with A upwards. Two rings start simultaneously to slide from A down the rods. Show that after any time the distance between the rings is equal to the vertical distance either would have fallen freely in the time.

9. Over a smooth pulley which is at a height $2h$ above a table passes a cord with masses M and m at its ends. On the table vertically below m is a body of mass $M - m$ attached to m by another cord of length h . If m is held at a height $\frac{1}{2}h$ above the table and the system is then let go,

(1) find the acceleration of m before the second string becomes tight:

(2) describe carefully both the effect on the velocity of m of the jerk when the second string becomes tight and the effect produced on the acceleration of m by the mass $M - m$:

(3) find the final velocity of m .

(M descends clear of the table.)

9. Two masses of 5 lb. and 8 lb. moving in opposite directions with velocities of 12 ft. per second and 3 ft. per second respectively come into direct collision. If $e = \frac{3}{4}$, find the velocities after impact.

CONICS.

(Two hours.)

1. PN is an ordinate of a parabola, S the focus, and A the vertex; prove that $PN^2 = 4 \cdot AS \cdot AN$.

A chord drawn through A parallel to the tangent at P meets PN in Q . Find the locus of Q .

2. If a chord PP' of an ellipse meets a directrix in K , and S is the corresponding focus, prove that SK bisects the exterior angle between SP and SP' .

3. If the sides of a parallelogram touch an ellipse, prove that the diagonals are conjugate diameters of the ellipse.

4. Find the equation of the straight line joining the points $(a, 3a)$, $(-3a, 6a)$, and the length of the perpendicular on it from the origin.

5. Find the equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $(a \cos \theta, a \sin \theta)$.

Find the equations of the tangents to the circle $x^2 + y^2 = 9$ that are parallel to the line $3x + 4y = 0$.

6. What do the following equations represent?

(1) $xy - 2x = 0$; (2) $x^2 + y^2 - 6x - 4y + 13 = 0$.

7. Through the point $(h, 0)$ is drawn any line to meet the parabola $y^2 = 4ax$ in the points P, P' and N, N' are the feet of the ordinates from P, P' . Show that $AN \cdot AN' = h^2$ where A is the origin of co-ordinates.

8. Show that the locus of the middle points of the portions of tangents to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cut off by the axes is the curve $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 4$.

OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD.

July 1910.

ARITHMETIC.

Pass Paper for Higher Certificates.

(Two hours.)

See Note after Question 11.

GENERAL PAPER, No. 2.—Questions 1—7 and A—D.

1. Find the difference between $\frac{3}{4}$ of £67 5s. 2½d. and $\frac{2}{3}$ of £30 9s. 8½d.
2. Find the prime factors of 952, 4284, and 1666, and find their L.C.M. as a product of prime factors.
3. Find the value of $(.51)^2 \div .084$ to three places of decimals.
4. Simplify the expression

$$\frac{1\frac{1}{2} + \frac{3}{4} - 1}{1\frac{1}{2} - \frac{3}{4} + 1} \div \frac{1\frac{1}{2} - \frac{3}{4} + 1}{1\frac{1}{2} - 1\frac{1}{4}}$$

5. The outside dimensions of a box without a lid are 3 ft. 2 in. long, 2 ft. 6 in. broad, and 2 ft. 5 in. in height, and the material of the box is 1 inch thick. What is the inside content of the box in cubic feet?

6. A man's earned income is £540 on which he pays Income tax at the rate of 9d. in £1, his income from investments is £420 on which he pays Income Tax at the rate of 1s. 2d. in £1. What is his net income after deducting Income Tax?

7. One clock gains 1 minute a day and another loses 25 seconds a day. They are both set to the true time at noon on January 1st. On what day will they differ by 1 hour?

8. A man spends two equal sums in buying two stocks, one at 93 and the other at 104. Afterwards he sells the stocks at 96 and 103 respectively. What is his gain per cent. on his total original outlay?

9. What sum of money, to the nearest pound, paid at once is equivalent to three sums each of £200 paid one at the end of 1 year, the second at the end of 2 years, and the third at the end of 3 years, allowing 4 per cent. compound interest payable annually?

10. A rectangular field is twice as long as it is broad, and its area is 2 acres 2200 sq. yards. Find its length to the nearest foot.

11. In a certain election there were both men and women qualified to vote, the number of women being 10 per cent. less than the number of men. Of the men 47 per cent. voted for candidate A, 43 per cent. for candidate B, and the remainder did not vote. Of the women 36 per cent. voted for A, 41 per cent. for B, and the remainder did not vote. The successful candidate had a majority of 75. How many votes did each candidate get?

[Candidates for Certificates may substitute any number of the following questions for an equal number of the questions 8—11, but lower marks are assigned to them.]

A. Find in yards the difference between 5 miles and 8 kilometres. (1 metre = 39·37 inches.)

B. A man borrows £250 on the 27th May at 4½ per cent. interest per annum and repays it with interest on 1st January following. How much interest does he pay?

C. How many revolutions does a bicycle wheel of 28 inches diameter make per mile?

(Take the circumference of a circle as ¾ times its diameter.)

D. A train starts at 10.5 a.m. and reaches a station 90 miles away at 11.45 a.m. If it travels at a uniform rate, at what time does it pass a station 72 miles from its starting point?

ALGEBRA.

Pass Paper for Higher Certificates.

(Two and a half hours.)

NOTE.—(1) In order to pass in Elementary Mathematics, Candidates must satisfy the Examiners in Part I.

(2) In order to pass in Additional Mathematics, Candidates must satisfy the Examiners in Parts I. and II. taken together.

GENERAL PAPER, No. 3.—Questions 1—6, 7—12.

GENERAL PAPER, No. 4.—Questions 1—6, A—D.

PART I.

1. Find the value when $a = 2\frac{1}{2}$, $b = -2$, $c = \frac{1}{2}$, of

$$\frac{a^2}{(a+b)(a-c)} - \frac{b^2}{(b+a)(b+c)} + \frac{c^2}{(c-a)(c+b)}$$

2. Express as a single fraction

$$\frac{1}{1+x} - \frac{3}{(1+x)^2} + \frac{2}{1-2x}$$

and simplify the expression

$$\frac{\left\{ \left(\frac{x}{y}\right)^4 - \left(\frac{y}{x}\right)^4 \right\} \left\{ \frac{x-y}{y-x} \right\}}{\left\{ \left(\frac{x}{y}\right)^3 - \left(\frac{y}{x}\right)^3 \right\} \left\{ \left(\frac{x}{y}\right)^3 - \left(\frac{y}{x}\right)^3 \right\}}$$

3. Multiply $3x^2 + 2x - 3$ by $2x^2 + x^2 + 4$,
and divide $4(x-y)^2 - 27xy^2$ by $(2x+y)^2$.

4. Solve the equations:

$$(1) \frac{x-1}{2x+1} = \frac{3x-1}{6x-5}$$

$$(2) a(x-a)^2 + b(x-b)^2 = (a+b)(x-a)(x-b),$$

$$(3) 3x^2 + xy = 24, \quad 2x + 5y = 1.$$

5. Draw on the same scale (1 inch as unit of x and y) and with the same axes the graphs of the equations

$$y = x^2 \text{ and } 2y + x = 4.$$

What quadratic equation is solved by the intersections of these two graphs? Obtain its solution as accurately as you can from the graphs and verify it by an algebraic solution.

6. A man who is walking from A to B , a distance of $22\frac{1}{2}$ miles, meets at the end of 4 hours a man who has been walking for 8 hours from B to A . Were they each walking a quarter of a mile an hour faster they would meet a quarter of an hour sooner. At what rates are they walking?

[The following questions are to be attempted ONLY by Candidates for Certificates who do NOT attempt Part II.]

A. Factorise:

$$(1) 12(x^2 - y^2) - 7xy,$$

$$(2) (a-b)^2 - (a-b).$$

B. Find the quotient and remainder when $x^4 - 5x + 2$ is divided by $x^2 + x - 2$.

C. Solve the equations:

$$(1) 3ax^2 + 4ax - 3bx - 4b = 0,$$

$$(2) \begin{cases} \frac{x+y}{a} = \frac{x-y}{b}, \\ 5x + 3y = (4a^2 + b)x. \end{cases}$$

D. Show that the sum of any four consecutive numbers is equal to the difference between the product of the two greatest of the numbers and the product of the two least.

PART II

7. Find the condition that the roots of the equation $ax^2 + 2bx + c = 0$ should be real.

If the roots of this equation are real, show that the roots of the equation $a^2x^3 + 2abx + 2b^3 - ac = 0$ are imaginary.

8. A man bought 16 cigars of one kind and 9 of a more expensive kind and paid altogether 14 shillings. If for the same money he had bought only the more expensive kind, he would have obtained 7 less than if he had bought only the cheaper kind. What was the cost of a cigar of each kind?

9. Interpret the expression $a^{\frac{3}{2}}$, and justify your interpretation.

Divide $a^{\frac{3}{2}} - 5a^{\frac{1}{2}}b^2 + 12b^3$ by $a^{\frac{1}{2}} + 3b^{\frac{1}{2}}$.

10. Show that $\log mn = \log m + \log n$.

Find the value of $\sqrt[3]{0.321} \times 3.753 \div \sqrt{1.582}$.

11. Find the sum of n terms of the progression

$$a, ar, ar^2, \dots$$

If a is the first term of the above progression, and b is the second term, and c is the last term, show that the sum is equal to $(bc - a^2)/(b - a)$.

How many terms of the progression 1, 1.2, 1.44, ... must be taken to secure that the sum shall exceed 400?

12. Draw the graph of $\frac{1}{2}(x^2 - 3x)$ and find its gradient at the point where $x = -1.5$.

Employ the graph to obtain approximately the roots of the equation $x^2 - 3x - 1 = 0$.

GEOMETRY.

Pass Paper for Higher Certificates.

(Two and a half hours.)

NOTE. (1) In answering questions 1, 2, 3 and 9 Candidates must state briefly in words how the problem is solved.

(2) In order to pass in Elementary Mathematics, Candidates must satisfy the Examiners in α and β .

(3) In order to pass in Additional Mathematics, Candidates must satisfy the Examiners on the whole paper.

GENERAL PAPER, No. 2.

α .

1. Draw a straight line 6 inches long and divide it accurately by a geometrical construction into two parts which are in the ratio of 8 to 5.

2. Draw a rhombus of which one side is 5 cm. and of which the area is 17 sq. cm. Measure its acute angle.

3. Construct a quadrilateral $ABCD$ in which AB is 2 inches, BC is 1.5 inches, CD is 3 inches, DA is 2.5 inches and the angle ABC is 106° .

Construct a square equal in area to this quadrilateral and measure the length of its side.

β .

4. ABC and DEF are two triangles in which AB is equal to DE , the angle ABC is equal to the angle DEF , and the angle BAC is equal to the angle EDG . Show that the side AC is equal to the side DF .

If we were given that AB is equal to DE , the angle ABC equal to the angle DEF , and the side AC equal to the side DF , show that the angle BAC will not necessarily be equal to the angle EDF .

5. Show that the opposite sides and angles of a parallelogram are equal.

OA and OB are two given intersecting straight lines; show how to draw a straight line in a given direction to meet OA at P and OB at Q so that PQ is a given length.

6. ABC is a triangle in which BAC is an obtuse angle. Show that the square on BC exceeds the sum of the squares on AB and AC by twice the rectangle $CA \cdot AM$, where M is the projection of B on AC .

State the corresponding theorem for the case in which the angle BAC is acute.

In a quadrilateral $ABCD$, if the sum of the squares on AB and CD is equal to the sum of the squares on BC and AD , show that AC must be perpendicular to BD .

7. Show that the angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Two circles intersect at A and B and a chord through A meets one of them again at P and the other at Q . If C is the centre of the circle on which P lies and D the centre of the circle on which Q lies, show that CP and DQ intersect on the circle which can be described through B, C , and D .

8. If two chords of a circle intersect outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

O is a point on a circle and XY is a fixed straight line at right angles to the diameter through O . A straight line drawn through O meets the circle again at P and XY at Q , and the diameter through O meets the circle again at A and XY at B . Show that the rectangle $OP \cdot OQ$ is equal to the rectangle $OA \cdot OB$.

 γ .

9. Show that the locus of a point at which a circle subtends a given angle is a concentric circle. (The angle subtended by a circle at a point is that angle between the tangents drawn to the circle from the point in which the circle lies.)

Draw two intersecting circles whose radii are three inches and one inch. Find by means of a carefully drawn figure two points lying on the straight line joining their intersections, at each of which the larger circle subtends an angle of 120° ; and find the angles which the smaller circle subtends at these points.

10. Show that a straight line drawn parallel to one side of a triangle divides the other two sides proportionally.

AB is a straight line bisected at C ; an equilateral triangle ABD is described on AB , and on AC and CB , on the side of AB remote from D , equilateral triangles ACE and CBF are described. Show that DE and DF trisect AB .

11. Show that the ratio of the areas of similar triangles is equal to the ratio of the squares on corresponding sides.

From a point P on the side BC of a triangle ABC , PM is drawn parallel to AB to meet AC at M and PN is drawn parallel to AC to meet AB at N . Show that the area of the triangle NPC is a mean proportional to the areas of the triangles NPB and MPC .

STATICS AND DYNAMICS.

For Higher Certificates.

(Two and a half Hours.)

NOTE.—(1) In order to pass in *Additional Mathematics*, a Candidate must satisfy the Examiners either in Part I. or in Part II.

(2) *Distinction in Mathematics* will depend upon the Candidate's work in both Parts.

(3) See notes after questions 6 and 12.

PART I. STATICS.

1. Assuming the parallelogram of forces, prove the polygon of forces, which gives the resultant of any number of forces acting at a point.

Three forces of 10 lbs., 50 lbs., 90 lbs. weight act at a point, the angle between the directions of any two forces being 120° . Find the magnitude of their resultant, and the angle its direction makes with that of the greatest force.

2. A light rod is pivoted at a point between its ends at distances a and b from them, and is kept in equilibrium by two forces acting at its ends. Find the relation between the forces (i) when they act at right angles to the rod, (ii) when the forces are at right angles to one another, one making an angle of 60° with the rod, and the other an angle of 30° .

A uniform ladder 24 feet long and weighing 60 lbs. rests against the top of a wall with 8 feet projecting beyond the wall. How far up the ladder may a boy weighing 100 lbs. safely ascend, provided that the ladder does not slip?

3. A body is placed on a rough plane inclined to the horizontal at an angle α . Show that it can remain in equilibrium if the coefficient of friction is greater than $\tan \alpha$.

If α is 60° and the coefficient of friction is $1/\sqrt{3}$, prove that no horizontal force however great can move the body up the plane.

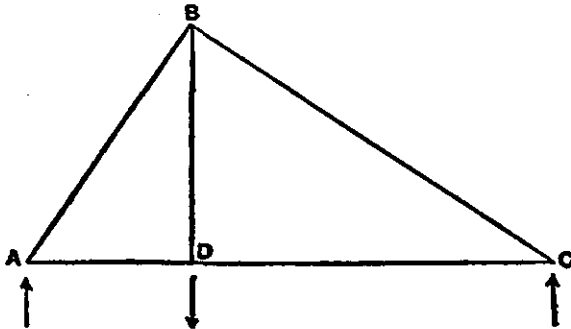
4. Prove that any force in the plane of a triangle can be replaced by three forces acting along the sides of the triangle.

If the triangle is equilateral and the force is parallel to the base at the same distance from it as the vertex and on the opposite side, find the three equivalent forces along the sides of the triangle.

5. Show that a system of forces in a plane can be reduced to a single force acting at a given point and to a couple.

$ABCD$ is a square with a side of 1 foot. Forces of 4 lbs., 7 lbs., 4 lbs. weight act along AC , CD , DB respectively in the directions AC , CD , DB . Find the single force at A and the couple, which are equivalent to this system of forces.

6. The framework $ABCD$ consists of five rods smoothly jointed, the length of AD , DB , and DC being 4 feet, 6 feet, and 9 feet respectively. A



weight of 130 lbs. is supported at *D* and the framework is supported at *A* and *C*. Find, graphically or otherwise, the tension or compression in each rod.

[One (or both) of the questions *A* and *B* MAY be substituted for one (or two) of the questions 4, 5, 6, but lower marks are assigned to them.]

A. Find the centre of gravity of a uniform triangular plate.

If the triangle is equilateral, of 3 foot side and weighs 36 lbs., how far will the centre of gravity of the triangle with a 12 lb. weight at one vertex be from the centre of gravity of the triangle alone?

B. Define the moment of a force about a point and show that the moment may be represented by a certain area.

A uniform beam 20 feet long and weighing 100 lbs. is supported at its ends. If weights of 30 lbs., 40 lbs., and 50 lbs. are placed on the beam at distances of 6 ft., 8 ft., and 12 ft. from one end, find the pressure on each support.

PART II. DYNAMICS.

7. Define uniform acceleration and prove the formula

$$s = ut + \frac{1}{2}ft^2.$$

If a particle is projected with a velocity of 20 feet per second and if there is a constant retardation of $1\frac{1}{2}$ feet per second per second, after what time will it return to its initial position?

8. Show how Newton's law connecting momentum and force enables us to define a unit of force.

A train, whose mass is 240 tons, increases its speed from 15 miles an hour to 45 miles an hour in 44 seconds. If the pull of the engine on the train is constant during this time, find the magnitude of the pull in tons weight.

9. Two masses m and m' moving with velocities v and v' in the same direction along the same straight line impinge and afterwards move as one mass. Find the velocity after impact.

Show that the loss of kinetic energy at the impact is

$$\frac{1}{2} \frac{mm'}{m+m'} (v-v')^2.$$

A bullet, of mass $\frac{1}{2}$ oz., is fired with a horizontal velocity of 1,800 feet per second into a mass of 10 lbs. and becomes imbedded in it. If this mass was hanging at the end of a string, find the energy lost at the impact in foot pounds.

10. A tug tows a ship at a rate of $7\frac{1}{2}$ miles an hour and this requires 200 horse-power in addition to what is required to keep the tug itself moving at that rate. Find the tension of the cable by which the ship is towed.

11. A body projected with a certain velocity reaches the ground again at a distance of 1,000 feet in 8 seconds. With what velocity and at what angle was it projected?

Two bodies are projected from a point on a hillside, one up the hill and one down the hill. If the two velocities of projection are the same and each body is projected in a direction which makes 45° with the horizontal, show that, if the inclination of the hillside with the horizontal is 15° , the range of the one body on the hillside is $\sqrt{3}$ times that of the other.

12. Prove that the total change of velocity, for a point moving uniformly in a circle, as it passes from a position *P* to a position *Q*, is at right angles to the chord *PQ* and is represented in magnitude by *PQ*, on the scale on which the radius represents the velocity at *P*.

Two bodies of masses 7 lbs. and 3 lbs. are connected by a string 5 feet long. If each moves in a circle on a smooth table with the string tight, find the radius of the path of each.

[One (or both) of the questions C and D MAY be substituted for one (or two) of the questions 10, 11, 12, but lower marks are assigned to them.]

C. A body projected vertically upwards reaches a maximum height of 100 feet. What is the velocity of projection? At what times is it at a height of 50 feet.

D. A mass of 40 lbs. is moved on a smooth table by a horizontal force of 2 lbs. weight. With what acceleration does it move and how many foot pounds of work are done in the first second of the motion?

STATICS AND DYNAMICS.

(Compound Paper.)

For Higher Certificates.

(Two and a half hours.)

1. Show how to find the components of a force along two given lines (not necessarily at right angles), through a point on the line of action of the force.

ABC is a triangle and D is a point in BC such that

$$nBD = mDC.$$

Show that a force along AB equal to nAB and a force along AC equal to mAC have a resultant along AD equal to

$$(m + n) AD.$$

2. If three coplanar forces acting on a body are in equilibrium, show that either they are parallel or they meet in a point and are parallel and proportional to the sides of a triangle.

A uniform rod is in equilibrium, with one end pivoted on the ground, in a position making an angle of 45° with the horizontal. The upper end of the rod is supported by a string at right angles to the rod. Prove that the reaction at the ground is equal to $\sqrt{5}$ times the tension of the string.

3. Show how to find the position of the centre of gravity of a number of bodies, when the mass and the position of the centre of gravity of each is known.

A triangle ABC is right-angled at B and has $AB = 2$ feet, $BC = 3$ feet. On AB and BC squares are described which lie outside the triangle. If a body of uniform thickness has the shape of the figure so formed, find the distance of its centre of gravity from AB .

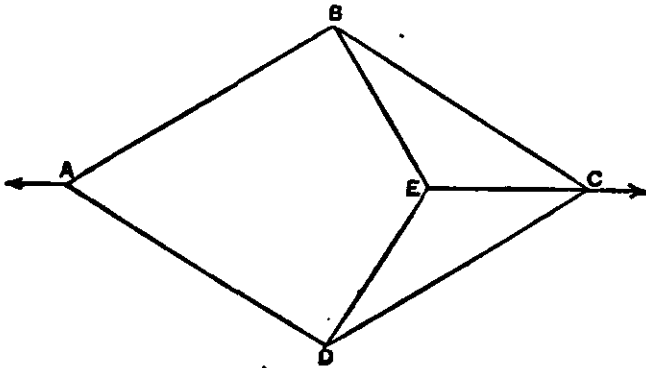
4. Two light rods AC and BC are jointed at C and the ends A and B are jointed to fixed supports on the same level, so that ABC forms an equilateral triangle in a vertical plane. A weight of 6 lb. hangs from C . Find by moments or otherwise the horizontal and vertical components of the reaction at the hinge at B .

5. A body of weight W rests on a rough plane of inclination α , the coefficient of friction between the body and the plane being μ ($< \tan \alpha$). Find the least force parallel to the plane which will move the body (i) up the plane, (ii) down the plane.

Two weights, W and W' , connected by a string, rest in limiting equilibrium on a rough plane of inclination 45° . W and W' are on a line of greatest slope, W being below W' . The coefficient of friction between W and the plane is $\frac{1}{2}$ and the coefficient of friction between W' and the plane is 2. Prove that $W = 2W'$.

6. A framework is formed as follows:— AB, BC, CD, DA are four equal rods, and the angle BAD is 60° , BE and DE are two equal rods at right

angles to AB and AD , and a rod joins C and E . At A and C there act equal and opposite forces of 10 lb. Find, graphically or otherwise, the stresses in each of the rods, distinguishing between tension and compression.



7. State and prove the theorem known as the parallelogram of velocities.

An airship can fly at a rate of 8 miles an hour relatively to the wind. Show by a diagram the region on the level of the airship which contains all the positions which can be reached by the airship when the velocity of the wind is 12 miles an hour.

8. Three particles of weights 2, 3, and 4 lb. lie together on a smooth table. The first and second particles are connected by an inelastic string 1 foot long, and so are the second and the third. The first particle is started along the table with a velocity of 10 feet per second. When and with what velocity does the last particle begin to move?

9. A body is projected with velocity V in a direction making an angle, with the horizontal. Find the greatest height that it reaches and its horizontal range.

If a body is projected with horizontal velocity 150 feet per second and with vertical velocity 100 feet per second, draw a diagram to show its position at the end of each of the first 6 seconds after projection.

10. If a body moves in a circle of radius r with linear velocity v , prove that its acceleration is directed to the centre of the circle, and is of magnitude $\frac{v^2}{r}$.

A body of weight 10 lbs. moves on a smooth table in a circle at the end of a string 3 feet long, which can stand a tension of 2 lbs. weight, the other end of the string being attached to a point on the table. What is the greatest number of revolutions which the body can make per minute without breaking the string?

TRIGONOMETRY.

For Higher Certificates.

(Two hours.)

NOTE. A Pass in Additional Mathematics may be secured by (i) satisfactory work in Part I, or (ii) satisfactory work in Parts I and II, or (iii) satisfactory work on the whole paper.

* N.B. Candidates who take Part III are recommended to omit Part I and question 7 of Part II until they have attempted the other questions.

GENERAL PAPER, No. 2.—Questions 1—10.

PART I.

1*. Give a definition of $\cos \theta$ that holds for all angles from 0° to 180° .

Show that $\cos (180^\circ - \theta) = -\cos \theta$.

2*. Show that $\sec^2 \theta = 1 + \tan^2 \theta$.

Draw the graph of $1 + \sin 3x^\circ$, where x lies between 0 and 60.

3*. Construct an acute angle whose cotangent is 2, an obtuse angle whose sine is $\cdot 3$, and an obtuse angle whose secant is $-3\cdot 5$. Measure these angles as accurately as you can with the protractor, and verify your results by means of tables.

4*. (i) Verify that 30° , 45° , and 60° are solutions of the equation $\sin 3x + \cos 3x = 2 \cos 2x$.

(ii) Show that

$$(\operatorname{cosec} A + \sec A)^2 + (\operatorname{cosec} A - \sec A)^2 = 2 \operatorname{cosec}^2 A (3 \sec^2 A - 2).$$

5*. Show that in an obtuse-angled triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

A man observes that the angular elevation of the foot of a tower on a distant hill side is α , and that the angular elevation of the top of the tower is β , and he knows that the height of the tower is h feet. Show that his horizontal distance from the tower is

$$h \cos \alpha \cos \beta \operatorname{cosec} (\beta - \alpha).$$

PART II.

6. Draw the graph of $\cot x^\circ$ between the values -180 and $+180$ of x , taking the unit of x to be $\frac{1}{2}$ inch and the unit of y to be one inch.

Find an acute angle to satisfy the equation $x = 60 \cot x^\circ$.

7*. Show that

$$\sin (A - B) = \sin A \cos B - \sin B \cos A,$$

taking A and B to be acute angles of which A is the greater.

If $\tan x = k \tan (A - x)$, show that

$$(k - 1) \sin A = (k + 1) \sin (2x - A).$$

Use this result and tables to solve the equation

$$\tan x = 2 \tan (50^\circ - x).$$

8. In the triangle in which $a = 72$ feet, $B = 40^\circ$, and $C = 55^\circ$, find c .

9. Find in terms of a , b , and c the radius of the circle escribed to the side BC of the triangle ABC .

If I_1 is the centre of this circle, show that

$$aAI_1^2 - bBI_1^2 - cCI_1^2 = abc.$$

10. AB is a diameter of a circle whose centre is O ; on AB an equilateral triangle ABC is described, and a point D is taken in AB such that $7BD = 2AB$; CD is produced beyond D to meet the circle at E . Show that $\tan \angle ADC = 7\sqrt{3}$, and that $\sin \angle OED = 3\sqrt{52}$.

Hence, or otherwise, show that the error made in taking the arc BE to be one-seventh of the circumference of the circle is less than $\cdot 2$ per cent.

PART III.

11. The lengths of the sides AB, BC, CD, DA of a quadrilateral $ABCD$, in which AB is parallel to CD , are a, b, c, d respectively, and c is greater than a . Show that the area of the quadrilateral is equal to

$$\frac{1}{4} \left(\frac{c+a}{c-a} \right) \sqrt{(b+c+d-a)(a+b+d-c)(b+c-a-d)(d+c-a-b)}.$$

12. Sum to n terms the series

$$\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \cos (\alpha + 3\beta) + \dots$$

Show that the arithmetic mean of the squares described on the distances of a fixed point from the corners of a regular polygon inscribed in a given circle is the same, whatever the number of sides of the polygon and its position in the circle.

13. Show that for all values of x one value of $\tan^{-1}x$ is

$$\frac{1}{2i} \log \frac{1+ix}{1-ix}.$$

14. Resolve into real factors $x^n + a^n$, where n is a positive integer and x and a are real, distinguishing between the cases of n odd and n even.

Show that $(x + \sqrt{1-x^2})^{2r+1} + (x - \sqrt{1-x^2})^{2r+1}$

$$= 2^{2r+1} \left(\cos \frac{r\pi}{2} - \sin \frac{r\pi}{2} \right) x^n \left(x^2 - \sec \frac{r\pi}{2n+1} \cos^2 \frac{r\pi}{4n+2} \right),$$

where r has the values $1, 3, 5, \dots, 2n-1$.

ALGEBRA.

Distinction Paper.

For Higher Certificates.

(Two hours.)

1. If $2s = a + b + c$, prove that

$$(s-a)^2 + (s-b)^2 + (s-c)^2 - 3(s-a)(s-b)(s-c) = \frac{1}{4}(a^2 + b^2 + c^2 - 3abc).$$

If $x + y + z = 0$ show that

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

can be written in the form $Px^2 + Qy^2 + Rz^2$ where P, Q, R are independent of x, y, z .

2. Solve the equations

$$\begin{cases} x - y = 4, \\ x^2 - y^2 = 163. \end{cases}$$

3. If b and c are unequal and if

$$\mu(a^2 + b^2) + a^2b^2 + ab + 1 = 0,$$

and

$$\mu(a^2 + c^2) + a^2c^2 + ac + 1 = 0,$$

then will

$$\mu(b^2 + c^2) + b^2c^2 + bc + 1 = 0,$$

provided $\mu^2 = \mu + 1$.

4. If A and B are two polynomials in x whose H.C.F. is G , show that polynomials L and M can be found such that

$$LA + MB = G.$$

Find the polynomials L and M of lowest degree which satisfy the equation

$$(1+x^2)L + (1+x^2+x^2)M = 1.$$

5. Develop a method by which a sum can be found for the series $1^r + 2^r + \dots + n^r$ where r is a positive integer.

Evaluate

$$1.3.5 + 2.4.6 + 3.5.7 + \dots \text{ to } n \text{ terms.}$$

6. If $\frac{p_n}{q_n}$ is the n th convergent of $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots$, show that $p = a_n p_{n-1} + p_{n-2}$ and that $q_n = a_n q_{n-1} + q_{n-2}$.

Show that the $2n$ th convergent of $\frac{1}{a} + \frac{1}{b} + \frac{1}{a} + \frac{1}{b} + \dots$ is the same as the n th convergent of

$$\frac{b}{(ab+1)} - \frac{1}{(ab+2)} - \frac{1}{(ab+2)} - \frac{1}{(ab+2)} \dots$$

7. If $y = x + \frac{1}{x}$, prove that

$$\begin{vmatrix} y & 1 & 0 & 0 \\ 1 & y & 1 & 0 \\ 0 & 1 & y & 1 \\ 0 & 0 & 1 & y \end{vmatrix} = \left(x^5 - \frac{1}{x^5}\right) / \left(x - \frac{1}{x}\right).$$

8. Explain what is meant by the convergence of a series and find any test for the convergence of a series of positive terms.

Examine the convergence of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

9. Explain Horner's method of calculating the roots of an algebraical equation numerically.

Find to two places of decimals the root of $x^3 - 5x - 3 = 0$ which lies between 2 and 3.

10. If α is a root of $x^3 + p_1x + q_1 = 0$ and β a root of

$$x^3 + p_2x + q_2 = 0,$$

find the equation whose roots are the four possible values of $\alpha\beta$.

PURE GEOMETRY.

Distinction Paper.

For Higher Certificates.

(Two hours.)

1. Show that the rectangle contained by the sides AB and AC of a triangle ABC is equal to that contained by the perpendicular drawn from A to BC and the diameter of the circle circumscribing the triangle.

Two circles touch one of their common tangents at the points P and Q and one of the points of intersection of the two circles is the point A . Show that the diameter of the circle circumscribing the triangle PAQ is a mean proportional to the diameters of the two circles.

2. A straight line meets the sides BC , CA , AB of a triangle ABC in the points P , Q , R respectively. Show that

$$BP \cdot CQ \cdot AR = CP \cdot AQ \cdot BR.$$

The radii of three circles C_1 , C_2 , C_3 , are r_1 , r_2 , r_3 respectively. S_1 is the external centre of similitude of C_2 and C_3 , S_2 that of C_3 and C_1 , and S_3 that of C_1 and C_2 . Show that $S_1S_2S_3$ are collinear.

$$\text{Show also that } \frac{S_2S_3}{r_1(r_2 - r_3)} = \frac{S_3S_1}{r_2(r_3 - r_1)} = \frac{S_1S_2}{r_3(r_1 - r_2)}.$$

3. Show that the lengths of the tangents drawn to two circles from any point on a circle coaxial with them are in a constant ratio.

One circle completely encloses another and L is that one of their limiting points which is inside both circles. A variable chord PQ of the first circle is a tangent to the second. Show that the straight line joining L to the point of contact bisects the angle PLQ and that the locus of the centre of the circle inscribed in the triangle PLQ is a circle.

4. Show that two intersecting curves, not necessarily in the same plane, intersect at the same angles as their inverses with respect to any point.

Two circles, not necessarily in the same plane, intersect at the points A and B . Any point P is taken on the circumference of one of the circles and any point Q on that of the other. Show, by inversion with respect to A or in any other manner, that the circles APQ and BPQ intersect at the same angles as the other two circles.

5. From a point P of a parabola, of which the vertex is A , a perpendicular PN is drawn to the axis. Show that the rectangle contained by AN and the latus-rectum is equal to the square on PN .

If PN produced meets the parabola again at P' and from Q , any point of the parabola, QM is drawn perpendicular to PP' , show that the rectangle $PM \cdot MP'$ varies as QM .

6. From a focus S of a central conic a perpendicular is drawn to any tangent to the conic. Show that the locus of the foot of this perpendicular is a circle.

A system of conics have the same foci, S and H . Show that the envelope of the tangents drawn to them at their intersections with a fixed straight line through S is a parabola of which H is the focus and the fixed straight line the directrix.

7. Show that the area of the triangle formed by the asymptotes of an hyperbola and any tangent is constant.

A tangent to an hyperbola meets the asymptotes in the points P and Q . The circle on PQ as diameter meets the asymptotes again at the points P' and Q' . Show that the envelope of the straight line $P'Q'$ is an hyperbola having the same asymptotes as the given hyperbola.

8. Define an anharmonic ratio of (1) four collinear points, (2) four concurrent straight lines.

Show that, if an anharmonic ratio of the four straight lines AB, AP, AQ, AE is equal to the corresponding anharmonic ratio of the four straight lines BA, BP, BQ, BE , then the three points $P, Q,$ and E are collinear.

9. Show that the shortest distance between two straight lines which do not intersect is perpendicular to both of them.

The lengths of the edges of a rectangular parallelepiped are $a, b,$ and c . Find the length of the shortest distance between an edge of length a and one of the diagonals of the parallelepiped which does not intersect it.

ANALYTICAL GEOMETRY.

Distinction Paper.

For Higher Certificates.

(Two hours.)

1. The co-ordinates of two points A and B are (x_1, y_1) and (x_2, y_2) and a point F divides AB so that $AF : FB = m : n$. Find the co-ordinates of F .

The co-ordinates of A, B, C the vertices of a triangle are $(-1, 8), (1, -1), (6, 1)$; D, E, F are points on BC, CA, AB such that $BD : DC = 2 : 3, CE : EA = 3 : 4, AF : FB = 2 : 1$. Find the equations of the lines AD, BE, CF ; show analytically that these lines are concurrent and find the co-ordinates of the point of concurrence.

2. Show that $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines and prove that the equation of their bisectors is

$$h(x^2 - y^2) = (a - b)xy.$$

Find the pair of lines through (p, q) each of which makes an isosceles right-angled triangle with these bisectors.

3. Find the condition that the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ and } x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

should be orthogonal.

Find the equation of the circle through the point $(2, 3)$ which is orthogonal both to

$$x^2 + y^2 + 4x + 4y + 3 \text{ and to } x^2 + y^2 + 4x + 8y + 18 = 0.$$

4. Prove that three normals to a given parabola pass through any given point.

Prove that two lines, other than the common axis, can be drawn which are normals both to $y^2 = 4ax$ and to $y^2 = 6a(x + 3a)$ and that their equations are $2x \pm y = 12a$.

5. If P is a given point, and Q a variable point such that PQ is divided harmonically by a given ellipse, show that the locus of Q is a straight line which is the chord of contact of the tangents to the ellipse from P .

Prove that the triangle formed by the lines $x + y = 1$, $3x + y = 9$, $2x - y = 1$ is self-conjugate with respect to the ellipse $3x^2 + 2y^2 = 6$.

6. Find the condition that the lines

$$lx + my = 0 \text{ and } l'x + m'y = 0$$

may lie along a pair of conjugate diameters of the conic

$$ax^2 + 2hxy + by^2 = 1.$$

Find the equation of a pair of conjugate diameters of $2x^2 + 3y^2 = 6$, which coincide in direction with a pair of conjugate diameters of $6x^2 + 4xy + y^2 = 1$.

7. If $S = 0$ represents a conic and $\alpha = 0$ and $\beta = 0$ represent straight lines, interpret the equations (i) $S - \lambda\alpha\beta = 0$ and (ii) $S - \lambda\alpha^2 = 0$, where λ is arbitrary.

Find the equations of the two conics which pass through the four points $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(2, -2)$ and which have their centres on the line $2y + 1 = 0$.

8. Find the equation of the tangent at the vertex of the parabola

$$(lx + my)^2 + 2gx + 2fy + c = 0.$$

Write down, in terms of two parameters, the general equation of a parabola which has the line $x + y = 0$ as the tangent at its vertex; and determine the values of the parameters for the parabolas which touch the line $y = 0$ and pass through the point $(3\frac{1}{2}, \frac{1}{2})$.

9. Find the lengths and the direction of the principal axes of

$$7x^2 - 12xy - 2y^2 - 28x + 24y + 12 = 0,$$

and sketch the position of the curve.

ELEMENTARY DIFFERENTIAL AND INTEGRAL CALCULUS.

Distinction Paper.

For Higher Certificates.

(Two hours.)

1. If x and y are functions of each other, show that

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1.$$

From the result $\frac{d}{dx} e^x = e^x$, deduce the result

$$\frac{d}{dx} \log x = \frac{1}{x}.$$

2. Find the differential coefficients with regard to x of

$$\frac{x}{(a^2 + x^2)^{\frac{1}{2}}}, \tan^{-1} \sqrt{\frac{a-x}{a-b}}.$$

3. If s is the distance described in time t by a particle moving in a straight line give the meanings of $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$.

If $t = as^3 + 2bs + c$, where a , b , and c are constants and a is positive, show that the particle undergoes a retardation which at any instant is proportional to the cube of its velocity.

4. If $f(x+h)$ can be expanded in a series of ascending powers of h , find the coefficient of h^n .

If $\sin \theta = (1+k) \sin \phi$, show that, if powers of k higher than the second may be neglected,

$$\theta = \phi + k \tan \phi + \frac{1}{2} k^2 \tan^3 \phi.$$

5. Show that a value of x , which makes $f'(x) = 0$, in general makes $f(x)$ a maximum or a minimum.

If x is restricted to lie between a and b and $f'(x)$ does not vanish for any value of x between a and b , how are the maximum and minimum values of $f(x)$ to be found?

If a uniform beam of length $2a$ is supported at its two extremities A and B and has a weight attached to a point C in its length, where $CB = \frac{1}{2}a$, it can be shown that the bending moment at any point of the rod between A and C is measured by $(2+n)ax - x^2$, where x is the distance of the point from A and n is the ratio of the attached weight to the weight of the rod, and that at any point between C and B by $(2+3n)ax - x^2$, where x is the distance of the point from B . Show that the bending moment at any point of the rod is greatest at C if $n > 1$, but is greatest at some point between C and the middle point of AB if $n < 1$.

6. Draw an arc at every point of which x , y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ are all positive; and another arc at every point of which x , y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ are all negative.

Give reasons for your results.

7. Integrate the functions

$$\frac{1}{\sqrt{a^2-x}}, x \sin^2 x.$$

Show that

$$\int_0^{\frac{\pi}{2}} \frac{x \cos \theta d\theta}{1-x^2 \cos^2 \theta} = \frac{\sin^{-1} x}{\sqrt{1-x^2}}.$$

8. Trace the curve $3ay^2 = x(a-x)^2$, and find the length and area of its loop.

9. Find the co-ordinates of the centre of gravity of that portion of the area of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in which x and y are both positive.

OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD.

EXAMINATION FOR SCHOOL CERTIFICATES.

July, 1910.

ARITHMETIC.

(Two hours.)

1. Find the number nearest to 9999 that can be divided exactly by 3, 4, 5, 7.
2. Simplify $\frac{2}{3}(1\frac{1}{2} - \frac{1}{2} \div \frac{1}{2}) + \frac{1}{2}(\frac{1}{2} - \frac{1}{2} \text{ of } 1\frac{1}{2})$.
- Divide $\cdot 003125$ by $\cdot 02048$, obtaining the answer correct to three decimal places.
3. Find the cost of 3 tons 7 cwt. 2 qrs. 11 lb. at £4 8s. 8d. per cwt.
4. If it costs 6 dollars 30 cents to carry $4\frac{1}{2}$ cwt. a distance of 500 miles, find, to the nearest cent, the cost of carrying 8 cwt. 360 miles.
5. A bankrupt owes £6,000. He pays 12s. 6d. in the £, and defrauds his creditors by concealing $\frac{3}{8}$ of his assets. Find the value of his estate.
6. Find to the nearest penny, the compound interest on £2,525 at $2\frac{1}{2}$ per cent. per annum for 4 years.
7. A square field contains 2 acres 3 roods, and is to be fenced with hurdles 6 feet in length. What is the smallest number of hurdles that can be used for each side of the field?
8. A rectangular tank is 35 cms. deep, 62 cms. long, and 27 cms. wide. If the tank is filled with water, how many times could a bucket holding 8.5 litres be filled from it, and what would remain over?
9. A man buys oranges at 5s. a hundred and sells them at 8d. a dozen. What is his gain or loss per cent.?
10. A man wishes to obtain £1,000 by selling $2\frac{1}{2}$ per cent. consols at $82\frac{1}{2}$. What amount of stock must he sell, and what would be the quarterly income derived from it? (Answers correct to the nearest penny.)

ALGEBRA.

(Two hours.)

NOTE.—(1) In order to pass in Elementary Mathematics Candidates must satisfy the Examiners in Part I.

(2) In order to pass in Additional Mathematics Candidates must satisfy the Examiners in Parts I. and II. taken together.

(3) Candidates for exemption from the Army Qualifying Examination must satisfy the Examiners in Parts I. and II. taken together, and should NOT attempt the questions A, B, C, D.

PART I.

1. If $a = 2$, $b = -3$, $c = 0$, $d = \frac{1}{2}$, find the value of

$$\frac{(2a + b)(c + 4d) + cd}{a + b}$$

2. Find the continued product of $b - c$, $c - a$, $a - b$.

Divide $4x^4 - 9x^3 + 6x - 1$ by $2x^2 + 3x - 1$.

3. Simplify :

$$(1) \frac{3}{x-1} - \frac{2}{x+2} - \frac{x+3}{x^2+x-2};$$

$$(2) \frac{\frac{a}{b} - \frac{b}{a}}{\frac{a^2}{b} - \frac{b^2}{a}} \div \frac{a+b}{\frac{a}{b} + \frac{b}{a}}$$

4. Prove that the L.C.M. of two quantities is equal to their product divided by their H.C.F.

Find the H.C.F. of

$$x^4 + x^3 + x^2 + x + 1 \text{ and } x^3 + x^2 - x + 2.$$

5. Solve the equations :

$$(1) \frac{1}{2}(x-2) - \frac{2}{3}(x+2) = \frac{1}{4}(x-6) + 1\frac{1}{2};$$

$$(2) \frac{x^2+1}{x} = \frac{x+5}{3}.$$

6. The difference of two fractions, one (the greater) formed by adding a certain number to both the numerator and the denominator of $\frac{2}{3}$, the other (the less) formed by subtracting this number from both the numerator and the denominator of $\frac{2}{3}$, is $\frac{1}{4}$. Find the number.

[The following Questions A, B, C, D are to be attempted ONLY by Candidates who do NOT attempt PART II.]

A. Factorize :

$$(1) 6x^2 + x - 12;$$

$$(2) a^3 + b^3 - c^3 - 2ab.$$

B. Solve the equations :

$$ax - by = a^2 + b^2, \quad x + y = 2a.$$

C. Show that the sum of the squares of three consecutive integers is greater by 5 than 3 times the product of the greatest and the least.

D. Solve graphically the equations :

$$\frac{1}{2}x + \frac{1}{3}y = 1.2, \quad 4x = 5y.$$

[Take 1 inch as your unit.]

PART II.

7. Solve the equations :

$$y^2 = 5x + 1, \quad 3x = 2y + 1.$$

8. If $\frac{a+b}{a-b} = \frac{c+d}{c-d}$, prove that $\frac{a}{b} = \frac{c}{d}$.

If a, b, c are in continued proportion, prove that

$$\frac{a-b}{b-c} = \left(\frac{a}{c}\right)^{\frac{1}{2}}.$$

9. Find the value of

$$(34.97)^{-2} \div (1.08)^{7.2}.$$

Obtain a formula for the sum of 27 terms of the series

$$1 + .9 + .81 + .729 + .6561 + \dots$$

and show that the sum is very nearly 9.42.

10. Draw the graphs of $2y - x = 3$, $y = 3 - x^2$ for values of x lying between -1 and 4 .

Find their gradients when $x = 1$, and show that the lines cut at right angles at the point $1, 2$.

GEOMETRY.

(Two hours.)

NOTE.—(1) *In order to pass in Elementary Mathematics, Candidates must satisfy the Examiners in a.*

(2) *In order to pass in Additional Mathematics or to obtain exemption from the Army Qualifying Examination, Candidates must satisfy the Examiners in α and β taken together.*

(3) *Figures should be drawn accurately with a hard pencil, and all 'constructions' clearly shown.*

α

1. Draw a triangle ABC having $BC = 3$ in., $ABC = 32^\circ$, and $ACB = 105^\circ$. Construct the bisector of the angle BAC meeting BC in D . Measure BD .

2. Draw a circle of radius 1 inch and take a point P 2.5 inches from its centre. Construct a circle of radius 1.3 inches to touch the former circle and also pass through P .

3. If two triangles have two sides of the one equal to two sides of the other, and also the angles contained by those sides equal, prove that the triangles are congruent.

If the sides AB, AC of a triangle are equal, and equal lengths AE, AD are cut off on AB, AC , prove that CE, BD are equal.

4. State an axiom relating to parallel straight lines, and use it to prove that when a straight line crosses two parallel straight lines it makes alternate angles equal.

Prove that the opposite angles of a parallelogram are equal to one another.

5. Prove that the square on a side of a triangle opposite to an obtuse angle is greater than the sum of the squares on the other two sides by twice the rectangle contained by one of these two sides and the projection on it of the other.

6. Prove that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

Two circles intersect in A, B . C, D are any two points on one circle and CA, DB cut the other circle in P, Q , respectively. Prove that PQ is parallel to CD .

β

7. Prove that the tangent at any point of a circle and the radius through the point are perpendicular to one another.

Two circles whose centres are A and B touch externally at C . The common tangent at C meets another common tangent in D . Prove that ADB is a right angle.

8. Construct a mean proportional to two given straight lines, and prove the construction.

$ABCD$ is a rectangle such that the square on AD is twice the square on AB . BE is drawn at right angles to AC to meet AC in E . Prove that AE is one-third of AC .

9. If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, prove that the triangles are similar.

10. Draw a straight line five inches long and divide it into three parts proportional to 37 : 46 : 53.

ELEMENTARY TRIGONOMETRY.

(Two hours.)

1. Define the tangent of an angle.
Construct an acute angle whose sine is $\cdot 6$, and find its cosine and its cotangent.
2. Prove that $\cos(180^\circ - a) = -\cos a$.
Arrange the angles α, β, γ in order of magnitude, if
 $\sin \alpha = \cdot 8211$, $\cos \beta = \cdot 7738$, $\tan \gamma = -0\cdot 6104$,
the angles being positive and each less than 180° .
3. What is the length of the shadow of a man 5 ft. 8 in. high cast by the sun when its altitude is $55^\circ 30'$?
4. Draw the graph of $10 + 10 \cos 2x$ for values of x between 0° and 60° .
Find a value of x to satisfy the equation
 $x = 10 + 10 \cos 2x^\circ$.

[Take one-tenth of an inch as unit along both axes.]

5. Prove that in any triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

If $A = 63^\circ$, $B = 49^\circ$, $a = 50$ in., find b to the nearest tenth of an inch.

6. Prove that:

$$(1) \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} + \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = \frac{2}{1 - 2 \sin^2 \theta};$$

$$(2) (\sec \theta + \tan \theta)(\operatorname{cosec} \theta - \cot \theta) = (\operatorname{cosec} \theta + 1)(\sec \theta - 1).$$

7. If
- $2 \sin \theta + 5 \cos \theta = 5$
- , prove that

$$\tan \theta = 0 \text{ or } \frac{3}{4}.$$

8. Prove that
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- , where
- A
- and
- B
- are both acute angles and
- A
- is greater than
- B
- .

$$\text{Prove that } \frac{\sin 5A + \sin A}{\sin 3A - \sin A} = 1 + 2 \cos 2A.$$

9. Show that in any triangle
- ABC

$$\frac{b + c}{a} = \frac{\cos \frac{1}{2}(B - C)}{\sin \frac{1}{2}A}$$

If $b + c = 24\cdot 8$ cm., $a = 11\cdot 89$ cm., $A = 39^\circ$, find B and C .

10. A lighthouse is observed from a ship which is steaming due N. to bear
- 62°
- W. of N.; after the ship has sailed 10 miles, the lighthouse is observed to bear
- 40°
- W. of S. Calculate the distance of the ship from the lighthouse when it was nearest to it.

STATICS AND DYNAMICS.

(Two and a half hours.)

[Assume $g = 32$ for foot-second units.]

1. State and prove the triangle of forces.
Find graphically, or otherwise, the resultant of the following forces acting at a point, viz. 5 lb. due N., 10 lb. 30° S. of W., 15 lb. 60° N. of E., 12 lb. SE. [Scale 1 cm. to 1 lb.]
2. A form, 10 ft. long, weighs 2 stone; its legs are 1 ft. from each end. A boy weighing 8 stone sits 2 ft. from one end, another weighing 7 stone sits 3 ft. from the other end. Find the pressures of the legs of the form on the ground.

3. Prove that, if three coplanar forces which are not parallel are in equilibrium, their lines of action meet in a point.

A stiff light rod ABC , bent so that ABC is a right angle and pivoted to a fixed point at A , has a weight of 12 lb. attached to it at C and is maintained with AB vertical and BC horizontal by a horizontal string attached at C . If the tension of the string is equal to the weight of 8 lb. and BC is 4 in., prove that AB is 6 in.

4. O is the centre of a rectangular piece of cardboard $ABCD$; the piece AOB is cut out and then fitted to and gummed on to DC so that AB lies along DC . Find the centre of gravity of the cardboard in its new form.

5. A mass of 14 lb. rests on a rough plane inclined at 30° to the horizon. If the coefficient of friction is $\cdot 8$, find the least force which, acting directly down the plane, will just move the mass.

6. Prove the formula $v^2 = 2fs$ for uniformly accelerated motion.

A point moving in a straight line with uniform retardation describes 7 ft. in the fifth second of its motion and 5 ft. in the seventh second. Prove that it will be at rest at the end of $11\frac{1}{2}$ seconds.

7. Define *force* and show how it is measured.

Find (1) in poundals, (2) in pounds' weight the force which will bring to rest in 8 seconds a mass of 3 lb. moving at the rate of a mile a minute.

8. Explain the meaning of the terms *work*, *energy*, and *power*.

Find the number of foot-pounds of work done by a man who picks up a stone weighing half a pound and throws it through a window 20 ft. above the ground, so that it passes through the window with a velocity of 8 ft. per second.

9. A bullet is fired with a velocity whose horizontal and vertical components are 200 ft. per second, and 40 ft. per second respectively. Find the greatest height to which it will rise and its distance from the firing-point when it reaches the ground again.

TRIGONOMETRY, STATICS AND DYNAMICS.

(MATHEMATICS I.)

(Three hours.)

[Assume $g = 32$ for foot-second units.]

1. Draw the graph of $5 \cos x$ for values of x from 0° to 60° ; and employ it to find the angle whose cosine is $\cdot 8$.

2. Prove that $\sin^2 A = 1 - \cos^2 A$.

Prove that

$$(\sin A + \sin B)(\cos B - \cos A) = \sin(A - B)(1 - \cos(A + B)).$$

3. Show that in any triangle:

$$(1) \quad b = c \cos A + a \cos C;$$

$$(2) \quad \tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

If $b = 60$ ft., $c = 72$ ft., $B = 52^\circ$, find the possible values of C .

4. Two men at A and B observe a balloon situated at a point vertically above the line AB (which is horizontal) and between A and B ; the angles of elevation of the balloon are observed to be $68^\circ 12'$ and $78^\circ 41'$, and the distance AB is 240 ft. Show that the height of the balloon is nearly 400 ft.

5. State and prove the triangle of forces.

Find graphically, or otherwise, the resultant of the following forces acting at a point, viz. 5 lb. due N., 10 lb. 30° S. of W., 15 lb. 60° N. of E., 12 lb. SE. (Scale 1 cm. to 1 lb.)

6. A form, 10 ft. long, weighs 2 stone; its legs are 1 ft. from each end; a boy weighing 8 stone sits 2 ft. from one end, another weighing 7 stone sits

3 ft. from the other end. Find the pressures of the legs of the form on the ground.

7. Prove that if three coplanar forces, which are not parallel, are in equilibrium, their lines of action meet in a point.

A stiff light rod ABC , bent so that ABC is a right angle and pivoted to a fixed point at A , has a weight of 12 lb. attached to it at C , and is maintained with AB vertical and BC horizontal by a horizontal string attached at C . If the tension of the string is equal to the weight of 8 lb. and BC is 4 in., prove that AB is 6 in.

8. A mass of 14 lb. rests on a rough plane inclined at 30° to the horizon. If the coefficient of friction is $\cdot 8$, find the least force which, acting directly down the plane, will just move the mass.

9. Prove the formula $v^2 = 2fs$ for uniformly accelerated motion.

A point moving in a straight line with uniform retardation describes 7 ft. in the fifth second of its motion and 5 ft. in the seventh second. Prove that it will be at rest at the end of $11\frac{1}{2}$ seconds.

10. Define force, and show how it is measured.

Find (1) in poundals, (2) in pounds' weight the force which will bring to rest in 8 seconds a mass of 3 lb. moving at the rate of a mile a minute.

11. A body is attached to a fixed point by a string. If the body is let go when the string is taut and horizontal, prove that, when the body is passing through its lowest position, the tension of the string is three times the weight of the body.

ARITHMETIC, ALGEBRA, GEOMETRY, AND GEOMETRICAL DRAWING.

(MATHEMATICS I.)

(Three hours.)

1. Three bells begin tolling together at the rates of 50, 55, and 65 times per minute respectively, how soon will they next toll together?

2. Having given that 1 cubic centimetre of water weighs 1 gramme, that
1 kilogramme = 2.2046 lbs.

and 1 foot = 30.48 cms.,

find the weight of 1 cubic foot of water to the nearest ounce.

3. Solve the equations:

$$(1) (x^2 - 1)(x + 2) = x^3;$$

$$(2) x - y = a,$$

$$x^2 - ax + by = 0.$$

4. Interpret the expressions $a^{\frac{1}{2}}$, a^{-2} , and justify your interpretation.

Simplify $\sqrt[3]{a^{-2}b^2c^{\frac{1}{2}}} \div (a^{-1}b^{-2}c^{\frac{1}{2}})^{\frac{1}{3}}$.

5. Find the sum of n terms of an arithmetic progression, whose first term and common difference are given.

Prove that, if n is an odd number, the sum of n terms of the progression is n times the middle term.

Find the fifth term and the sum of 5 terms of the progression

$$\sqrt{2} + 1, 1, \sqrt{2} - 1 \dots$$

6. Draw the graphs of the equations

$$x + y = 1, \quad y(1 - x) = 1$$

for the values of x between -2 and 3 .

Show that the second curve has equal gradients at the two points where it is cut by the first.

7. Prove that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

Two circles intersect in A, B . C, D are any two points on one circle and CA, DB cut the other circle in P, Q respectively. Prove that PQ is parallel to CD .

8. Construct a mean proportional to two given straight lines and prove the construction.

$ABCD$ is a rectangle such that the square on AD is twice the square on AB . BE is drawn at right angles to AC to meet AC in E . Prove that AE is one-third of AC .

9. If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, prove that the triangles are similar.

10. Draw a straight line 5 inches long and divide it into three parts proportional to 37 : 46 : 53.

UNIVERSITY OF LONDON MATRICULATION
EXAMINATION.
September 1910.

ARITHMETIC AND ALGEBRA.

(Three hours.)

1. If

$$\pi = 3.14159265 \dots$$

and

$$e = 2.71828182 \dots$$

find (i) to 3 places of decimals, (ii) to 3 significant figures, the values of

$$\pi + e, 10\pi - e, \pi \times e, \frac{\pi}{e}, \frac{e}{25\pi}.$$

2. The rainfall in a certain district is 5.76 cm. How many litres per hectare is this? (A litre is 1 cubic decimetre; a hectare is 10,000 square metres.)

Taking an inch as 2½ cm. and a cubic foot of water as weighing 62½ lbs., how many tons per acre is this? Give your answer to the nearest ten tons.

3. From April 1, 1908, to March 31, 1909, the numbers of letters, &c., delivered by the Post Office were estimated as follows:

England and Wales	-	-	-	-	2,482 $\frac{4}{5}$ millions.
Scotland	-	-	-	-	256 $\frac{2}{5}$ "
Ireland	-	-	-	-	167 $\frac{2}{5}$ "

and the average numbers of letters, &c., delivered per head of the population as:

England and Wales	-	-	-	-	-	70
Scotland	-	-	-	-	-	53
Ireland	-	-	-	-	-	38

From these data estimate the number of letters delivered per head of the whole population of Great Britain and Ireland combined.

4. A man has £ a in one bank and £ b in another. If he has £ c more to deposit, how should he divide it between the two banks so that (i) the two accounts may be equal, (ii) the first account may be double the second?

5. If

$$x^2 + ax^2 + bx + 1$$

is exactly divisible by

$$x^2 + cx + 1,$$

what must the quotient be?

A certain algebraical expression is exactly divisible by $x - 2$, the quotient being $x^2 - x - 6$. Show that the expression is divisible by $x + 2$, and find the quotient.

6. What is meant by an equation, and what is meant by solving an equation?

Solve the equations

$$(i) \frac{x - a + b}{x + a - b} = \frac{1 - a + b}{1 + a - b};$$

$$(ii) \frac{x^2 - 6x + 2}{x^2 + 6x - 2} = \frac{1 - 6 + 2}{1 + 6 - 2};$$

$$(iii) \left. \begin{aligned} x - ay + b &= 1 \\ ax - y + b &= 1 \end{aligned} \right\}.$$

7. In each of the following, y is given in terms of x . Find, in each case, x in terms of y .

$$(i) y = mx + c;$$

$$(ii) y = \frac{2 - 3x}{x + 3};$$

$$(iii) y = \frac{-1 \pm \sqrt{3x - 2}}{3}.$$

8. Prove the formula for the sum of n terms of a geometric progression.

What is the error made in taking the sum of the infinite series 1, 0.2, 0.04, 0.008, ... as being 1.248?

Find, to three decimal places, the sum of the square roots of the terms of this series (i) taken as all positive, (ii) taken as alternately positive and negative.

9. Continue the series 1, $1\frac{1}{2}$, ..., in harmonic progression, and find the n th term.

Show that if

$$\frac{a}{1 + ak}, \frac{b}{1 + bk}, \frac{c}{1 + ck}$$

are in harmonic progression for any one value of k , they are so for all values of k .

10. Running a certain course, starting level, A finishes 176 yards in front of B . With a minute's start, B would have finished 160 yards in front of A ; and with 160 yards start he would have finished three seconds later than A . What is the length of the course, and what are the times of each for the full course?

GEOMETRY.

(Three hours.)

[In every case where a construction is made a complete explanation of the work must be given.]

1. If two triangles have two sides of the one equal respectively to two sides of the other, but the contained angles not equal, prove that that triangle in which the contained angle is greater has the greater third side.

In the triangle ABC , it is given that $AB = 2.5$ in., $AC = 1.9$ in., and the angle BAC is known to be less than 125° but greater than 70° . Find graphically the limits between which the third side BC must lie.

2. Draw a quadrilateral $ABCD$ having $AB = 5$ cm., $BC = 6$ cm., $CD = 7$ cm., $DA = 8$ cm., $AC = 7$ cm., and then construct a square of equal area, measuring its sides. Every step of the method you employ must be explained.

3. State clearly (with the aid of a figure) the special properties of the interior angles made by a straight line falling across two other lines which (1) are not parallel, (2) are parallel. (No proofs are required.)

In any triangle ABC , prove that the internal bisector of the angle B and the external bisector of the angle C cannot be parallel, and also prove that their point of intersection lies on the same side of BC as A .

4. If the sum of the squares on two sides of a triangle is equal to the square on the third side, prove that the triangle is right angled.

Prove that a triangle the lengths of whose sides are $x^2 - 1$, $2x$, $x^2 + 1$ has a right angle whatever x may be. Deduce a geometrical construction for finding the square root of any number, e.g., 5. A drawing to scale is not required.

5. If a , b are lengths, give a geometrical interpretation of the theorem $(a + b)(a - b) = a^2 - b^2$, and prove the theorem so enunciated.

6. Show how to construct on a given line a segment of a circle containing a given angle, and indicate clearly in a diagram whereabouts a point must be if the angle subtended at it by the line is greater than this given angle.

The base AB of a triangle is 2.5 in., the area is 4 sq. in., and the vertical angle is greater than 80° . Whereabouts must the vertex lie?

7. Prove that the line joining the centre of a circle to the middle point of any chord is perpendicular to that chord.

Draw a circle of radius 2.7 in. and take a point P which is 2.2 in. from the centre O . Construct a chord of the circle passing through P , and having its middle point 1.4 in. from P , and measure its length.

8. The tangent at A to the circle ABC meets the diameter BC at D , and AL is drawn perpendicular to BC . Prove that

$$DB \cdot DC = LB \cdot LC = DL^2.$$

9. If A , B , P , Q are any four points on a circle, prove that the angles APB , AQB are either equal or supplementary.

If one pair of opposite sides of a cyclic quadrilateral are equal, prove that the other pair are parallel.

10. Explain how, without the use of a protractor, you could construct angles of 18° , 15° , 9° . The constructions need not be accurately performed, but proofs must be given.

MATHEMATICS (MORE ADVANCED).

(Three hours.)

1. Draw a line of length one inch and then, without making any further use of the graduations on your ruler, construct an equilateral triangle of area four square inches.

2. If two equal circles pass each through the centre of the other, find, to four significant figures, the proportion of the area of either which is common to both.

3. Prove the Binomial Theorem for a positive integral index, and apply it to find an approximation for $(1 + x)^7$ which is accurate to 10 decimal places if x is not greater than 0.001.

4. Prove the theorem

$$\log x^n = n \log x,$$

and find the fifth roots of

$$0.24967, \quad -\frac{1}{0.024967}, \quad 0.24967 - \frac{1}{0.024967}.$$

5. Interpret a^x , a^{-x} and show that the statements

$$a^x \times a^y = a^{x+y}, \quad (a^x)^y = a^{xy}, \quad (xy)^z = x^z y^z$$

are consistent with your interpretation.

6. Show that if the sine of an acute angle is greater than its cosine, the tangent is greater than the cotangent and the secant than the cosecant.

Find, without using the tables, all the trigonometrical ratios of the angles of an isosceles triangle whose altitude is one-third of its base.

7. Prove the identities

$$(i) \quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$(ii) \quad \cot \frac{A}{2} = \cot A (1 + \sec A),$$

and solve the equation

$$\begin{aligned} x + y &= 41^\circ \\ \sin x + \sin y &= 0.69 \end{aligned}$$

8. A railway ascends from A to B at an inclination $0^\circ 30'$ to the horizontal, and then descends from B to C , which is on the same level as A , at an inclination $0^\circ 45'$ to the horizontal. If the horizontal distance between A and C is a mile, find to the nearest foot the height of B above AC .

9. The co-ordinates of the angular points of a triangle, referred to rectangular axes, are $(0, 0)$ $(24, 10)$ $(12, -9)$. Find the equation of its sides, their lengths, and the tangents of the angles of the triangle.

10. Show that the equation

$$ax^2 + ay^2 + 2gx + 2fy + c = 0$$

represents a circle, and give the co-ordinates of its centre and the length of its radius.

MECHANICS.

(Three hours.)

1. Two railway tracks Ox and Oy enclose an angle of 60° . A train moves along Ox with uniform velocity of 60 miles an hour, while a second train moves along Oy with equal speed, passing through O two minutes after the first. Find the velocity of the second train relatively to the first, and indicate in a diagram the shortest distance between the trains.

2. Describe an experiment to verify the law that the velocity acquired by a mass under the action of a constant force is proportional to the square root of the distance traversed from rest.

3. A man who weighs 160 lbs. slides down a rope, that hangs freely, with a uniform speed of 4 feet per second. What pull does he exert upon the rope, and what would happen if at a given instant he should reduce his pull by one half.

What do you understand by the term *work*?

An engine working at the rate of 6,000,000 foot pounds per minute draws a train at the rate of 60 miles an hour. With what force does it pull the train.

5. If a body is in equilibrium under the action of three forces, how could you find the magnitude and line of action of the third force from a knowledge of the magnitudes and lines of action of the other two forces (a) when the lines of action of these two forces are parallel, and (b) when their lines of action meet in a point?

How could you illustrate the conditions of equilibrium in each case by experiment?

6. Find the force, acting parallel to the plane, which would be required to draw a mass of 100 pounds uniformly up a smooth inclined plane of which the height is half the length. Find also the normal pressure between the mass and the plane.

7. A stone hangs in a tank of water, being supported by a string so that it is completely immersed without touching the bottom of the tank. The water is allowed to drain away and at a certain stage, when the stone is partly unimmersed, the string breaks. Explain this.

8. What is the relation between the pressure and volume of a quantity of gas when the temperature is constant?

How would you test the relation experimentally?

THE UNIVERSITIES OF MANCHESTER, LIVERPOOL,
LEEDS, AND SHEFFIELD.

MATRICULATION EXAMINATION.

July, 1910.

ARITHMETIC AND ALGEBRA.

(Two hours.)

1. The earth's circumference, measured along a meridian, is 40 million metres, or 21,600 nautical miles. Assuming that a metre = 39·370 in., find the number of feet in a nautical mile.

2. What is the present value of an estate bringing in an annual net rent of 275*l.* 15*s.* per annum, the first payment being due a year hence, and compound interest being reckoned at 3*l.* 6*s.* 8*d.* per cent. ?

Also, what is the present value if the first payment is to be made three years hence.

3. Find the factors of

$$6x^3 - 5xy - 6y^2, x^4 - x^2y^2 + 16y^4, x^4 - 9x^2 + 30x - 25.$$

4. (i.) A certain sum of money would pay *A*'s wages for *a* days, or *B*'s wages for *b* days, or *C*'s wages for *c* days. How long will the money last if *A*, *B*, *C* are employed together ?

(ii.) A man rows down a stream a certain distance and returns. His speed of rowing relative to the water is *m* miles an hour, and the current is *n* miles an hour. Find the ratio of the time he takes to what it would have been if there had been no current.

5. State and prove the rule for forming the square of an expression of the form

$$a + b + c + \dots$$

Find the square root of

$$(x^2 + 1)^2 - 4x(x^2 - 1).$$

6. At an election there were 25 electors, each of whom voted for two out of the three candidates *A*, *B*, *C*. *A* received 18, *B* 17, and *C* 15 votes. How many electors voted for each pair of candidates ?

7. (i.) Prove that the expressions $3x^2 - 5x - 1$ and $2x^2 - 3x - 4$ cannot be equal for any real value of *x*. For what value of *x* will the difference be least ?

(ii.) Solve the equations :

$$\left. \begin{aligned} xy + 2(x - y) &= 4, \\ x + y - 3xy &= 5. \end{aligned} \right\}$$

8. The distances which a man travels in successive days are in arithmetical progression. At the end of the fifth day he has gone 100 miles, and at the end of the seventh 150 miles. How long will he take to complete 300 miles in all ?

9. (i.) Find the ratio of the sum of the squares of the first *n* terms of the series

$$a + ax + ax^2 + \dots$$

to the sum of the first *n* terms.

(ii.) Prove that

$$1 - \frac{a^n}{1-a} + \left(\frac{a^n}{1-a} \right)^2 - \dots = 1 - a$$

for all negative values of *a*. For what positive values of *a* does the statement hold ?

ALGEBRA AND TRIGONOMETRY.

(HIGHER ALTERNATIVE PAPER.)

(Three hours.)

Tables of logarithms and sheets of squared paper are provided.

1. What is the present value of an estate bringing in an annual net rent of 275*l.* 15*s.* per annum, the first payment being due a year hence, compound interest being reckoned at $3\frac{1}{2}$ per cent.?

Also, what is the present value, if the first payment is to be made three years hence?

2. Graph the expression $(x-2)(x-4)(x-6)$, and calculate other values of x for which the expression has the same value that it has when $x=2.5$.

3. Prove that a quadratic expression, ax^2+bx+c , attains its least (or greatest) value when x is equal to the mean of the values for which the expression vanishes.

AB is a given straight line and P a point in it, find the position of P in order that $3AP^2+4BP^2$ may be the least possible.

4. (i.) If $x=\sqrt{3}-\sqrt{2}$ find the value of $x^3+\frac{1}{x^2}$.

(ii.) Solve the equation $\sqrt{x+9}-\sqrt{x-10}=1$.

5. Obtain from first principles the formula for nC_r , (the number of combinations of n things r at a time).

Prove ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$.

Find the total number of selections that can be made from n different things.

6. Prove that the general term in the expansion of $(1+x)^n$, where n is a positive integer, is ${}^nC_r x^r$. How would you express the result if n is not a positive integer?

Prove that the $(n+1)^{\text{th}}$ term in $(1-x)^{-\frac{1}{2}}$ is the middle term of $\left(1+\frac{x}{4}\right)^{2n}$.

7. Explain the expression "sum to infinity," and illustrate your answer by the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots$

Sum to n terms the series

$$(i.) 1^2+2^2+3^2+\dots$$

$$(ii.) \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots$$

8. Express (i.) $\sin 2\theta$ and $\cos 2\theta$ in terms of $\tan \theta$; (ii.) $\cos^2 \theta$ in the form $A \cos 4\theta + B \cos 2\theta + C$.

Solve the equation $\sin \theta + 3 \cos \theta = 2$, giving both the least positive root and the general solution.

9. A stationary balloon is observed from a point A due south of it on a horizontal plane and the elevation is a . From a point B due west of A and at distance c from A the balloon is observed at an elevation β . Find the height of the balloon above the horizontal plane. Work out the case in which $a=33^\circ$, $\beta=21^\circ$, $c=\frac{1}{2}$ mile, giving the result in feet as accurately as your tables permit.

10. Draw the graph of $\tan x$, and obtain an approximate solution of the equation $\tan x^\circ = x/30$.

11. The sides of a triangle are 13, 14, 15 units. Calculate (i.) the radius of the circumcircle, (ii.) the acute angle between the side 14 and the median which bisects it.

12. Prove the formula

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$$

for any triangle.

Solve the triangle in which two sides are 17 and 10 units, and the difference of the angles opposite to them is 20° .

G E O M E T R Y.

(Two hours.)

1. State and prove a construction for drawing a line which shall be perpendicular to a given line from a given point without the line.

P, Q are given points on opposite sides of a given line *KL*. Find a point *R* in *KL* so that *KL* shall bisect the angle *PRQ*.

2. Define a *parallelogram*, and prove that (1) each diagonal bisects the figure, (2) the diagonals bisect one another.

Show how to trisect a given parallelogram by lines drawn through a corner.

3. *ABC* is a triangle, right-angled at *A*, and *AD* is perpendicular to the hypotenuse *BC*. Prove that the square on *AB* is equal to the rectangle whose sides are equal to *BC, BD*.

If the sides *AB, AC* are 2.8, 9.6 centimetres respectively, calculate the lengths of *BD, DC, AD*.

4. Divide a given straight line into two parts so that the square on one part may be equal to the rectangle contained by the whole line and the other part.

Prove that, if the line be so divided, the sum of the squares on the whole line and on one part is equal to three times the square on the other part.

5. Investigate the locus of the mid-points of (1) parallel, (2) equal, (3) concurrent chords of a circle.

6. If *PQ* and *RS* are two chords of a circle intersecting at a point *O* within the circle, prove that the rectangles *PO.OQ* and *RO.OS* are equal.

7. *ABC* is a triangle, *AD* bisects the angle *BAC* and meets *BC* in *D*. Prove that *BD : DC = BA : AC*.

If *I* is the centre of the circle inscribed in the triangle, prove that *AI : ID = BA + AC : BC*.

8. Prove that the areas of similar triangles are in the same ratio as the squares on corresponding sides.

ABC is a triangle and *BE, CF* are drawn perpendicular to the opposite sides. Prove that the triangle *AEF* is similar to *ABC*; and if the angle at *A* be 60° compare the areas of these triangles.

PURE AND ANALYTICAL GEOMETRY.

(HIGHER ALTERNATIVE PAPER.)

(Three hours.)

1. Inscribe a rhombus in a given triangle *ABC* so that one angle shall coincide with the angle *A* and the opposite corner shall lie in *BC*.

2. A plane quadrilateral is made of four unequal rods hinged together at their ends. Prove that if the diagonals be at right angles in any position they are always at right angles.

3. In any parallelogram the difference of the squares on the diagonals is equal to four times the rectangle contained by one of the sides and the projection on it of an adjacent side.

4. Two equal circles intersect in A, B , and a straight line perpendicular to AB meets the circumferences in P, Q , on the same side of AB . Prove that P is the orthocentre of the triangle AQB , and Q that of the triangle APB .

5. Find a mean proportional to two given straight lines.

A straight line AB is bisected in O and produced to C , and circles are described on AC and OC as diameters. Perpendiculars to AB at O and B meet the two circles in P and Q respectively. Prove that $OP = OQ$.

6. Prove that the diagonals of a parallelopiped meet in one point and bisect one another.

If OA, OB, OC be three edges of a parallelopiped, and D the corner opposite to O , the diagonal OD will meet the plane ABC in the intersection of medians of the triangle ABC .

7. Prove that, of the plane angles which form the faces of a trihedral angle, any two are together greater than the third.

If each edge of a tetrahedron be equal to the opposite edge, prove that all the angles of the triangular faces will be acute angles.

8. Find the conditions of parallelism and perpendicularity, respectively, of the straight lines

$$A_1x + B_1y + C_1 = 0, A_2x + B_2y + C_2 = 0.$$

The co-ordinates of two vertices of a triangle are $(3, -2)$ and $(-1, 3)$, respectively, and the orthocentre is at the origin. Find the co-ordinates of the remaining vertex.

Give an accurate figure.

9. How does it appear analytically that a circle is determined by three conditions? Give examples.

Find the equations of the two circles which pass through each of the points $(1, 0)$ and $(2, 0)$, and touch the straight line $y = x$.

10. Prove that in a parabola the subnormal is constant.

The line drawn through the middle point V of a chord of a parabola, perpendicular to the chord, meets the axis in G ; and the ordinate of V meets the curve in P . Prove that PG is the normal at P .

11. Prove that the equation

$$y = Ax^2 + Bx + C$$

represents a parabola; and find the equations of its axis and directrix.

Find the equations of the common tangents to

$$y = x^2 - 2x + 4, y = -x^2 + 4x - 1.$$

12. Prove that the locus of the intersection of perpendicular tangents to an ellipse is a circle.

Prove also that the length of the chord of contact varies as the square on the diameter parallel to it.

MECHANICS.

(Three hours.)

1. What is meant by the term *acceleration*?

A man standing at the top of a tower 36 feet high throws up a stone with a velocity of 64 feet per second. How much later must he drop a second stone if the two stones are to reach the ground simultaneously? [The gravitational acceleration is 32 feet per second per second.]

2. Define the terms *moment*, *couple*.

A uniform bar 14 feet long, weighing 160 lbs., is supported on two props 2 feet and 4 feet from either end respectively. A string is attached to one end from a point vertically above it. What tension must be put on the string if the two props are to support equal weights? To which end must the string be attached?

3. What is meant by the statement that one horse-power = 550 on the foot, pound, second system of units?

A car weighing 10 tons is drawn up an incline of 1 in 50 at the rate of 300 feet per minute. If the resistances to motion due to friction, etc., be 20 lb. per ton, what will be the horse-power of the engine required?

4. Define the terms *momentum*, *energy*.

A shot of mass 1,000 lbs. is projected from an 80-ton gun with a velocity of 1,200 feet per second. Calculate in foot-pounds the work done by the gun and shot respectively before coming to rest.

5. What is meant by the *centre of gravity* of a body?

A cylinder of lead one inch long is cemented on to the end of a cylinder of wood 10 inches long of the same cross-sectional area. Find the centre of gravity of the body so formed if the density of lead is 11.4 and that of wood .6.

6. Upon what factors does the time of swing of a pendulum depend?

At a certain place the time of swing of a pendulum is 1 second, and at another one five-hundredth of a second longer. At the first place a mass of 300 grammes is weighed on a spring balance; what mass will be required at the second place to stretch the spring to the same extent?

7. How does the pressure vary at different depths in a liquid?

At what depth in the sea will the pressure be 10 atmospheres, given that the atmospheric pressure is 1,035 grams weight per square centimetre and that the density of sea water is 1.025?

8. Define the term *specific gravity*.

A hollow cube, the length of whose edges is 10 centimetres, is made of lead 1 millimetre thick. How much of the cube will be immersed when thrown into water?

[Specific gravity of lead = 11.4.]

9. Describe an experiment to compare the densities of air at two different pressures.

An exhausting pump the volume of whose cylinder is 100 cubic centimetres is attached to a vessel of one litre capacity containing air at atmospheric pressure. What will be the pressure in the vessel at the end of four strokes?

10. Explain the principles upon which an airship and a "heavier than air" flying machine keep above the ground.

MECHANICS AND PHYSICS.

(HIGHER ALTERNATIVE PAPER.)

(Three hours.)

Candidates must satisfy the Examiners in SECTION A and in ONE AT LEAST of SECTIONS B and C.

A.

1. What do you know of the position of a body suspended freely from a point?

A uniform square board weighing 40 lbs. is suspended from a point by means of four equal threads (of length equal to the diagonal of the square) attached (a) to the four corners, (b) to the mid points of the sides. Find the tension of the threads in each case.

What is the effect of cutting one of the threads?

2. A lift starting from rest descends in three stages:—(1) with uniform acceleration of 2 feet per second per second for 5 seconds, (2) with uniform velocity, (3) with uniform retardation of 2 feet per second per second. A 10 lb. weight is suspended from the roof at a distance of 8 feet from the floor. Find the tension of the thread at each stage.

If the thread were able just to support a tension of 10 lbs. weight, how long would the weight take to reach the floor of the lift after breaking the

thread, and what would be its velocity relative to the lift at the instant of striking the floor?

[$g = 32$ feet per second per second.]

3. A body is projected with a velocity of 64 feet per second up a smooth inclined plane (inclined at 30° to the horizontal) in a direction making an angle of 45° with the line of greatest slope. Find the time taken by the body to return to the original level and the horizontal distance traversed.

4. Define coefficient of friction, angular velocity.

A body is placed on a circular table at a distance of 3 feet from the centre. With what angular velocity may the table be rotated about a vertical axis through the centre before the body begins to slip?

How would this velocity depend on the place on the earth's surface at which the experiment is performed?

[Coefficient of friction between body and table = .6.]

5. What is meant by "pressure at a point" in a fluid, "centre of pressure"?

A cubical box of edge 2 feet with one pair of faces horizontal is half filled with water and half with oil of specific gravity .8, the oil floating on the water. Calculate the total pressure on the bottom and on one of the sides of the box; and the position of the centre of pressure on the side.

B.

6. What do you understand by "total reflexion"? Describe a method of determining the refractive index of any substance by total reflexion.

Determine the critical angle at the surface of separation of glass and water.

$$\mu_{\text{glass}} = 1.55 \quad \mu_{\text{water}} = 1.33.$$

7. How has the velocity of light through air been determined?

Given the velocity through air under atmospheric conditions, what do you know of the velocity through the specimen of glass referred to in question 6?

Does the velocity of light depend on the colour of the light?

8. What is meant by the "mechanical equivalent of heat"?

A body weighing 10 lbs. starts from rest to slide down a rough inclined plane. When it has descended a vertical distance of 50 feet its velocity along the plane is 20 feet per second. Find the quantity of heat generated by friction.

One water pound degree Fahrenheit = 772 foot-pounds.

9. Define the coefficient of linear, superficial, and cubical expansion, and find the relation between them.

Discuss the effect of a rise of temperature on (a) the cross-sectional area of a barometer tube, (b) the height of a barometer.

10. Describe and explain the changes in the appearance of the image of an object which is gradually moved from the surface of a concave mirror along the axis when viewed from a point near the axis at a considerable distance from the mirror.

C.

11. Describe how you would experimentally study the variation of magnetic force along the line of the axis of a bar magnet. From your observations what would you learn of the law of force between two point poles?

12. Define potential, capacity, condenser.

Describe an experiment showing how the capacity of a condenser depends on (a) the distance apart of the plates, (b) the medium between the plates.

13. Define resistance, specific resistance.

A battery of E.M.F. = 2 volts has its terminals connected by a wire of resistance 3 ohms. It is found that a current of .4 ampere flows round the circuit. Find the resistance of the battery, and the difference of potential between its terminals when the current is flowing.

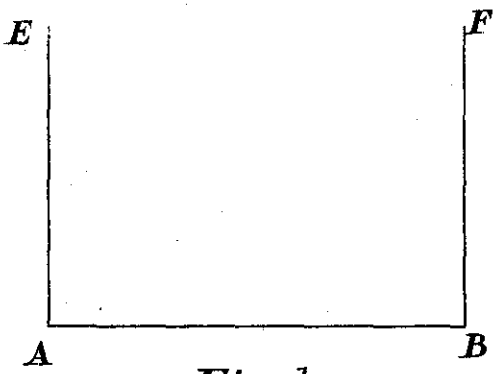


Fig. 1.

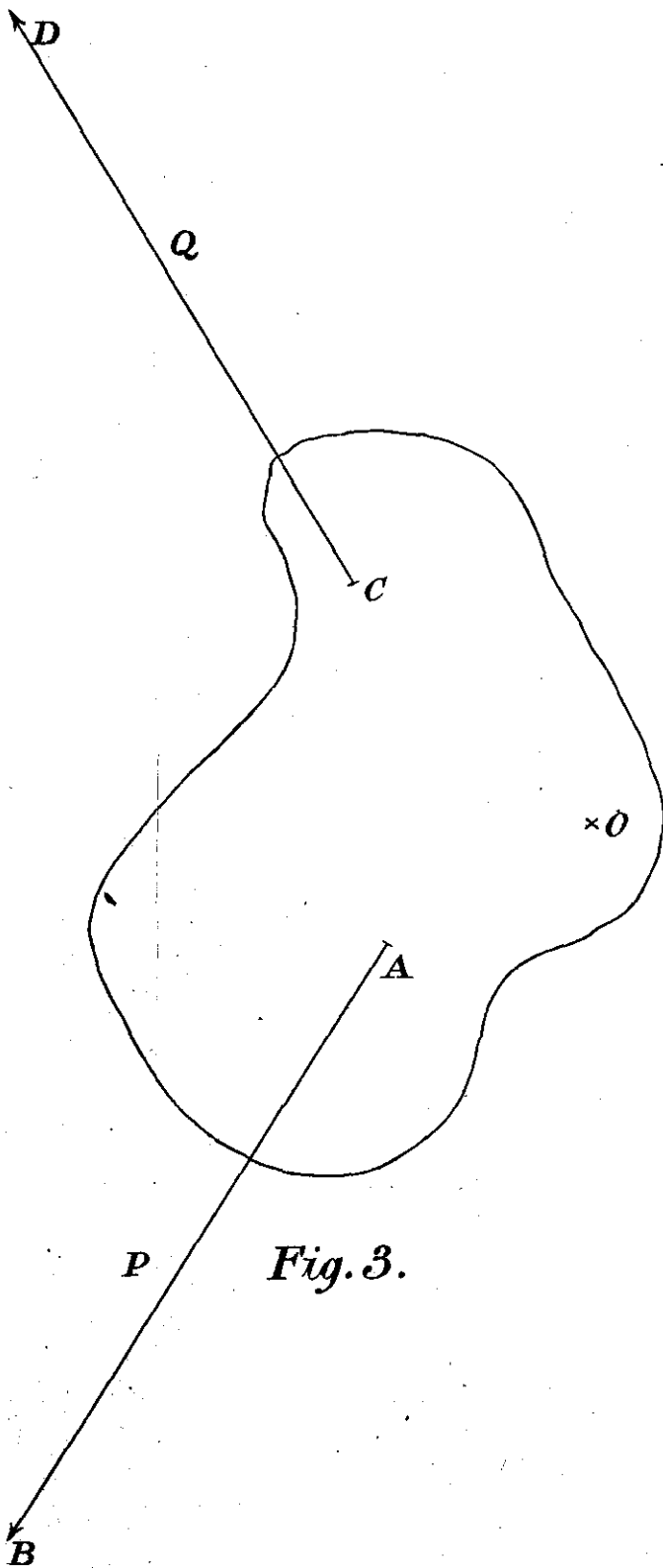
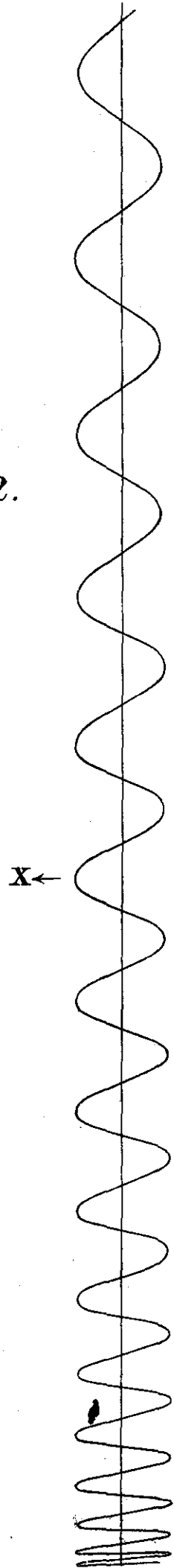


Fig. 3.

Fig. 2.



14. Explain the terms electrolysis, electrolyte, ion, electro-chemical equivalent.

State the laws of electrolysis.

An electric current is passed through a solution of copper sulphate and a tangent galvanometer in series. The galvanometer coil consists of 10 turns of wire of 20 centimetres radius. The needle is suspended at its centre. How long must the current be passed to get a deposit of copper of average thickness $\frac{1}{100}$ millimetres on a kathode 100 sq. cm. in area when the deflexion of the galvanometer needle is 30° ?

Electro-chemical equivalent of copper = .00032 gram per Coulomb.

Density of copper = 9 grams per cc.

Horizontal magnetic intensity = .18 absolute units.

15. Give a general account of the phenomena of electro-magnetic induction.

Describe and explain the action of an induction coil.

CIVIL SERVICE COMMISSION.

MILITARY ENTRANCE EXAMINATION.

November and December 1910.

MATHEMATICS I. PAPER I

(Three hours.)

For full credit your work must be immediately intelligible, and your results given only to the degree of accuracy that is asked for or is justified by the data.

The slide rule may be used whenever it allows of sufficient accuracy, and in each case a note of its use should be made.

1. A small tank has a square base $ABCD$ measuring 30 cm. each way, and vertical sides $AE, BF, \&c.$, 18 cm. high, and contains water to a depth of 16 cm. (Fig. 1, see opposite page). How many c.cm. of water are there in the tank?

The tank is gradually tilted about the edge BC and the water allowed to run out. What volume of water remains when the corner A is in the surface of the water? Draw the tank half-size, and measure with your protractor the angle which the edge AB then makes with the horizontal.

Give an expression for the volume of the water remaining, as the tank is gradually tilted, (1) in terms of the length of AE under water, for cases in which the corner A is covered, and (2) in terms of the length of BA under water, for cases in which A is above the surface.

Find, by drawing, the inclination of the edge AB to the horizontal when a quarter of the water has run away, and when half the water has run away.

2. A piece of cardboard is pinned to the table at the point O and is free to rotate about the pin (Fig. 3, see opposite page). A force P of 1 kilogram is applied at the point A in the direction AB . Another force Q is applied at C in the direction CD . Prick the figure through into your book and find the magnitude of the force Q which will hold the cardboard still, and the magnitude and direction of the force exerted by the pin.

3. AOB is the diameter of a circle whose centre is at O . At C in the circumference a tangent CE is drawn. OE parallel to AC meets this tangent in E . Prove that EB touches the circle, and find an expression for the distance EB , in terms of " r " the radius of the circle, and " θ " the angle COA .

4. Two observations taken in one vertical plane at a point *A* give the elevation of a mountain summit as $7\frac{1}{2}^\circ$, and that of a point *B* between *A* and the mountain as 5° . A straight road 3 miles long runs from *A* to *B*, and at *B* the elevation of the summit of the mountain is $16\frac{1}{4}^\circ$. Find its height above the level of *A*.

5. The following table gives observations of the Centigrade temperature of some water in a vessel. The temperature taken at intervals of 5 minutes was:—

80°, 67°, 56°, 47°, 40°, 35°, 31°.

Plot the temperatures with the times and join by a smooth curve.

Estimate, as nearly as you can, the rate at which the temperature was falling at the moment when the temperature was 60° , and when it was 50° , and when it was 40° . Taking the room temperature as 20° , compare these three results and state whether the rate of cooling was proportional to the excess of temperature above that of the room.

6. Fig. 2 (*see above*) is the trace made by a pen attached to a spring which vibrated backwards and forwards in $\frac{1}{2}$ second, while a strip of paper was drawn under it. The straight line in the figure was made by keeping the pen still while the paper was drawn back again.

From measurements of the figure show in a graph the whole distance moved by the strip of paper up to any time. Take 1 cm. to represent $\frac{1}{2}$ sec., and show the distance full size.

From the figure measure the distance moved in $\frac{1}{2}$ sec. at various times, and use it to give a graph of the velocity of the strip of paper at any time. Represent $\frac{1}{2}$ sec. by 1 cm., and a speed of 1 cm. per sec. by 1 cm.

Find the average value of the acceleration, and taking this as the true value throughout, calculate the velocity of the strip of paper at the moment when the pen was at *X*. Check this value for the velocity by measuring the distance travelled in a short interval just before and in a short interval just after.

CONSTANTS.

1 metre = 100 cm. = 1000 mm. = 0.001 km.

1 yard = 3 ft. = 36 in. = 0.9144 metre.

1 mile = 8 fur. = 80 chains = 320 po. = 1760 yds. = 8000 links.

1 hectare = 100 ares = 10000 sq. metres.

1 acre = 4 roods = 10 sq. chains = 4840 sq. yds. = 0.4047 hectare.

1 litre = 1 c.dm. = 100 cl. = 0.01 hl. = 0.001 c. metre.

1 gallon = 4 qt. = 8 pt. = 0.1606 c. ft. = 4.546 litres.

1 quarter = 8 bush. = 32 pk. = 64 gal.

1 kilogram = 1000 g. = 0.001 tonne.

1 lb. = 16 oz. = 7000 grains = 453.6 grammes.

1 ton = 20 cwt. = 80 qr. = 2240 lb.

1 litre of water weighs 1 kg. 1 c. ft. of water weighs 62.3 lb.

A circle of radius *r* has a circumference of length $2\pi r$ and has an area of πr^2 , where $\pi = 3.1416$.

Base of natural logarithms is $e = 2.7183$.

The acceleration of a falling body is 32.2 feet per sec. per sec. or 981 cm. per sec. per sec.

1 kilowatt = 102 kilogram-metres per second. 1 horse-power = 550 foot-pounds per second.

1 atmosphere = 760 mm. of mercury = 1.03 kg. per sq. cm. = 14.7 lb. per sq. inch.

MATHEMATICS I. PAPER 2.

(Three hours.)

For various data see p. 536.

For full credit your work must be immediately intelligible, and your results given only to the degree of accuracy that is asked for or is justified by the data.

The slide rule may be used whenever it allows of sufficient accuracy, and in each case a note of its use should be made.

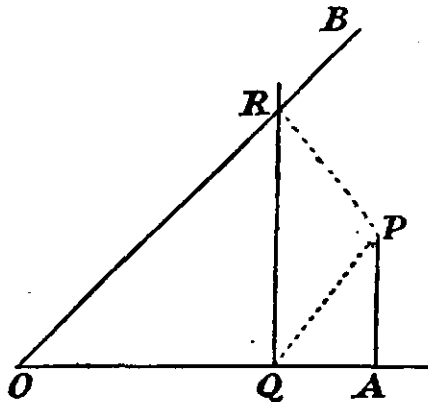
1. The following table gives the scheduled times of a certain train:—

Station A	-	-	-	-	-	depart 5.35
„ B (48 miles from A)	-	-	-	-	-	arrive 6.30
„ B	-	-	-	-	-	depart 6.32
„ C (90 miles from A)	-	-	-	-	-	arrive 7.28

Find the average value of the scheduled speed of running for each section of the journey and for the whole journey. The average speed for the whole journey is the quotient of the distance by the difference between times of departure and arrival.

In an actual journey the train leaves A 5 minutes behind time, arrives at B 10 minutes behind time, stops 3 minutes at B, and arrives at C 4 minutes behind time. Find the average value of the speed of running for each section, and for the whole journey, as before.

2. In the figure the angle between the fixed straight lines OA and OB is 45° , and P is a fixed point such that A , its projection on OA , is distant a from O . Through any point Q in OA , distant x from O , a straight line QR is drawn perpendicular to OA , meeting OB in R . Find an expression for the area of the triangle PQR in terms of a and x .



Taking OA as equal to 4 inches, plot a graph showing how the value of this area varies as Q moves from O to A , and from the graph deduce (1) the position of Q for which the area, PQR , has a maximum value, (2) the positions of Q for which the area, PQR , is equal to 1 square inch.

3. A smooth circular cylinder 120 cm. in diameter lies on the ground and is held fast. A uniform plank 3 metres long and weighing 100 kilograms is placed against the cylinder, and rests with one end on the ground at a distance of 240 cm. from the lowest line of the cylinder.

Draw a figure carefully, representing a metre by the side of a large square in your ruled answer book. Draw the lines of action of (1) the weight of the plank, (2) the pressure of the cylinder on the plank.

Hence find the line of action and the magnitude of the force exerted by the ground on the plank. Find also its vertical and horizontal components.

If the plank, in this position, is just on the point of slipping down against the friction of the ground, find the value of the coefficient of friction, and determine what horizontal force must be applied to the lower end of the plank to make it just begin to slip up on the cylinder.

4. To find the distance of an inaccessible point P from his position at A , a man measures off AB at right angles to AP and 50 metres long. He then goes to a point C in line with A and P and 100 metres from A . From C he walks at right angles to CP until he reaches a point D in line with B and P . The distance CD being 82.6 metres, find the distance AP by drawing to scale.

Also calculate the distance AP .

5. A point moves so that at any time t seconds its co-ordinates x and y are given by $x = \sin t$ and $y = \sin 2t$, the angles being measured in degrees. Find the values of x and y for a number of values of t and draw the curve, using the side of a large square in your answer book as unit.

How long does the point take to describe the curve?

6. A uniform plank AB , 20 feet long and weighing 180 lb. rests horizontally on two supports, one at A and the other at C , 15 feet from A . A rolling weight of 300 lb. can be moved along the plank. Find expressions for the pressures on the two supports when the rolling weight is distant x feet from A .

Plot a graph showing how the pressure on A varies as the rolling weight is moved from A towards B , and find the position of the weight when the plank just lifts off the support A .

MATHEMATICS I.

LABORATORY EXAMINATION.

(One hour.)

The Candidates were examined in batches, each batch having one of the following questions 1, 2, 3, 4.

Your work will be valued according to its accuracy, the completeness of your record of it, and your care of the apparatus. In graphs the axes of co-ordinates and the scales used are to be shown; the scales should be large. Every observation made should be recorded.

1.

Call the three strings knotted together OA , OB , OC . Fix A to a screw-eye near the left-hand end of the top-rail, and a weight of 3 kilos to B , and tie the end C to the hook of a spring balance. Attach a cord to the ring of the balance, and fasten the cord by a clip on one of the vertical pillars so that OC is horizontal. Measure the tension T indicated by the spring balance, and the angle θ which AO makes with the vertical.

Vary the position of OC , and so the value of T , keeping OC always horizontal, and for each position observe the values of T and θ .

The load of 3 kilos should not be changed.

Give a graph co-ordinating T and $\tan \theta$.

Under what circumstances would the tension in OA be a minimum, and what would be the minimum value?

2.

Observe the suspended metal plate, and note the hole by which it is suspended.

Find the centre of gravity of the metal plate.

Suspend the plate in its original position. To the hole at the narrow end fasten a cord, pass the cord over the fixed pulley and attach a weight of 400 grams to its free end.

Find, by equating moments, the weight of the plate.
Sketch the arrangement.

3.

A weight of 1 kilo is attached to each end of a cord passing over two fixed pulleys. You are to add from 100 to 300 grams to the load on one side so that the larger weight can fall in an observed time through a distance to be measured in centimetres. State the actual load used.

Write down the relation between the time, the distance fallen, and the acceleration, and so find the acceleration on the C.G.S. system.

Hence find the velocity of the weights at the instant when the heavier has descended 100 cm., and the total kinetic energy of the two weights.

Find how much energy has been absorbed by friction, etc., when the heavier weight has descended 100 cm. [$g = 981$ cm. per second per second.]

4.

Use the board with the attached pulley as an inclined plane, setting it at an angle of between 20° and 30° with the horizontal. Suspend suitable weights by a cord passing over the pulley and attached to the loaded trolley, so as to cause the trolley to travel slowly up the board when given a slight starting impulse. Observe the pull in the cord.

Also find the pull under which the trolley runs slowly down.

Assume that the average of these pulls is that under which the trolley would remain at rest if it were frictionless, and from this average and the measured inclination of the board, construct a force diagram and find the weight of the loaded trolley.

Also find the resultant thrust of the cord on the pulley.

MATHEMATICS II. PAPER 1.

(Three hours.)

For various data see p. 536. For full credit your work must be immediately intelligible, your diagrams distinctly drawn, and your results given only to the degree of accuracy that is asked for or is justified by the data. The slide rule may be used wherever it allows of sufficient accuracy, and in each case a note of its use should be made.

1. I have two bottles of equal size, one filled with water and another with a liquid called rodinal. I pour out one-tenth of the water and fill up the bottle with rodinal. When the two have been mixed I repeat the process, and do this over and over again till the bottle of rodinal has been emptied. Find the proportion of rodinal to water in the mixture.

Do the same when only one-twentieth of the liquid is poured out each time, and when one-fiftieth is poured out, verifying that the quantity of water ultimately left in the bottle is always less than $1/5$ of the original amount, but becomes more nearly equal to $1/5$ when the amount removed at each operation is made less.

2. Find the slope of the tangent to the hyperbola $x^2 - y^2 = 4$ at the point $\left(\frac{5}{2}, \frac{3}{2}\right)$, and draw the curve and its tangent.

If O is the origin, P any point (x', y') on the curve, prove that the tangent at P makes the same angle with the axis of x that OP makes with

the axis of y . If this tangent meets the lines $y = x$ and $y = -x$ in Q and R , find the co-ordinates of Q and R and the area of the triangle QOR . Verify for the case of the tangent drawn in the first part of the question.

3. An airship P , 1,300 yards distant from the observer O , is observed to be at an angle of elevation 15° above a point of the horizon 33° north of east. Another airship Q , 1,700 yards distant from O , has an elevation of 20° above a point on the horizon 75° south of east. Draw a plan on the scale of 1 centimetre to 100 yards showing the projections of the airships on a horizontal plane. If OE and ON , drawn respectively due east and north, and OZ drawn vertically upwards, are taken as axes of co-ordinates xyz , find the co-ordinates of P and Q , and the equation of the plane POQ , expressing z in terms of x and y . (The coefficients in the equations need only be given to two significant figures.)

4. A rigid body is in equilibrium under four forces acting along the sides of a plane quadrilateral. Show that the resultant of the forces in two adjacent sides must be along the diagonal which is concurrent with them.

Show that a force acting along one side DA of a plane quadrilateral $ABCD$ can be balanced, as regards effect on a rigid body, by forces acting along the other sides AB , BC , CD , in one and only one way, and examine whether the same is possible in the case of a triangle. If $AB = 5$, $BC = 4$, $CD = 2$ in., and the angle $ABC =$ the angle $BCD = 90^\circ$, find the forces along AB , BC , CD , which will balance a force of 10 lbs. along DA .

5. A rectangular tank is filled with water and placed with very nearly one half projecting beyond the edge of a table. I have a body attached to a string which I can raise or lower into the water. Discuss under what circumstances the tank will overturn when the body is lowered into the projecting part, examining what differences, if any, it makes whether the body is lowered quickly or slowly, and whether the body is heavier or lighter than water, and in the former case whether the body is lowered till it touches the bottom.

In the second place, suppose the body has been lowered into the part that rests on the table, and examine whether, and if so, when the tank can be overturned by raising the body out of the water. You are advised to classify the different cases which arise under separate headings.

MATHEMATICS II. PAPER 2.

(Three hours.)

For various data see p. 536. For full credit your work must be immediately intelligible, and your results given only to the degree of accuracy that is asked for or is justified by the data. The slide rule may be used wherever it allows of sufficient accuracy, and in each case a note of its use should be made.

1. Draw the curve $y = x(4 - x^2)$ using as unit the side of a small square in the answer book.

Suppose (a, b) to be the co-ordinates of a point P on the curve, and $(a + h, b + k)$ of a neighbouring point Q , so that h and k are small quantities. Express b in terms of a , and k in terms of a and h . Give an expression for the tangent of the angle which PQ makes with the x -axis, and deduce an expression for the tangent of the angle which the tangent at P makes with the x -axis.

Give in degrees the inclination to the x -axis of the tangent at the point where $x = 0.3$. Find the positive abscissa for which the inclination is zero, and the corresponding ordinate.

Draw the part of the curve from $x = 0$ to $x = 2$ on a large scale, using the side of a large square as unit.

By counting squares find the area A included between the x -axis and this part of the curve. Write down an expression for the area included

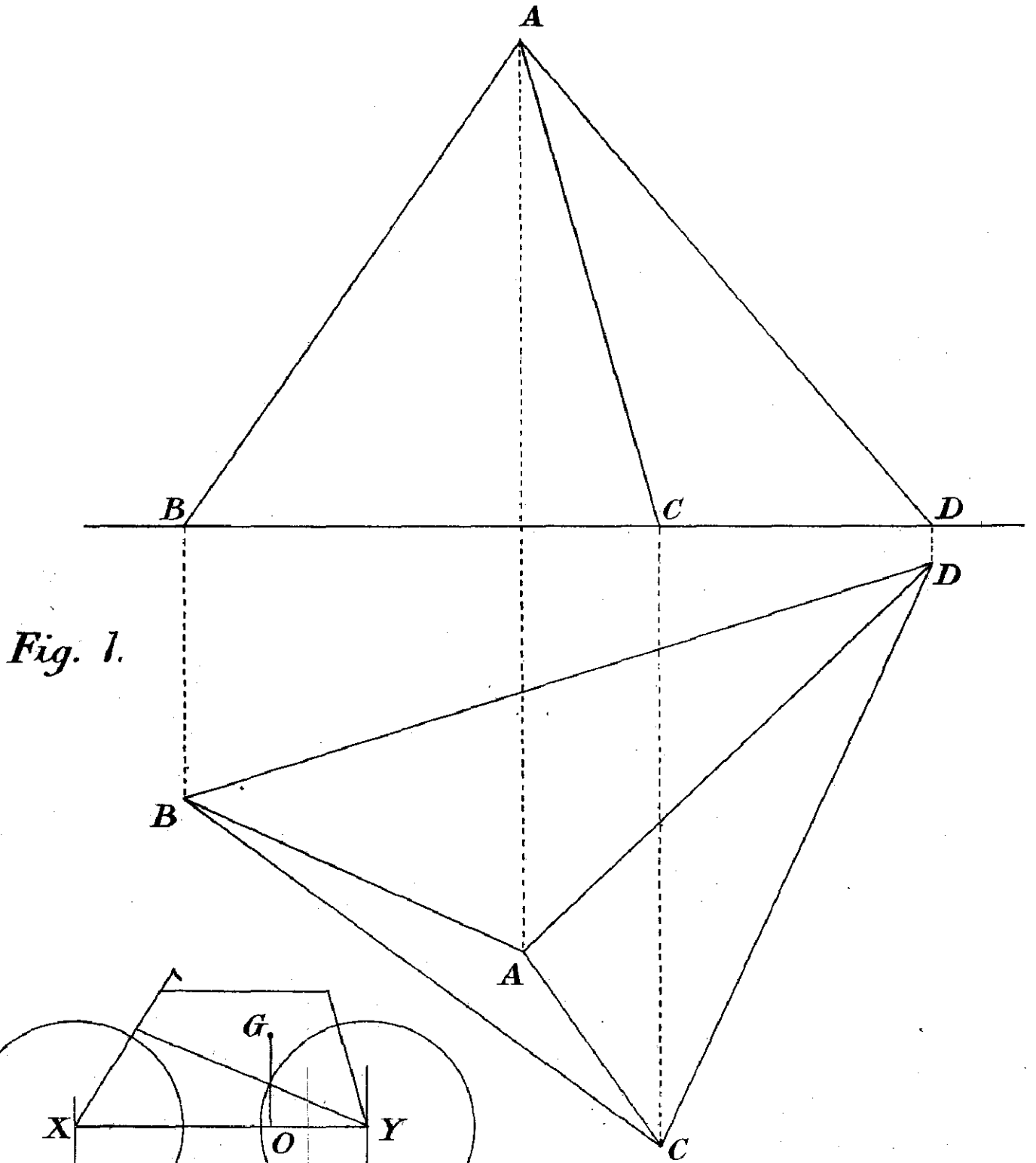


Fig. 1.

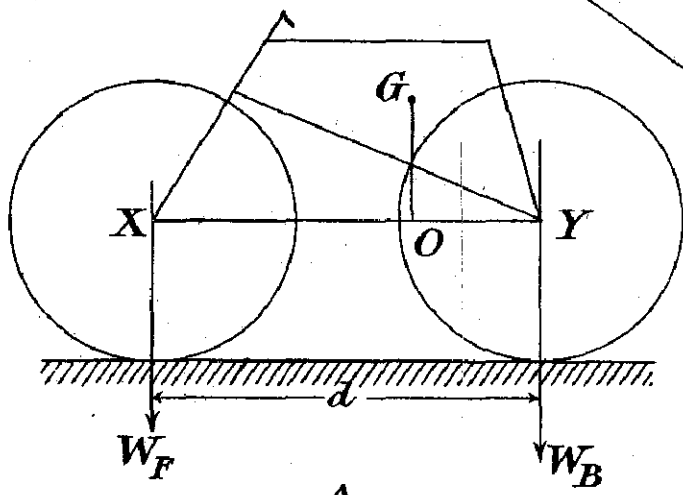


Fig. 2.

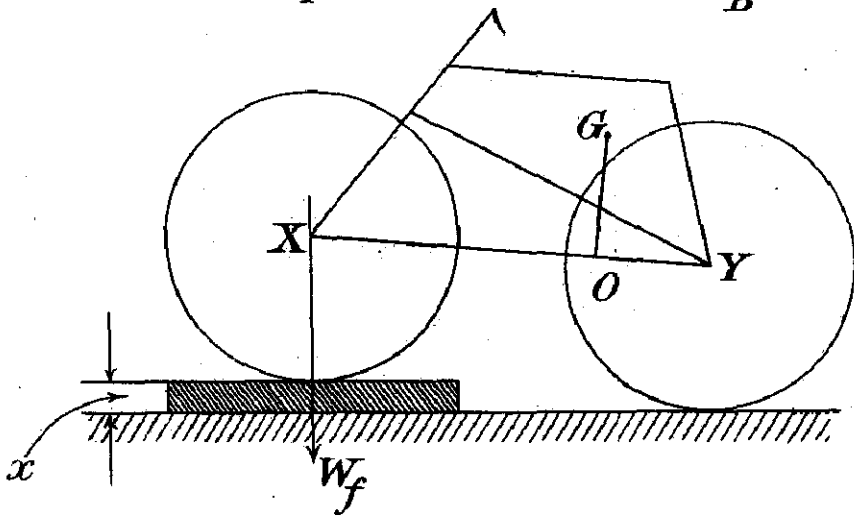
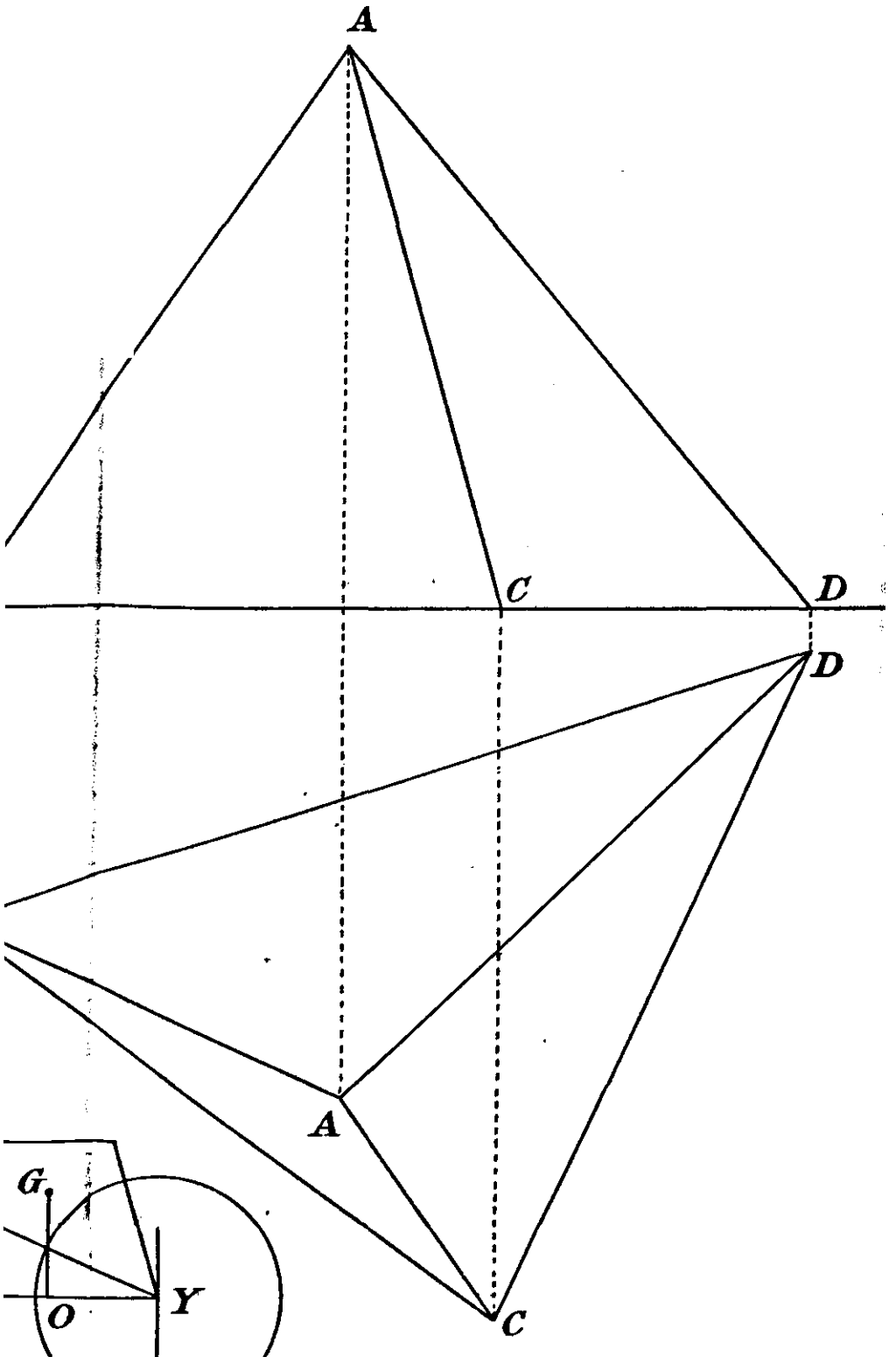


Fig. 3.



between the two ordinates at $x = a$ and $x = a + h$, h being a small quantity. Deduce an expression for the area A as an integral and evaluate it.

2. A pulley has all its mass, M lbs., in its rim, which has a radius of a ft. Give an expression for the kinetic energy of a particle of the pulley and of the whole pulley when it is revolving at ω radians per second.

Two masses, P lbs. and Q lbs., of which P is the greater, hang at the ends of a string, which passes over the pulley and moves with it (without slipping). Give an expression for the kinetic energy of the whole system.

From the energy equation, or otherwise, find an expression for the speed of the mass P when it has fallen a distance S from rest, friction being negligible.

If the pulley has a radius of 5 inches, and a mass of 10 oz., and the masses on the string are 17 oz. and 16 oz., how many revolutions per second is the pulley making when P has fallen 6 ft.

Being given the relation $v^2 = ks$ connecting the velocity and distance traversed by a body moving in a straight line, deduce (by the calculus or otherwise) an expression for the acceleration.

3. Fig. 1 (see opposite page) shows a triangular pyramid $ABCD$ in plan and elevation. By pricking through, or otherwise, transfer the figure to your book. Draw the three slant faces of the pyramid and find the area of each in sq. cm. Find the volume of the pyramid in c.cm.

Find the inclination of the slant face ABC to the horizontal. The pyramid being turned about the edge BC until the vertex A lies in the horizontal plane, show the pyramid in plan and elevation.

4. A bicycle weighs 34.3 lb., and the centres of its wheels are 42.5 in. apart. When the whole weight is borne by the wheels and the front wheel alone is placed on the platform of a weighing machine, the machine registers a weight of 15 lb. What are the distances of the vertical line through the centre of gravity from the centres of the two wheels?

Now suppose the front wheel is 5 in. higher than the back, and that in this case it registers 14.3 lbs. on the weighing machine, find the perpendicular distance of the centre of gravity from the line joining the centres of the wheels.

If the diagrams (Figs. 2 and 3, see opposite page) represent the bicycle in the two positions, the pressures on the ground and dimensions being as indicated, G being the centre of gravity and GO perpendicular to XY the line of centres, give expressions for XO and GO in terms of W_F , W , W_B , d and α .

MATHEMATICS II.

LABORATORY EXAMINATION.

(One and a half hours.)

The Candidates were examined in batches, each batch having one of the following questions, 1, 2.

You are advised to read through the paper before beginning to work.

Your work will be valued according to its accuracy, the completeness of your record of it, your care of the apparatus, and the suitability of the methods you adopt.

Should you find the time in the laboratory insufficient for the completion of your record and your calculations, you will be allowed half an hour more in another room.

1.

Clamp a metre stick horizontally to the table with 10 cm of its length on the table and 90 cm. projecting free. Fix another metre stick vertically in such a position as to enable accurate readings to be made of the deflections of the free end of the horizontal stick.

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Tie a weight on a loop of string that can be slid to any part of the horizontal stick between the 25 cm. and the 100 cm. mark.

Place the weight in turn at the graduations 25, 40, 55, 70, 85, 100, and observe with the greatest possible accuracy the deflection of the end of the stick (the 100 mark) produced by the weight at each position.

Tabulate the observations and make a graph, using as abscissas the distances of the weight from the edge of the table, and as ordinates the corresponding deflections of the end of the stick.

The graph will be of the form $y = Ax^2 + Bx^3$. Determine A and B by substituting in this equation the co-ordinates of two points of the graph.

2.

Fix two lengths of wire horizontally between the table uprights, one about 50 cm. above the table and the other a little higher. The lower one is to be deflected by weights; the upper wire merely affords a zero from which to measure the deflections of the lower, and should be loaded slightly by a middle fixed weight.

Apply loads centrally to the lower wire so as to get deflections of between 1 and 10 cm. In all about 6 values of load should be used, and observations of the deflections should be made both while the load is being increased and while it is being decreased.

From the observations calculate the tension in the wire and the stretch of the wire at each observation. Tabulate these results, and from them obtain a mean value of Young's modulus of elasticity.

THE ORGANISATION OF THE TEACHING OF MATHEMATICS IN PUBLIC SECONDARY SCHOOLS FOR GIRLS.

INTRODUCTORY.

Rather more than half a century has elapsed since Professor Maurice, speaking on the occasion of the opening of Queen's College, London, uttered his often-quoted belief that women students were not likely to advance far in Mathematics.

Mathematics was, however, introduced into the curriculum of Queen's College, and of Bedford College, shortly afterwards founded, because, to quote from the same speech, "we believe " that if they learn really what they do learn, they will have got, " not what is dangerous, but what is safe " ; or, as Professor Hudson worded it thirty years later, "because it (algebra) is a " part of mathematical truth and no one ought to be wholly " alien from that important department of human knowledge."

Mathematics was thus, in 1853, formally recognised as a subject not altogether suited to the capacity of women, but one of which the more highly educated women should not be wholly ignorant. We find, however, that in the curriculum of the schools founded in the same decade by the two famous Queen's College students, Miss Beale and Miss Buss, the only branch of Mathematics studied was arithmetic. The teaching of the subject in the schools of that time was in the hands of poorly-equipped, but keen, teachers who gave a remarkable impetus to arithmetical work in girls' schools between the years 1862 and 1865. In 1862, sixty-eight per cent. of the girls examined informally by the Cambridge Local Examiners failed in arithmetic ; in 1865, three years later, three times the number of candidates were presented, this time formally, for the same examination, and seventy-one per cent. passed, the remaining twenty-nine per cent. either failing in or not entering for that subject.

In 1867, the report of the Schools' Inquiry Commission (Lord Taunton) was published, condemning the superficiality and want of thoroughness of the education in girls' schools. The immediate result was the foundation, four years later, of the " National Union for improving the Education of Women of all Classes," and the first work of this Union was the establishment, in the following year, of the Girls' Public Day School Company. For the first time Mathematics became one of the recognised subjects of the curriculum of girls' schools, beginning with the High Schools, which were rapidly started all over the kingdom

by this Company ; it was realised that this branch of science would best correct the faults condemned in the Report of the Commission of 1865-7, and its inclusion as a school subject was strongly urged by the famous inspectors of the sixties and early seventies. Sir Joshua (then Mr.) Fitch wrote in 1875 : " Girls are more in danger of falling into desultory habits of reading and of thought, and therefore need more the bracing and the mental precision furnished by studies of another kind. For this reason I should like to see some of the time now devoted to ancient history rescued for the study of *science* in some form or other. Whether it would be better to give greater prominence to ' pure science ' including advanced arithmetic, algebra, and geometry, or to some branch of applied or experimental science, I should be disposed to leave for determination. But it is clear that for the formation of habits of exactness and for the purpose of bringing the intellectual character of the pupils under the best possible discipline it is of the highest importance to give greater prominence to scientific teaching."^{*}

Mathematics has thus been recognised as a subject for study in girls' schools for some thirty or forty years, but at first the difficulties were great owing to the fact that the supply of well-qualified teachers was wholly inadequate. The Honours Examinations of the Universities were not open to women until 1881, when Cambridge formally admitted students of Newnham and Girton Colleges to the Tripos Examinations, and, in consequence, the equipment of those teaching before that time was in the main meagre: a smattering of arithmetic and algebra, with little knowledge of principles, and perhaps two books of Euclid learnt more or less by rote. In the twenty years following the opening of the Tripos Examination to women, 250 students took honours in Mathematics at Cambridge alone, and it was with the entrance into the teaching profession of this band of highly qualified women that the organisation of mathematical teaching may be said to have begun.

At the present day, Mathematics is taught—with remarkable uniformity of matter and method—in all public secondary schools for girls and in all the best private schools. The two factors which have mainly determined the special form of the organisation of mathematical teaching in girls' schools in England are: (1) the training and qualifications of the mistresses teaching Mathematics, and (2) the syllabuses of the various "outside" examinations taken by school-girls. The latter have made for uniformity among the various schools, as the same or corresponding examinations are taken at corresponding ages and stages in school. With regard to the former, the fact that up to the present a large proportion of the Senior

* Report on the Royal School for the Daughters of Officers of the Army, Bath, 1875.

Mathematical Mistresses have taken Honours Examinations at the Universities, that nearly all hold some degree or equivalent, but that an extremely small proportion have had any training in the practice of their profession, has made the curriculum in Mathematics at girls' schools, as well as at boys', a preparation for the future mathematical student at the Universities, rather than an education for the average pupil. That the pedagogic, as well as the academic, side of the question needs recognition, is shown by the Report on the Teaching of Algebra and Trigonometry issued by the Mathematical Association Committee,* and by the recent Circular No. 711 of the Board of Education on "The Teaching of Geometry and Graphic Algebra in Secondary Schools."†

CLASSIFICATION OF SCHOOLS.

For the purpose of this enquiry circulars were sent to the Head Mistresses of 275 schools, all members of the Association of Head Mistresses; the schools are, therefore, all recognised as "Public Secondary Schools for Girls," as this is a condition of membership of the Association. To these circulars 180 sets of answers were received, with much useful information; thus the facts given below represent information obtained from 66 per cent. of the chief secondary schools for girls in England.

The classification of these schools is roughly as follows:—

	Years of age.
Preparatory, or Kindergarten, containing	—
children of from	5 to 7 or 8
Form I. do. -	7 or 8 to 10
Form II. do. -	10 to 12
Form III. girls of from	12 to 14
Form IV. do. -	14 to 15½
Form V. do. -	15½ to 17
Form VI. do. -	17 to 19

In the larger schools there are two, or even three, Fourth or Fifth Forms as the case may be. For instance, in Notting Hill High School there are two Second Forms (Lower Second and Upper Second), one Third Form, two Fourth Forms, two Fifth Forms (the Lower Fifth being the Senior Cambridge Form,‡ the Upper Fifth the Matriculation Form§), two Sixth Forms (the Lower Sixth working for London University Intermediate Arts or Science, and the Upper Sixth consisting of two or three girls doing Scholarship work). In a few schools of large size

* The Teaching of Elementary Algebra and Numerical Trigonometry. Report of the Mathematical Association Committee. (London: G. Bell and Sons. 1911. 3s.)

† Board of Education. Circular 711. March 1909. (London: Wyman and Sons. 1909.)

‡ Form in which the Cambridge Senior Local Examination is taken.

§ Form in which the London University Matriculation is taken.

there are as many as three Fourth Forms, while in the smaller schools there is often only one. Reclassification takes place, as a rule, once a year, in September, and in the larger schools all but the slower girls obtain annual "removes"; in the smaller schools the smaller number of Forms renders a two years' stay in some Form or Forms inevitable, even for the brighter girls. In a few schools the system of classification is somewhat different, albeit with the same nomenclature, and is known as the "parallel" system. At every stage there are two Forms, IV^a and IV^b, &c., but these, instead of being consecutive, are in certain subjects parallel, and differ in certain others. Thus a girl passing out of Form III, instead of going first into Form IV^b and then into IV^a, would be definitely on either the "A-side" or the "B-side," and her progress up the school would be either III^b, IV^b, V^b, or III^a, IV^a, V^a. In schools of this type, as a rule, the girls on the "A-side" work towards a University entrance examination, and learn two languages (generally French and Latin), Mathematics, and Physics or Chemistry; while girls on the "B-side" take modern languages (not Latin), more English, probably no Mathematics except arithmetic and that of a practical nature, Chemistry and Physics with special reference to Domestic Science work. Other schools, notably Cheltenham Ladies' College, have a special classification of their own. A fair number of schools, notably some of the newer County Secondary Schools, have no forms below Form III, at which stage children from the elementary schools begin secondary work.

In the majority of the schools each Form is under the special care of a "Form-mistress," who teaches in her own Form certain "Form-subjects," while taking in other Forms the special subject which is her strong point. For example, a mathematical specialist may be Mistress of Form IV; she would probably take with her own form Mathematics and Scripture, or Mathematics and Geography, while teaching Mathematics in several other forms and having the general supervision of the mathematical work throughout the school.

MATHEMATICAL STAFF.

In the 180 schools from which replies to the circular have been received, there are 681 mistresses occupied either for full time or part time in the teaching of Mathematics. The Senior Mathematical Mistress takes, as a rule, the upper forms, often the beginners in Mathematics (this is most useful as regards future work), and any scholarship coaching, and is responsible for the general supervision of the mathematical teaching throughout the school, the arrangement of syllabuses, and the unification of method. With regard to qualification, 34 per cent. of the Senior Mistresses have taken the Cambridge Mathematical Tripos, 31 per cent. are Pass Graduates in Arts

or Science of London University, 10 per cent. hold Honours Certificates of Oxford University, 5 per cent. are Honours Graduates of London University, and the remaining 20 per cent. are Pass Graduates of the Scotch, Irish, and newer English Universities, or have some lesser academic qualification. An extraordinarily small proportion have received any training in the practice of their profession.

STANDARD ATTAINED.

The standard attained is best measured by the highest examinations taken in schools, exclusive of scholarship work, which is purely individual. In all but 2 per cent. of the schools from which returns have been received the standard attained is that recognised for entrance to a University, or a higher standard. Roughly speaking, the syllabus for Matriculation, or entrance to a University, includes: (1) Arithmetic generally; (2) Algebra, including simple and quadratic equations with one or two unknowns, ratio and proportion, elementary laws of indices, and the progressions; (3) the substance of Euclid I—IV treated by modern methods. This is not in itself a high standard, and in the better schools the Matriculation examination or an equivalent is taken in the Upper Fifth Form, while more advanced work is taken in the Sixth Form. Eighty-four per cent. of the schools attain a higher standard than that required for Matriculation, including logarithms and the binomial theorem, the substance of Euclid VI. and XI. 1–21, and some elementary trigonometry. Of these, in considerably more than half, the best girls reach the standard required for the First Degree examination of the University of London (Intermediate Arts or Science), while two schools prepare successfully for the London B.A. Degree (External). In these latter schools elementary trigonometry is studied for at least two years, while in some schools Applied Mathematics (statics, dynamics, and hydrostatics) is studied up to the standard required for the Intermediate Arts or Science examination, and such other additional mathematical subjects as co-ordinate geography, modern plane geometry, conic sections, and occasionally the elements of the calculus, are begun.

GENERAL ORGANISATION.

The stage at which Mathematics other than arithmetic is begun differs somewhat in the different schools. In 62 per cent. of the schools the Third Form is the starting point for the formal teaching of Mathematics, while about an equal number of schools start this work in the Second and Fourth Forms, and a very few in the First Form. In about one-third of the schools geometry (mainly practical) is started one year earlier than algebra, while in many the formal mathematical work is

preceded by a course of practical arithmetic in the First Form, including possibly cardboard modelling and plasticine-work as well as measuring, weighing, &c. Arithmetic, algebra, and geometry are then taught simultaneously throughout the school, not consecutively as in America, and recently much has been done by the foremost teachers of the subject to break down the barriers between the three divisions, and to show them all as parts of the same science—algebra as generalised arithmetic and “written geometry,” geometry as “pictorial algebra.”* It is therefore increasingly the rule for the same teacher to take the three branches of the subject with a Form.

The amount of time given to the subject averages about three hours a week in school with $1\frac{1}{2}$ hours' homework below the Fourth Form; above that 4 hours in school with 2 to 3 hours' homework.

Mathematics is often taught as a “form subject,” and is not infrequently one of the chief subjects used as a basis of classification, but in a considerable number of schools the basis is languages, and Mathematics is taught in “sets.” For example, Forms IV^a, IV^b, IV^c in a certain school have their mathematical lessons simultaneously, and are cross-divided for that subject, the weaker girls in the three forms being taught together in one class, the stronger girls in another. There is no subject in which girls differ more in ability than they do in Mathematics. This system is only possible where several teachers on the staff are well qualified in the same subject; it is especially useful to obviate the difficulty caused by new girls entering from elementary or private schools with less mathematical knowledge than is expected of girls of their age.

NOTES ON METHODS OF TEACHING THE VARIOUS BRANCHES OF MATHEMATICS.†

(1) *Arithmetic.*—Arithmetic is taught from the lowest forms, and is carried on at any rate as far as the Lower Fifth Form. A few schools drop it in the Upper Fifth, but the majority retain it until the Sixth Form. This protracted teaching of the subject is, in part, due to the fact that the Examiners of the Oxford and Cambridge Schools Examination Board do not at present see their way to setting a combined arithmetic and algebra paper for the Mathematical Division of the Higher Certificate, instead of devoting a whole paper to arithmetic; it has been suggested that they should at the same time raise the standard of the algebra required for a Pass and make the arithmetic questions more mathematical and less commercial in character. The whole question of commercial

* This must of course be understood to apply to only a very limited section of Geometry.

† See also Syllabuses in Appendix.

arithmetic is one which is being freely discussed at the present time, many teachers being of opinion that such purely commercial parts of arithmetic as equation of payments, exchange, "true" discount, compound interest (beyond the ability to find interest), complicated problems in stocks and shares, ought to be omitted from the school programme.

Another reason for the great number of years spent in the study of arithmetic in English schools is the cumbersomeness of our weights and measures. I think it not an over-statement to say that two years' teaching could be saved if we were to adopt the metric system of weights and measures, which is at present taught in our schools in addition to our own. In America, the Committee of Fifteen have declared it as their belief that "with the right methods and a wise use of time in preparing the arithmetic lesson in and out of school, five years are sufficient for the study of mere arithmetic—the five years beginning with the second school year and ending with the sixth; and that the seventh and eighth years should be given to the algebraic method of dealing with problems." In England the arithmetic course extends through ten or twelve years.

Great advance has been made in recent years in the methods of teaching decimal fractions. Until recently the study of decimals was postponed until after the study of more or less complex vulgar fractions, instead of in the logical place as an extension of the ordinary notation. It is now increasingly frequent to begin work with decimals as low as Form II, or even Form I, in which case the metric ruler is found to be of great help in conducing towards a clear understanding of the nature of a decimal. With the best teachers there is, of course, no rigorous exclusion of the notation of vulgar fractions, as with some extremists both in England and in Germany after the idea gained currency that decimal fractions should precede vulgar fractions. $\frac{1}{10}$ is understood by the pupil before $\cdot 1$, $\frac{1}{5}$ before $\cdot 5$, but the addition of decimals precedes the addition of vulgar fractions. In at least 50 schools, and probably more, decimals are taught first. This is of great advantage when practical science work is begun, usually in Form III. Another way in which arithmetical work is now being made to help work in science is by the earlier introduction of contracted methods giving approximate results, and approximation generally. It is, however, a matter of opinion whether it is advisable to spend much time in evaluating complicated multiplications and divisions of rational and surd quantities by contracted arithmetical methods, as, for example, in some of the questions set in the London Matriculation examination, or whether it would be better to introduce the use of 4-figure logarithms at a much earlier stage. In some schools, for instance the George Watson's Ladies' College, Edinburgh, logarithms are used by girls of 15½ and 16 with excellent

results. There is no reason why they should not be introduced as soon as some algebraical work has been done on the law of indices. Here, and wherever possible in arithmetic, work in algebra should aid work in arithmetic, and algebraic methods should be largely used in the solution of arithmetical problems.

As an introduction to geometry many schools (for example, Sheffield High School) have in Form I a practical arithmetic course, including mensuration, with an introduction to metric system, paper-folding, cardboard modelling, and work with plasticine. Several text-books have now been written on courses of this kind, and various educational publishers issue suitable sets of apparatus.

(2) *Algebra*.—The Report of the Mathematical Association Committee on the Teaching of Algebra and Trigonometry,^{*} and the suggestions of the Board of Education on "The Teaching of Geometry and Graphic Algebra in Secondary Schools," give a good idea of the tendencies of the present day in the teaching of algebra in English schools. The former lays insistence upon the educative value of algebra and aims at suggesting an elementary course in Mathematics which will "avoid sacrificing the interests of the ordinary pupil, while affording a suitable foundation for more advanced work." It has been felt that time has been spent upon topics of algebra which have no value or interest save for the future mathematician, *e.g.*, factors and fractions of certain unusual types, surds (except the evaluation of arithmetical surds), imaginaries, theory of quadratics; that these are best omitted from the course leading up to "a pass examination designed to test general culture"; and that the time gained thereby will be welcomed by the many teachers who "wish for opportunity to develop with their pupils mathematical ideas that they feel to be of better educative value—ideas drawn from mechanics, mensuration, solid geometry, infinitesimal calculus, and, more especially, from numerical trigonometry. Custom, represented by public examinations, has at present the effect of withholding that opportunity." Examinations for schools have been on too purely academic lines, and the work has followed closely the order of the text books framed to suit the syllabuses of these examinations. Many girls who would do fairly good work in a non-specialist algebra course, have been unduly disheartened by the difficulty of certain parts of the algebra taught them in accordance with these regulations, and would have been better employed in work on the earlier part of numerical trigonometry.

The question of graphs is one on which teachers differ. All possibly realise the importance of a certain amount of graphical work, but whereas some begin algebra by graphs, others postpone the use of graphical methods until much later. The chief educative value is that in no other way can the

* Published by George Bell and Sons, 1911.

average pupil realise so easily the idea of functionality and continuity. The Board of Education Circular on this aspect of algebra, and the publications of the Mathematical Association, have done much to stimulate interest in graphic methods which are now largely used in girls' schools. The traditional order of presentation of the subject of algebra in the first years is also in process of change. The equation is now frequently taught almost at the beginning, followed by the "first four rules." At the same time symbolic expression is taught, leading on almost immediately to simple equation problems.

An advance has been made in many text-books in the type of question set for "substitution." The artificial expressions formerly found have given place to physical formulæ such as $s = ut + \frac{1}{2}ft^2$, or some definite function of a variable, say x , whose value is to be calculated successively for different values of x . If these values of x proceed by small (equal) intervals and the results are graphed, the first steps have been taken in giving the pupil some idea of functionality, and keen interest is at the same time aroused. In the same way, at a slightly later stage some teachers find it possible to illustrate and verify geometrical measurements by substitution in algebraic formulæ: these exercises are always attended with interest, and serve to bring algebra and geometry into close relationship.

(3) *Geometry*.—The movement in favour of abandoning strictly Euclidean methods in geometry did not begin in England until about fifteen years ago. Perhaps too much insistence was at first laid upon the preliminary year of practical geometry and, to quote the same Circular of the Board of Education, "too frequently this stage is unduly protracted and the results achieved are not sufficiently definite, and, moreover, the theoretical work is developed on an independent basis, which might have been avoided were there proper correlation between theory and practice." In a large number of schools the children spend the first year of geometry (usually in Form III) mainly on practical work leading up to the discovery of the more important geometrical truths, finishing the year's work by formal proofs of the congruency theorems and a large number of exercises. In the following year, when they have, as a rule, two lessons of $\frac{3}{4}$ -hour each instead of one, the substance of the main propositions of Euclid I. is taken. Following on the suggestions of the Board of Education, however, in some schools the experiment is being tried of starting theoretical geometry—after the children have gained "familiarity with and clearness of perception of the fundamental geometrical concepts"—by the establishment of the "fundamental mental propositions" (Euclid I., 13–15, 27–29, and 32, and the congruency theorems Euclid I., 4, 8, 26) and the working of many exercises on them. The Euclidean proofs are not given; they are "out of place with beginners, and attention must be

“concentrated not on formal proofs but on vivid presentation, and accurate, firm apprehension of the propositions themselves.” Throughout the course theoretical and practical work go hand in hand. It is not possible to state at present what effect this will have on the mathematical knowledge of the girls, as the work is still in the experimental stage.

(4) *Trigonometry*.—In a few schools trigonometry is begun in the Upper Fifth Form, but in the majority it is deferred until the Sixth Form or not taken at all. The usual school course includes the logarithmic solution of triangles, and is practically the syllabus of the Intermediate Arts examination. Of late, numerical trigonometry has formed the bulk of the work.

(5) *Higher Work*.—In a number of schools Applied Mathematics is taught to a few girls up to the standard of the Intermediate Arts and Science examinations of London University. In perhaps the majority of these the work is mainly theoretical, but others, such as South Hampstead High School, are fully equipped with apparatus for successful practical work, and in many cases hydrostatics is taught in the laboratory.

It is a matter for regret that co-ordinate geometry is not more universally taught, but is as a rule—together with modern plane geometry, conic sections, the calculus—reserved for the scholarship candidate.

PRACTICAL WORK.

The practical work done in the lower forms as a preparation for geometry has been already mentioned, as well as practical work in Applied Mathematics in the higher forms. At Kensington High School, and a few other schools, geometry is not started until Form IV, but is preceded in Form III by a course of “Boole Curve-sewing,”* the aim being “to stimulate the mathematical imagination prior to the formal treatment given to the subject in first-year geometry.” It introduces the children informally to the idea of loci, tangency, envelopes, limiting cases, and is found to be of much value in later geometrical work.

CORRELATION OF MATHEMATICS WITH OTHER SUBJECTS.

The subject with which Mathematics ought to be most closely correlated is, of course, Physics, and valuable suggestions as to how this may best be effected are contained in a Report on “The Correlation of Mathematical and Science Teaching,” by a Joint Committee of the Mathematical Association and the Association of Public School Science Masters.† A number of

* See “A Rhythmic Approach to Mathematics,” by Edith L. Somervell. (London: G. Philip and Son, 1906.)

† Published by George Bell and Sons, 1909. Price 6d.

teachers in the various girls' schools are in sympathy with these suggestions and are endeavouring to carry them out more or less fully. Others are working on their own lines with the same end in view, and much care is taken to avoid overlapping of syllabuses, and to frame the arithmetic syllabus of the lower forms in such manner that the science teacher may have as much help as possible. This is done, as has already been stated, by means of an early practical course in arithmetic, by the early introduction of the decimal notation, by the use of approximation and contracted methods, and in many cases by putting Elementary Physics teaching of the first year into the hands of the mathematical mistress of the form.

Another subject with which the mathematical work of some schools is more or less closely correlated is geography. Drawing to scale connects it with geometry, barometer-charts and other charts with graphical algebra, and a certain amount of mensuration is done in connection with map-work.

EXAMINATIONS.

The number of examinations for which girls may be entered is so great that it is difficult to give any clear idea of them. They may be divided roughly into three kinds: (1) School Examinations (taken at certain intervals in the school course; (2) Examinations qualifying for entrance to the Universities; and (3) Scholarships, and other more advanced examinations.

(1) The chief school examinations are: the Oxford Local Examinations, Senior, Junior, and Preliminary; the Cambridge Local Examinations, Senior, Junior, and Preliminary; and the examinations conducted by the Oxford and Cambridge Schools Examination Board, commonly called the "Joint Board."* The Oxford and Cambridge Local Examinations, conducted by University Examiners, are "outside" examinations: candidates offer a number of subjects, chosen according to rule, are examined at a certain "centre" in the town or district, and receive a certificate with the seal of the University as the reward of passing. The Preliminary Examination is taken by candidates under 14, the Junior by those under 16, and the Senior by those under 19 years of age. The Preliminary and Junior Local Examinations are taken now by far fewer candidates than the Senior, as the tendency is to restrict, in the interests of health, the strain of outside examinations to the later years of school life. These examinations, which have done incalculable good in raising the standard of efficiency in girls' schools, seem now to be giving place to such tests as those of the Oxford and Cambridge Schools Examination Board. This Board examines, by means of either general papers, or papers specially prepared

* To this list should be added the more recently instituted School and Leaving Examinations of London University.

for the school in question, the work of the year in any subject and in any form or forms presented for examination. The examinations are conducted in the school, and may be marked either by the examiners or by the school staff. They are thus examinations which tend to standardise the efficiency of the school without causing in the candidates the intense nervous excitement, so prejudicial to health, which often accompanies "outside" examinations. In addition, the Board conducts a Lower Certificate and a Higher Certificate Examination open to candidates of about 16 and 18, and corresponding fairly accurately with the Junior and Senior Local Examination. With regard to the mathematical papers of the Joint Board in arithmetic, it is generally felt by teachers of the subject that the examiners have not fully realised the changes in recent arithmetical teaching noted above, and a more extended syllabus in algebra on the lines of the Mathematical Association Report would be welcomed. These examinations, Oxford Senior Local, Cambridge Senior Local, and Higher Certificate, admit, under certain conditions, to some of the Universities.

(2) The principal examinations qualifying for entrance to the Universities are: the Matriculation Examination of London University; Responsions admitting to the Degree courses of Oxford University; the Previous Examination admitting to those of Cambridge University; the Matriculation Examination conducted by the Joint Board of the Universities of Manchester, Liverpool, Leeds and Sheffield; the Matriculation Examinations of Bristol University, Birmingham University, and so on. In all these, Mathematics is one of the compulsory subjects. The standard in the Matriculation Examinations is that stated above; the examination consists, as a rule, of two three-hour papers, one on algebra and arithmetic, the other on geometry. No trigonometry is at present required, as for the Scotch Universities; otherwise the examinations are, on the whole, on sound modern lines. Responsions and the Previous Examination are on a somewhat lower standard.

(3) The scholarship, and other more advanced examinations, are many, and differ considerably in standard. The scholarships may be either those offered for competition by a County, Town or City Council, or the "open" scholarships offered by the Women's University Colleges, or the newer Universities. The papers set in the former are on a distinctly lower standard than those set in the latter, a standard approximating more or less closely to that of the Matriculation examinations and differing in different localities; as an example, the Intermediate Scholarships of the London County Council are now offered on the result of the London Matriculation Examination, but most other Councils set their own papers. The standard of the "open" University Scholarships is considerably higher, and the competition is very great. For some Universities and Colleges (*e.g.*, Bristol University, Bedford and Westfield Colleges) the

standard is very approximately that of the London Intermediate Arts Examination and includes (1) Algebra up to the binomial theorem, (2) the substance of Euclid I.-IV., VI. and XI. 1-21 with mensuration of solids, (3) Trigonometry to the solution of triangles, and (4) either Elementary Coordinate Geometry or Elementary Calculus (Bedford College) or Applied Mathematics (Westfield). Another group of Colleges, including Newnham College (University of Cambridge), the Royal Holloway College (University of London), and the Oxford Colleges, have a more extended syllabus, including Analytical and Geometrical Conics, Applied Mathematics, and somewhat more Algebra and Geometry. The Girton College (University of Cambridge) Scholarship Examination is the most difficult of all, and competition is very severe: in addition to much wider syllabuses in all the above-mentioned branches of Mathematics, candidates are also examined in the Elements of Differential and Integral Calculus. In this connexion it is considered a matter for regret by many teachers that elementary study of the calculus does not precede the harder parts of the branches of Mathematics now studied first. Scholarships are also awarded by the London Inter-Collegiate Scholarships Board, some of these scholarships being open to men and women on equal terms, and the examination being a difficult one; and others on the result of the Cambridge Higher Local Examination, the papers for the Mathematical Group (Group C) being much the same standard as those for the scholarship examinations of the majority of the Colleges. Candidates for any of the open scholarships are examined as a rule in one subject on an advanced standard (for example, Mathematics), but have either to take additional pass papers in other subjects or to claim exemption from these on account of having passed such an examination as the London Matriculation. Candidates in Mathematics are, at many Colleges, expected to take, in addition to their mathematical papers, a language translation paper, chiefly with a view to testing their ability to work from German and French books if necessary during their Honours course at the University.

Among higher examinations taken by school girls should also be reckoned the Intermediate Arts or Science (First B.A. or B.Sc.) Examinations of London University. The standard in Mathematics has been already defined; four subjects have to be taken in addition to Mathematics by Arts candidates, three by Science candidates.

It will be seen from the above description of a selected number of examinations for which girls are prepared that the need of the present time is for a simplification of the whole examination system. In some schools all the following examinations are taken: (1) the whole school is examined annually in Arithmetic by the Oxford and Cambridge Schools Examinations Board, and in alternate years in Algebra and Geometry; (2) the Cambridge Senior Local Examination is taken in the Lower

Fifth Form by such girls as are finishing their school course at the age of about 16; (3) the London Matriculation is taken in the Upper Fifth Form; (4) the girls in the Lower Sixth who are intending to proceed to College take the London Intermediate Arts or Science Examination, while those in the same Form who are not intending to pursue their studies further take the Higher Certificate of the Oxford and Cambridge Schools Examination Board; (5) in the Upper Sixth Form special coaching is given for scholarships and possibly for the Cambridge Higher Local (Group C.).

It is interesting to compare this most complicated system with the splendid system in force in Scotland at the present time. The Scottish Education Department examines annually for Intermediate Certificates, and Leaving Certificates, awarded to candidates of about 16 and 18 years of age, the standard approximating to the standard of the Lower and Higher Certificates of the Joint Board in most subjects; in Mathematics the standard is higher, and for the Leaving Certificate includes Trigonometry up to the solution of triangles. The result of the examination depends on three sets of marks: (1) the marks assigned on class-work, some six or eight weeks before the examination takes place, by the teachers of the candidates presented; (2) the marks assigned on the working of written papers set and corrected by the Examiners of the Department; (3) the impression formed by these Examiners (who are the Inspectors of the schools), after a *viva voce* examination of the class presented and of individuals in the class, subsequent to the correction of the written examination. By these means the examiners are kept in touch with the methods of teaching employed in the schools, and the teachers, as is right, have some responsibility for the marking of their own candidates. I think that practically all teachers who have anything to do with these examinations are agreed that the method and the papers set are thoroughly satisfactory. Scope is given, moreover, for individuality in work, such as is desired in the preamble to the last Report of the Mathematical Association quoted above, by "Additional Papers." After taking, or while taking, the Leaving Certificate, candidates may be presented for examination in any further branch of Mathematics selected by the teacher; e.g., Modern Plane Geometry and Elementary Conic Sections, Co-ordinate Geometry, Statics and Dynamics. The certificates obtained are recognised throughout Scotland: the Intermediate Certificate is the *sine qua non* for all girls who wish to take up elementary school work, domestic economy teaching, gymnastic teaching, &c.; and the Leaving Certificate admits to any of the Universities of Scotland. They are, therefore, the only examinations taken in the schools except the Bursary Examinations of the Universities, which are of the same standard as the Leaving Certificate Examinations. On the result of these are awarded all scholarships or bursaries to the Universities. There are,

I think, few educationalists in England who would grumble at the introduction and organisation of a scheme so simple and withal so satisfactory.

I have endeavoured to trace the development of the organisation of mathematical teaching in girls' schools, and to show the trend of the present day. It cannot be denied that the work done in the past half-century has vindicated the right of Mathematics to a place in a curriculum designed for girls. It has been a period of growth, and the growing stage is not yet over. We have still many questions to answer; for example, the much-debated question as to whether all girls should learn Mathematics beyond the first stages, and if not, what proportion can continue the study and derive from it real benefit. Much also remains to be done in the direction of simplifying the examination system and placing the teaching of Mathematics on a sound pedagogic basis.

LOUISA STORY.

APPENDIX.

SYLLABUSES.

Note.—Syllabus "A" is a type of syllabus very generally in use.

Syllabus "B" is one slightly more on the lines of recent suggestions and is representative of the syllabuses of a good many schools.

Both would represent the work of schools in which a fairly high standard is maintained.

The times given are exclusive of "home-work," and represent only the time given to actual teaching.

<i>Form.*</i>	SYLLABUS "A."
Preparatory.	Numeration and notation. First four rules for numbers under 100. Mental work. Simple problems. Five lessons of 20 minutes to 30 minutes.
I.	Revision of four simple rules; extension to numbers of five or six figures. Multiplication and Division by factors. Money sums. Five lessons of 30 minutes.
II.	Recapitulation and extension of above. Weights and measures. Easy reduction. H.C.F. and L.C.M. Easy decimals. Five lessons of 30 minutes (one practical).
III.	<i>Arithmetic.</i> —Decimals and easy Vulgar Fractions. Metric system. Decimalisation of money. Reduction. Unitary method. Two or three lessons of $\frac{1}{2}$ hour.

* For ages in the various forms, see above, p. 545.

Form.

- III. *Algebra*.—First four rules, brackets, simple equations and problems. Easy factors as converse of multiplication.
One lesson of $\frac{1}{2}$ hour.
- Geometry*.—Practical work leading up to discovery of geometrical truths. Congruency theorems and deductions.
One lesson of $\frac{1}{2}$ hour.
- IV. *Arithmetic*.—Proportion, practice, areas, percentages and simple interest.
Two lessons of $\frac{1}{2}$ hour.
- Algebra*.—Simultaneous equations and linear graphs, graphs of statistics, factors, fractions, quadratic equations.
Two lessons of $\frac{1}{2}$ hour.
- Geometry*.—Substance of Euclid I and III. (principal theorems).
Two lessons of $\frac{1}{2}$ hour.
- V. *Arithmetic*.—General Arithmetic including Compound Interest and Stocks and Shares.
One lesson of $\frac{1}{2}$ hour.
- Algebra*.—Quadratic Equations and graphic solutions, elementary surds and indices, the progressions, proportion.
Two lessons of $\frac{1}{2}$ hour.
- Geometry*.—The substance of Euclid I–IV. Introduction to the geometrical treatment of proportion.
Two lessons of $\frac{1}{2}$ hour.
- VI. *Algebra*.—To the Binomial Theorem including logarithms.
Two lessons of $\frac{1}{2}$ hour.
- Geometry*.—Substance of Euclid I–IV., VI., and XI. 1–21, with solid mensuration.
Two lessons of $\frac{1}{2}$ hour.
- Trigonometry*.—Introductory to solution of triangles.
Two lessons of $\frac{1}{2}$ hour.
- VII. Scholarship work by coaching.

SYLLABUS "B."

- Preparatory. Number lessons, simple sums in connection with Tillick's Bricks (or other Kindergarten apparatus), first four rules for numbers under 100. Elementary tables built up and written out.
Five lessons of 20 minutes or 30 minutes.
- I. Four rules with larger numbers, and with first-place decimals (addition or subtraction only), taught with help of metric ruler. Easy money sums. Practical measurement involving some paper-folding, cardboard modelling and plasticine work.
Five lessons of 30 minutes.
- II. First four rules using decimals to hundredths (or further). Easy examples in metric system. Money sums. Introduction to simple graphs by the plotting of charts. Simple properties of triangle, circle, cube, cylinder, sphere.
Five lessons of 30 minutes.

Form.

- III. *Arithmetic*.—Decimals and metric system, decimalisation of money. Short methods. Weights and measures. Reduction. H.C.F. and L.C.M.
Two or three lessons of $\frac{3}{4}$ hour.
- Geometry*.—Measurement of lines, angles, &c. Establishment of fundamental conceptions of Geometry. Establishment of fundamental propositions practically (see Board of Education Circular No. 711). Exercises on congruence of triangles.
One lesson of $\frac{3}{4}$ hour.
- Algebra*.—Simple equations, symbolic expression and problems. Substitution in physical formulæ and functions of a single variable giving introduction to graphs. Rules of sign. First four rules gradually developed. Practice in easy factors as converse of multiplication.
One lesson of $\frac{3}{4}$ hour.
- IV. *Arithmetic*.—Proportion, areas and volumes, percentages, profit and loss, interest. Graphic methods used in teaching proportion concurrently with arithmetic method.
Two lessons of $\frac{3}{4}$ hour.
- Geometry*.—Substance of Euclid I. and III. (principal theorems).
Two lessons of $\frac{3}{4}$ hour.
- Algebra*.—Factors, quadratics (by factors), fractions, simultaneous equations and linear graphs. Harder factors and fractions. Graphs of functions of a variable (within certain limits) and of statistics.
Two lessons of $\frac{3}{4}$ hour.
- V. *Arithmetic*.—General, including use of 4-figure logarithms.
One lesson of $\frac{3}{4}$ hour.
- Geometry*.—Substance of Euclid I.–IV. Introduction to geometric theory of proportion, concurrently with algebraic treatment.
Two lessons of $\frac{3}{4}$ hour.
- Algebra*.—Quadratics (by completing square), graph of quadratic function with introduction to idea of maximum and minimum values, turning-points, real and imaginary roots. Elementary work on Index laws followed immediately by logarithms to base 10. Evaluation of surds. Progression. Proportion.
Two lessons of $\frac{3}{4}$ hour.
- Trigonometry*.—Introductory. Solution of right-angled triangles.
One lesson of $\frac{3}{4}$ hour.
- VI. Intermediate Arts standard in Algebra, Geometry, Trigonometry, Elementary Co-ordinate Geometry.
- VI. Scholarship work.

N.B.—Some of the subjects given in the above syllabus are due to examinations requirements, and would be better omitted or put at a different stage.

THE VALUE OF THE STUDY OF MATHEMATICS IN PUBLIC SECONDARY SCHOOLS FOR GIRLS.

For the purposes of this paper it is sufficient to state at the outset, that Mathematics occupies at present a fairly good position among the important subjects in girls' schools, at any rate in the public secondary schools under consideration. It is probably safe to say that in schools of this type, all the pupils are expected to learn something of elementary Mathematics beyond arithmetic, and that work at this subject is usually continued until within one—or in some cases two—years of the leaving age; girls who specialise in Mathematics usually pass through the ordinary course, and then devote additional time to the subject during the last year or two of their school career. Further details of the amount of time allotted to Mathematics at the various stages of school life will be found in one of the other papers of this series. The point with which I am chiefly concerned is that the position of Mathematics in girls' schools is being challenged. School curricula are already overcrowded, and the pressure is increasing rather than decreasing; each subject must be ready to justify its position, or it will surely lose it, and that in the near future. I propose to consider first some of the objections which are urged against Mathematics, and then some of the arguments which may be brought forward in its favour.

The objections to the position of Mathematics as a subject of cardinal importance for all or nearly all girls may be summarised as follows:—

- (1) That the subject is uninteresting, relatively at least, if not absolutely, to many girls;
- (2) that its utilitarian value is negligible—an argument which might explain the lack of interest;
- (3) that from its difficulty it puts a strain on the pupil out of all proportion to the benefit received.

To consider the objections in detail:—It is said that girls do not find Mathematics interesting. The reason suggested is that it has little connection with their present-day life or later activities. The deduction drawn is that its study should be reduced to a minimum in girls' schools, and that it should not be compulsory for girls in University Entrance Examinations, but an alternative among certain other subjects, such as Harmony and Physics.

It is at least open to question whether girls are really bored with Mathematics; it certainly has not been my own experience, nor is it the experience of others from whom I have made inquiries, though I have not been able to gather general statistics of opinion. Again, it might be debated at some length, how far this alleged lack of interest, where it does exist, is real and inherent in the nature of the subject, and how far it is accidental and due to a bad presentation of it. Mathematics, like other subjects, is often taught in a dry and uninteresting way, and teachers let slip the opportunities which they ought to seize, for establishing connections with everyday life. A girl will spend a long time, with keen interest, calculating the cubical contents of her desk, awkward in shape, and inconvenient in measurements though it may be, and will learn a far more effective lesson from her experience, than from working a dozen text-book examples on the contents of boxes and tanks, lidless or otherwise. She will also take a personal interest in her result; and, though it would be rash to say such a thing would never occur, she will be far less likely to announce that her desk will accommodate a few hundred cubic feet of books, than if she had merely taken down a set of measurements from her text-book. Similarly, a statement that 7,000 picture postcards can be pinned upon a form notice board will be greeted with derision by girls who might cheerfully make equally ridiculous mistakes in working out examples on the rule, $\text{area} = \text{length} \times \text{breadth}$. No doubt such examples take time, but, however pressed for time we may be, we need not fall into the error of thinking that the number of examples worked by a new method is a measure of the power gained over it. Often one or two examples of practical interest, if possible suggested by the class, worked out in full by every member in such a way that all the data are brought home, are of more value than a large number worked from the book with superficial attention. Especially is it important for the weaker members to take their own measurements; the remembrance of the actions performed will often prove a help in recalling a forgotten process, which their intelligence can then justify, even though it was not strong enough to suggest it.

It may be said that I am merely pointing out differences between good teaching and bad. I do think, however, that bad, or at least inadequate, teaching is mainly responsible for the lack of interest complained of. Sometimes it is due to want of skill in the teacher, who rarely receives any training for her profession, and who has not always the ability to draw the right conclusions from her experience instead. Often, especially in the case of girls, the teaching the pupils receive in the early stages is so mechanical and deadening, that they are rendered almost proof against more enlightened methods later on. Very often, examination pressure allows none but a strong teacher to find time to consider the individuality of her pupils. It is of

course a truism to say that bad teaching will make any subject distasteful, and good teaching almost any subject acceptable ; but it might be well to try to devise and spread good methods of teaching Mathematics, suited to the needs of girls, before adopting the somewhat drastic remedy for lack of interest, namely, elimination of the subject.

It is said, however, that at any rate Mathematics is relatively uninteresting, that the main interest of girls lies in the humanities, and that their studies should be based on those subjects. This idea is very widespread in Germany, and Mathematics is at a discount there in girls' schools. The broad statement that most girls prefer the humanities is probably true. If the tastes of all the girls specialising in secondary schools and in the Universities were ascertained, it would almost certainly be found that a smaller number were working at Mathematics than at literature or history. This may be partly due to the fact that good teaching in literature and history has been available for girls for a much longer time than in Mathematics. The latter subject has fought its way to the front in girls' schools, but it has had many difficulties to contend with, and the supply of properly qualified teachers has never been sufficient to go round. Still it may be granted that Mathematics is likely to remain a less popular subject for advanced study with girls than it is with boys, especially perhaps in the South of England. But it does not immediately follow that Mathematics should be ousted from its hard won place in the front rank of school subjects ; it would be a pity to organise school work so that those girls with mathematical ability should get too slight a preliminary training to enable them to specialise with profit later on. Moreover the question of the best subjects for girls to specialise in, is an entirely different one from the question of the best subjects for the foundation of a general education on which future specialisation should be based. The first question should, in my opinion, be decided entirely by the girl's personal taste, or the results will be evanescent. The second problem is far more difficult to solve. The results of collective experience must be known, and those who try to interpret them should hold broad views of the kind of men and women they wish to see grow up. The objection, that whatever may be the value of Mathematics for boys, its utilitarian value for girls is very slight, has been frequently urged. Connection of the school curriculum with everyday life is rather a fetish in the educational world at present. It is, perhaps, not always borne in mind that the everyday life of the average English child is often very narrow and uninspiring, and tends to produce a very narrow man or woman. It is the duty and the privilege of the school to widen that life. We hear much of the value of agricultural education in country districts, and there is no doubt that its utilitarian value is greatest there ; at the same time it may be that a fine school garden is most appreciated in a city school, and that by

it instincts are reached and fostered which would otherwise be starved out of existence. Similarly we may esteem Mathematics highly, even if we have to admit that its utilitarian value is less for girls than for boys. But this admission must be made with reservations. If left to themselves girls may not attempt to make so many things for which their Mathematics might come in useful as boys do. But the constructive instinct is not lacking, as anyone who has set a class making mathematical models of geometrical solids knows, and it often needs only a slight stimulus to make it function. We may indeed recognise that in working out the higher education of girls, the traditional educational of boys has been too slavishly copied, and that the mental and physical differences between the sexes have been completely ignored. There is a tendency now showing itself, however, to rush to the other extreme, and to assume that the minds of all boys are fashioned after one model, and the minds of all girls after another. Boys have certain main interests which must be highly cultivated, to the exclusion of others, which some boys at any rate certainly possess. The main interests of girls are less well defined, but at least we are to assume, according to this school of thinkers, that they are different from those of the boys. So much is asserted dogmatically, and if the evidence is still to seek, we may find the subject fully discussed in the pages of German educational literature, though even there the phrase "die weibliche Eigenart," which occurs almost *ad nauseam*, is supposed to be so well understood that any attempt at definition is rarely made. Personally I find it hard to believe that the difference between an ordinary boy and girl is necessarily greater than the difference between two individual boys or two individual girls. At any rate each individual should have an opportunity to develop on many sides; if a boy or girl has a special aptitude we should not develop that only, and crush all less assertive instincts. We do not wish that our schools should produce one-sided boys and girls with all their interests along different lines, and there is a distinct danger of this, if all the weaker instincts are allowed to die of inanition just because they are the weaker.

It is sometimes taken for granted that Mathematics can have no intrinsic interest for girls because it will not help them to cook a dinner or to darn stockings. But girls have minds as well as bodies, and so have their future husbands. We live in an age of marvellous scientific advance, and it is difficult for an educated person to be without some interest in natural science. Mathematics underlies all science, and it is only when mathematical methods are introduced, when distinctions become quantitative rather than qualitative, that the study rises from the mists of imaginative descriptions and can be dignified by the name of Science. "All science as it grows towards perfection becomes mathematical in its ideas."* Physics is, of

* Whitehead (A. N.), *An Introduction to Mathematics*, p. 14. (London: Williams and Norgate. 1911.)

course, the science which is entirely based upon Mathematics, but biology, psychology, economics, and the rising study of sociology all depend upon it in a greater or less degree. The formulæ of Mendelism, the tabulation of the results of psychophysical experiments, the graphical representation of economic statistics, all these are a sealed book to one who has no acquaintance with Mathematics. The idea of a formula, the generalisation of innumerable apparently isolated results, is not easy to grasp unless it has first been met with in the simple cases of elementary Mathematics. The fruitful idea of variation, again, has shown enormous development in late years; the very word is pregnant with significance to those who have met it in strict mathematical relations, and connotes what is beyond the outlook of the person to whom there is no distinction of meaning between "varying with" and "varying as." In a statement such as "a young man can do more work than an old man," the interesting question is, "How much more?" and in the attempt to answer this question we are drawn into measurement and Mathematics. The results of experiments in strength of grip, or in power of lifting, curves of fatigue, curves of expectation of life, and many other mathematical illustrations rise to the mind at once in considering questions of this sort. Surely it would be difficult to maintain that no women are interested in such questions, or even that a few may be, but necessarily at the expense of their domestic activities.

Leaving these objections to retain such weight as may be due to them, we may fairly urge that, apart from the intrinsic interest of Mathematics, there are certain advantages to be gained from its study, which we should be ill-advised to forego in our girls' schools.

All girls ought to grow up reasonable beings, and many of them do not; all girls ought to acquire a knowledge of the meaning of language, and a power of using it accurately, and many of them do not. Mathematics, properly taught, will help to both ends. In all branches of Mathematics, though perhaps more especially in Geometry, it is necessary to be clearheaded; we have to pick out exactly what it is required to do, to decide what portions of previous knowledge are relevant, and what are not, and to set down clearly the steps in the procedure. It is often urged against women that they talk off the point, that they do not know what argument is. In Mathematics it is constantly necessary to run over facts in the mind and to sort and select. Whether we have to cudgel our brains for fragments of geometrical knowledge to help us to solve a rider, or whether we have merely to study and classify data in order to form an algebraical equation, the same process of selection and rejection has to be gone through. Any wandering from the point is fatal, and usually the futility of a wrong selection shows itself at once. It will be seen that in this view of Mathematics little stress is laid upon manipulative skill, or rapidity of calculation; these things may be useful as a

means to an end, the first is certainly required by the future mathematical specialist, the second, no doubt, by the future commercial clerk; but they are not in themselves of educational value, and they should be at a minimum in a non-specialist course designed to give the average pupil the foundations of a liberal education. In the recently issued Report of the Mathematical Association on the Teaching of Elementary Algebra and Trigonometry,* it was intended to emphasise the more thoughtful side of mathematical study, and it is therefore recommended that the mechanical work needed to produce manipulative skill should be done at a later stage than the elementary course, and only by those who will need it. Mathematics must teach a girl to think, and be taught with that end in view, if it is to retain its honourable place in our school curriculum. In Geometry it may be quite valueless that a girl can write out a neat and tidy proof of a proposition with all the steps put in and references duly given in the margin. The value depends on the extent to which her work is a conscious reasoning process. Does she realise that a step left out is not merely something omitted, as a detail in the description of a flower might be, but a vitiation of the whole proof? Does she realise that the omission of the reason for any statement means that, for her, the structure is unsupported? In correcting elementary work in Geometry, immense stress should be laid upon this; a piece of work might suitably receive no marks if a step were omitted and slurred over, and three-quarter marks if a statement were made that the step was required, but the pupil could not prove it. Pupils should be encouraged to show up incomplete proofs, that the point at which they fail may be discussed and cleared up. Girls like to show up neat complete work, and will often give in nothing rather than an unsuccessful attempt; this should be strongly discouraged, as success comes largely through an investigation of the causes of failure. In the early stages of Geometry the teacher's main business is to say "Why?" If she does this sufficiently indefatigably then, by the time the later stages arrive, she should be able to reduce herself almost to a state of superfluity—the Nirvana of the good teacher. If a girl gains nothing from her mathematical lessons but a living conception of the words "because" and "therefore," she will have gained a possession for life, and a possession which many women, and, I venture to think, many men also, go through life without: "One of the chief ends served by Mathematics, when rightly taught, is to awaken the learner's belief in reason, his confidence in the truth of what has been demonstrated, and in the value of demonstration."† It may

* Report of the Mathematical Association Committee on the Teaching of Elementary Algebra and Numerical Trigonometry, 1911. (London: G. Bell & Sons. 3s.)

† Russell (Bertrand), *Philosophical Essays*, p. 75 (*The Study of Mathematics*). (London: Longmans, Green & Co., 1910.)

be argued that cause and effect can be studied equally well in the history and geography lessons. They may, and of course should, be so studied, but the circumstances are far less simple, especially in history. We can less often say that a certain effect is a necessary consequence of a certain cause, or even of a certain combination of causes, and even if we can, the reasons may very well be beyond the appreciation of a child.

No doubt, in Mathematics, the circumstances are unnaturally simplified, and not such as occur in everyday life, but when we have anything complex to study, we begin by analysing it, we isolate the simplest elements and try to understand them. Well for us if we first learn what argument means in a study where human thought is pure and passionless, not clouded by emotionalism nor yet weighted by utilitarianism.

It would hardly do to base a claim for the retention of Mathematics mainly on its effect on style in composition, especially as this is a point which has too often been neglected in mathematical teaching; at the same time the clearness and brevity so desirable in Mathematics are qualities often conspicuously absent from girls' written work. They tend to be diffuse, fond of adjectives and phrases, and apt to converse pleasantly about many things whose relevancy to the subject in hand is not obvious. It is good for them to have to condense their thoughts and to explain their proceedings in the briefest and clearest way. Much harm is done by rating too highly a girl's power to produce—somehow, anyhow—the correct answer to a given sum, and more attention should be paid to style, and more credit given for connected and intelligible explanations, both oral and written.

Women are said to be inconsistent, and, moreover, to be quite unable to recognise their inconsistency. There is no place where the penalties of inconsistency are more striking than in the mathematical class-room. Here, as elsewhere, much depends upon the teaching; if a pupil gives a wrong answer and the teacher merely passes on and gets the right one from someone else the opportunity is lost. The wrong statement should be accepted, and the pupil made to follow it out to its consequences, the more ridiculous the better. People will say that this method would be intolerably tedious and that progress would be much retarded thereby. In large, or very uneven classes it will no doubt be necessary to modify it in practice, but the ideal is that a wrong statement should never go unexposed, and in this way progress, if slow, will be real and not fictitious. Also if this method be rigorously pursued it will tend to work itself out; if we can only train our pupils to be self-critical, our criticisms will gradually become works of supererogation. It is the fashion now to consider geometrical proofs leading up to "*quod est absurdum*" as too hard for children, and direct proofs are usually substituted for them. This may be wordy-wise, certainly, in so far as it is advisable

to have an alternative proof for the weaker candidates for examination, but it would be a great pity if this most instructive method of proof were banished from the class-room. The opportunities offered by Mathematics for self-criticism are probably unrivalled. In arithmetic and algebra answers can nearly always be tested, and they can always be classified as possible or impossible, probable or improbable, and it is a very important part of the teacher's work to get the pupil to apply the necessary tests. We do too much correction ourselves in schools; a pupil gains more by being set to work a sum and to prove it, or to work a sum in two different ways and compare the answers, than if she works two sums and gives them in, waiting placidly with a greater or less degree of interest, as the case may be, to be told by her teacher whether the results are right or wrong. Incidentally the work of the teacher would be largely reduced to the consideration of style if this method were generally adopted. It is a method much enjoyed by the pupil if she is accustomed to it early. A child of eight will prove a multiplication sum by division and *vice versa*, and take a pride in being able to dispense with the teacher.

One great advantage of Mathematics in the education of girls is that it offers unique opportunities to the teacher for recognising and encouraging independent thought. Girls are said to have less initiative than boys, they are more docile, follow more easily where they are led. At the same time they do offer suggestions, they do ask questions and show desires of following up certain lines of thought; if the opportunities are less frequent it is all the more important that they should not be missed. Girls, perhaps, need a greater stimulus to independent thought than boys do, they are less contradictory, less likely to react in a direction opposite to that which was intended. For instance, an unwise teacher frequently proceeds in the following way; she writes the enunciation of a geometrical proposition on the blackboard—at the worst, proves it herself; if she has some sense, gets the class to do it for her—and then says, "Learn it in the book for homework, you will have to write it out in class next time." I believe this method is not uncommon, but to my mind it is radically wrong. A boy, very probably, would omit to learn it, and would at least have the opportunity of wrestling with it afresh next time. The girl, being diligent, learns it carefully and writes it out correctly, and is left with a comfortable sense of work conscientiously done and knowledge acquired, where the recording angel would note down another opportunity for thought lost, another blow struck at the girl's individuality. Surely all propositions should be presented first as riders, the fact discovered by the class from a study of the figure, where possible, and the proof attempted by all the individual members before anyone is allowed to go through it; when it has been worked on the board it should be written out by everybody before access to a book is allowed.

No doubt many will fail, but those who succeed will have done something far more worth doing than any amount of learning from a book, and those who fail may succeed next time, and will at least know that the aim set before them is to do their own thinking, and not to assimilate somebody else's. A teacher of girls is, perhaps, too easily satisfied when her pupils are working steadily and conscientiously along the lines which she has laid down for them; a boy is almost certain to go off at a tangent sometime, and force himself and his teacher, if not too hidebound, out of the rut; routine does him less harm, because he is less susceptible to its influence. Probably one of the weaknesses of girls is that they will submit to so much dullness without resentment. Hence the neat, careful, and utterly unenterprising examination papers of which bored examiners complain. It is true that young children, whether boys or girls, do not as a rule find mechanical work dull, but if this be an advantage it is double-edged. Mechanical work can train them to rapidity and accuracy, two useful qualities which it is difficult to acquire after the age of twelve; on the other hand it is fatally easy for the teacher to proceed calmly without stirring the impulse to mental effort, which may remain latent and atrophy in a surprisingly short time. It is vital to rouse the mind to exertion, and for this reason, among many others, junior external examinations are to be deprecated. Success in these examinations is almost bound to be more easily obtained by speed and accuracy in mechanical work than by mental effort, and the effects of teaching mainly directed to such an end can only be described as disastrous. A girl's independent thought may be steadily crushed down without consciousness of either teacher or taught; her self-assertiveness is weaker than a boy's, her individuality more delicate. Many girls who are apparently good workers are really mentally lazy, they reproduce but they do not produce. A teacher needs to be alive to this danger, and to realise that it is her business to stimulate intellectual curiosity, the desire to know, and not only to know, but to find out at first hand; she has every opportunity in Mathematics, where hardly any of the work should be reproductive. If for no other reason, Mathematics should remain in the front rank of educational subjects, because it helps the growth of the instinct to do independent work, which awakes in every child, boy or girl, at some period, but is sometimes stifled among the many excellencies of our modern educational system.

Another ground, allied to the last, for holding Mathematics valuable in the school curriculum is that it is hard. It really requires that power of concentrating the attention which should be one of the most important results of education. Naturally inattention is harmful in any lesson, but it is far more difficult to pick up the threads after a few minutes of wandering thought in Mathematics than in other cases. As Bacon says, "If a

“Man’s Wit be Wandring let him Study the Mathematicks; For in Demonstrations, if his Wit be called away never so little, he must begin again.” It seems probable that a woman’s wit also may wander, and that therefore the advice which Bacon offers man should be extended to woman, unless indeed a wandering wit is held to be an excellent thing in woman. Concentration is essential to good work in any subject, but in Mathematics the whole attention is needed to produce anything at all. This statement is borne out by studies in fatigue among school children; it is the mathematical lessons which take most out of them, and we are advised to put them early in the day, when the children are fresh. There can be little doubt that the majority of girls find Mathematics hard; by this it is not necessary to understand that they feel it beyond their powers, but that it taxes their powers to the utmost. Still less is it necessary to assume, as is often done, that hard is synonymous with distasteful; more shame to our education if it were so. Power is to be exercised and not to be spared, and a curriculum of soft options could only produce an invertebrate generation of dilettantes. Some people who write about the education of girls seem to wish them to glide over the surface of many subjects, so that they may be able to take an intelligent interest in them and be well informed, but consider hard work to be detrimental to a girl’s health, or—in some unexplained way—to her womanliness. The question of health is a difficult one, and girls have to be carefully guarded against overwork. Probably a girl should do less work at all times than a boy, and she should be protected against strain during the years of development (another argument against junior examinations), but hardly anyone would advocate that no girl should be taught anything she found difficult. We have made mistakes in the past, and it is to be feared that many girls have suffered in health through their zeal for higher education; we need not, therefore, return to the days of our great-grandmothers, when the domestic and fine arts, together with a little French, were considered to make sufficient demands upon what intellect a girl had, and when it was correct for a woman to swoon on occasions which would appear to us to call for instant action. Let a girl work, and work hard while she is at it, let her learn what effort really means, and taste of “the joy of most glorious striving,” and we shall have fewer idle discontented women who live in enervating ease, and breed weaklings to the nation.

Probably exception will be taken to some of these suggestions on the ground that the idea of training is old-fashioned and that any theory based upon it is untenable; that no subject has any disciplinary value whatever, that it must be taught for its intrinsic value, too often identified with its utilitarian value, or not at all. Psychology tells us that there is no such thing as a generalised habit, and it has been proved by experiment that,

for instance, a marked improvement in neatness in working arithmetic papers is not accompanied by a corresponding improvement in neatness in paper work in other subjects. It is impossible to resist the theoretical evidence that a habit—due to the subconscious reaction to a specific stimulus to which we have often reacted before—cannot be carried over from one field of life to another. And yet experience tells us that something is carried over; the character of a man is the outcome of his conduct, and his past habits do in a very real sense determine his future habits. If a philosopher and a professional boxer were each set down on a desert island, it is unlikely that they would develop identical habits in their new circumstances. In teaching we do find that certain subjects are more educative than others, and though I do not propose to attempt a definition of the word educate, most people would probably agree that a well-educated man is not merely a man who is well informed, and who has acquired desirable habits; he is a man whom we should expect to acquire good habits in whatever new circumstances he might find himself. It has been suggested, that what is really carried over is not the habit but the ideal behind the habit, and an ideal which has been consciously developed. “It is something that functions “ in the focus of consciousness, and hence cannot be identified “ with habit, which always functions either marginally or “ subconsciously. This ideal furnishes a motive, and this “ motive holds me to conscious persistent effort until the new “ habit has become effective, until the distracting influences “ no longer solicit passive attention. If I had acquired a “ specific habit of work in one field without at the same time “ acquiring a general idea of work, my acquisition of a specific “ habit of work in another field would probably not be “ materially benefited.”* Let us see to it, that high ideals of work and style are set before the girls. Slipshod work should never be accepted, and anything in the nature of “cooking” is dishonest and should be considered a serious offence. I do not feel competent to enter into a psychological discussion nor to take up the cudgels on behalf of training, strongly as I believe in it, so I will only quote the words of a well-known educationist: “Mathematics is above all other subjects of “ school study an exact and precise science, its reasonings form “ a standard towards which all other sciences look with envy. “ Hence the study of Mathematics will consciously and un- “ consciously set up an ideal of valid thought more perfect “ than that given by the study of less exact bodies of know- “ ledge. On this ground alone Mathematics would hold its “ own as a mental discipline.”† As to the best way of securing

* Bagley (W. C.), *The Educative Process*, p. 213. (London: Macmillan & Co., 1905.)

† Welton (James), *Principles and Methods of Teaching*, p. 409. (London: W. B. Clive, 1906.)

the advantages which have just been enumerated, opinions will naturally differ. The following remarks are based upon my own experience, which has not been confined to one type of school.

There is one common custom which, in my opinion, does much to detract from the value of mathematical teaching. In most schools the time-honoured divisions of the subject—arithmetic, geometry, algebra—are kept up, and separate lessons, perhaps two a week, are allotted to each branch. This is, of course, a minor matter if the class unit is the same, and the teacher the same in each case; even if she is not free to divide the time at her discretion, she can at least make the subjects work into one another, and can illustrate each from the others; she also knows the exact stage of mathematical development of each of her pupils, and the previous knowledge upon which she can count. If, however, all the lessons are not given by the same teacher this is no longer the case. We are just emancipating ourselves from the idea that school Mathematics is of necessity divided into watertight compartments. Even examiners have been brought to permit, and in some cases to encourage, the solution of problems by algebra rather than by arithmetic when the method is simpler. Geometrical exercises are no longer exclusively drawn from the most abstract sources, but efforts are made to show pupils how little problems which arise in their everyday work and play may be solved by the combined help of geometry and arithmetic. It is most important that Mathematics should be taught as a whole; it is a science and not an art, and the main emphasis should not be laid on the performance of certain operations, but on the significance of the operations, and on the scheme of which they are interdependent units. For instance, graphs of equations can be taught in a purely mechanical way, and the value of plotting points, as such, is practically nil, as has been pointed out by critics of the prevalent passion for graphs. If the subject is taught in connection with loci in geometry, light is shed upon both. Again a pupil who learns a practical geometrical method for finding a square equal to a given rectangle should be able to use it to find the solutions of a quadratic equation. These are two instances only, but any teacher could increase them almost indefinitely out of her own experience.

It may not be absolutely impossible, but it is clearly much more difficult, to teach the subject as a whole, that is, to teach Mathematics from the arithmetical, geometrical, algebraical points of view, rather than to teach arithmetic, geometry, algebra, if the work is put into the hands of different people. It sometimes happens, especially in schools where the mathematical mistress is pressed for time, that some other mistress takes up—in the old days probably Euclid, now-a-days more likely arithmetic—and teaches it through the greater part of the school. In schools

which are redivided into sets for Mathematics it is sometimes the class unit which varies; a girl arrives at school backward in geometry, is put in a lower division for it, and is allowed to remain in the same relative position for the rest of her school career. It is certainly best, if it can be managed, to re-divide the school for Mathematics, but in any case all the mathematical lessons of a given division should be in the hands of the same teacher, who should have power to allot the time as she thinks fit. Of course, each teacher will have her own plan of lessons, but she should be free to vary them whenever she likes; those who have tried it know how stimulating a week's lessons continuously given to geometry may be; a grip can be obtained on a certain piece of the subject of a strength which scattered lessons will not give. Very possibly two lessons may usually be given to arithmetic, two to geometry, and two to algebra, and only occasionally a whole lesson taken from one subject and given to another, or any radical change made. Even in this case, however, it is often worth while to spend a few minutes at the beginning of a lesson discussing any point in the homework of the night before which may have presented special difficulty, instead of waiting until the next lesson on the subject normally occurs; the solution of a problem which is still fresh in the minds of the pupils will be received with greater eagerness, and grasped with correspondingly greater quickness, than after they have had time to forget all about it.

Again, the order of teaching mathematical facts should depend far more upon the interests and aptitudes of the class, and even upon the illustrations which arise in the course of teaching, and far less upon the order adopted in the various text-books, or stereotyped by examination requirements, than it does at present. Those teachers who have not made a special study of Mathematics, but are unfortunately obliged, by the exigencies of school time-tables, to teach it, are probably well-advised in adopting a good text-book and abiding by it. But those who are at the same time mathematicians and trained teachers should show more confidence. The teacher who puts off an answer to a child's question because it does not occur in its orthodox place, or because it interferes with the order of the lesson she has prepared, may lose an opportunity which will not come again.

There are many practical difficulties in the way of treating Mathematics as a single subject on the time-table, chiefly connected with the entry of new pupils. Some have been pushed on in algebra, but have done no geometry; others have worked laboriously up to stocks in arithmetic, can do complicated examples in profit and loss and true discount, but are helpless before the simplest of simple equations. Girls who are sent to boarding schools have often been under the charge of governesses who are quite innocent of Mathematics; the Mathematics of girls coming from elementary schools is usually confined to an

extraordinarily over-elaborated arithmetic. The worst difficulties may be overcome by reclassifying the school for Mathematics, though this involves difficulties of its own in small schools. Two or three divisions must have their lessons at the same time in order that girls may readily move up and down. In a large school where there are several mathematical mistresses this arrangement presents little difficulty, but in a smaller school, where there is perhaps only one, it is not so easy. Sometimes only part of the school can be re-divided; the science mistress will probably be willing to take a division; if the head mistress is mathematical she will naturally do something; often a junior mistress with a pass degree including Mathematics can be appointed. Sometimes the lessons need not be at the same time, but can be interchangeable with science, or drill, or some other subject. Of course all the work must be under the supervision of the mathematical mistress. In any school, large or small, the organisation of the work should be in the hands of one person. There are obvious drawbacks to the teaching of Mathematics by people who are not mathematical specialists, but they are probably less on the whole than those which occur when girls are taught in forms irrespective of previous knowledge or mathematical ability. But it should be clear that the non-specialist mistress must take the division in all subjects, and not merely in one, and must feel responsible for its progress. It might cause her less trouble to get up algebra only; but she could not teach it properly without the others. In some schools arithmetic is treated almost as a form subject, and all sorts of people teach it, with all sorts of aims and methods, and little or no knowledge of its connections with algebra or geometry, to say nothing of higher branches. This is fatal to the idea of mathematical unity.

The most revolutionary suggestion which has been made by the opponents of Mathematics is that it shall lose its place among the compulsory subjects in University Entrance Examinations. If this happened, I think the result would be disastrous, not only to mathematical education, but to that general education of which it is an integral part. There may be, and no doubt usually are, girls in a school who seem incapable of grasping more than the elements of arithmetic, a minimum of practical geometry, and the use of the simple equation. But these are not the girls whom we wish to send to the University. The mathematical requirements for entrance to the various Universities are not very high, and though there is a good deal of dead matter and mere manipulation included among some of them, which might well be omitted, I still hold that the girl who has not the ability to tackle the necessary work, and to do it successfully, is not fit for a University education. I do not consider the alternative of physics, as I cannot see how any satisfactory standard in that subject could be reached unless the work were based on a knowledge of Mathematics at least as wide as that

required in Responsions at Oxford. General elementary science should be taught to younger children, but when they come to years of discretion let them either eschew physics altogether, or do something of more value than the kind of physics which "would not give a headache to a caterpillar," to quote the verdict of an eminent physicist on the physics taught in many secondary schools. Probably more attention relatively should be given to the humanities in girls' schools than in boys', but it is doubtful whether any differentiation need take place until the age at which specialisation begins. As it is, girls under 16 usually spend more time on English subjects than their brothers, the time being gained from the third foreign language, which most boys learn and most girls do not. In the earlier stages of school life we are training the mind to work, and our main object is to lay a sure foundation for the future, and as wide a one as may be possible. It is my conviction that the number of girls who can derive real intellectual pleasure from such theorems of pure mathematical form as abound, for example, in the elementary theory of numbers, is far greater than is generally supposed. School teaching will not of necessity produce an appreciation of the true beauty of Mathematics, but if we drop the teaching we lose what chance there is of doing so. Many may seem untouched, but there are always some girls who derive keen æsthetic enjoyment from the symbolisation of Mathematics, and once their eyes are open they may come to see something of its beauty—"a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show."*

E. R. GWATKIN.

* Russell (Bertrand), *op. cit.*, p. 73.

THE PLACE OF MATHEMATICS IN THE EDUCATION OF GIRLS AND WOMEN.

The best method of briefly treating this important subject is to approach it historically and see how Mathematics came to be a part of girls' education. The facts are by no means obscure, since they have occurred within the lifetime of many who are now teaching in girls' schools.

Fifty or sixty years ago, as we all know, girls did not learn Mathematics, and if we go back to the days of Queen Elizabeth and Lady Jane Grey, when young women of the upper classes did receive a good education, we find classics, not Mathematics, as the staple. Indeed the introduction of Mathematics as an ordinary and compulsory subject (not an extra) in all boys' schools is comparatively recent. The importance of the subject has grown with the developments of science, and the value in active life of Applied Mathematics.

In the eighteenth century and earlier, girls were of course taught very elementary arithmetic for the practical purposes of housekeeping, just as the boy who was going into business was taught in a Writing or Commercial School separate from the old Grammar School. That this arithmetic teaching was purely for practical purposes is clear from the absence of it in the curriculum of those young gentlewomen who could rely, in their future life, on having a housekeeper or land steward to manage their affairs.*

Emphasis on Mathematics as an essential part of a girl's education is one of the most characteristic features of the reform movement that followed, in the "seventies," on the publication of the Taunton Commission Report. There, it will be remembered, were exposed the inaccuracy and superficiality of the current methods, and it was thought that the best way to reform these evils was to improve and develop the teaching of arithmetic in the girls' schools, and to introduce geometry and algebra. These subjects, it was thought, would make girls accurate and thorough; would teach them to reason; would develop their intelligence; and would be the cure for the evils which the Taunton Commission revealed. Some of the best schools, which were at that time feeling their way towards an improved condition

* In the report of the Schools' Inquiry Commission (Lord Taunton), Vol. IX., p. 793, we find quoted the statement of a Schoolmistress as to the indifference of parents even as late as 1865. "If I report that another (girl) does not try to improve herself in Arithmetic, her mother says 'It really doesn't matter; her husband will be able to do all her accounts for her, you know.'"

of things, eagerly seized every opportunity of obtaining good teachers of Mathematics—men or exceptional women. Qualified women teachers of Mathematics were available at a very early stage, since it was at the University of Cambridge that the pioneers established important women's Colleges as early as 1871.

The example of Mrs. Somerville undoubtedly influenced public opinion, and those who were in the movement themselves will remember how Mathematics for girls became a kind of test subject, which could be used to prove that women possessed intellectual powers not inferior to those of men and that, therefore, if girls could take good places in mathematical examinations, opportunities of higher study at the Universities and elsewhere should be afforded to their sex. Such opportunities were given. At the same time, in the examinations which were compulsory for admittance to University study, Mathematics had taken a place side by side with classics, and so all the girls who wished to go to College were obliged to study geometry and algebra. There was and is a movement against compulsory Latin and Greek in University Entrance Examinations for boys; this movement has prevented classics being made as much a matter of course in the modern curriculum for girls as Mathematics is. Latin is studied by girls with success; but not by all girls. Mathematics has become inevitable.

Of late years a reaction has shown itself. The schools have now some practical experience of what teaching Mathematics to Matriculation standard involves for the average girl. The old fashioned faculty psychology can no longer be unquestionably quoted in an argument to the effect that "Geometry trains the logical faculties"; which are, it is thought, deficient in women. At the same time there has been a revolt, led by the engineers, against the actual methods and material of mathematical teaching in the schools which have been found too academic and unpractical for the needs of real life, that is, the need of boys. Euclid is dethroned. Practical Mathematics of various kinds is introduced, and school Mathematics is more definitely directed towards the demands of careers which do not concern girls. At the same time we are beginning to believe that the curriculum should be differentiated for different types of pupils: vocational education is urged, and even where the truth of it is not accepted, academic studies are so followed in school as to have a direct bearing on future occupations.

These things directly affect the place of Mathematics in the education of girls. It is closely related to the characteristic industries and activities of *men*, engineering in all its forms, war, navigation, building, land surveying, finance, and some of the higher forms of commerce. It has no such relation to the characteristic activities of women: it is not required for any vocation which the ordinary woman will follow. Its only value is as a mental discipline, a point well brought out in the

article on Mathematics by Miss Sheldon in the recently published book of the Head Mistresses' Association, "Public Schools for Girls."^{*} The difficulty and dislike felt by many girls for the subject are probably due to this fact. "What is the use of it"? they say. It is divorced from reality, from experience: it makes no appeal to their special instincts as women. Biology, on the other hand, does make such an appeal, so that botany and (wherever it is taught) zoology are very successful subjects in girls' education. The same is true of history and literature, particularly English literature: subjects traditional in girls' education, and directly related to the emotional and altruistic interests of women.

Another matter must also be considered,—the crowding of the curriculum which has become so serious in boys' schools, and which is far more serious with girls. They are physically not so strong at the secondary school age; many of them have domestic duties which do not fall on the boys; many are expected to study music, and to give time to social duties in a fashion that is not expected of their brothers. Thus the girl with less physical energy, is required under existing systems to do more, not less, than the boy. This cannot go on: something must be given up. Since Mathematics was originally arbitrarily introduced into the girls' schools, and has no direct relation to the future activities of girls' lives, it is the opinion of many, including the present writer, that the amount of Mathematics demanded of girls must be materially reduced, and that it should be no longer compulsory, as it is at present, in all examinations of matriculation standard.

At all events, whether we take this view or not, it is clear that the problem must be reconsidered from the beginning. The present writer would venture to put forward her contribution to the discussion as the result of more than thirty-five years' experience as a student and teacher of Mathematics, and as one who is responsible for the curriculum of a large girls' school. It appears to her that in this matter girls fall into three groups, and that to state this briefly in detail is the best way of formulating what her experience leads her to think.

(1) There is a small but very important group of girls and women who have a real taste for Mathematics; who enjoy this study; who find it comparatively easy, and who, it is not too much to say, revel in it, as a delight, as a new world of thought and power. Its highly abstract character appeals to those faculties which make women delight in the study of philosophy, and also—though this must be said with diffidence—to those more spiritual qualities that make religion so much to women. For girls of this type Mathematics is an admirable training. Some of them possess powers which may enable them to do research. Mrs. Somerville presumably was one of these.

* Longmans, Green, and Co, London, 1911.

Others have gifts for administration and organisation, and the sort of training that they obtain by studying for such examinations as the Mathematical Tripos at Cambridge makes everything of the kind that they may have to do in after life easier. Mathematics, in such cases, has a distinct relation to the occupations of later life.

Now the opening of opportunities of mathematical study at school and college to such women as these has been of the highest importance. It has made them happier as human beings, and more useful as citizens, and it would be a loss and an injury if anything were done to prevent such women from enjoying in the future the opportunities they have had during the last forty years. What was won for us by the pioneers we can never consent to surrender. Those mathematical studies must be preserved in the schools and, *a fortiori*, in the colleges for the few, who will return to the community in adult life more valuable service of one kind or another, because of their mathematical education.

(2) Secondly, there is another small group, rather larger than the first, but at the other end of the scale, of girls and women who cannot do Mathematics at all, or, if they are able intellectually, do it with a large and wasteful expenditure of power. To get such girls through compulsory mathematical examinations on their way to College is like making diamonds in a laboratory. You secure the results, but they are wretchedly small, and cost much more than they are worth. There are able boys who find Mathematics equally difficult; certainly Lord Macaulay found the subject repulsive. But this paper is not concerned with the needs and difficulties of boys.

It may be thought that the numbers in this group are so small that it is not worth while, for their sakes, to limit the compulsory Mathematics for examinations. Everyone must form his own opinion in this matter, but the writer has seen enough of hard cases coming under this rule, to make her an ardent advocate for the removal of the mathematical barrier in College Admission Examinations.

(3) In the third group comes the great bulk of girls who can and do study Mathematics with some satisfaction to their teachers and examiners, if not to themselves, and who pass their examinations more or less successfully. About this there is room for considerable variety of opinion. Undoubtedly mathematical study is of value for the average girl; it does train accuracy and thought, power of grasp, and a certain sense of form. The value of the logical training is more doubtful. What is really wanted for girls is the training in logic as applied to concrete things, and this is found perhaps more usefully in the study of history or of the new geography, as well as in science.

Undoubtedly the average girl finds Mathematics harder than the average boy, and is less successful in it, in spite of her greater industry and earnestness about her work. The testimony

of teachers of mixed classes, and of the examination lists, is quite clear on this point; it is really a commonplace. When an equal degree of success is secured by the average girl it has cost a great deal more. This point should be emphasised. There is only a certain amount of power available in school, just as an engine will only develop a certain number of units of work, since a pound of coal can only give out a certain number of units of heat. It is found in practice by those who teach girls that the mathematical work takes up a greater number of units of energy than anything else. Girls with less physical energy, or vigorous girls at times when their energy stock is low, find mathematical work suffers first; conversely, to take the subject out of a girl's course of study, when she is over pressed, gives relief at once. Again, any serious or even trivial occurrences which may tell against effective school work, such as, heat, fog, bad ventilation, slackness of discipline, emotional stress, all tell on the Mathematics first; and, since girls are on the whole naturally more susceptible to physical and emotional disturbance, they are much more likely not to do themselves justice in mathematical examinations, and to fail even when they deserve to pass.

It may be said of course that a subject is valuable in proportion as it is difficult, and that the training in courage and perseverance gained by going through with a difficult matter against obstacles, is of the highest value. So it is, but even that may be bought too dearly, and it is bought too dearly by many a conscientious girl. One cannot but feel that the time and energy given to mathematical study, by the average girl who matriculates, could be spent to better purpose in other ways. Has she, for her four or five years' study with a lesson every day, and homework every day, gained anything that is adequate to the expenditure that has been made?

To this question some authorities will answer "Yes," though they admit that the progress made in actual knowledge and skill is not at first sight adequate. They hold that the girl *has* won something by her striving; has gained in reasoning power and intelligence by the hours she has spent in grappling with problems she hardly could solve. She will be a more useful woman; one less likely to make mistakes; more amenable to right reason, because she has persevered in her mathematical studies up to Matriculation standard, poor as her result seems to the mathematician. Such a belief is of faith, rather than susceptible of definite demonstration, but it is held strongly by experienced teachers, whose opinion is entitled to respect. Miss Sheldon, a mathematical specialist, takes this view, in the article already referred to, recommending that the average girl should take Mathematics to the Matriculation standard (approximately) but not beyond; she should not, *e.g.*, take trigonometry.

It is also the view of certain educated men, not teachers or mathematicians, that we must avoid any serious divergence

between the intellectual interests and standards of men and of women: that as women are by nature more intuitive and emotional, their school training should supply the harder, more rationalistic element; should give balance, and steadiness; should teach them to think, if not as men do, at any rate so that they can understand men's way of thinking. This is of course a sound argument: the only question is, how much mathematical teaching is required for this end. It is a matter of syllabus, not principle.

The writer would suggest, as a solution of the problem, that Mathematics should be compulsory in girls' schools for say a three years' course, from twelve to fifteen, but that it should not necessarily be taken up to Matriculation standard. Anyone who has studied Mathematics knows that the subject does not present a slope that rises regularly in difficulty, like the inclined plane of the text-books. It is rather like a great mountain, with precipitous steeps every here and there, and with long slopes of comparatively easy going in between. Most of us find that we stop at the foot of one of these precipices somewhere or other in our course, and never get up it. The average school girl soon comes to the precipice that she cannot climb (without an excessive amount of assistance from her teacher), generally somewhere about the Upper Fourth or Lower Fifth Form. It would be a saving of energy and time to let her go no further. The simpler course here recommended would include good arithmetic, elementary geometry, and simple algebra of the kind which Mr. Godfrey, of the Royal Naval College at Osborne, recommended—in his paper* in this series—for the ordinary boy or girl student. Much of the elaborate manipulation of algebraic symbols taught in school is quite useless to the average person. If this simpler course of mathematical study was taught with no reference to any examination, teachers could pursue it with an originality and a reference to real life, and to some elementary philosophical ideas, which would make it truly valuable in itself. For examination purposes a school record might be handed in, showing that a course of this character had been followed for a certain number of years, under qualified teachers.

For the actual Matriculation examination the necessary guarantee of sound intellectual work can be furnished by making either Latin or Mathematics compulsory. Latin does not present the difficulties to girls that Mathematics involves. They take to it more readily; they assimilate it better, since it is more human. It has a direct relation to the study of modern languages, a congenial and successful subject, of practical value, in a girl's education.

The present writer is of opinion that much more might be made of harmony as a school subject than is generally the case, as a study giving that element of austere intellectual effort which

* See p. 280 above.

is a real part of education. It is not an easy subject, and, taught with constant reference to the actual sounds on the voice or an instrument, is found to be an admirable intellectual training. Most girls have to learn the piano: many have a real taste for music, and if they took harmony in school instead of algebra, it would be much more real to them, since it would be related to something in their daily life, and in their thought and interest. The study of harmony, in a serious and definite fashion up to a good Matriculation standard, would not only train a girl's intelligence and reasoning power, thus doing for her what Mathematics does: it would also have a valuable effect on her musical taste, and would enable her to appreciate better music, and to enjoy music in a higher way.

For the exceptional girls, as we have said, more advanced mathematical study would be retained in the school curriculum and in the Universities, just as it is now, though in College Scholarship competitions more use might be made of the school record. But in the case of these exceptional girls, who are making Mathematics their main study, there is one serious danger to be guarded against,—narrowness, hardness, ossification. Women lead narrower lives than men, and follow with greater intensity and rigidity any purpose they have set before themselves. Thus a young woman at College, who is following a mathematical honours course, is much more likely to be hardened and narrowed by it than is her brother. In her leisure she will play games, but will hardly have enough surplus intellectual vigour spontaneously to read literature, or to talk politics and philosophy. If these women are to do all they should later on as teachers, administrators, organisers, they must have in their College life humanising influences. To take another tripos, or an Arts course at another University; to make oneself read poetry and history; to do social work; to develop the whole nature of the woman through relationships with friends and family: all these are correctives of that petrification, that stoniness of heart, which higher mathematical study may induce. The schools would be encouraged to do their part, if a high standard of requirement in a literary subject were compulsory for all mathematical scholarships, just as good Latin is required for historical scholarships.

For the ordinary girl, a moderate amount of mathematical study; Latin or Harmony as an alternative to Mathematics in Matriculation examinations: for those who will never go to College, and whose gifts are practical rather than academic, a very limited course of generalised arithmetic and elementary geometry: for the exceptional girl Mathematics as it is now, but with literary study compulsory to the end: such, briefly stated, are the lines on which the present writer would solve the problem of the place of Mathematics in the education of girls.

SARA A. BURSTALL.

HIGHER MATHEMATICS FOR WOMEN.

I have been asked, in connection with the work of the International Commission on the Teaching of Mathematics, to write a paper on the above subject, and my attention has been specially invited to the question whether, for instance, the study of Mathematics by women should have a special bearing on the subsequent study of Economics, or of statistical inquiries into sociological questions, and therefore perhaps differ in kind from the Mathematics required by, *e.g.*, a future engineer. I understand the expression Higher Mathematics to be used, not in the sense that mathematicians might give it, but merely to mean Mathematics as studied at the Universities by those who are at least to some extent specialising in the subject as distinct from the Mathematics which form a part of general education at secondary schools.

It will perhaps conduce to clearness if I state at once that, for reasons set forth below, it does not appear to me necessary or advisable to have separate programmes of mathematical study for men and for women at the Universities. If an inherent difference in mathematical ability between the two sexes were established it might seem at first sight to furnish a reason for a difference of programme. But experience has shown that some women have sufficient ability and sufficient liking for Mathematics to justify their making it their principal subject of study at the University, and this being so, the inquiry how their powers compare with those of men, either on the average, or as regards the highest degrees of mathematical ability becomes irrelevant to the question before us. For the great differences in ability among the men who specialise in Mathematics afford a range amply wide enough to include all the women who could on any hypothesis be reasonably advised to do so.

In considering the nature of the demand for mathematical education on the part of women, I may take as fairly typical the present (1911-12) students of Newnham College at Cambridge. This is convenient, both because information about them is easily accessible to me, and because the extent to which specialisation is carried at Cambridge, and the consequent sharp differentiation between the different Triposes, makes classification comparatively easy. I should explain that, with the exception of a few graduates from other Universities, and a few other students taking special courses, the women students

at Cambridge, whether at Girton or Newnham, are reading for one or other of the Tripos examinations, *i.e.*, examinations which, for men, lead to degrees in Honours. Out of 213 students at Newnham in November 1911, 30 were reading Mathematics; 2 Engineering; 35 Classics; 3 Moral Sciences; 1 Law; 33 Natural Sciences (including Geography); 39 History; 61 Medieval and Modern Languages; 9 Economics. Over 14 per cent. of the students therefore were working at Mathematics, and this proportion is somewhat less than has been usual in other years. Of the 30 reading Mathematics, 9 were in their first year, 13 in their second year, and 8 in their third year, at the University. A certain number will not continue to work at Mathematics through their whole course, but after obtaining Honours in Part I.* of the Mathematical Tripos at the end of their first or second year, will turn their attention to some other subject, *e.g.*, Science, especially Practical Physics for the first or second part of the Natural Science Tripos; or Geography for the Diploma in Geography; or possibly Economics for the Economics Tripos. They might even take some subject unconnected with Mathematics, such as History or Languages. The proportion of students of Mathematics is usually much the same at Girton as at Newnham; but probably the proportion at Cambridge is in excess of that among women students at the Universities generally, as the prestige of Cambridge in Mathematics no doubt tends to attract mathematical students there.

The next question to be considered is what leads these women to make a special study of Mathematics. It is almost always, I think, that they have liked the subject and succeeded fairly well in it during their school education; and have consequently been advised by their school teachers to go on with it at the University as a subject they are likely to do well in. If, as is frequently the case, they intend to become teachers, they think it is the subject they will like best to teach, and which they are most likely to teach well and to find profitable employment in. The estimate thus formed of their powers and tastes, either by the students themselves or their teachers, does not always prove correct. It happens occasionally that facility in manipulating algebraical formulæ has been mistaken for mathematical grasp. It happens, occasionally, that taste for the subject diminishes as the difficulty increases. It happens also, sometimes, that the amount of previous preparation required for success in the Cambridge course† by any but persons of unusual ability is under-estimated, and that a student who has come to the University with the intention of studying Mathematics, but insufficiently prepared, has to be advised to change her plans. It does not often happen that, on the other hand,

* The schedule of subjects for this examination is given as an Appendix to this paper.

† It must be remembered that in the Cambridge Honours course there is a strict time limit.

a student who comes up intending to devote herself to some other subject, changes to Mathematics.

If I am right in the above diagnosis of the motives that lead women to the study of Mathematics at Cambridge, the subject is in almost all cases studied mainly for its own sake—not because it is useful as an adjunct or stepping-stone to something else. And educationally it is very important that this should be so. A subject which is studied, not for its own sake, but because it is useful for something else, is almost always degraded in the process, and loses much of its educational value, whether the ultimate object be merely to pass an examination or to acquire the minimum knowledge necessary for dealing with some other and different subject of study. It is, no doubt, quite possible to learn Mathematics after a fashion, and up to a certain point, purely as an instrument for some ulterior purpose. But probably for everyone, and certainly for anyone with any mathematical ability, the loss from such a method of study would be great. There would be loss in mental training, loss in knowledge, and loss in interest. A recent writer in the Engineering Supplement of "The Times" advocating a continued attention to Mathematics for its own sake on the part of engineers, said:—"Its importance in education is to form deliberate judgment, to assign a measure to results, to disclose fallacies, to discourage narrowness, to reveal the unity of whatsoever things are true, and generally to exalt the mind."* I should be inclined to add, to stimulate the imagination. The study of Mathematics reveals to the eyes that can see whole vistas of knowledge, whole aspects of the universe, unsuspected by those without mathematical perception.

I would not, therefore, limit the opportunities of pursuing the study of Mathematics for its own sake in any way, either for women or for men. It must not, of course, be forgotten that among those who study Mathematics at the Universities a few—one here and there—may have sufficient power and originality to carry the subject in some direction or other beyond the present limits of our knowledge. But quite apart from this, it is important to keep before everyone, and before women especially, the value of knowledge for its own sake. I say before women especially, because I think the education of women has suffered more than that of men from what I may call, for shortness, the commercial or utilitarian point of view. The old bad education so prevalent in the girls' schools and the private schoolrooms of 50 or 60 years ago—drawing-room accomplishments, needlework, a little arithmetic and something of general information—was a kind of degenerate technical education for domestic life. The standard was low because neither thoroughness nor a high standard were needed for the

* The Teaching of Mathematics. "The Times," Engineering Supplement, October 5th, 1910.

purposes aimed at, and the standard tended to get worse rather than better because the teachers, being taught in the same schools, had no better opportunities of learning than their pupils. As a general result extremely little intellectual training or cultivation was, as a rule, attained, and very little interest was developed in anything beyond trivialities. I fear a return towards a similar state of things when I hear it suggested that Science should be taught to girls and women as the handmaid of Cookery and Hygiene, Mathematics as the handmaid of Economics, and Economics again studied because a knowledge of Economics may be useful in social work.

I do not, however, mean to imply that practical utility can be left out of account in education. The evils of doing this were exemplified in the arid education imposed on many boys by the almost exclusive study of Latin and Greek in boys' grammar schools long after these subjects had lost their pre-eminent position as keys to knowledge. In planning a scheme of general education regard must be had to the probable future work of the learners, both because preparation for this is necessary and because, where intellectual interest is difficult to stimulate, the hope of practical advantage may partly take its place as an incentive to the application required to make learning profitable. Besides, life is not long enough to learn thoroughly even all the subjects for which the student may have some taste and ability, and even in the selection of subjects for academic study sometimes one and sometimes another may have to take a purely subservient position, and Mathematics among the rest.

I suppose that the subjects of study directly subserved by a basis of Mathematics are Science—especially Physics—and Astronomy and Economics; and that the professions which may be prepared for at the Universities for which some knowledge of Mathematics is similarly needed are those of engineer, architect, actuary, statistician, accountant, computer, and last but not least, especially in the case of women, that of teacher of Mathematics. Among these subjects the study of Economics and social statistics associated with it is, of course, not unlikely to be taken up by women intending to engage professionally, or at any rate seriously, in philanthropic or social work; and it is quite possible that some women might wish to engage in any one of the studies and professions named. What is highly improbable is that any of them except teaching will be followed by as many women as men. It does not therefore seem necessary to make any special provision for the mathematical teaching of women with a view to future occupations. Any arrangements of this kind found advisable in the case of men will also serve in the case of women, if the latter are allowed to avail themselves of them freely. I do not think it can even be said that Mathematics suitable as a basis for Economics is more likely to be useful to women than the Mathematics required for Physics, considering the interest of the latter subject and its bearing on domestic affairs.

But among occupations and professions pursued by women who have had a university education, that of teaching is undoubtedly the most prevalent. If statistics were taken, I should expect to find that considerably over half of those who specially study Mathematics afterwards teach them. It will hardly be questioned that for good teachers of Mathematics as wide and thorough a knowledge of Mathematics as possible is desirable, for the teacher must be the source not only of knowledge but of enthusiasm and inspiration. By teachers especially is it important that the subject should be cared for for its own sake; and teachers must have a wider outlook and fuller grasp than they can hope to impart to any but their best pupils.

It may, perhaps, be objected that in considering the question of mathematical studies for women, at the Universities, and especially for those intending to be teachers, I have assumed too much that the mathematical foundation to be laid at school, and consequently the preparation required of the teachers who lay this foundation, should be the same in the case of girls as of boys. It may be argued that as girls are expected to learn more subjects than boys at school and as the hours spent weekly in school are generally fewer, the time devoted to some subjects must be less. If the time given to Mathematics be thus curtailed, may it not be advisable to split up the subject as it were longitudinally, still taking the pupils as far along certain lines. Would it be desirable, for instance, that girls should learn chiefly Pure Mathematics at School? Or should they learn chiefly Applied Mathematics? And should there be differentiation in the Mathematics taught in the highest forms, to suit the possible later studies or other work of the different pupils? Differentiation of a kind there will be, for some girls will have the time, the taste, the ability, to pursue their studies further than others. But the kind of differentiation implied in the above questions is to be deprecated. Pure and Applied Mathematics must go hand in hand at school or we lose much of the educational value of both. We should ask therefore how far can each girl progress in Mathematics, not in which direction should she progress. Without attempting to decide the important and difficult question whether the average girl should be expected to go as far as the average boy of the same age—a question which cannot be answered satisfactorily without more experience than I possess—it is, I think, safe to say that the general course for all who can go beyond the most elementary geometry and algebra should include elementary measurement of angles, elementary mechanics correlated with practical physics, the plotting of curves, while in the highest forms the essential underlying ideas of the calculus should be at least introduced. For this programme it cannot be doubted that it is a good general knowledge of Mathematics that the teachers should possess.

One point needs, perhaps, to be made clear. In saying that Mathematics should so far as possible be learnt for its own sake. I do not mean to imply that in all cases the whole available

time at the University should be spent in the study of Mathematics, as used to be necessary at Cambridge under the old regulations for the Mathematical Tripos. Those who are concerned with teaching Mathematics to women have welcomed the new plan which, by putting the examination for the first part of the Tripos at the end of the first or second year, leaves it open to the student to decide whether it is worth her while to proceed further with the subject, or whether she would gain more by widening her range of interests and devoting herself to some other study during the remainder of her University course. A woman who has done well in Part I. (of which the subjects are given in the Appendix below) should have the knowledge necessary for a good teacher of Elementary Mathematics in a secondary school, and if she means to be a teacher may find it to her advantage, from a purely professional point of view, to qualify herself to teach some second subject, rather than strive to make herself more of a mathematician than she is perhaps capable of becoming. For Mathematics is eminently a subject in which a student may reach a limit beyond which it is of little use for her to attempt to go.

To sum up, my conclusion is that in the first place it is highly desirable that women with mathematical ability, even if it be not of a very high order, should be encouraged to study the subject for its own sake, and not with limits prescribed by the utility of Mathematics for something else, for it is thus that they will get the utmost value out of it both in pleasure and in mental training. And I conclude in the second place that so far as it may be necessary to make arrangements for the study of Mathematics merely as a stepping stone to other studies, there is no need to consider the case of women separately from that of men.

I should like to add that in writing this paper I have had the advantage of knowing the views of the directors of mathematical studies both at Newnham and Girton Colleges, and that, I believe, they are both in substantial agreement with what I have said.

ELEANOR MILDRED SIDGWICK.

APPENDIX.

SCHEDULE FOR THE MATHEMATICAL TRIPOS.—PART I.

Pure Geometry.—Lines, planes, circles and spheres; including questions on reciprocation, cross-ratios, and inversion. The representation of solid objects on a plane by means of orthogonal projection and the methods of perspective.

Algebra and Trigonometry.—Exercises involving simple algebraic computation, the use of the slide rule and logarithmic tables being allowed.

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Elementary properties of equations. Use of the Binomial, Exponential and Logarithmic expansions; applications to trigonometric functions. Use and properties of trigonometric functions. Trigonometry of simple figures.

Analytical Geometry.—Lines, planes, circles, and spheres, excluding questions requiring the use of oblique co-ordinates. Elementary properties of conic sections, and of an ellipsoid referred to its principal axes, including purely Geometrical Methods in suitable cases.

Differential and Integral Calculus.—Differentiation and simple integration, partial differentiation. Simple applications to plane curves. Use of Taylor's and Maclaurin's Theorems. Maxima and minima for one independent variable. Curvature: plotting of curves from their equations. Simple linear differential equations.

Dynamics.—Mass, momentum, force, energy, moment of momentum, and their fundamental relations. Equilibrium without and with friction. Easy exercises on conservation of energy, of momentum, and of moment of momentum. Motion under gravity, and under simple central forces. Simple and compound pendulums. Elementary uniplanar applications to equilibrium and stability. Simple graphical Statics; funiculars, frameworks. Statics of liquids and gases.

Elementary Electricity.—The elementary parts of electricity; namely, properties of the potential; lines of forces; air-condensers; Ohm's law with simple applications; galvanometers.

Optics.—Reflection and refraction of light; applications to mirrors, prisms, lenses and simple combinations, excluding spherical and chromatic aberrations.

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PREFATORY NOTE TO VOLUME 27 OF SPECIAL REPORTS ON
EDUCATIONAL SUBJECTS.

The present volume is the result of the action taken by the International Congress of Mathematicians at their meeting in Rome in 1908, when, at the proposal of Dr. David Eugene Smith, Professor of Mathematics in Columbia University, New York, it was resolved to appoint an International Commission on the Teaching of Mathematics. This Commission was entrusted with the task of securing a series of Reports on the state and progress of mathematical instruction in various countries of the world.

There were certain difficulties in arranging for the collection of Reports in this country, and the Board of Education were approached and asked to give assistance in the matter. The Board were glad to avail themselves of the opportunity which thus presented itself of adding to their series of Special Reports on Educational Subjects a collection of papers dealing with mathematical teaching in this country. In order to assist the Office of Special Inquiries and Reports in the collection of suitable material, the President appointed an Advisory Committee consisting of:—

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This Committee acted at the same time as the British Sub-Commission of the International Commission on the Teaching of Mathematics. The Reports which have been secured through their assistance are therefore intended to serve a double purpose: on the one hand they enrich the series of Special Reports, while on the other they serve as the Report of the British Sub-Commission, which will be submitted to the International Congress of Mathematicians meeting in Cambridge this year.

The Board desire to take this opportunity of expressing their cordial thanks to the members of the Committee for the assistance they have rendered and to the various authors who have contributed papers to this and the accompanying

volume. In particular they wish to express their acknowledgment of the invaluable service which has been rendered by Mr. C. S. Jackson, Secretary to the British Sub-Commission, on whom has fallen the main work in connexion with the collection and revision of the papers included in this and the accompanying volume. Mr. Jackson has placed his time and knowledge most freely at the disposal of the Board.

The Board's thanks are also due to the authorities of those Colleges at Oxford and Cambridge who have courteously given permission for the reprinting of their Scholarship Examination papers. The Board are also indebted to the Delegates of Local Examinations, University of Oxford; the University of Cambridge Local Examinations and Lectures Syndicate; the Oxford and Cambridge Schools Examination Board; the Senate of the University of London; the Joint Matriculation Board of the Universities of Manchester, Liverpool, Leeds, and Sheffield; the Delegates of the Clarendon Press, Oxford; and the Syndics of the Cambridge University Press, for a similar service.

It may be said that the immediate cause which led to the formation of the International Commission was a divergence between mathematical and pedagogic requirements in the schools and Universities. This divergence is well illustrated by the papers which appear in the present volumes. It is hoped that their publication may contribute towards the diminution of this divergence, a task in which all who are interested in mathematical instruction are eager to co-operate.

Though the papers are published by the Board on the recommendation of the Committee, it must be clearly understood that neither the Committee nor the Board accept any responsibility for the statements and opinions contained in them.

Office of Special Inquiries and Reports,
August 1912.

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THE TEACHING OF MATHEMATICS IN EVENING TECHNICAL INSTITUTIONS.

The spread of Technical Education has been accompanied by a largely increased demand for mathematical teaching. There is no training more valuable to the technical student than that which enables him to perform the simpler mathematical operations, and to apply them to the work with which he is mostly concerned. But this fact will not be realised by the average student (who has no natural liking for Mathematics) until his student days are over, unless the class work is so conducted that his interest in some useful application of a process is enlisted to aid his struggle to understand the process itself. In technical classes, as in classes of other kinds, the good teacher adapts his teaching to the conditions of the students with whom he has to deal. For this reason the mathematical work done in the day, and more especially in the evening, classes of Technical Colleges and Schools will be found to differ in method from that adopted in the older and more academic institutions.

The University professor has his students provided for him, and may rest confident that he will retain them till the end of the session. He has a free hand to arrange his course, and need consider little but the interests of the subject itself. To some extent these considerations apply also to the mathematical lecturer in the day classes of a Technical College, but two new conditions modify his position. In the first place the majority of his students will be of a type not hitherto much attracted to University Colleges. The object of their studies is more utilitarian than that of ordinary undergraduates. Their educational outlook is in many cases narrower, and is certainly differently directed. It follows that the mathematical, and other theoretical, training cannot be carried to so advanced a stage, or be made so complete; while on the other hand the training which is given must be more thorough in the sense that the student must be made to realise the application to concrete examples of the various processes taught. Again this tendency is accentuated by the fact that the technical student has to spend time on many subjects which are absent from the course of the pure science student, and less time is therefore available for the more theoretical studies.

The above considerations are important in regard to the day classes of Technical Colleges, but they are much more so in connection with the evening classes held in the various Technical Institutions and Schools throughout the country. These evening students attend in very large numbers, and they vary to an extraordinary extent in requirements and in state of

2 *Teaching of Mathematics in Evening Technical Institutions.*

preparation ; while the limitations imposed by the conditions of evening work are most serious. In what follows I propose to confine my remarks entirely to the case of evening students.

The evening classes of Technical Institutions attract students of all social, and also of all industrial, grades. Sets of classes are arranged in the interest of students of different types. The sorting of these students into appropriate groups of classes presents no difficulty in practice. The student is left complete liberty of choice, provided he can satisfy the teachers that he is fit to follow the class-work of the course selected. But the ages and occupations of the different students vary so much that no clear idea can be conveyed of the work done in the mathematical or other classes without first referring to the three or four main groups into which we may sort the students. Each of these groups calls for separate consideration.

We may for the present purpose ignore the large number of evening students taking classes in languages, art, and other subjects not involving mathematical calculations. The remaining students may be divided into three groups, which from a mathematical point of view may be graded as low, medium, and high ; and the corresponding students may for convenience be respectively called artisans, engineers, and mathematical students. The lines of division between these groups are, of course, not sharply defined. By an artisan is simply meant a student aiming at perfecting his qualifications for some mechanical occupation. The word engineer is used in a very wide sense and implies anyone whose industrial work involves to some extent the use of precise theory and quantitative forecasting, while his post may be of any grade of importance. By the mathematical student is meant one with some natural aptitude for mathematical work, one who has overcome the initial difficulties, and who is willing to study Mathematics as a more or less isolated subject.

The percentages of the evening students belonging to the above three groups vary somewhat with the locality and size of the Technical Institution, but approximately one-third of the whole students may be classed as artisans, and about double this number as engineers. The mathematical students form but a small percentage—about 5 per cent.—of the total. But, though the percentage may be small, the actual number of such students is a large one. There are many Technical Institutions in the country dealing with over 3,000 evening students, and it is quite safe to say that in such schools there are more than 150 students who can be classed as good in the mathematical sense, and whose state of preparation is distinctly better than that of the ordinary undergraduate of a university. It must, of course, be borne in mind that these students are the best out 3,000. Even in regard to the 3,000 there is no legitimate reason for assuming natural ability below the average, and, moreover, it must be borne in mind that these students represent not the youths in the city who most need instruction,

but rather those who are willing to make a considerable personal effort to gain it.

ARTISAN STUDENTS.

The mathematical work done in the artisan classes of Technical Schools is of a primitive character. It is, indeed, so elementary that it would hardly be worth while referring to were it not for the fact that the methods on which these classes are usually conducted are well defined, and are worthy of note because to a certain extent they are also applied to the classes for engineering students. Experience has shown that these methods are both possible to carry out and essential to the success of artisan classes. They are most desirable though not quite so necessary for engineering students, and for such students they cannot be so completely put in practice. They can be most clearly explained in connection with the artisan classes.

Evening students attend in such huge numbers for the more elementary subjects, that in many cases it is possible, and when so it is always advantageous, to group them in accordance with the nature of their work, as well as to grade them in correspondence with the state of their training. It is much easier to do this with artisan students than with others, because the group of subjects desirable for a particular type of artisan to study is clearly defined, and is fairly evident to the student as well as to the teacher. Individual requirements do not vary to anything like the extent found in the more advanced work of the engineering students. Whenever sufficient students can be obtained to justify the holding of a special type of workshop class, there will also be applicants enough to justify special theory classes restricted to these students. In classes so arranged the teacher has the great advantage that all his students are interested in a particular kind of work, and can illustrate his theory by means of examples taken from the work itself. The hold of the theoretical teacher over the class is greatly increased if, other things being equal, he has some connection with the industry itself, or, failing this, if he can show a sympathetic interest in it. Experience shows that students, however utilitarian their views, can be interested in theoretical principles if useful applications of the theory can be shown at the time the principle is first explained. After this has been done the students are willing to work at examples of a more abstruse type, if they serve to make clearer the principle whose importance has been recognised. But if the teacher tries to teach the abstract principle before showing any concrete cases, his efforts often end in failure. These remarks are of general application, and apply to the mathematical classes of engineering students as well as to those of artisans. Indeed, it must be recognised that in all these classes something more than Mathematics is being taught.

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The actual work done in the artisan classes is most elementary, and in general consists of little more than arithmetical calculations concerning areas, volumes, weights, &c., such as are likely to occur in the daily work of the student. But in some cases the work is more advanced. Thus in the classes dealing with the metal plate industries the geometrical problems that arise, although simple as regards the main principles, involve at times most intricate detail in graphical working out.

ENGINEERING STUDENTS.

The students who for present purposes are alluded to as engineers form by far the largest portion of those attending evening schools. It must be remembered that they are of all ages above a lower limit, which is usually 14 years. The younger students for the most part do not attend the main Technical Institution of the district, but smaller evening schools working in association with it. At the larger Institutions the bulk of the students are over 18 years of age. The students, as far as possible, are urged to take definite courses of classes occupying three evenings per week, one of these being devoted to Mathematics. These arrangements can be more effectively carried out with the younger students (under 18) whose work is not much specialised, than with the older ones whose evenings tend to be more occupied, whose individual wants vary greatly, and for whom it is more difficult to arrange a course which will exactly fit the needs of sufficient students to form a class.

The youngest students (14 to 16 years of age) have in most cases recently left an elementary school, and are required to take a non-specialised course of classes the object of which is a training in *accuracy*, in the use of English, in the use of drawing instruments, in calculations, and in simple physical measurements. The mathematical work includes decimals, contracted methods, the elements of algebra and trigonometry, geometrical drawing, graphical methods applied to problems involving mechanical forces, and simple problems involving the relationships between the various physical quantities dealt with in the science class. In nearly all cases the results have to be worked out numerically. This course lasts for two years and is essentially the same in all Technical Schools.

Courses of the next stage are not quite the same in different Technical Schools. They are to a certain extent specialised, and are commonly associated with one of the departments of the School. They are in general suitable for youths who have recently left a secondary school. Normally the ages of the students vary from 16 to 18 years. The mathematical work occupies one evening per week, and is much the same whatever Department of the School the course may be associated with, except that the teacher can with advantage illustrate his work by selecting examples in which his students are likely to be

specially interested. It is developed on the lines of the syllabus of Practical Mathematics laid down by the Board of Education, though the stage is quite elementary. The main principle is that Mathematics is taught as one subject, and not as many. No attempt is made to carry any section of the subject far, and no section is avoided if it can be usefully introduced. Results are only recognised when reduced to numerical form. Every aid to obtain numerical results, or to express them clearly, is made use of. Thus the use of logarithms, the use of the slide rule, the graphical representation of results on squared paper, are all made prominent in the work, the range of which covers the elements of algebra, trigonometry, dynamics, and statics. The average student takes two sessions to complete the work of the course, but it not infrequently happens that the brighter and more industrious students do sufficiently well to be allowed to omit the work of one of the years.

Beyond the stage just indicated great variations are to be found in the methods adopted by different Technical Schools for dealing with the mathematical classes laid down for engineering students. There are many reasons for this. With the older students the attractions of their special technical subject are so strong, and their state of preparation in the theoretical subjects which should be associated with them is so varied, that though it is easy enough to draft good courses of classes, it is impossible in many cases to get the student to follow any one of the courses exactly as laid down. Some students make rapid progress with their engineering subjects, but are backward at Mathematics. These students are best dealt with by arranging special mathematical classes in immediate association with the chief engineering subjects. Other students who have some natural bent for Mathematics will join in preference one of the classes in Practical Mathematics dealing with the subject from a more general point of view. It must not be forgotten that several of the recognised sections into which engineering studies are divided are little more than Mathematics clothed in a garment of engineering terms, and that the teaching in such classes consists partly of Mathematics, and partly of instruction in exact ideas regarding the relations of various engineering quantities. The tendency in evening schools is to increase the number of classes so conducted, the syllabus being arranged primarily in the interest of the engineering subject, and the mathematical processes being introduced and explained as they are wanted for the examples which naturally occur in the class-work. While it is true that for the mathematically minded student such classes are but little wanted, it is also the fact that many students can only be induced to work at Mathematics in some such way, and that an appreciable portion of these are willing during the next session to join a class more exclusively devoted to Mathematics. The general effect of such classes is not to diminish, but to increase, the demand for mathematical classes of the ordinary type. The larger

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Technical Schools provide a number of mathematical classes of this associative type, in immediate connection with one or other of the chief departments. In one of the largest Technical Institutions in London all the classes are of this type, and no specific mathematical department exists although a fair proportion of the classes deal with Mathematics of quite an advanced stage. But this arrangement is exceptional, and is due to the fact that London contains many large evening Institutions among which the various sections of the work can (within limits) be divided whenever an agreement can be arrived at between the governing bodies. The provincial city contains but one Technical School, and this must provide for students of all types. There are always sufficient students in the larger schools to enable mathematical classes to be established of almost every kind that can be mentioned, and in many cases also to give a student a choice of nights on which he may join a class of a suitable stage and type. Thus at the Manchester School of Technology, besides the mathematical classes of the associative type just mentioned, and in addition to classes in Pure and Applied Mathematics suitable for undergraduate students (whose case is alluded to later), there are no fewer than twelve evening classes in Practical Mathematics. This subject is taught in four stages. The classes in Stage 1 are held on five evenings per week, those in Stage 2 on four evenings per week, those in Stage 3 on two evenings per week, while one class is provided in Stage 4. A similar statement is true of the Technical Schools in Birmingham and other cities. It is fortunate that the large demand for this subject renders possible extensive duplication of classes, for this greatly simplifies the time-table problem for the individual student.

There would be little use in giving a personal opinion of what the syllabus of class-work in Practical Mathematics should be, and still less in attempting to state what it actually is in the average case. The work does not depend on the syllabus, but on the teacher and on the students in the class. It varies in different schools, and with different teachers, and even with the same teacher in successive sessions. Whatever the subject taught there is no doubt for every type of student a best method of teaching, but this best method always demands the best teacher to carry it out, and always seems to depend more than any other method upon the teacher's personality. I have no doubt that the most useful and effective teaching in evening mathematical classes is done under the name of Practical Mathematics. I am equally confident that some of the worst teaching is done under that name. The syllabus may be the same in the two cases.

The teacher should have some knowledge of, and sympathy with, engineering problems, as well as a good knowledge of Mathematics. That certain personal qualities are desirable goes without saying. Originality and independence of text-books are the chief qualifications wanted, and are the most difficult

to obtain. The engineering student wants to study Mathematics as one subject, not as many. But mathematical studies have been divided by the text-books into sections which have been kept too rigidly apart and have been developed as if they were quite separate subjects. They are arranged in an arbitrary order of sequence, and it is generally considered necessary to proceed to a somewhat advanced stage in each before proceeding to the next. There is no harm in this—quite the contrary,—if it can be assumed that the student is going on. But for one student wanting Mathematics of whom this assumption is true, there are at least a hundred who need Mathematics in their work of whom it is not true. In dealing with the majority of such students everything turns upon the examples chosen to illustrate the processes. If an ordinary text-book is examined critically, it will be found to contain sections which are not really needed for the sequence of the subject. They contain groups of examples of a particular type, many of which are needlessly complicated, however excellent as exercises in manipulation. But expert manipulation is the last thing the student wants. He most needs confidence in working simple examples, together with a clear grasp of some physical application of them. He wants his results put into an easily intelligible form (graphical or numerical). There is no need to show an application for every example, or to reduce each result to a quantitative form, but many such examples must be given, and the rate of progress must be correspondingly slow. The distinctions between the various parts of Mathematics must as far as possible be ignored. Thus the subject of vectors should not be treated as advanced. Only the elementary properties are needed, and these may be made to illustrate the simpler properties of straight line figures in two and three dimensions in a way which is easily intelligible to the average engineering student. The subject should really be treated as an elementary part of geometry. The scalar and vector products of two vectors can be illustrated in engineering and physical science by many important examples which are of immediate interest to students.

The calculus should also be treated as a simple subject, and its study should begin at an early stage. The differential calculus, the integral calculus, and differential equations should be studied simultaneously and not in succession. The ideas of a rate, and the slope of a curve, can be introduced from the very first. Newton's notation for a differential coefficient can with advantage be used at first in preference to the ordinary one. (Indeed electrical students should make more use of the original notation of Newton than they commonly do, because it is simpler and more convenient for most of the examples in which they are interested.) In the earliest stage of the subject the simplest algebraic function is the only one which need be studied. In the later stages the exponential and sine functions should be added. The ordinary engineering student rarely

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deals with problems involving functions other than these three. Such students take so readily to graphical methods that in many cases the best way to introduce the subject of the calculus is a graphical one. Although this method has been described in several text-books it is not yet commonly realised how much can be done with the aid merely of a couple of set-squares and the squared paper now so much used in evening schools. The parallel lines of the squared paper should be made use of to save work. The student can be readily taught how to integrate any curve, and to solve a simple differential equation, without having to draw any construction lines, or any lines at all except the curve which is actually wanted. A very good preliminary acquaintance with the calculus can in this way be imparted to students whose previous mathematical studies have been most elementary. Where this method is taught I consider it preferable to introduce it at an earlier stage than is usually chosen. I have always found it a popular subject with engineering students. But like every other mathematical method, it is not to be recommended in all cases. In some classes it is not advisable to spare time for the half-dozen lessons needed to effectively explain and apply it. The best way of introducing the subject of the calculus is debateable, and dependent both on the class of student and on the individual teacher. As regards the higher stages of the subject the examples chosen should be restricted to those involving comparatively little manipulative skill, and needing only the simpler devices for reaching the solutions. Useful and interesting illustrations can easily be found in mechanical and electrical engineering. The subject of vibrations should receive a good deal of attention because of the increasing number of its important applications.

MATHEMATICAL STUDENTS.

In every Technical School there are some students who fully realise the importance of Mathematics, and are willing to study it without any direct reference to its applications. They form a small proportion of the total, but in the larger Institutions their actual number is considerable. They include some who have a natural bent for such studies, and others, such as teachers, who are working to obtain certain examination successes. They also include an increasing proportion of students who might never have been attracted to mathematical work at all had it not been for the influence of the more "practical" classes already referred to, but who, having attended such classes and realised the importance of the subject, are anxious to improve their manipulative skill, and to pursue the subject further than is possible in the slowly moving classes of the practical type. There is an impression among some teachers of Pure Mathematics that these practical classes are very poor things, and that if they affect mathematical teaching at all, they do it harm. Quite the reverse of this is the real truth. The

common experience of Technical Schools is that the enormous increase of students in the classes in Practical Mathematics has resulted not in a decrease, but in a very large increase, in the number of students attending classes in Pure and Applied Mathematics of the ordinary academic type. The "practical" classes do not draw students who would otherwise go to these academic classes, except in the more elementary stages. The fact is they bring into existence large numbers of extra students of Mathematics, and some of these, after passing through the classes of the former type, proceed to the more advanced classes of the latter kind.

In the larger centres there have always been a number of students working for the external science degree of the London University, and, now that engineering degrees have been established, the number of students working for degrees is rapidly increasing. Indeed there is a growing tendency for the more advanced engineering classes in Technical Schools to be carried on in such a way as to form a suitable preparation for a technical degree.

The students now under consideration form excellent material for mathematical teaching. In most cases they have had the benefit of a good secondary education, many of them are working for university examinations, and in a few cases the student already possesses a degree. Such students are really superior to ordinary undergraduates. They are older and more experienced. Evening class work conditions impose serious limitations to the rate of progress, but on the other hand there is no limit to the number of sessions the student can attend.

As regards the syllabuses of the subjects taught these follow the ordinary lines and there is no need to describe them. The usual methods of treating the various sections of Mathematics, the order of sequence in which these portions are commonly taken, and the ordinary text-books dealing with them, can be criticised from the point of view of the engineering student, and pronounced unsuitable in certain respects, but hardly any fault can be found from the standpoint of the students now in question. The objections and difficulties of the former type of student do not exist for the latter. It only remains to add that in the larger Technical Schools there are always to be found sufficient students to justify the existence of all the mathematical classes necessary for a University degree, and in certain schools the senior classes deal with work of the honours stage of such a degree.

THE PRELIMINARY MATHEMATICAL TRAINING OF TECHNICAL STUDENTS.

I.—INTRODUCTORY.

It is probably correct to say that nobody has done more to influence the teaching of Mathematics in this country during the last 15 or 20 years than Professor Perry; and in no branch of the work has he brought about greater changes than in the teaching of Mathematics to technical students. Until the introduction of his "Practical Mathematics," the mathematical teaching of students in Technical Institutions followed, for the most part, the ordinary conventional academic routine, the guiding syllabus being that devised by the Board of Education for their ordinary examinations in Pure Mathematics. The result was that very few of the lower grade of technical students took Mathematics at all, especially as at that time the modern system of group courses in technical schools was very little in vogue. But all this was changed by the introduction of Professor Perry's "Practical Mathematics" and its incorporation in the Board of Education's system of examinations. At the present time a certain amount of Mathematics forms an integral part of most group courses, certainly in all Engineering and Building courses. The number of students taking the Board of Education's Examinations in Practical Mathematics has steadily increased, while those taking Pure Mathematics has diminished. In 1910, in England alone, 6,964 students presented themselves in Practical Mathematics while only 2,841 presented themselves in the first three stages of Pure Mathematics.

It will be well, perhaps, to make clear at the outset what kind of work is included in the early part of this course in Practical Mathematics, as laid down in the Board of Education syllabus. This will also serve to indicate the usual first year's course in Practical Mathematics in most Technical Institutions, and should be borne in mind subsequently when the character of the previous training is under consideration.

The syllabus of the first stage* of the subject of Practical Mathematics is of an elementary character. The arithmetic consists chiefly of the manipulation of decimals, contracted methods, ratio, proportion and variation, with applications to percentages, interest, and the usual rules, together with the

* The examination in the first stage has now been abandoned by the Board of Education.

use of logarithms. The algebra includes elementary operations with symbols and formulæ, the solution of easy equations, easy factors and manipulations; there is a fair amount of elementary mensuration, of practical geometry and also of graphical work, with applications. This brief outline of the syllabus shows that there is not apparently a great deal of ground to cover if the time taken is, as usual, a year. Progress, however, has to be fairly rapid, or the Mathematics will not keep pace with the applications of it to the students' main subjects. The work also becomes somewhat disconnected, and unless great care is taken the student, after a time, is apt to be working in the dark, applying his Mathematics mechanically and, perhaps, successfully, but finding it more and more difficult to progress to higher work. This constitutes a real difficulty in teaching the subject, a difficulty which will always be present unless the preliminary training has been sound. There is always a danger that the superstructure in Practical Mathematics will be shaky unless it is based upon solid foundations laid before the student begins the study of "Practical" Mathematics.

The above remarks apply in the main to evening students. The position of day students is somewhat different. In general they have had a better and a more advanced preliminary training. They start at a higher level, and they progress more rapidly, since, of course, they give more time to the work. Although in some cases the syllabus of Practical Mathematics previously alluded to is followed, in general a more extended treatment is given, more "Pure" Mathematics being introduced. This is, of course, necessarily the case where the students work for a University degree, say, in engineering. But the problem of the preliminary training is equally important, and requires careful consideration.

It is the object of this paper, in both cases, to deal with the preliminary mathematical training received before entering the Technical Institution, and also to some extent with the earlier training within the Technical Institutions themselves, more especially in the cases of those Institutions which find it necessary to make provision for this preliminary training within the Institution. It is proposed, also, to deal only with those classes of students to whom Mathematics is of real importance, such as engineers, and not with those, such as textile students, who require, in most cases, very little Mathematics, and that of a special character.

In much of the following, more especially when dealing with the mathematical knowledge of students entering Technical Institutions, the writer has not relied merely on his own experience, but has been careful to obtain the views of a number of other mathematical teachers in all parts of the country. To those who have thus assisted him he is greatly indebted, and he offers to them his sincere thanks.

II. DAY STUDENTS.

1. *Character of Students and the training required.*—It is proposed here to deal only with those students who are taking whole-time courses, and not with those who attend for two or three half-days per week in such classes as “apprentices” classes, &c. These full-time students are those whose training should fit them to occupy the higher positions in industry—the future captains of industry. The character of their mathematical training must be regulated accordingly. It must aim at something more than mere ability to apply their Mathematics intelligently to the solution of problems which confront the engineer or the architect in his ordinary routine work. It must provide an instrument which, in the ever-changing conditions of modern industry, will always be available for the investigation of new problems, and whose flexibility will admit of adaptation to new conditions and new purposes. For this no mere knowledge of formulæ or ability to apply mechanically the knowledge obtained will serve; the student’s grasp of fundamental principles must be clear; he must not only have knowledge of processes, he must have insight into the reasons of processes; he must not be content to accept results, but, in most of his earlier work at least, should be trained to investigate proofs which should be of a fairly rigorous character. If these results are to be attained satisfactorily the preliminary training must be sound and comprehensive. The brief period of training in the Technical Institution and the limited time which can be there given to Mathematics, amid the wide range of subjects demanded by modern industrial conditions, do not allow of much attention being given to fundamental work; a certain amount of previous knowledge must be postulated, not necessarily extensive in amount nor specialised in character, but accurate, compact, and ready for further development.

2. *Previous training under present conditions.*—The majority of these day students come to the Technical Institution from secondary schools, while a few come from public schools. In certain provincial colleges where most of the students come from one or two large secondary schools in the same locality, there is a certain amount of homogeneity in the character of their work, but in general there is more variation, not only in the amount of mathematical knowledge attained, but also in the nature of their training. In a fair number of cases the previous training is satisfactory, especially where the teaching has been on modern lines, but in others the differences are so wide, and the deficiencies so marked, that some preliminary course within the Institute itself is necessary before a beginning can be made with the technical course proper.

Where faults exist they consist in the main of a lack of accuracy, both in working and thinking, inability to apply

knowledge to new problems, hazy notions as to fundamental principles, and a tendency to regard Mathematics as something aloof from the phenomena of every-day life. Their training has frequently been too academic in character; there has been too much stress on manipulation and too little on application. A few specific examples of criticisms which have reached the writer may be quoted:—

“Students admitted are very unequal in mathematical attainments. Arithmetic is generally satisfactory; about one-half can do algebra to quadratics, the other half know little. Perhaps 10 to 15 per cent. have done a little trigonometry.”

“The great fault which I always have to find is that apparently the students have never, or very rarely, been taught to think for themselves, and are greatly lacking in initiative in solving problems. Far too much reliance is placed on the use of formulæ. In the majority of cases also the students have little idea of arranging their calculative work concisely, systematically and clearly.”

“It is necessary in the schools from which these pupils come to be very particular about numerical accuracy and fundamentals generally. I might suggest also that those who teach algebra to boys should employ other symbols than x and y . The problem: ‘If $R_1 = R_0(1 + at)$; and $R_1 = 74.28$, $R_0 = 73.94$ and $t = 83$, find a ,’ will floor many a student who would be aggrieved if told that he could not solve a simple equation.”

“In many cases the teaching has been far too abstract. Knowledge of the rules is usually good and the handling of figures and expressions also, but I have found that many students do not recognise stereotyped expressions when occurring in a new connection.”

3. *Suggestions as to previous work.*—The criticisms in the previous paragraph carry with them certain suggestions as to methods, but recommendations as to the amount of previous work are, perhaps, desirable at this stage.

It is probably true to say that technical teachers do not desire that there should be any form of specialisation in the work of those who come to them from secondary and other schools. Apart from the impracticability of arranging for any such specialisation, it is undesirable on educational grounds; the technical teacher would also prefer that any specialised work should be done in the Technical Institution itself, and that the student should come to him with a sound training in the elementary parts of what is termed “Pure Mathematics.”

It would not be possible to draw up a syllabus of work which would meet the ideas of every technical teacher, but the following probably represents the amount of mathematical training which most would like to see accomplished before entry to the Technical School:—

Arithmetic.—The ordinary compound rules taught experimentally, accuracy in measurements being essential; vulgar fractions without

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elaborate manipulations; thorough treatment of decimals, approximate measurement, approximations, contracted methods properly taught, decimalization of money; averages; ratio, constant ratios, proportion and variation, with applications to percentages, interest, &c.; square root; logarithms; proportional division. Accuracy in computation is most important.

Mensuration.—Taught in conjunction with arithmetic and geometry, and to include elementary mensuration of quadrilaterals, triangles, polygons, circles, spheres, prisms, and pyramids.

Algebra and Trigonometry.—The syllabus outlined in the report of the Mathematical Association (1912) would probably meet the approval of technical teachers, though most would like the addition of progressions, variation, and possibly an elementary treatment of imaginaries on lines suggested by Mr. G. St. L. Carson in an address to the Mathematical Association in January 1912.

The amount of trigonometry in the syllabus alluded to is not made clear; probably it would not be sufficient to please the technical teacher who would like the pupils to be acquainted with methods for the solution of triangles, circular measure of angles, sum and difference formulæ, formulæ connected with $\sin 2A$, $\cos 2A$, &c. Identities should not be included. The work in the main should be concrete, and applied to practical problems. The student should also be able to use tables rapidly and correctly.

The technical teacher welcomes with satisfaction the early treatment of the calculus on sound and rational lines, which is now finding a place in the curriculum of many of our best secondary schools.

Geometry.—The modern treatment of geometry meets with the emphatic approval of most technical teachers who find in it an excellent basis for future work. There are, of course, complaints as to the poor results in many cases, but this is perhaps inevitable in a time of transition. There is a general feeling that solid geometry should find a place in the curriculum of every secondary school, even if the treatment is mainly experimental. A great deal can be accomplished in connection with the mensuration.

Simple but clear ideas on plans and elevations are also desirable, but not essential.

Mechanics.—It is desirable that students on entering should have a knowledge of the fundamental principles of statics, dynamics, and hydrostatics, but the subject should have been treated experimentally.

The intelligent use of squared paper, and a proper knowledge of graphs, not mere curve tracing, is most desirable in all branches of the work.

If the amount of Mathematics contained in the foregoing syllabus is considered too much for the average student leaving the secondary school at the age, say, of $16\frac{1}{2}$ or 17, then the parts which can best be dispensed with are the mechanics and part of the trigonometry. In most of the American Technological Schools, the entrance examination requirements consist of arithmetic, algebra (of about the amount indicated above), plane and solid geometry, and in some cases, trigonometry. In view of the fact that so much of the dead wood which has hitherto encumbered our elementary mathematical teaching has now been cut out, the syllabus indicated above does not seem to be too extensive; but if it is, then some of it may be sacrificed so long as we can get thoroughness and sound knowledge.

4. *Entrance Examinations.*—A number of Technical Institutions require a student to pass an entrance examination before taking up an organised course of work. This examination is usually excused if the student can show that he has passed some standard examination, such as the matriculation examination of a University. Some Institutions content themselves with stating that the candidate must give evidence that he has had a good general education. The institution of some sort of standard school leaving certificate for secondary schools would prove very useful in this respect. Where entrance examinations are imposed, Mathematics is always considered essential, though the amount required is usually very small. It would seem to be hopeless at present to set up a standard which in any way approached the course described above; if such an amount were required the number of day students admitted would be very seriously diminished. One of the most comprehensive syllabuses for such entrance examinations is that of the Manchester School of Technology which is briefly as follows:—*Arithmetic*—fundamental rules, including vulgar and decimal fractions, and square root. *Algebra*—to quadratic equations, elementary theory of indices, surds. *Geometry*—the substance of Euclid I.—III., and VI. (1–19). *Trigonometry*—trigonometrical ratios, functions of angles of any size, relations between the sides and angles of triangles—use of four-figure tables. There is also an examination in elementary mechanics, including hydrostatics.

When it is considered that the standard of this entrance examination is much higher than in most other cases it will be seen that at the present time the general standard for entrance is very low and is, to some extent, a measure of the amount of Mathematics generally taught in secondary schools.

The entrance examination for the Imperial College of Science and Technology is as follows:—*Arithmetic*—ordinary rules, decimals, square root, use of logarithms; special value attached to exact calculations with decimals, especially by abridged methods. *Algebra*—to quadratics, elements of permutations and combinations, progressions, indices, logarithms. *Geometry*—subjects of Euclid I.—IV., similar figures. *Trigonometry*—to solution of plane triangles. *Mechanics*—velocity, acceleration, their units, their composition, their moments. Mass, force, weight. Rectilinear motion under gravity. Work and energy. Composition and resolution of forces at a point. Equilibrium of forces. Special importance will be attached to clearness in the conception of mechanical quantities.

There is also an optional paper, of a more advanced character, in co-ordinate geometry of the straight line and circle, and elementary calculus.

The syllabus for this examination approximates somewhat to the one drawn out above. No solid geometry is included however.

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5. *Preliminary training within the Technical Institute.*—For reasons dealt with above most Institutes make provision for a preliminary training in Mathematics for their students. This preliminary course is not imposed upon all students, those who give evidence that their previous training has been adequate, being excused. The number of such students is, however, small. The preliminary course followed consists generally of a review of the work supposed to be done before entering the college, but, naturally, it is of a more technical character, and the applications are more specialised. In some cases the slide rule is introduced, and special attention is devoted to mensuration and to the use of squared paper. This preliminary work, as has been previously mentioned, is in most cases largely influenced by Professor Perry's methods. It is often modified, too, to meet the requirements of universities in those institutions where preparation for university work is undertaken. The following are outline syllabuses of a few typical institutions:—

Manchester School of Technology.—Approximations and computations; logarithms; use of slide rule; areas; volumes; graphs, equations of straight line and circle; exponential and logarithmic functions; trigonometry, to solution of triangles; simple series and expansions.

This is a first year's course, common to the Engineering, Chemical, and Textile Departments, the subsequent courses depending upon the work followed.

Battersea Polytechnic (London).—Algebra, to quadratics; graphs; logarithms; geometry, the important results of elementary plane geometry; trigonometry, trigonometrical ratios, with applications to heights and distances; plotting of statistics.

Bradford Technical College.—Arithmetic, contracted methods, approximations, logarithms, slide rule; mensuration of plane figures and solids; algebra, to quadratics; trigonometry, trigonometrical ratios with applications to heights and distances; graphs of easy functions.

[Some students are allowed to take this preliminary course in half a year.]

Halifax Technical College.—Arithmetic and mensuration, the usual course, with the addition of Simpson's rule; algebra, to progressions; logarithms; graphs; geometry, the substance of Euclid I-IV.; trigonometry, trigonometrical ratios, application to solution of right-angled triangles.

Swansea Technical College.—Arithmetic, computations, logarithms; algebra, to quadratics, graphs, ratio, proportion, variations; trigonometry, to solution of triangles, heights and distances.

Sunderland Technical College.—Algebra, to progressions, variation, logarithms; geometry, Euclid I-VI.; trigonometry, to solution of triangles.

South Western Polytechnic (Chelsea).—Arithmetic, short course, including contracted methods, proportion, logarithms; algebra, to simultaneous equations; geometry, elementary, including easy solid geometry; use of squared paper.

The Imperial College of Science and Technology requires that students taking the Associateship in chemistry, botany, zoology, geology, mining, metallurgy, mechanics and physics should take the course given below. This includes the minimum amount of mathematics necessary for the first six subjects, and is a preliminary course

to four others which must be followed by students in mechanics and physics.

Rapid methods of making approximate numerical calculations; use of logarithms, algebra as far as quadratic equations with special reference to the simplification of algebraical expressions; practical uses of squared paper; finding empirical formulæ from tabulated observations; the elementary parts of plane trigonometry; the solution of triangles; heights and distances; the simplification of trigonometrical expressions involving the sums and differences of angles; the slope of a curve at a point on it; meaning of "Rate of Increase" and "Rate of Decrease," and how these are measured; the meaning of the symbols $\frac{dy}{dx}$ and $\int y dx$, with simple applications.

This last course is given in full as being of special interest, since it is Professor Perry's own idea of a preliminary course.

III.—EVENING STUDENTS.

(Including part-time day students.)

1. *Character of the Students and the Training required.*—In the case of evening students there is even greater diversity in the ages and attainments of the students on entering the Technical School than is the case with day students. The evening technical students are, for the most part, of the artizan type, who are working at or learning their trade during the day. Their ages on entry vary considerably from boys of 14 or 15 to adults of 30 or even more. In the provinces the number of adult students thus entering is generally less than in London, for reasons which will appear below.

The Mathematics required by these students is in general of a different character and less in amount than in the case of day students. The number who proceed to higher work is comparatively small and consequently the amount of Mathematics required by most is not great. It must be sufficient to enable the students to grapple with the mathematical work which occurs in their collateral subjects, as well as the calculations and applications which may be incidental to their everyday work. It is therefore of a different character from that required for day students and requires different handling. It is more practical in its character, more direct in its applications and more disconnected in its treatment. The difficulties are increased by the limited amount of time at the disposal of both teacher and student and also by a certain amount of reluctance on the part of many students to take a subject whose utility is not immediately apparent to them. The difficulties arising from these circumstances are enormously increased if the earlier training has been unsatisfactory, indeed it is probably true to say that in many cases they are never overcome and that there is a continual leakage of students from the Mathematics classes due to the fact that their previous training has not enabled them to follow the course in the Technical Institution.

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2. *Their previous Training.*—The majority of these students have been trained in the first instance at elementary schools, the number coming from secondary schools not being large. Where they do come direct from secondary schools their work is in the main, satisfactory. The careers of those from elementary schools between the time they leave those schools and enter the Technical College, vary very greatly, this variation is in large measure due to the lack of continuity in our educational system and to the variations in the local organisation of evening school work. In the absence of any system of compulsory continuation schools there is, in a large number of cases, a gap between the time of leaving the day school and the taking up of work again at an evening school. In most of our provincial towns provision is now made for continuity of the work. The boy on leaving the elementary school is urged to join the evening continuation school at once; the course he then takes depends upon his work. If he will ultimately enter the engineering or building or a similar course in the technical school, he is urged to take a preliminary technical course. In some towns this course is followed by a junior technical course in a branch Technical School. In these preliminary technical courses Mathematics of a practical character is an essential feature; further particulars of the course will be given below. This continuous and properly organised work is admirable; but unfortunately it is seldom compulsory; a youth is *strongly advised* to follow out a training of this character but it is not enforced. At Manchester, where there is an admirably co-ordinated system of evening continuation schools and preparatory technical classes leading up to the School of Technology, attendance at them does not seem to be compulsory; a student may not enter the School of Technology below the age of 16, but once over that age he is admitted and if he is not in a fit state to begin on the technical course proper, he is placed in preparatory classes held in the school itself. Apparently then he may have been taking no classes at all between the times when he left the elementary school and joined the School of Technology. In one or two other schools, *e.g.*, Birmingham, the age of entrance is also 16 and attendance at the preparatory technical classes is required or else a student must satisfy the Head of the Department concerned that his education has been such as to prepare him to take up the course he requires. Other towns provide an admirable system of preparatory work but make no effort to enforce it on the student desiring to enter the Technical Institution.

In London, with the exception of a self-contained district like Woolwich, there is no system at all. There are some classes in the evening continuation schools where work of a preparatory technical character is done, but there is no co-ordination or linking up whatever with the Polytechnics and Technical Institutions. It is, I believe, correct to say that no entrance examinations are insisted upon, and apparently all

students enter at once on the first year's course, or if not fitted for this, on a preparatory course within the building itself. Thus a large number of the students have received no education between the times of leaving the elementary school and joining the polytechnic, consequently, as the average age of entry is probably well over 16, and perhaps nearer 18, there has been a big gap in the education of the youth. During the time covered by this gap most of the work done in the elementary school has been forgotten, or is remembered in such a vague and inaccurate way that it would be better forgotten altogether. This brief general account of the previous training is offered here as it is obviously of the utmost importance to get some idea of the general problem before proceeding to the discussion of the mathematical training alone.

3. *Mathematical Training before entering the Technical School.*—(a) *In the Elementary School.*—For reasons given above it is not always easy to judge of the mathematical work done in the elementary schools from the point of view of the technical school, though this is often possible in the case of part-time day classes. In general, this previous work comes in for severe criticism, more severe than in the case of secondary schools. The general opinion seems to be that there is a very grave lack of thoroughness and accuracy, ascribed by some to the wide range of subjects which the teacher is expected to take in these schools. A few examples are given of criticisms received from experienced and responsible teachers. The first four are from teachers in large London technical institutions:—

“The chief defects in the mathematical work of students coming from primary schools are absolute lack of accuracy, extremely untidy work, and the absence of any habit of checking a result in any way. At present, the hopeless floundering of the ex-elementary school boy in dealing with figures is the most important and disagreeable fact to be faced.”

“They have no idea as to accuracy in arithmetical or algebraical work, and do not know how to check results.”

“I am of opinion that they are all deficient in arithmetic, especially decimals and fractions. I think that primary schools would do well to devote more time to fundamental principles and accuracy of work, and leave much of the other work to the evening schools.”

“Arithmetic is unreliable, and they have no idea of how to apply ‘rules’ to actual problems met with in Engineering and Building.”

Criticisms from the provinces are not quite so emphatic, though all agree that the arithmetical training is not satisfactory. A few actual criticisms are appended:—

“The chief defects consist in a lack of concrete knowledge. The work has been too bookish, and there is no fundamental grip of arithmetical principles. Everything points to a lack of experimental work.”

“The usual mathematical training seems to leave the students lacking in the power to apply their work even in the simplest of cases. Again, too much stress is always laid on vulgar and too little on decimal fractions.”

“ There is little correlation of arithmetic with practical mensuration and geometry; the students have no notion of approximation in results or methods of checking.”

These criticisms are given as typical, and might easily be multiplied. They indicate a consensus of opinion that the teaching of arithmetic in elementary schools cannot be considered satisfactory, if we are to judge by the product, as we have them in the technical schools. The test may not, of course, be a fair one, but the important and serious fact remains that the teacher of Mathematics in the evening classes of technical schools is at the present time seriously handicapped by the defective earlier mathematical training of the student.

There are welcome signs, however, that the teaching of arithmetic in elementary schools is now being placed on a more satisfactory basis. In an increasing number of schools it is evident that the teaching is proceeding upon good modern lines, and the results are very satisfactory to the technical teacher. There is no suggestion that there should be any attempt at specialisation, so as to meet the requirements of any subsequent technical training; the work which the technical teacher would like to see done is of a character which would meet with the approval of most modern teachers of Mathematics. He would like to see a better grip obtained of the fundamental principles of arithmetic, with increased power to apply them to new conditions and new problems. Decimals should be treated very fully to the exclusion of much of the work now given to vulgar fractions. Mensuration should be developed with as much practical work as the circumstances of the school will allow. Many of the so-called “ rules,” *e.g.*, discount, stocks and shares, the H.C.F. and L.C.M., &c., might profitably be abandoned, or where they are at all necessary, cut down to the minimum. A clear understanding of a ratio and a proper grasp of what is meant by proportion are very important afterwards and, if treated properly, may be mastered at an early age. Most technical teachers would probably be well content if pupils came to them with a clear idea of a fraction, a thorough knowledge of decimals, a good grasp of proportion, with the ability to work accurately and neatly, and with the power to tackle new problems.

(b) *In Branch Technical Schools.*—The mathematical work done in the preparatory technical classes, whether in branch technical schools, or evening continuation classes, is, on the whole, satisfactory and the pupils who go through these classes are able to enter at once on the first year's course in the Technical Institution. The teaching is generally carried out by elementary teachers, often under the supervision of the Technical Institution, and the work is usually well done. There is some complaint that the methods are not always good, and the work is inclined to be bookish, but, on the whole, the superior work done by pupils who have passed through these classes as compared with those who have not, emphasizes the necessity of

insisting upon all students passing through such a course, the only exception, perhaps, being in the case of students who have recently left a secondary school at the age of 16 or upwards. A few specimen syllabuses are given showing the kind of work done in these preparatory classes. As the pupils in these classes come, as a rule, direct from the elementary schools, the syllabus of work, which is considered necessary, is some indication of the work previously accomplished.

Manchester.

First Year.—Decimals, English and metric measures; vulgar fractions—easy operations, various rules illustrated by measurements from solids and drawings. Powers and roots—simple exercises. simple averages and percentages. Areas, rectangle, triangle, circle, surfaces of simple solids, volumes. Algebra—use of symbols; meaning and use of signs, simple operations; easy simple equations. There is also a course in practical drawing involving easy geometrical constructions and simple solid geometry; also hand-sketching is practised.

Second Year.—Revision of decimals, contracted methods of multiplication and division. Revision of vulgar fractions; powers and roots. Harder exercises in areas. Ratio, proportion and variation, treated arithmetically; volumes of regular solids; algebra—revision, simple factors, fractions; simple equations and problems, graphs. There is a further course in practical drawing. Three hours weekly is given to this work in each year.

The above courses are taken in evening continuation classes. Third and fourth year technical courses follow at branch Technical Schools, before proceeding to the School of Technology.

Third Year.—A thorough course in logarithms; elementary algebra to factors and fractions; a harder course in mensuration, meaning and use of trigonometrical ratios, solution of a right-angled triangle; solid geometry, plans, elevations and sections, similar triangles.

Fourth Year.—The use of logarithms, contracted methods; further mensuration including frustra, practical determination of areas (Simpson's rule), &c.; graphs; elementary trigonometry, including solution of right-angled triangles with problems; algebra to quadratic equations and indices.

The student who works through this course comes to the Technical School excellently equipped for his work there, but there are very few Technical Institutions in the country which can hope for students who have thus had a thorough four years training before entering the institutions. If the above course has a fault it is that the amount of ground covered does not seem very large for four years' work. It may be presumed that most of the pupils come direct from the elementary school and in that case the greater part of the first year's course should be unnecessary. All students are not, however, compelled to begin with the first year's course, but are placed according to their attainments.

Birmingham.—Here students under 16 years of age must take a two years' preparatory course at a branch Technical School and the course is explicitly intended for those who have recently

left an elementary school. The Mathematics course is called "Experimental Mathematics," and the syllabus is as follows:—

"Use of decimals and contracted methods of multiplication and division. The working out of simple formulæ; the elements of algebra and trigonometry. The use of graphical methods in working out numerically the relations of force, &c. The measurement of areas and volumes, &c. Simple experiments in mechanics and the numerical working out of results."

This syllabus is somewhat vague as to the amount of work to be covered, but judging by the syllabus of the elementary classes within the technical school itself, it is evidently not anticipated that a great deal of work will have been done in the course.

Halifax.—The organisation in Halifax is similar to that of Manchester, in that the preliminary course, before entrance to the Technical School, is a four years' one. It is also intended for boys leaving the elementary schools. The first two years are called preparatory courses, and are common to all kinds of students; they are followed by two years' specialised courses, "Industrial," "Commercial," &c. The Mathematics is termed "Experimental Mathematics," and is of a very elementary character. In the first two years the syllabus includes:—

Vulgar and decimal fractions, square root, simple graphs, unitary method and percentages, simple mensuration and simple equations. The third and fourth years include easy graphs and their equations, simultaneous equations, multiplication and division in algebra, factors, indices and logarithms, mensuration of simple plane figures and solids, similar triangles, trigonometrical ratios, speed velocity, parallelograms of displacements and velocities, freehand sketches of plans and elevations.

The above three syllabuses are given as typical of what is probably the best work of the kind in the country, at least as far as organisation and co-ordination are concerned. In most cases the syllabuses cannot be considered as being entirely satisfactory. Of course much depends on the way the matter is handled by individual teachers, but generally speaking the courses seem somewhat scrappy. They do not appear calculated to give the student that grasp of fundamental principles which so many technical teachers declare to be wanting in the students who come to them from other sources. Too often it seems as if isolated topics are picked out and put into the syllabus merely because of their subsequent utility. This may often be necessary, but where the course is taken in leisurely fashion, as it obviously is in some of the four year courses, it ought to be more closely knit together. In particular the treatment of algebra cannot always be considered satisfactory. Too often it consists apparently in that conventional, academic treatment of the beginning of algebra which has obtained for so long in our educational system, and which is so meaningless to the beginner. Substitution, meaning and use of + and —, addition, subtraction, multiplication, division, use of brackets, &c.—

the usual lifeless procedure—all exist still in places where one would hardly expect to find them, where one ought to find the algebra made into a living thing, an efficient tool. Even graphs frequently receive that unsatisfactory treatment which is only too often to be found in other classes of schools at the present time. “Simple simultaneous equations treated graphically and algebraically,” “Graphical solution of quadratic equations” are items all too common in these syllabuses, and seem very frequently to sum up the complete treatment of graphs at this stage.

4. *Preliminary Training in the Technical School.*—As indicated above, there are, unfortunately, some parts of the country, notably London, where no proper organised effort has been made to link up the evening work of the Technical Institutions with that of the evening continuation schools, and through them with the elementary schools. In London, with the possible exception of Woolwich, branch Technical Schools are non-existent, and such work of a preliminary technical character as exists in the evening continuation schools is not in any way linked up with the Polytechnics or Technical Institutions. The result is that there has been a deplorable gap in the education of a number of students who enter these Institutions; in many cases they have no mathematical knowledge whatever, except, perhaps, an unreliable smattering of arithmetic. In the absence of any branch Technical School the needs of these students have to be met by preliminary classes within the Technical Institution itself. Even in towns where the branch Technical Schools are efficiently organised, students, especially older students, are not compelled to pass through them, and preliminary classes are, again, provided within the Technical Institution. This means an overlapping of classes, but it is difficult to see how it can be avoided unless attendance at continuation classes is made compulsory.

There is one great advantage in having the preliminary work done within the Institution itself, viz., that there is continuity in the work. There is no break of system, such as is in some degree inevitable, in passing from one school to another. The work done dovetails in better with the work which follows, but, on the other hand, it has to be done more hurriedly. The syllabus of work is usually similar in form to that in the branch Technical School, though, perhaps, slightly less academic in character.

IV.—GENERAL SUGGESTIONS.

The preliminary training of technical students in Mathematics is a matter of great importance to the Technical Institution. At the outset of his career the young technical student, an engineer for example, does not realise the value of Mathematics in his work. Most of the other subjects of the course appeal

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directly to him, for their immediate application to his work is more or less obvious. But there is a tendency to shirk Mathematics, and when the student is pressed for time or overburdened with other studies, this is usually the first subject to be thrown aside. More especially is this the case with those students who have had a gap in their work. During this time the youth has been engaged at the works and has been gaining some practical acquaintance with some of the subjects that he begins to study on entering the Technical School, but his Mathematics has usually been neglected all the time. There is therefore considerable danger of a large number of students falling behind the others, and as soon as this happens, and they cease to follow the work, the class sees them no more. It is therefore essential that the work should be made as attractive as possible, as practical as possible, and as much individual assistance given to backward students as circumstances will allow. To effect this the following suggestions are put forward. They are methods which have been tried by the writer and found to be fruitful in good results.

In the first place it is desirable that some of the work should be carried out in a mathematical laboratory, or, if that is not available, under circumstances which will admit of practical work being done. The equipment need not be elaborate and much of the apparatus required can be contrived within the Institution itself or obtained with ease. For example, the objects used for the work in mensuration should as far as possible be objects in common use, and they should be actually handled and measured by the students. The measurement of a piece of gas-pipe by the student to illustrate a hollow cylinder is far more effective than any amount of blackboard work, and far more convincing to the student. The principles of proportion are best driven home by simple experiments with stretched springs, home-made levers, swing of a pendulum, &c. In such ways as these the subject can be rendered more attractive and will arouse the genuine interest of the student.

To render the work of immediate and practical value, the co-operation of colleagues taking collateral subjects should be sought, and as far as possible there should be co-ordination with other subjects in the technical course which the student is following. It is very helpful if the Mathematics teacher can obtain from the teacher of the principal subject of the course, say electrical engineering, problems on which he is engaged from time to time, so that the Mathematics involved should be treated concurrently.

The difficulty of the backward pupil can in large measure be surmounted by the institution of the "tutorial class." This should be a class in which there is no lecturing, and generally speaking, no collective work, but in which the whole time is devoted to giving individual assistance. Difficulties arising out of the lectures, out of home work, or encountered in the

students own reading, may be dealt with in this time. The best time for the tutorial hour is immediately following the lecture, when difficulties of the lecture can be dealt with at once and the chief points emphasized by suitable examples. At the Polytechnic, Regent Street, there has been instituted a series of "preparation papers" for use in the tutorial hour. These consist of sheets of problems and examples carefully selected to drive home the work covered in the lecture. They are distributed to the students who select those parts of the work in which they are weakest. There is an additional advantage in the fact that the better students proceed at once to the more difficult problems, while the weaker ones start with the easier ones. If the tutorial class is to be really effective extra assistance should be given to the lecturer in charge of the class. When the number of students is large, it is obvious that very little time can be given to each individual student and so for this hour additional teachers should be put on, according to the numbers. The extra outlay will be well repaid.

A plan adopted by some is to combine the lecture and tutorial class, and two or two and a half hours are occupied in this way. The lecturer takes a fresh piece of work and after the necessary explanations have been given, he at once sets examples for the students to work. Then with the aid of one or more assistants he goes round among the students and helps them over their difficulties. When he sees that the work is mastered by the students he proceeds to the next step. This method is very effective if properly handled; difficulties are dealt with immediately they arise, the interest of the student is maintained, and progress is more sure.

V.—CO-OPERATION OF TEACHERS.

In the preliminary training of technical students there are two factors which are essential for ultimate success. One, the co-ordination of the work in Technical Institutions with that of the evening continuation schools has already been dealt with. The other is almost as important; it is the co-operation of the various classes of teachers who are affected. There are four such classes, elementary teachers, secondary teachers, teachers in evening continuation schools, and technical teachers, and in the opinion of the writer, no completely satisfactory solution of the problems involved will be reached until co-operation between these different classes has been secured. It is most desirable that in each locality the technical and secondary teachers should confer on problems affecting the day technical students, while co-operation between technical teachers, evening continuation school teachers, and elementary school teachers is essential if we are to obtain a proper sequence of work for evening students, with a minimum amount of disturbance on transference from one kind of institution to another.

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To provide for this co-operation, I would suggest the formation, in each locality, of an advisory committee of studies for Mathematics, composed of representatives of the different classes of teachers concerned. If such committees can be got to work, many misunderstandings will be cleared away, and many of the difficulties in the preliminary training of technical students will be removed.

P. ABBOTT.

GEOMETRY FOR ENGINEERS.

Much has been written on the "Education of Engineers," and many divergent views have been expressed as to the relative importance of the many subjects which have a more or less direct application to the work of engineers. There is, however, a general consensus of opinion that a fair knowledge of Mathematics is essential to all engineers and that there is a field in engineering for men having a wide and profound knowledge of that subject.

Of the various subdivisions of Mathematics, geometry is one of the most important to engineers. In fact some knowledge of geometry is necessary to the proper understanding of many workshop operations, and since geometry is the foundation of mechanical drawing, without which the great achievements of the engineer would have been impossible, it follows that the importance of this subject can hardly be overrated.

As a simple illustration of the use of a knowledge of geometry in the workshop, consider the operation of turning a conical plug in a lathe. In this operation the material is placed between the lathe centres and rotated, while the point of the cutting tool is made to move in a straight line inclined to the axis between the centres. Now it is found that if the point of the tool is not exactly at the same level as the lathe centres, the finished plug does not have a straight or regular taper, but is curved in the direction of its length. This has mystified many a skilled workman, but it is easily explained. The surface of the plug is a surface of revolution, and, in order that it may be conical, the line described by the point of the cutting tool must, if produced, intersect the axis of the plug, otherwise the surface of the plug will be an hyperboloid of revolution which is curved in the direction of its length or axis.

Granted then that some knowledge of geometry is indispensable to the engineer, it becomes of interest to consider over how much ground it is desirable that the student who intends to become an engineer should be taken, and, incidentally, how the subject should be presented to him. In considering this question two points must be kept in view, (i) the great educational value of a proper study of geometry, and (ii) the requirements of the engineer in the matter of geometry.

Bearing on both of these points is the necessity for careful and accurate mechanical drawing, which should accompany the study of the principles of geometry from the very beginning. In this paper geometry is being considered from the engineer's point of view, but the writer would venture to suggest that in connection with the mathematical department of every school and college there should be a drawing office which should be to

that department partly what laboratories are to the chemical and physical departments. By mechanical drawing is here meant, not machine drawing, but the drawing of geometrical figures, and the working out of geometrical problems with mathematical instruments on paper. It is rather curious that the tools which are called mathematical instruments should be used to such a small extent as they are by mathematicians.

In this mechanical drawing the greatest possible accuracy should be insisted on, and, in the earlier stages, the student should have a considerable amount of drawing practice in working examples involving very elementary geometrical principles, in order to give him facility in using the instruments, and to test the degree of accuracy attained. The very first exercise might be to draw as many parallel and separate straight lines as possible in a width of, say, one-fifth of an inch. With a sharp chisel pointed pencil, an ordinary draughtsman can draw good serviceable lines which are about 0·005 of an inch wide. Allowing that the distance between adjacent lines is equal to their width, it follows that about twenty separate parallel lines can be drawn distinctly in a width of one-fifth of an inch. A similar exercise on the use of the compasses might then be worked, namely, to draw as many separate concentric circles as possible lying between two concentric circles, one of, say, 1 inch radius, and the other of, say, 1·2 inches radius. Of course with young students the question of eyesight must not be overlooked, and fine work must not be attempted in indifferent conditions of light, nor should it be excessive in amount.

The use of set squares and the methods of testing the accuracy of their angles should now be explained. The student will at this stage be studying the first book of Euclid, or a modern equivalent of it, and the propositions in that book which are made use of in the well-known tests of the accuracy of the set square angles should be specially pointed out to him, although he may not yet have proved these propositions.

The purpose of the mechanical drawing which is here insisted on is threefold: (1) It is a preparation for the engineering drawing which is so important to the engineer; (2) it familiarises the student with important propositions in geometry; and (3) it enables him to verify propositions which he may not have the time or talent to fully demonstrate by strict deductive reasoning.

The construction of angles by means of a protractor or a scale of chords, or from a table of trigonometrical ratios of angles having been explained, a great variety of examples for drawing practice can easily be devised by the teacher. A very good exercise is to first draw a number of straight lines OA, OB, OC, &c. of given lengths radiating from a fixed point O and making given angles with a fixed straight line OX, then to draw a polygon whose sides are respectively parallel and equal to OA,

OB, OC, &c. The length of the closing side of this polygon is then to be measured. A number of different polygons may be drawn by taking the lines in different order, but the length of the closing side should be the same in each case. The length of the closing side, as got by construction, should be compared with the length obtained by calculation.

Special attention should be given to the construction of angles from a table of trigonometrical ratios of angles.* The most convenient ratio to use is the tangent of the angle.

It will be seen that, in a very short time, the mechanical drawing is got well under way without waiting until the student has advanced beyond the most elementary principles in geometry.

In connection with the teaching of elementary geometry the writer would here strongly recommend that teachers should carefully study "Circular 711" of the Board of Education, entitled "Teaching of Geometry and Graphic Algebra in Secondary Schools." The suggestions in this circular are admirable, and should prove most helpful to teachers.

It has already been indicated that the student has commenced his study of the principles of geometry as contained in the first book of Euclid or a modern equivalent of it. Throughout his study of this book it is most important that the student should have ample practice in the solution of riders. There are two good reasons for this. First, because of the splendid training which it gives to the reasoning and inventive faculties, and, secondly, because there are numerous riders to the propositions, which are quite as important as the propositions themselves, if not more so. These remarks on riders apply to the whole of the student's course in geometry.

The theorems in the second book of Euclid might with advantage be proved algebraically, and modern solutions of the problems should be given.

Euclid's third book is probably the most interesting of all the books of his geometry, and many of the propositions have numerous practical applications. After mastering this book the student is in a position to study a considerable number of propositions in modern geometry relating to triangles and circles.

Coming next to Euclid's fourth book: this book in particular may, as is now customary, be treated rather as giving exercises in drawing than for the sake of deductive reasoning. To the problem on the inscribed circle of a triangle must be added the construction for the escribed circles, and then the various propositions relating to the inscribed and escribed circles of a triangle not given in Euclid should be taken. Practical methods for constructing a regular polygon having any number of sides should be considered. The various propositions relating to the nine point circle should then be studied.

* Such a table may be used intelligently without any knowledge of trigonometry; even the word "tangent" need not necessarily be used.

Up to this point the student in studying the principles of geometry has demonstrated the various propositions, excepting, perhaps, the theorems of Euclid's second book, in the rigorous manner required by the mathematician. He has also by mechanical drawing confirmed the demonstrations of certain propositions, and has, on the drawing board, made numerous and in some cases elaborate applications of the principles which he has studied.

In the further study of geometry the average engineering student should not be required to give rigorous geometrical proofs of all the propositions brought to his notice. There is yet a considerable amount of geometry which the engineering student must know, and he must be convinced of the truth of the theorems and the correctness of the constructions in the problems which are placed before him. But the demonstrations should now be abbreviated as much as possible, and any method allowed which will convince the student. In some cases it may even be sufficient that the student should test the truth of a proposition by actual mechanical drawing.

To illustrate what the writer means, take the sixth book of Euclid. The substance of this book may be given to the student in a very short time and in a convincing manner. Begin by showing that the area of a rectangle is equal to the product of two adjacent sides. Then, since a triangle on the same base as a rectangle and having the same altitude has half the area of the rectangle, it follows that the area of a triangle is equal to half the product of the base and altitude. Hence the truth of the first proposition of Euclid's sixth book follows at once. Consider the 19th proposition, namely, "Similar triangles are to one another in the duplicate ratio of their homologous sides." This is equivalent to the following:—"Similar triangles are to one another as the *squares* on their homologous sides," which is a more practical form. Let B and b denote the lengths of homologous sides of two similar triangles whose areas are A and a respectively. Call these sides the bases of the triangles, and let H and h denote the respective altitudes of the triangles. It will have been previously proved that—

$$\frac{H}{h} = \frac{B}{b}. \quad \text{Now } A = \frac{1}{2} BH, \text{ and } a = \frac{1}{2} bh.$$

$$\text{Therefore } \frac{A}{a} = \frac{BH}{bh} = \frac{B}{b} \cdot \frac{B}{b} = \frac{B^2}{b^2}$$

At this stage graphic arithmetic, including addition, subtraction, proportion, multiplication, division, and square root should be introduced. Multiplication and division are simply cases of the finding of a fourth proportional when one of the terms is unity, and the extraction of the square root is simply the finding of the mean proportional to two quantities, one of which is taken as unity. Graphic arithmetic is not of much

practical importance, but it is simple, and should be known to engineering students.

There are many geometrical problems which admit of easy solution by first drawing a locus. This method, in general, readily suggests itself, and may be quicker and quite as accurate as a direct solution, if there is one, because the direct solution may be troublesome to discover. The locus may be used without knowing what the curve may be, or what other properties it may have besides the particular one suggested by the problem. An example is shown in Fig. 1. ABC is a three-sided figure, one side being straight and the others curved. It is required to inscribe in this figure an equilateral triangle of given size. PQR is the equilateral triangle having one angular point Q on AC and another angular point R on AB. Keeping Q on AC and R on AB, the triangle may be drawn in a number of different positions and the locus of P determined. The intersection of this locus with BC determines the required position of one angular point of the triangle, and the triangle is then readily constructed. It is obvious that only that part of the locus of P which is in the neighbourhood of BC need be constructed.

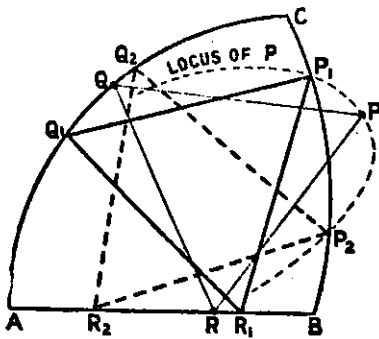


Fig. 1.

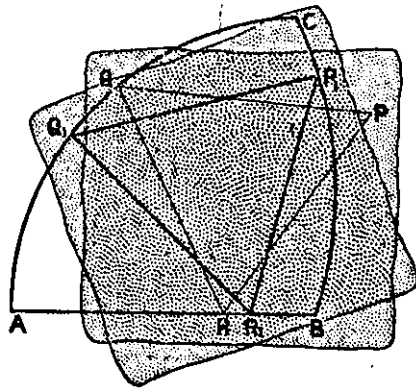


Fig. 2.

The foregoing problem furnishes a good example to illustrate the use of tracing paper in solving certain problems. Referring to Fig. 2, the triangle PQR is drawn on a piece of tracing paper. The figure ABC is drawn on the drawing paper. The tracing paper is placed on the drawing paper so that Q is on AC and R on AB. The tracing paper is now moved about until a position is found in which P is on BC, Q on AC, and R on AB as shown at P_1, Q_1, R_1 . This can be done very quickly. The points P_1, Q_1 and R_1 are then pricked through on to the drawing paper, and the points so obtained being joined, the required triangle is determined. It is obvious how the locus of P shown in Fig. 1 may be determined, if required, by means of the moving tracing paper.

This tracing paper method is one of very great value to engineers, because it can be readily applied, and the drawing is not confused by a number of construction lines.

The use of tracing paper in solving a problem which would otherwise be very difficult is illustrated by Figs. 3 and 4. ABCD (Fig. 4) is a given figure in which it is required to inscribe a figure similar to another given figure EFHK (Fig. 3), the angular points of the required figure corresponding to E, F, H, and K to be on the sides AB, BC, CD, and DA respectively. The figure ABCD is supposed to be drawn on the drawing paper. Draw the figure EFHK on a piece of tracing paper. Take a point on the tracing paper as a pole (preferably one angular point of the figure EFHK, say, E) and join this point to each of the angular points of the figure on the tracing paper. Graduate these lines, the divisions being proportional to the distances of the pole from the angular points of the figure. In the example illustrated EF, EH, and EK are each divided into

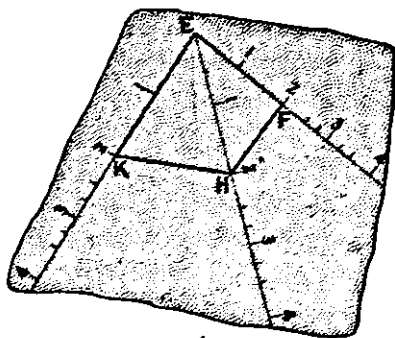


Fig. 3.

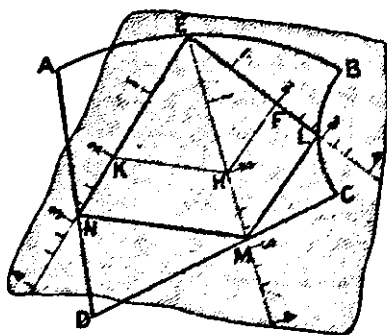


Fig. 4.

two equal parts, and the graduations are extended on these lines produced. Place the tracing paper on the drawing paper so that E is on AB, and so that the lines EF, EH, and EK, or these lines produced, cut BC, CD, and DA respectively. Now move the tracing paper about, keeping E on AB until the lines through the pole on the tracing paper cut the sides of the figure ABCD on the drawing paper at corresponding points L, M, and N, as shown in Fig. 4. The points L, M, and N are then pricked through on to the drawing paper, and the points so obtained being joined the required figure is determined. It may happen that the corresponding points L, M, and N are not at points of graduation on the lines through the pole. In that case the graduations must be made finer in the neighbourhood where they appear to be required. If the corresponding points required are still not at points of graduation, it may be possible, without further subdivision, to judge by the eye whether the

points where the polar lines cut the sides of ABCD are corresponding points.

A further illustration of the use of tracing paper in geometry is to be found in the drawing of roulettes. A roulette is the curve traced by a point connected to a moving curve which rolls on a fixed curve, the describing point and the curves being in the same plane. If the tracing paper method of drawing a roulette, now to be described, be compared with the ordinary method, it will be seen that the former has great advantages over the latter, especially when the rolling curve or the base curve or both of them are curves other than circles.

Referring to Fig. 5, let ACB be the directing curve or base, and let CDP be the rolling curve. (In Fig. 5 the base and rolling curve are circles, but they may be any curves whatever.) Three tracing or describing points are shown, P on the rolling curve, Q within, and R without the rolling curve. In Fig. 5 the tracing points are on a straight line through O, the centre of the rolling circle. The base is drawn on the drawing paper and the rolling curve is drawn on a piece of tracing paper. The describing points are also marked on the tracing paper.

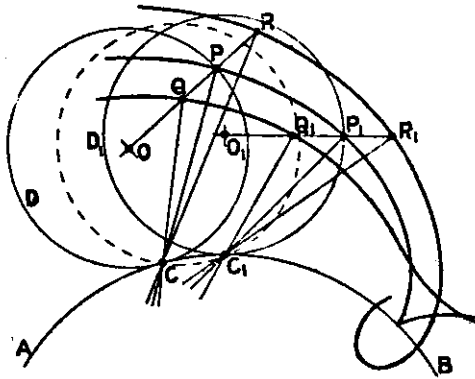


Fig. 5.

Place the tracing paper on the drawing paper so that the rolling curve is tangential to the base, say, at C. Place a needle through the tracing paper and into the drawing paper at C. Prick through the points PQ and R. Turn the tracing paper round about the needle at C until the rolling curve on the tracing paper cuts the base at a near point C₁. This position of the rolling curve is shown dotted. Transfer the needle from C to C₁, taking care that the tracing paper is not moved while changing the position of the needle. Next turn the tracing paper round about the needle at C₁ until the rolling curve is tangential to the base at C₂. The tracing points will now have moved to P₁, Q₁, and R₁. Prick through the points P₁, Q₁, and R₁. This completes the cycle of operations, and by repeating

the cycle any number of times any number of points on the required roulettes may be found, and fair curves through the corresponding points determined are the roulettes required. (In Fig. 5 the roulettes described by P, Q, and R are an epicycloid, an inferior epitrochoid, and a superior epitrochoid respectively.) For the sake of clearness in the illustration (Fig. 5) the distance CC_1 is greater than would be the case in actual practice.

P, Q, and R (Fig. 5) being joined to C, these lines are the normals to the roulettes at P, Q, and R respectively. So also, the lines joining P_1 , Q_1 , and R_1 to C_1 are normals to the roulettes at P_1 , Q_1 , and R_1 respectively.

The cycloidal curves, including the cycloid, the hypocycloid, the epicycloid, and the involute of a circle, which are all roulettes, are of great interest to the engineer, and the engineering student should have considerable practice in drawing these curves. He should note particularly the straight line form of the hypocycloid, and also the fact that there are two different rolling circles which may be used to describe the same epicycloid or the same hypocycloid. The student should also draw the evolutes of the cycloidal curves.

It may be mentioned here that in mechanical drawing, any approximate geometrical construction, the error in which is less than the probable error of the drawing, should be permitted. Such constructions, for example, as are given in Rankine's rules for the rectification of circular arcs, are sufficiently accurate for all practical purposes, when the angle subtended by the arc at the centre of the circle is not too large.

At an early stage in his study of geometry the student should be introduced to the conceptions of vector geometry, and as he proceeds in his study opportunities will occur of introducing and explaining the nomenclature of that subject. Ultimately he will apply the principles and methods of vector geometry to problems in kinematics and statics.

A considerable part of that section of geometry known as Geometrical Conics is important to engineers, and the study of this section of geometry becomes much more attractive to the student when it is accompanied by mechanical drawing. The problems in conics afford excellent practice in drawing, and the correct drawing, by the students, of the various figures required in studying the properties of conics makes the demonstrations easier to follow and less likely to be forgotten.

The conics should first be defined with reference to a focus and a directrix. A general construction should then be given for drawing a conic having given the focus, directrix, and eccentricity. This construction should be applied, on the same figure, to three different values of the eccentricity which will produce an ellipse, a parabola, and an hyperbola respectively.

This has been done in Fig. 6. F is the given focus and XM the given directrix.

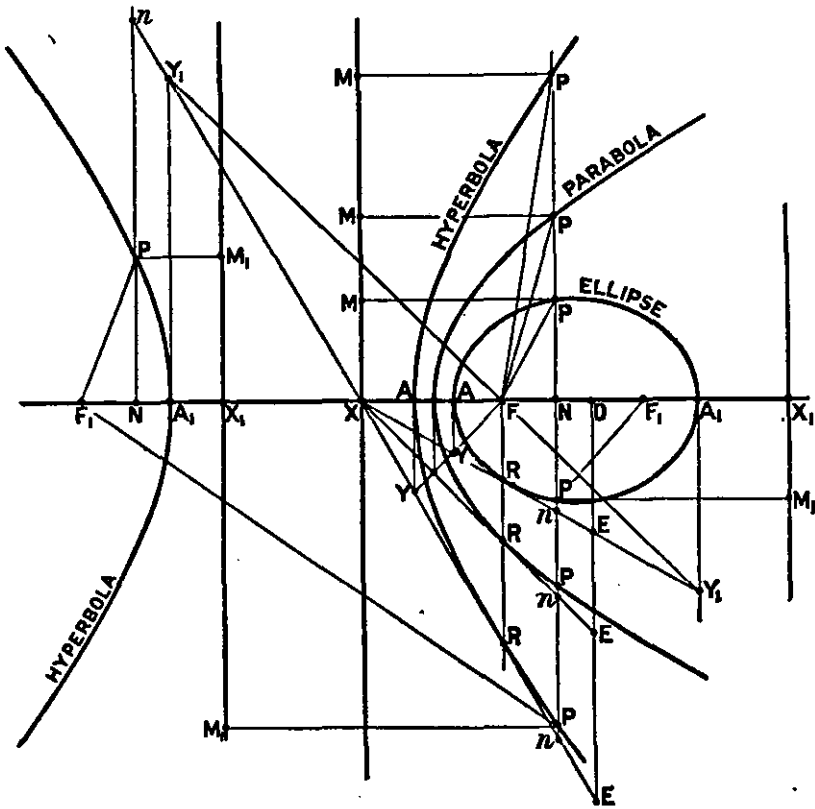


Fig. 6.

The construction used is as follows. Take a point D on the axis XD and draw DE at right angles to XD . Make DE such that $DE : XD$ is equal to the given eccentricity. Join XE . Draw any straight line Nn parallel to the directrix XM , cutting XD at N and XE at n . With centre F and radius equal to Nn describe arcs of a circle to cut nN and nN produced at P and P . P, P are points on the required conic. In like manner any number of points may be determined, and a fair curve drawn through them is the conic required. It is obvious that the ratio of Nn to XN is the same as the ratio of DE to XD , and, therefore, the ratio of FP to PM is the same as the ratio of DE to XD , which was made equal to the given eccentricity. The line XE will touch the conic at a point R obtained by drawing FR perpendicular to the axis.

It is obviously unnecessary to draw the line XE in the case of the parabola, but it is instructive to notice that the construction given applies to all the conics.

If lines FY and FY_1 be drawn, making 45° with the axis and meeting XE at Y and Y_1 , then perpendiculars from Y and

Y_1 to the axis determine the points A and A_1 , where the conic cuts the axis.

It is unnecessary here to further discuss Fig. 6, but by a study of the figure and the construction given, much may be learned as to the relations of the conics to one another, and their characteristics.

The conics should then be defined as plane sections of a cone, and it should be demonstrated that if a sphere be inscribed in the cone and touching a plane of section, the point of contact of the sphere and plane is the focus, and the line of intersection of the plane of the circle of contact of the sphere and cone with the plane of section is the directrix of the conic, and that the conic section has the property used in the definition previously given.

The student should here be taught how to draw the true form of a plane section of a cone. The construction for an elliptic section is shown in Fig. 7 and needs no description here. The same figure also shows how the foci and directrices of the ellipse are found from the inscribed spheres. The student should draw such a figure for each of the conic sections, and he should see how the conic is altered by varying the inclination of the plane of section.

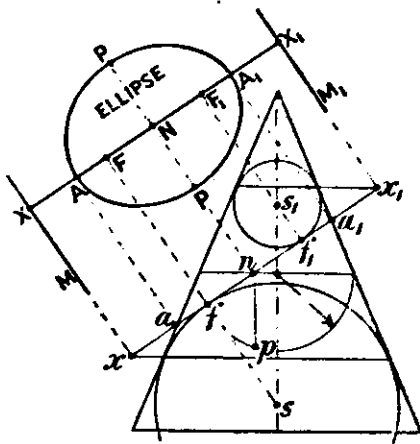


Fig. 7.

A systematic study of the properties of the conics may now be taken up according to the following syllabus:—General properties of conics. Properties of the parabola. Construction of the parabola. General properties of the central conics. Properties of the ellipse. Construction of the ellipse. Properties of the hyperbola. Construction of the hyperbola. Centre of curvature at any point of a conic. Evolutes of conics. So far as engineering students are concerned only the more important theorems and problems on conics need be considered.

Before the student has taken up the study of conics he will have commenced the study of the orthographic projection of

points, lines, and figures in space on horizontal and vertical planes of projection. He should therefore be able to follow the demonstration of the theorem that the orthographic projection of a circle on a plane inclined to it is an ellipse, and he should then study some of the properties of the ellipse considered as the projection of a circle. For example, take a well-known construction for drawing an ellipse, having given a pair of conjugate diameters. Referring to Fig. 8, PP_1 and QQ_1 are two diameters of a circle at right angles to one another. The circle is supposed to be lying on the horizontal plane. A square $RSTU$ is shown circumscribing the circle, the points of contact being P, Q, P_1 , and Q_1 . The radius CP is divided into any number of equal parts, in this case three, at the points 1 and 2, and RP is divided into the same number of equal parts at the points $1'$ and $2'$. If the lines joining Q_1 to the points 1 and 2 be produced, it is easy to show that they will intersect the lines joining Q to $1'$ and $2'$ respectively at points on the circle as shown.

Now imagine the circle, with all the lines connected with it, to turn about an axis MN in its plane into an inclined position, and let a projection of the whole be then made on the horizontal plane as shown. The square $RSTU$ projects into the

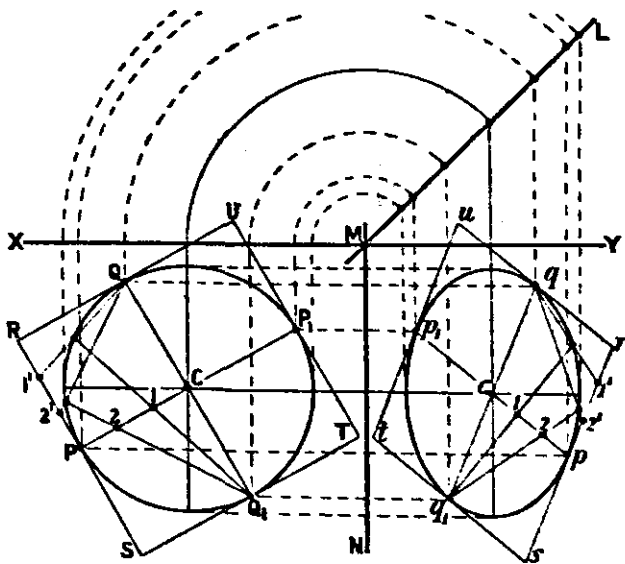


Fig. 8.

parallelogram $rstu$, and the circle projects into an ellipse which touches the sides of the parallelogram at p, q, p_1 , and q_1 . Also pp_1 and qq_1 will evidently be conjugate diameters of the ellipse. Lastly, the lines through Q_1 and Q which intersect on the circle project into lines which intersect on the ellipse and divide cp and rp into three equal parts. Hence the construction for drawing the ellipse having given two conjugate diameters.

The writer considers that while, for instance, many of the properties of the ellipse may be most advantageously obtained by treating it as the projection of a circle, a course whereby the student's grasp both of solid geometry and geometrical conics is strengthened, yet that modern projective theories of involution, anharmonic ratio, &c. have not yet been brought into a form in which they can usefully replace a detailed study of geometrical conics for the engineer student of advanced geometry.

The study of conics should be followed by a brief consideration of the principal properties of miscellaneous plane curves, such as the archimedean spiral, the logarithmic spiral, the catenary and the tractory or anti-friction curve.

So far this paper has dealt almost entirely with plane geometry, but before the student has advanced very far in that section of the subject he should commence the study of the representation of points, lines, and figures in space, by means of projections of them on horizontal and vertical planes of projection. This section of the subject is generally called Descriptive Geometry or Practical Solid Geometry. Preliminary to the subject of Descriptive Geometry the student should have explained to him the fundamental principles of radial, parallel, and perpendicular projection as applied to points and lines in a plane, the projectors being in the same plane.

A course on descriptive geometry should begin with the orthographic projections of points and lines on horizontal and vertical planes of projection, and after working a few

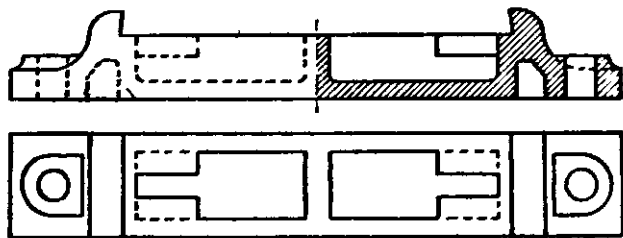


Fig. 9.

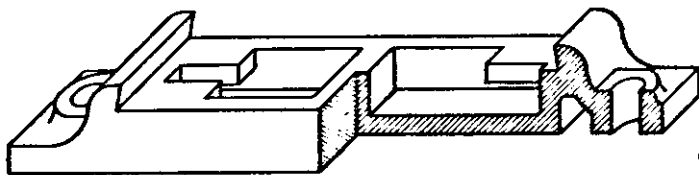


Fig. 10.

simple problems on the projection of points and lines, the application of the method of orthographic projection to the representation of solids should be taken up. The representation of a solid by means of an oblique parallel projection of it on one plane of projection should now be explained. This

method of representing a solid is one which might be used with great advantage more frequently than it is in engineering drawing. The advantage of this method of projection over the ordinary orthographic projections on two planes is illustrated by Figs. 9 and 10. In Fig. 9 two orthographic projections (a plan and an elevation partly in section) of a sole plate for a shaft bearing are shown. The same object is shown by one oblique parallel projection in Fig. 10. It is at once seen that the form of the object is more readily understood from the projection in Fig. 10 than from the two projections in Fig. 9. Moreover, since the front face of the solid is placed parallel to the plane of projection, it has exactly the same outline in Fig. 10 that it has in the elevation in Fig. 9, and is therefore as easily drawn.

In Fig. 10 the projectors have been so inclined to the plane of projection and to a horizontal edge of the front face of the object that the edges which are perpendicular to the front face project into lines which are to the same scale as the edges of the front face and are inclined at 45° (a set square angle) to the horizontal lines on the drawing. The drawing in Fig. 10 may therefore be made almost as readily as the one in Fig. 9, and the former may be used instead of the latter as a working drawing.*

Oblique parallel projection is also extremely useful in exhibiting the relations between three variable quantities such as P , V , and T in the well-known equation $PV = RT$, where P is the pressure of a given mass of a gas, V its volume, T its absolute temperature, and R a constant. This equation is represented by a surface which may be depicted by contours, which are hyperbolas, for different values of T .

In general it will not be necessary for the engineering student to write out rigid proofs of the theorems upon which the various constructions in descriptive geometry are based, but it is nevertheless most important that he should clearly understand the principles which he applies. In connection with lines and planes the theorems in Euclid's eleventh book should be read, and the student should be able to demonstrate such theorems as—"If a line is perpendicular to a plane P , the orthographic projection of the line on a plane Q is perpendicular to the trace of the plane P on the plane Q ;" and, "If two straight lines are at right angles to one another, their orthographic projections on a plane parallel to one of them are also at right angles to one another."

It only remains to indicate briefly the remaining parts of descriptive geometry which are important to engineers. These are:—Sections of solids. Horizontal projection including the representation of a portion of the earth's surface by a contoured

* Some authorities prefer "isometric" to the "axometric" projection of Fig. 10, but the writer considers the projection of Fig. 10 to be much simpler and more useful than the isometric projection.

plan, and problems on cuttings and embankments. Generation of curved surfaces. Tangent planes to curved surfaces. Development of surfaces. Projection of helices and screw threads. Intersection of surfaces. Perspective projection, isometric projection, and the determination of shadows are not important except as affording good examples for the application of the principles and methods of descriptive geometry.

The student's work in descriptive geometry is mainly drawing, and only by working out a large number of examples on the drawing board can he hope to acquire a useful knowledge of that subject.

The examples which are placed before the engineering student in his study of geometry should not be too technical. The technical applications of geometry are better considered in connection with the subjects to which they refer. For example, it should be sufficient, in a course on geometry, to consider the properties of the cycloidal curves and the methods of constructing them without taking up problems on the teeth of wheels, which are more properly considered in a course on the theory of machines.

No attempt has been made in this paper to specify in detail those parts of geometry which may be of most importance to the civil engineer, the mechanical engineer, or the electrical engineer. Experience has shown that it is undesirable for a student of engineering to specialise in one branch of engineering until he has had a good general training in the scientific principles which underlie engineering as a whole, and the course in geometry outlined in this paper should be taken before the student has begun to specialise, or at all events before he has proceeded very far in such specialisation. The branch of engineering which an engineering student will ultimately adopt as his life's work is often very uncertain until after his college course is finished. In most cases the branch finally adopted will depend on the opportunities which may present themselves, and on circumstances over which the student has no control. Moreover, no well-defined lines can be drawn between civil, mechanical, and electrical engineers, the work of one is continually overlapping the work of the others. Again, an engineer does not cease to be a student after he has left college. An engineer who endeavours to advance the special branch of engineering in which he is engaged must be an ardent student, and he will continue to extend his knowledge of those subjects which bear on his daily work.

In urging the importance of the early introduction of mechanical drawing in the study of geometry the writer is well aware that much has been done already in this direction, and great credit is due especially to the Board of Education for its efforts towards the improvement of the teaching of geometry. But there is still need to emphasise this point, and the writer, having had great experience in the education of engineers,

ventures to hope that the views here expressed on the importance of mechanical drawing in the study of geometry may help forward the movement for the better teaching of that subject.

It may seem to some teachers that the course on geometry for engineers which has been outlined in this paper is the traditional course on geometry with additions which will require considerably more time than has hitherto been given to the subject in school and college courses. To some extent this is true, but it must be left to individual teachers to adjust the amount of teaching by the strict deductive reasoning process to the capacity of his students and the time available. A student may make a good engineer and may yet be unable to get up much enthusiasm for the strict deductive reasoning process in geometry, but he may nevertheless be able to acquire a good working knowledge of the subject by devoting sufficient time to it on the drawing board.

In conclusion, it is extremely difficult to specify exactly those parts of geometry which are essential and those which are not essential to the engineer, but the writer hopes that this paper may be of some help to those who may be engaged in teaching geometry to pupils who are looking forward to becoming engineers. The teacher must, however, keep in view the great educational value of a course on geometry properly taught, and he must not give undue consideration to the question of practical utility.

D. A. Low.

MATHEMATICS WITH RELATION TO ENGINEERING WORK IN SCHOOLS.

The subject of Mathematics considered in relation to engineering work in schools is intimately connected with the whole problem of industrial education. In the following paper it has seemed best to divide what I had to say into two parts: the first, summarising the work actually being done in a typical secondary school possessing an engineering laboratory and discussing the Mathematics essential for the proper development of the work of that laboratory, is mainly descriptive of personal experiences, but is critical in so far as it discusses how far existing school courses of Mathematics are capable of satisfying the demands of such laboratories; the second, being an attempt at a constructive policy, is frankly a piece of special pleading for a complete reorganisation of the work of our schools.

The school I have in mind consisted of an upper and a lower division*; the former was subdivided into Literary, Commercial and Technical Sides, the latter into Latin and non-Latin Sides. The Technical Side, for the organisation of which I was responsible, was kept very small for various reasons and consisted normally of two forms, Form IV. and Form VI., there being from 12 to 20 boys in each; the average age ranged from 13 to 14 in Form IV. and from 16 to 17 in Form VI. Boys drifted or were drafted into Form IV. with vague inclinations towards "Engineering" and, with few exceptions, remained on the Technical Side about three years.

Let me consider the case of a typical boy, briefly before he enters the Technical Side and in some detail thereafter. Entering school at the age of 9 or 10, he did the work of either the Latin or non-Latin Side of the Lower School until he was about 13, when he could be admitted to the Technical IV. if fit for upper school work. The Mathematics he had done was of the usual type, not improved by the fact that for several years past the subject had been in the melting pot. Every experienced teacher will agree that the fact that a boy has "done" Arithmetic does not guarantee that he can calculate with any facility or accuracy. Such a boy has not realised that, as Herschel says, numerical precision is the soul of science: he may know a little Algebra, but it will be too formal, it will not economise his thought, not being really part of his own experience; he will not appreciate its value in generalisation, while he will not dream of applying what he has learnt, out of the mathematical hours: he has begun Experimental Geometry, but it will be the exception for him to feel the need for undertaking any of the experiments he has been called on to perform and of course he has not studied the subject long enough for its effect on his reasoning powers to be appreciable. This represents, fairly accurately, the mathematical equipment and frame of mind of the average

* The school referred to is St. Dunstan's College, Catford.

boy of 13* and of our typical boy when entering the Technical Side.* But this entry must not be regarded as specialisation. The boy was sent to the "Engineering Room" for some six or seven periods only; during the other periods he worked with the remainder of Form IV. at English, French, German, Mathematics, and Science, taking Engineering while the others did Latin or Commercial subjects as they were on the Literary or Commercial Side.

The problem before me was to utilise the seven periods allotted to Engineering in such a way as to avoid over-early specialisation and yet take advantage of the boy's natural inclinations and so prepare him for promotion to the upper set. Accordingly, three periods were set aside for metal work—the boy having already had some experience with tools, as the whole of the Lower School did Woodwork once or twice a week; two consecutive periods (1½ hours) were allotted to Experimental Mechanics; the remaining periods were given to Mechanical Drawing. The chief difficulty was in Experimental Mechanics, as the other work was practically individual. What was wanted (owing to the system of terminal promotions) were five or six short courses, each to last one term, each quite self-contained, capable of being treated *ab initio* without reference, except in special cases, to the other courses; simple and so well within a boy's expanding experience as to arise naturally from his own activities—in short, as educational as possible. The following courses were selected:—

1. On weights (as typical forces, carefully avoiding any definitions of mass, force, &c.—metaphysical difficulties); on equilibrium. The determination of the conditions of equilibrium in certain cases. The triangle and polygon of forces (by way of concrete problems on the crane, sheer legs, inclined plane, &c.). Parallel forces. Funicular polygon.
2. On motion. The idea of relativity. The estimation and measurement of time and distance. Speed. Velocity. Acceleration. The effect of force on speed change.
3. On stretching strings. Springs. Stress. Strain. Elasticity.
4. On machines. Effort and resistance. Advantage. Work loss. The idea of conservation of energy. Efficiency.
5. On friction.

The general routine was as follows:—At the beginning of each term we had an informal discussion in which a few leading questions started an inquiry. From this some problem arising quite naturally was selected and formally stated for investigation. Methods of solution were asked for and were tried, whether fruitful or not, so long as they were apposite—my object being primarily the cultivation of a scientific frame of mind, of the power of analysing stated conditions, the formation

* The age of entry to the Technical Side at St. Dunstan's has recently been raised to 14.

of methodical habits, the improvement of the power of self-expression, and only indirectly of teaching mechanics as mechanics. When, by the judicious elimination of unwise, unlikely or unsatisfactory methods, a scientific form of attack was agreed on, it was followed up systematically: but obviously no preconceived order of experiment could be adhered to rigidly. Where interest led we followed and it seldom happened that the same order of experiment would be adopted in repeating any course. Whenever the type of experiment permitted, all results were tabulated and exhibited graphically. In all cases the boys made sketches of the apparatus they suggested using or actually employed, discussed the methods of experimenting to be adopted, were asked to criticise their results and to suggest further tests which might be applied to their conclusions. I insisted that all data—no matter how apparently inconsiderable or unworthy of attention—should be incorporated in the notes. It was pure accident that led one little boy to observe that a piece of elastic behaved as if it *got tired*, another to discover the time effect in elasticity. Finally, where suitable, the results were expressed in symbolic form, *i.e.*, were generalised. Every course was at once a field for exercises in drawing, for scientific investigation, for research and for mathematical analysis. In the normal term of 13 weeks, fair—though varying—progress could be made with any course, and the boy accumulated some mechanical experience during the average 18 months that he spent in the Technical IV.

While the preceding note indicates the general routine adopted for each course, it may not be considered out of place if I give somewhat fuller details of the development of one course—say, that on “Motion”—during a particular term.

1. *Introductory*.—What is meant by motion. A man walks across a railway carriage or climbs the mast of a ship, specify his change of position. What is his change of position if the carriage or ship be in motion? The idea of the relativity of motion. How is position fixed? The position vector. Choice of origin, axes, &c. Net and gross change of position, displacement. The map of the path of a point. The fundamental concepts involved in position change in the case of simple linear motion. Measurement of distance and angle, measurement of time, choice of units, conversion of units. Practical consideration of time measurement. Experiments on: (a) the pendulum, (b) the water-clock, (c) the vibrating spring, (d) counting.

NOTE.—Suggestions for experiments and for suitable apparatus were invited, the latter being made in the workshop when they were such as could be prepared in time for use during the periods available; each member of the class contributed a share of the labour necessary, particularly if, as often occurred, some modification of existing apparatus was intended. In the case of experiment (d): if the words *one, two, three, four*, &c. be spoken rapidly, distinctly, and successively, it will be

found that as far as *six, seven, or eight* can be uttered in one second with such accuracy after short practice, that an error not greater than about one-fifth of a second would be made in the estimation of an interval of 10 seconds as measured with an ordinary stop-watch. Boys soon acquire facility, and I found that this proved a very convenient method of measuring short time intervals. For example, A holds a stop-watch, B gives signals, and C counts. B taps with a pencil and C says, "one, two, three, four, five, six, seven, one, two, three, four, five, six, seven," and so on, without making a perceptible pause between seven and one, checking complete seconds on his fingers until he has checked six, and continues—"one, two, three, four, five"—when he hears B signal again and A stops his watch. C estimates the time as 6.6 seconds, saying that he said "five" after he heard the second tap: A checks his result. The observers then interchange places and practice until groups of boys with approximately the same personal equation can be selected. The boys in such groups are in a position to check one another's time estimations—an acquirement of some value when only one or two stop-watches are available, and water-clocks are inconvenient. The power of estimating time is as useful as that of estimating distance.

2. *The Idea of Speed.*—Examination of a railway time-table. Difficulty and inconvenience of examining columns of figures. Representation of time and distance on squared paper, choice of scales, selection of axes, &c. Examination of various graphs. What is meant by the statement that the speed of a train is 30 miles per hour? Is the meaning that the train will go 15 miles in the next 30 minutes? Will go 1 mile in the next 2 minutes? 4.4 feet in the next 0.1 second? Comparison of map of path (in the case of straight line motion in the first place) and the space-time graph, both for regular (uniform) and irregular motion. The necessity for actual experiment on these kinds of motion. Realisation that measurement of speed involves measurement of time and distance, that velocity involves speed and direction, in the general case of motion.

3. *Experiments on Speed.*—Suggestions for experiments invited. Ball rolled along inclined plane, so adjusted that the motion was apparently uniform. Method of obtaining definite and easily reproduced small speed. Actual proposals for apparatus required. Sketches considered, the best form chosen after discussion. (The apparatus was, as stated before, made during the periods set aside for manual work if not at hand, or was selected from stock.) The first experiment must usually be suggested; for convenience, I express it symbolically:—A ball is allowed to roll down a distance d on a plane inclined at an angle θ , when it rolls on to a second plane inclined at a very small angle, so determined as to overcome frictional resistances as far as possible: the times t taken by the ball to roll distances s on this second plane are estimated and plotted. What is

noticed? (Note here the varying initial error, due to many causes but chiefly the personal equation.) What can be changed so as to vary the experiment? Leading questions lead to the suggestion that " d " and " θ " may be varied. Vary d and plot the graphs connecting s and t obtained in each case on the same sheet of squared paper, using the same scales and axes. What is to be noticed? Connect slope and speed. Vary θ and repeat. Refer to first notes on speed and ask for methods of measurement. "Rate" of motion.

4. *Varying Speed.*—The falling body. "Dilution" of motion. Experiment on motion of ball down inclined plane. The times t for a ball to roll down distances d on a plane inclined at an angle θ were noted and a graph drawn. Character of graph. Magnification of portion of graph. Comparison of graphs obtained for changes of θ (within a small range). Reference to general result of § 2. Average speed over given displacement. Average speed during given time interval. Average speeds during small and continually diminishing intervals. Actual speed. Connection between speed at bottom and average speed. Measurement of slopes of curves obtained in experiment on the variation of d in 3 §. Graph connecting speed and d . Graph of speed and t . "Rate" of change of speed. Acceleration. What does the "area" of the graph mean? Question as to the actual law of motion of ball in case of inclined plane. Method for determining "law" of given graph: (a) linear, (b) other graphs. (Note.—The general consideration of the problem, except in the case of the linear law, was postponed: the "necessity" for the work was merely indicated.) Test of the law connecting s and t by various methods: two, three, or more balls on inclined plane—the "penny" Attwood—&c. Study of motion of projectile. Map of path and determination of its equation. Reference to variation in acceleration produced by varying θ above.

5. *Experiments on Acceleration.*—Suggestions for methods of producing accelerated motion invited. Finite acceleration reproducible at will. The effect of falling body in producing motion in a second body. Method of eliminating effect of "weight" of second body. The Fletcher trolley. Measurement of acceleration from curve made by vibrating spring. Variation of acceleration with (a) varying "weight" of falling body, (b) varying total weight in motion. "Tension" of string during experiment. Algebraic form of "laws" discovered. The meaning of mass.

Once again one would insist that this is not to be regarded as a rigid syllabus: it is far from exhaustive, and is the barest outline of the direction taken by an investigation on "motion" during one particular term. Its value as a source of mathematical investigations is sufficiently apparent and it is linked with the other branches of the boy's work by the necessity for

the design and construction of new, or the modification of existing apparatus. In order to preserve the latter feature, apparatus was frequently dismantled or "scrapped" when of no great intrinsic value.

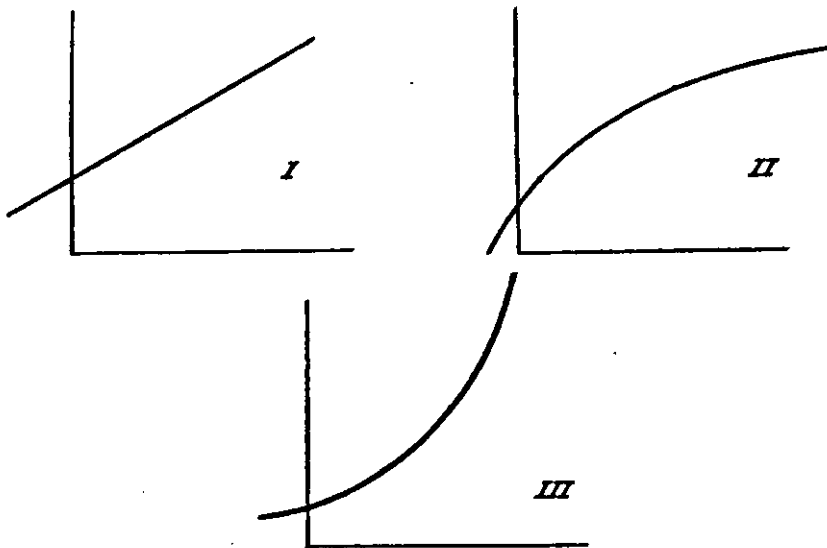
The subject of Mechanical Drawing was approached in the same spirit of inquiry: the boys took a keen interest in the apparatus they proposed to use in the solution of mechanical problems, in the model upon which they were engaged during the periods devoted to manual work, in the tools, in certain machine details which they came across in the workshop, and all served as elementary solids to be sketched, measured and of which working drawings were desirable and could be made. The necessity for drawing to scale was realised and the properties of similar figures investigated. Drawings of such geometrical models as the cube, prism, pyramid, cone, cylinder, &c. were not required at this stage, nor were the boys expected to work problems like those usually found at the beginning of a course on practical Solid Geometry. For the beginner to take a real interest in geometrical models as such is very difficult; for him to do so in the case of problems on the point, line, and plane is all but impossible. Broadly speaking, the course was on elementary machine drawing and it was intended to serve as a foundation for formal work in practical Plane and Solid Geometry and thus, through the methods required for the convenient representation of solids, the boys were led in their small way to the beginnings of what, in the hands of Monge, grew into the vast subject of Descriptive Geometry, thus illustrating yet again how this important and striking mathematical weapon was forged to suit the needs of the engineer. In this case I was able to assume that a certain amount of experimental Plane Geometry had been, or was being, done in the ordinary mathematical periods and this, of course, diminished the amount of "drill" in the use of mathematical instruments which would otherwise have been necessary. I do not contend, however, that technical drawing should be related to previous work in Geometry, but rather that the first stage should be to get the boy to see that for any practical work certain sketches are essential and a knowledge of the principles of mechanical drawing is advisable, and to use this desire to draw as a means for introducing the formal treatment of Geometry, the necessity for which then becomes self-apparent. The model first made is probably cuboidal in shape—the drawing of any face consisting of a network of parallel and perpendicular lines; in other words the use of simple mathematical instruments—in particular, the set squares—becomes necessary. Formal consideration of the theory underlying the use of such instruments should be postponed until facility has been acquired and interest has been aroused. In most schools a course of Experimental Geometry now forms the foundation of formal geometrical theory, but the need for the experiments undertaken is not urgently felt by the pupils,

at any rate, in the earlier stages: it is the utilitarian aspect which most appeals to the beginner. As more complex drawings are attempted, methods required for the production of such drawings can be invented or suggested and in the invention of such methods relations between simple geometrical concepts present themselves and short cuts are discovered, while the investigation of these relations forms a foundation for theoretical work. For example, any simple casting will probably give an example of a fillet or "rounding-off" between two surfaces: the representation of these surfaces on a plane involves investigation of the properties of tangents. In short, at every stage one is forced to realise the strength of the claim to consideration of mechanical drawing associated with manual work as introductory to, or even as a substitute for, Experimental Geometry.

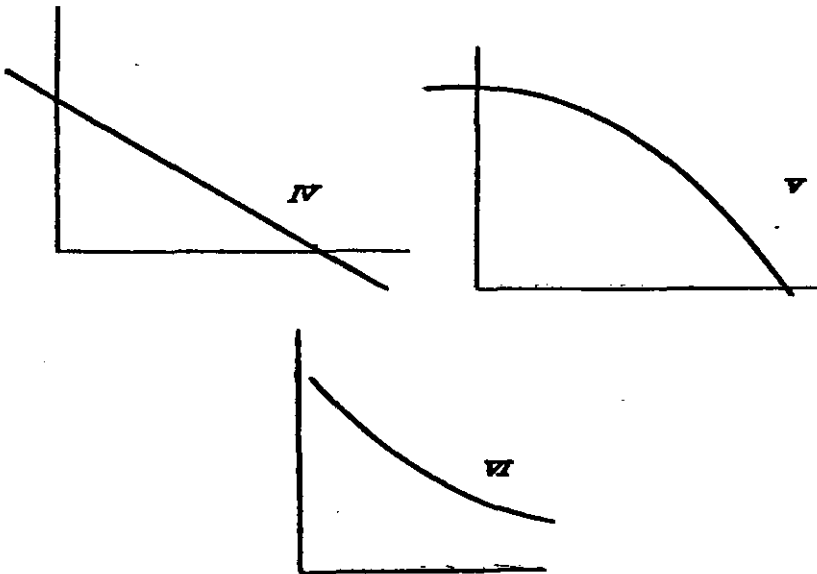
No periods of those allotted to Engineering were set aside for the study of pure Mathematics as such, although my greatest difficulty was the boys' failure to undertake the necessary calculations or analysis with speed and accuracy; but, as I have endeavoured to show, both the mechanical drawing and experimental mechanics as well as the manual work were rich in natural opportunities for mathematical investigation. I am sure that every boy began to realise, as never before, the value of algebraic expressions in economising space, time and thought, as well as the meaning of functionality; while he learnt to associate definite graphs with definite physical characteristics or meanings.

From a note-book kept by a beginner, I take the following illustration, the words italicised being my own corrections or additions:—

"We have found that curves representing *graphically the numbers obtained in various experiments* are of one or the other following typical shapes:—



"I also imagine that it might be possible to obtain graphs like this :—



"If we take x and y to denote quantities of some kind, generally physical, x being in most cases (what we choose) (at our own disposal) and y depending on x , the appropriate general inference in the several cases is :—

- I. As x increases steadily, y also increases steadily.
- II. " " " " but at a diminishing rate.
- III. As x increases steadily, y also increases, but at an increasing rate.
- IV. As x increases steadily, y diminishes steadily.
- V. " " " " at an increasing rate.
- VI. As x increases steadily, y diminishes at a diminishing rate."

Cases I. and IV. were also worked algebraically and every boy knew that if his experimental numbers lay approximately in a straight line the law connecting the variables, so far as his range of observation went, was of the form $y = a + bx$: or in words, that the value of the dependent (or subordinate) variable corresponding to any given value of the independent (or principal) variable could be calculated by multiplying the latter by one constant number and adding a second constant number to the result. Methods of determining the values of these constants were understood and their nature was appreciated, while proportion, variation, and the meaning of a rate seldom presented any difficulty when tackled in this way.

In developing the conception of a rate, the consideration of problems on growth was found helpful. Thus, following the

study of growth or increase of displacement in the course on motion, reference would be made to growth of population, growth of a sum of money out at interest, extension of material when heated, when loaded, &c. The conditions upon which such growth or increase depend were investigated and means were suggested or invented for measuring the average or mean growth and for indicating how it depended on the value of the independent variable under various conditions, while these ideas were found to be capable of graphical illustration. Appeals to intuition and to graphic representation then were used to show that as the x interval diminished the average rate of growth or increase in y per unit increment of x approached a definite limiting value—which was illustrated experimentally in various cases—depending on the value of x from which the interval was measured. Thus was developed appreciation of the idea that this limiting value defines the rate of change of y with respect to x , while the specification of both variables was seen to be necessary in order to distinguish the various types of rates considered.

I admit the logical difficulty of co-ordinating the characteristics of physical variables with the geometrical properties—slope, &c.—of the curves employed to illustrate those variables, but no boy ever realised this logical difficulty and no teacher would raise the point unless he had some ulterior motive.

It will be noticed that the graphical work done was such that the boys were practically compelled to choose plain scales which could be read accurately and rapidly: in other words, they realised that the whole utility of the graphical method for representing the simultaneous and correlated variation of two physical quantities depended on the suitable selection, as well as the clear and correct figuring, of the scales to be employed.

The course on weights and equilibrium, as well as that dealing with motion, led naturally to the adoption of the vector idea and notation. After simple graphical problems on displacements had been solved, I attempted to apply this calculus to pure geometry: the boys were not yet ripe for this however, and the attempt failed. Later, I gave some details of a course on vector calculus taken by a higher form. Its development showed that the failure in the case of the 'Technical Form IV. was due to the fact that the boys had not sufficient foundation of geometrical and physical experience and so were unable to realise what the term "vector" connotes.

I have described the work of a boy in the "Technical IV." during the periods assigned to "Engineering" in some detail, the point on which I wish to lay stress being that whatever pure Mathematics was done during those periods was incidental, natural and not formal, each problem arising in such a way that every boy realised its *practical* importance. Instruction began with the concrete aspect in all cases; the studies were wholly observational at first, but deductive methods were used as soon

as the mental development of the boys permitted, while certain new conceptual elements, without which the science of Mathematics could not exist, were introduced naturally and in their normal time. The study of any branch of work and its practical application were closely interdependent; the former was consequently more interesting and its development could be continuous. Each new discovery, each fresh conception, was a starting place for fresh investigation.

Our typical boy has now been in the "Technical IV." some 15 or 18 months; he can use his mathematical instruments neatly, sketch, measure and draw to scale a simple machine detail, producing plans, elevations, and sections when required—*i.e.*, he can see in the solid; he has acquired some manual dexterity; he knows what a graph is and—more important—what it is not, and he has some idea of what is involved in making an experiment; he begins to know too, how to study, correlate and test his results and how to make an inference. Meantime, with other boys in the normal mathematical periods, he has continued his formal algebraic work with so-called practical examples—in many cases perhaps really practical, but lacking interest because their connection with his self-activity is too often strained or entirely wanting. Compare, for example, the mental attitude towards graphs of a boy whose first experience of the linear graph is derived from the set obtained in an experiment performed by himself with that of the boy whose first knowledge of the linear graph is obtained from a text-book of algebra.

NOTE.—The graphical representation of statistics, particularly social statistics, may be interesting enough and so finds legitimate place in the mathematical scheme: but the author who in the desire to be "practical" introduces in his book graphs of results obtained in experiments which boys have not themselves performed or seen, runs the risk of being charged with attempting to force a false and artificial correlation between Mathematics and another science. A recently published text-book on Algebra introduces graphical work in the following order:—

1. Draw graphs of the form $y = a + bx$. Instructions given to determine the value of y for various values of x and plot corresponding values obtained.

2. Solution of linear equations, of simultaneous linear equations and determination of equation of line joining given points.

3. "Practical" linear applications—chiefly on conversion of units, redistribution of marks, &c., but with some examples of forced correlation.

4. Graphs of statistics—chiefly social, but with many bad examples. There are also examples on mensuration, which are legitimate enough if geometrical work is being done simultaneously.

5. Graphs of easy functions and approximate solution of quadratics.
6. Maxima and minima.
7. Simultaneous quadratics.

The scheme is clearly exhaustive enough for an elementary text-book, but it appears to be artificial and uneducational in its genesis—unless it is intended that a course of experimental geometry be worked concurrently—while its whole development is too purely mathematical to appeal to the average boy.

Considering our typical boy again, it must be admitted, however, that he has acquired some skill in algebraic manipulation in the mathematical periods; he has continued his experimental geometry and may have begun its formal logical treatment; he has done much arithmetic, and is probably in the region of stocks and shares; if he is fortunate, he can use his logarithm tables after a fashion, but he does not produce them in the science or engineering laboratory unless pressed to do so—logarithms are “maths,” not “science.” He has also done a course or two in pure science and may now be promoted to the Technical VI. with one of two objects before him; either to prepare for one or other of the Technical Colleges, or to go direct into some manufacturing industry, chemical, engineering, &c.

At the beginning of my work as a teacher, I was of the opinion—probably because it had been my own experience—that the ideal scheme was for a boy to leave school at the age of 16 and turn out into the “works,” keeping the normal day of 9 or 9½ hours and attending evening classes in order to keep up his theoretical work meanwhile. Now, and this in spite of much strong opposition, I am convinced that any break in the continuity of formal education is dangerous and—taking into consideration the strain on the undeveloped physique of a youth involved in undertaking the two or three hours’ hard mental labour necessary to maintain the continuity of studies after the prolonged physical effort required in the works—that the best plan is for the embryo engineer to continue his college course and round off his theoretical studies, at any rate to the graduate stage, before attempting to engage in practical work on an industrial scale. I have had no experience of the “sandwich” system and so can form no opinion as to its merits or demerits, but between the alternatives of formal apprenticeship with evening class work immediately after leaving school, and college work, there can be no hesitation; I am strongly in favour of the latter.

This being the case, it will be obvious that the general scheme of work for the Technical VI. was based on the standard required for entry to an engineering college. That is to say, the work must be so arranged that the standard of knowledge required for such an entrance examination forms its inferior

limit. As a matter of fact, the top boys of the Technical VI. were always fit to commence the second year's work of the average engineering college course and generally were allowed to omit the first year's work, which in most cases consisted merely of revision and was of quite an elementary nature.

The accompanying time-table shows when the Technical VI. worked with the other VI. forms and how they were differentiated at other times.

It will be seen that, for the "Technical," 10 periods were allotted to Engineering, 8 to Science—4 to Physics and 4 to Chemistry—7 to Mathematics, and the remaining 7 to literary work. As the time-table shows, in Mathematics the Technical Side worked with the other Sides, the whole upper school being divided into four sets which continued the more or less formal treatment of the subject as in the IV. and lower forms, until in the Upper VI. they aimed at reaching the standard required for scholarships at the older universities. The boys, being older, appreciate the æsthetic treatment of Mathematics to some extent, and in exceptional cases this method may be ideal: but for the majority it still lacks reality and further, the amount they can do formally when the subject is studied on the usual school lines, falls far short of what can be done, far short of what is absolutely necessary, if they are to be able to apply their knowledge with any degree of success to those practical problems so urgently requiring solution and so frequently occurring in their engineering and science work. That being the case, my first step was to take one of the engineering periods for Practical Mathematics, the other nine being divided as follows:—Geometrical and Mechanical Drawing, five periods; Experimental Mechanics, two periods; Metal Work, two periods. As it happened, I also took this form in Physics, so that the boys were with me for 14 of the 32 periods in a week. Further, I was particularly fortunate in that I was not compelled to adhere too rigidly to my time-table: if a boy needed more Physics, more Drawing, more time for special Mathematics, it could be arranged. This freedom was of the greatest advantage in developing the work of the Technical Side, and should be borne in mind when considering the standard of work done.

During the drawing hours, examples of machine details more complex than those studied in the lower form were drawn and these led to the necessity for a more advanced treatment of practical Plane and Solid Geometry. For example, interpenetrations would be required adequately to represent a casting: consideration of gears—toothed wheels, racks, &c.—would lead to the study of cycloidal and involute curves; the necessity for the representation of screw threads, propellers, &c. would require elementary work on helices; problems on the line and plane would arise in connection with some practical work; in fact, the subject of Descriptive Geometry was developed

FORMS VI. AND V.

L. = Literary Side. C. = Commercial Side. T. = Technical Side.

—	Monday.	Tuesday.	Wednesday.	Thursday.	Friday.	Saturday.
9.15—10.15.	L. } T. } Scripture. C. }	L. } C. } English. T. }	L. } C. } Mathematics. T. }	L. } C. } Chemistry. T. }	L. } C. } Mathematics. T. }	L. } C. } Mathematics. T. }
10.15—11.	L. } C. } German. T. Engineering.	L. } C. } French. T. }	L. } C. } Mathematics. T. }	L. } C. } Chemistry. T. }	L. } C. } Mathematics. T. }	L. } C. } Mathematics. T. }
11.15—12.	L. Latin. C. Commercial. T. Engineering.	L. } C. } History. T. Physics.	L. Latin. C. Geography. T. Engineering.	L. } C. } English. T. }	L. } C. } History. T. Engineering.	L. } C. } English. T. }
12—12.45.	L. } T. } French. C. }	L. } C. } German. T. Physics.	L. Latin. C. Commercial. T. Engineering.	L. } C. } French. T. }	L. Latin. C. Geography. T. Engineering.	L. Latin. C. Commercial. T. Engineering.
2.30—3.30.	L. } T. } Chemistry. C. }	L. } T. } Mathematics. C. }		L. Latin. C. Spanish. T. Physics.	L. } C. } German. T. Engineering.	
3.30—4.30.	L. } C. } Chemistry.	L. Latin. C. Spanish. T. Engineering.		L. } C. } German. T. Physics.	L. Latin. C. Spanish. T. Engineering.	

experimentally as a series of problems in Mechanical Drawing. The stage at which boys wanted to "design" details was very soon reached, and in this—as in continuing the Experimental Mechanics they had begun in Form IV., but which was now studied on slightly more formal lines, the experimental and theoretical being combined—I found that the boys quickly reached a point at which their mathematical knowledge proved but a broken reed. This was even more obvious in their work on Physics, which was on similar heuristic lines. (It is far too seldom realised, at any rate in schools, that Physics is as much a mathematical as an experimental subject.) Even in the highest form a boy seldom reached the calculus, and when he did, the study was too formal to be of much use in practice. The hour set aside for mathematical investigation found a number of problems awaiting solution and the boys soon found that the necessary course followed two main lines of development—(i) vector calculus, the need for which arose in connection with their study of motion and statics, work on structures, &c. ; (ii) the differential and integral calculus, arising from the study of the graphs they obtained in experiments of various kinds as well as directly from physical and mechanical investigations.

As a course of vector calculus is not often attempted in schools, I venture to give an outline of our work in some detail.

The idea of and the necessity for the vector concept being present, physical quantities may be grouped as vectors or scalars, the former group being subdivided into non-localised vectors and localised vectors or "rotors." The importance of the vector conception is probably first realised in connection with problems on motion (displacements) and forces. The idea most readily grasped is that conveyed in picturing a transference from a point A to a point B, *i.e.*, that a vector represents, in the first place, a transference through a given distance in a definite direction. The question of notation arises and it is found convenient to use the symbol P_{θ} in specifying a vector for graphical representation: for convenience in writing, Greek letters α , β , ρ , &c. may be used. A definition may be given from which it at once follows that all lines which are equal and are drawn in the same direction may be represented by the same vector symbol. Since parallel lines may be drawn in opposite directions, it is necessary to impose the convention of signs: thus, since $AB + BA = 0$, we have, if $AB = a$, $BA = -a$. Tests are applied and the laws of algebra are found to hold as far as the addition and subtraction of parallel vectors are concerned. For example, the distributive law $a + a + a = (a + a) + a = a + (a + a) = 3a$ is illustrated graphically. So far there is no difference between parallel vectors and scalars.

The next step is the deduction that if lines as vectors are represented by the same vector symbol, they are parallel:

further, if AB be denoted by α and BC be not in the same direction as AB, it cannot be represented by α , or by any multiple of α . A natural extension of the idea and graphical illustration (chiefly in connection with consideration of "net" and "gross" displacements and the hodograph) leads to full appreciation of the meaning of the fundamental vector equation $\alpha + \beta = \gamma$, where α, β, γ may be respectively vectors representing the sides AB, BC, AC of the triangle ABC. Next comes experimental verification of the validity of the commutative and associative laws as applied to the addition and subtraction of non-parallel vectors (not necessarily coplanar), with plenty of examples involving specific vector quantities (displacements, velocities, accelerations, momenta, forces at a point, &c.).

Along these lines of development a firm practical foundation is assured, on which to base the fundamental axiom of the calculus, viz. :—If two vectors have not the same direction, it is impossible that one neutralises the other: or symbolically, if $m\alpha + n\beta = 0$, then $m = 0$ and $n = 0$. It follows at once that if $p\alpha + q\beta = a\alpha + b\beta$, then $p = a$, and $q = b$. The "tensor" concept is then developed and used to illustrate first ideas of multiplication in connection with vectors, while facility in the application of the fundamental principles is acquired, by attempting the solution of geometrical examples. The following are typical :—

1. Find the vector of the line determined by two coinitial vectors. Find, also, the vector to the mid point of the line so determined.

2. The straight lines joining the extremities of equal and parallel straight lines towards the same parts are themselves equal and parallel.

3. The opposite sides of a parallelogram are equal and the diagonals bisect one another. The diagonals of a parallelepiped bisect one another.

4. The sides about the equal angles of equiangular triangles are proportional.

5. The medians of a triangle are concurrent and meet in a point of trisection.

6. The mid points of the lines joining the mid points of opposite sides of a quadrilateral (plane or gauche) are coincident (the mean point).

7. The point of bisection of the join of the mid points of the diagonals of any quadrilateral is the mean point.

8. The bisector of the vertical angle of a triangle divides the base in the ratio of the sides. The bisectors are concurrent.

9. If from a point within a parallelogram, parallels are drawn to the sides, the corresponding diagonals of the parallelograms so formed and of the original parallelogram are concurrent.

Next, the condition that the vectors a , β , γ may be coplanar is investigated and, by considering the triangle whose sides are parallel to the vectors, it is found that if a , β , γ are neither parallel nor coplanar it is impossible to find numbers a , b , c , not equal to zero, which can make $a a + b \beta + c \gamma = 0$, while if a , β , γ are coplanar the values of a , b , c which make $a a + b \beta + c \gamma = 0$ are unique. The difficulty of geometrical methods of proving theorems on collinearity being recognised, the value of the vector method is appreciated and the boys soon learn that if the free ends of three coplanar, coinitial vectors are collinear, the values of a , b , c which make $a a + b \beta + c \gamma = 0$, also make $a + b + c = 0$. The converse theorem is investigated and typical problems are attempted, e.g. :—

- (1) The mid points of the three diagonals of a complete quadrilateral.
 - (2) Triangles in perspective.
 - (3) Centre of gravity, orthocentre and circumcentre of a triangle.
 - (4) The bisectors of the external angles of a triangle meet the opposite sides in collinear points.
- &c. &c.

Applications to mean points, centres of figure, mass centres, &c. are next studied, and graphical problems on these points and on the link polygon are worked. In connection with the determination of the condition that the vectors a , β , γ may be coplanar, it is found that any vector ρ may be expressed as $a a + b \beta + c \gamma$ where a , β , γ are unit vectors (orts) along mutually perpendicular axes, and many examples are worked illustrating "resultant" and "component" vectors.

The validity of the operation of multiplying a vector by a scalar had been assumed in the work on tensors. In Mechanics too, the boys had learnt that the product of two vectors—force \times displacement—measured a scalar—work: that, under other conditions, a vector—force—resulted from the product of two vectors—magnetic field \times current: the concepts, moment, and momental area, had also been used. Here, then, is the nucleus for the formal consideration of vector multiplication. It is difficult to avoid anything like a *tour-de-force* at this stage, but I found it convenient to define the fundamental operation and work out its logical development as in Hamilton's work. As introductory to the definition, we have to bear in mind that any theory must be such that either our results are capable of two interpretations, or our products are dual in nature, since we have seen that two vectors may give either a vector or a scalar product. Again, since any vector ρ can be expressed in terms of orts a , β , γ along mutually perpendicular axes, we can confine our attention to such symbols in the first place and, as we are dealing with a new conception, we are at liberty to make

any convention we choose as to the nature of the operation "multiplication," with the proviso that the interpretation of our results will depend on the definition of the operation adopted. In the examples to which reference has already been made, the boys have noticed that if one vector multiplies a second, the result is to convert or *turn* the second into something else, either vector or scalar, according to circumstances. For example, "magnetic field" \times "current strength" = "mechanical force": a magnetic field operating on a current gives a force and this force is perpendicular to the plane of the first pair of vectors. These and similar considerations form a rough justification for the convention actually adopted: if a, β, γ be unit vectors along mutually perpendicular axes Ox, Oy, Oz , the result of the multiplication of a into β —*i.e.*, $a\beta$ —is defined to be the *turning* of β through a right angle in the positive direction (with the usual sign convention) in the plane perpendicular to a and β ; in other words, the operation of a on β is to *turn* it round to coincide with γ ; or, briefly, $a\beta = \gamma$. To be consistent, it follows at once that $\beta\gamma = a$ and $\gamma a = \beta$. We now test the product βa and, applying our rotation definition, find that $\beta a = -\gamma$; similarly, $\gamma\beta = -a$ and $a\gamma = -\beta$. Next, remembering that we may have to deal with vectors not units, we consider the product $aa \times b\beta$ (a and b being tensors). Referring to the practical cases already considered (*e.g.*, 1 ft. \times 1 lb. = 1 ft. lb.; 10 ft. \times 5 lbs. = 10 (1 ft.) \times 5 (1 lb.) = 50 (1 ft. lb.)), we find that we can retain the old laws of multiplication as far as mere numbers are concerned: thus $aa \times b\beta = ab.a\beta$. Further, the associative and distributive laws for the operation of multiplication are assumed temporarily—to be discarded later if we find that the results yielded are inconsistent—while we have found that the commutative law does not hold. Thus we assume that $a\rho\sigma = a\rho.\sigma = a.\rho\sigma$ and that $a(\delta + \rho) = a\delta + a\rho$, but have shown that $a\beta \neq \beta a$, since definition imposes on a the duty of turning β in a specified direction yielding γ , whereas the same definition imposes on β the duty of turning a in a specified direction, giving $-\gamma$ as the result.

Many examples on the operators are now worked and, as a result of the assumption of the associative law, the important discovery is made that the *square of a unit vector* = -1 . This is compared with and illustrated by the usual convention of signs adopted in trigonometry, &c.: the repetition of the operation of turning a vector through a right angle clearly reverses it. Experiment and analysis also lead to the discoveries that $a\beta\gamma = \beta\gamma a = \gamma a\beta = -a\gamma\beta$, &c. The development of the subject is now straightforward enough up to the limits imposed by the time at our disposal, our next step being the consideration of the product $a\rho$ where ρ is inclined at an angle θ to a . The boys are reminded that they know rules for

the multiplication of vectors which are perpendicular to one another or are parallel, and they are asked to evaluate $a\rho$ where both are unit vectors. If there is still difficulty, they are reminded of the fact that ρ can be resolved into components in any directions. We have then at once—

$$\begin{aligned} \rho &= \cos \theta . a + \sin \theta . \beta \\ \therefore a\rho &= \cos \theta . a^2 + \sin \theta . a\beta \\ &= -\cos \theta + \sin \theta . \gamma \end{aligned}$$

The right-hand side is seen to consist of a scalar term and a vector term, the latter being perpendicular to the plane containing $a\rho$. Extending the result we have—

$$aa \times r\rho = -\cos \theta . ar + \sin \theta . ar . \gamma$$

$$\text{Similarly } r\rho \times aa = -\cos \theta . ar - \sin \theta . ar . \gamma$$

The first result is now written, $aa \times r\rho = ar S(a\rho) + ar V(a\rho)$ and it at once follows that $S(a\rho) = S(\rho a)$, $V(a\rho) = -V(\rho a)$, $a\rho + \rho a = 2S(a\rho)$, $a\rho - \rho a = 2V(a\rho)$, while $S(a\rho)$ is compared with work and $V(a\rho)$ with the force due to field \times current and the results are found to be consistent. Further, the coefficient of γ in $V(a\rho)$ is seen to be the area of the parallelogram of which $a\rho$ are coterminous sides, which leads to consideration of the sign of areas and the "aspect" of a plane. Examples are now attempted, the following being typical:—

- (1) Evaluate $S(a\rho)$ where $\theta = \frac{\pi}{2}$. Illustrate mechanically.
- (2) Compare $(a\beta)^2$ and $a^2\beta^2$.
- (3) Interpret $S(a\rho\sigma)$ geometrically and find an expression for the volume of a triangular pyramid. Show that $S(a\rho\sigma) = 0$ is the condition for a, ρ, σ to be coplanar.
- (4) What is the connection between the vector representation of the position of a point and its Cartesian co-ordinates?
- (5) a, β are adjacent sides of a parallelogram, show that $V(a + \beta, a - \beta) = 2V(\beta a)$ and translate the result. Show, also, that $S(a + \beta, a - \beta) = a^2 - \beta^2$ and translate the result if $a = \beta$ in magnitude. Show also that $(a + \beta)^2 + (a - \beta)^2 = 2(a^2 + \beta^2)$ and translate the result where a and β are constant in magnitude.
- (6) ABC is a triangle: a, β, γ are unit vectors along CB, CA, AB which are of lengths a, b, c units respectively. Show that $aa = b\beta + c\gamma$. Now operate on both sides of this equation with aa and translate the result. Again, square both sides of the equation and translate the result.

Remember that in each case you have the dual nature of the product to consider.

- (7) Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals. Extend to the case of a quadrilateral.
- (8) The sum of the squares of the distances of a point from the angular points of a triangle is equal to the sum of the squares of its distances from the mid points of the sides together with the sum of the squares of half the sides.
- (9) Show that (i) $S \gamma \cdot S(\alpha \beta) = 0$. (ii) $S(\alpha \beta \gamma) = S(\beta \gamma \alpha)$.
 (iii) $S(\alpha \beta \gamma) = -S(\beta \alpha \gamma)$. (iv) $V(\alpha \beta \gamma) = V(\gamma \beta \alpha)$. (v) $V(\alpha \beta \gamma) + V(\gamma \alpha \beta) = 2 \gamma S(\alpha \beta)$.
- (10) Show that the sum of the vector areas of the faces of a polyhedron is zero.
- (11) Show that $V(\alpha \beta) \times V(\gamma \delta) + V(\beta \gamma) \times V(\alpha \delta) + V(\gamma \alpha) \times V(\beta \delta) = 0$, also that $V(\alpha \beta) \times S(\gamma \delta) + V(\beta \gamma) \times S(\alpha \delta) + V(\gamma \alpha) \times S(\beta \delta) = 0$, and translate these results, $\alpha \beta \gamma \delta$ being any four vectors.
- (12) Find the vector equation to a straight line and to a plane. (Hence we have applications to the methods of drawing a straight line due to Hart and Peaucellier, &c.)
- (13) Find the vector equation to a circle and to a sphere. Deduce that the angle in a semicircle is right and that a tangent is perpendicular to the radius. If ABCD are points of the circle and AB, CD intersect in P, show that $PA \times PB = PC \times PD$.
- (14) If $\alpha \beta$ are unit vectors prove that $S(\alpha \beta) \times S(\alpha \beta) + V(\alpha \beta) \times (\alpha \beta) = 1$ and translate the result.
- (15) Find the relative position at any instant of two points moving uniformly along straight lines and find the time when the distance is least.
- (16) Assume $\frac{\rho}{\alpha} \cdot \alpha = \rho$ and evaluate $\frac{\rho}{\alpha}(\alpha \rho)$. Now compare the statements $\frac{\rho}{\alpha} \cdot (\alpha \rho) = \frac{\rho}{\alpha} \cdot (-\cos \theta + \gamma \sin \theta)$ and $\frac{\rho}{\alpha} \cdot \alpha \rho = \frac{\rho}{\alpha} \alpha \cdot \rho = \rho^2 = -1$. Hence, or otherwise, deduce the value or meaning of $\frac{\rho}{\alpha}$.

In the last example $\frac{\rho}{\alpha}$ was found to be $\cos \theta + \gamma \sin \theta$.

If now $\frac{\rho}{\alpha}$ is defined as an operator which acting on α gives ρ , evidently $\frac{\rho}{\alpha}$,—i.e., $\cos \theta + \gamma \sin \theta$ —has the effect of turning α through the angle θ : it is a versor. A figure at once makes it

clear that $\frac{\rho}{a} = \frac{\rho}{\sigma} \cdot \frac{\sigma}{a}$ and we can deduce De Moivre's theorem,

while $\frac{\rho}{a}$ can be written $\gamma^{\frac{2\theta}{a}}$. This leads with ease to well-known trigonometric formulæ for the functions of the sum and difference of angles and hence to the product formulæ.

The resumé above may appear too formal, but it must be remembered that the boys attempting this work were older boys who had experimentally acquired a fair elementary knowledge of Geometry, Mechanics, Physics, &c., and that *every step was linked with their general mathematical and scientific work by direct appeal and by experiment where possible*. That the methods were understood and appreciated is evidenced by the fact that they were frequently used in preference to the ordinary text-book methods for establishing such formulæ as those connecting the sides and angles of a triangle—for example—and in solving geometrical problems. In fact, the boys learnt how to translate a vector equation into ordinary mathematical language.

The study of graphs developed, briefly, thus:—The linear graph; the family $y = a + bx^n$, the meaning and determination in particular cases of a , b , and n , geometrical and other methods of drawing the curve under various conditions; the family $y = ae^{bx}$, similarly treated; the use of logarithmic and semi-logarithmic paper; the family $y = a \sin(cx + d)$, the meaning and determination of the values of a , c , and d in many cases, periodicity—this curve, of course, occurring in connection with the study of harmonic motion; derived curves, first studied in connection with motion; integral curves, studied both as inverse operations and as operations rendered advisable if not necessary by investigations in mensuration, mechanics, physics, &c.; graphical and other methods of drawing derived and integral curves in general as well as in particular cases; the determination of the "laws" of the derived and integral curves of the families already studied, graphically in most cases, although algebraic methods were not despised or entirely neglected, but the "laws" were thoroughly tested arithmetically and, in fact, in every other way likely to impress the boys with their validity, importance and power: then came illustrations of successive differentiation, differentiation of complex and implicit functions and Taylor's theorem—treated, I fear, somewhat cavalierly. Throughout I gave plenty of examples, most of which were ready to hand and were not created *ad hoc*—maxima and minima, areas, volumes, centres of gravity, moments of inertia, work, physical illustrations, &c. The last led to the treatment of partial differentiation—chiefly in connection with the gas equation and elementary work on steam engines.

Mechanical and physical examples, too, called for the solution of differential equations—of a very modest kind. For

example, the boys could determine velocity and displacement when given the acceleration as a simple function of the time: the energy stored in a spring from the work equation, using Hooke's Law: the work done by a fluid expanding according to the "law" $pv^n = k$, &c., because they could integrate expressions like $y = ax^n$. They could investigate mathematically (making certain assumptions) the slipping of a belt on a pulley, the variation of atmospheric pressure with height, the variation in the form of Ohm's Law due to induction, viscosity, cooling, &c., because they recognised the "compound interest law" when expressed symbolically. In fact, when the meaning of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, &c. (x and y being physical quantities) was understood, the solution of equations like $\frac{d^n y}{dx^n} = ax^n$, $\frac{d^2 y}{dx^2} = ky$, $\frac{dy}{dx} = ky$, &c. could be attempted, trial methods being adopted in some cases. The genesis of such equations by elimination of the constants from algebraic equations was illustrated, but nothing was attempted in the nature of formal treatment, whether of equations of the first order and first degree, or of more complicated type. My object was to interest the boys in this branch of Mathematics and to show that the solution of certain forms could be accomplished without the necessity for much mechanical "drill," thus giving them confidence in their application of the calculus methods.

Occasionally I had one or two boys studying elementary harmonic analysis. Work on the pendulum and on elementary dynamo design led to the investigation of periodic curves. Simple sine curves were compounded with harmonics and the resulting graph seen to be periodic. Cases in which the amplitude of the octave was a multiple of that of the fundamental were first studied and the nature of the resulting periodic curve learnt. The combination of fundamental and third harmonic, was similarly treated. In this way certain inferences are led up to, *e.g.*—

- (1) The number of ripples or humps on the curve in one period corresponds to the order of the harmonic (although the ripples are not always easily recognisable).
- (2) Odd harmonics (without "lag") change the shape of the curve, but it is still symmetrical in each half-period.
- (3) Even harmonics (without "lag") change the shape unsymmetrically.
- (4) The curves resulting from the combination of even harmonics are such that the second half-period resembles the first half-period, but inverted and reversed left for right; hence, if the second half-period is moved back (underneath the first), the top

and bottom do not balance, *i.e.*, are not symmetrical about the x -axis.

- (5) If the curves resulting from odd harmonics are treated similarly (to 4), the top and bottom do balance.

The next step was to study "lag," and here the boys found that any sine curve could be resolved into two, differing from one another by a prescribed phase difference—generally into a sine and cosine curve without "lag." This question was also treated algebraically.

Considerations of this nature led to attempts on the converse problem, the analysis of periodic curves into simple sine curves, and hence to a first investigation into Fourier's theorem. The methods illustrated were Perry's graphic and arithmetical methods and Wedmore's method, while calculus methods were commenced.

In two years, giving on the average one period a week to the work, a boy was able to acquire some knowledge of the elements of the infinitesimal calculus and of vectors—and this knowledge proved to be a real working knowledge: the boy appreciated its importance, its necessity; he realised that without it his physics, his engineering, was too mechanical, was a sham. I think the only things he did not appreciate were its limitations and perhaps—though I admit the point is debateable—this is no real fault. It is unnecessary to insist that the six periods per week "ordinary" Mathematics he gets elsewhere in the school were of vital importance. I do not say that the boy could make such progress at "Practical Mathematics"—obnoxious term—if he were not doing so much formal work, so much drill, but one had only to compare the mathematical outlook of the average "Technical" with that of the average boy on another "side" of the school, to realise that, to the one, Mathematics was part of his mental equipment, to the other it was a subject to be got up; to the one it was a sequence of problems arising naturally from his actual needs, the solution of which he was able to attempt in many ways, to the other it was too often a collection of "sums" to be done in stock methods. Professor Dewey, in his essay on "Ethical Principles underlying Education," says: "The moment mathematical study is severed from the place which it occupies with reference to use in social life, it becomes unduly abstract, even from the purely intellectual side."* Herein lies the "pull" of the Technical boy: his work is emphatically on the constructive side, on the side of active research; the workshop atmosphere appears to contain the elements necessary for the development of a proper sense of perspective in his outlook on Mathematics.

Such is a brief summary of my experience in teaching boys specialising in engineering and applied science generally. I

* Dewey (John). "Educational Essays." Edited by J. J. Findlay. p. 55. (London: Blackie and Son. 1910.)

turn now to indicate some constructive principles which that experience has shown me should underlie such work.

In his "Study of Mathematical Education,"* Mr. Branford points out that there may be considered to be three main impulses or stimuli towards mathematical study—the practical, due to man's struggle with his environment, the impulse to systematise, and the impulse to perfect the formal creation of the human mind. The last belongs chiefly, if not entirely, to the post-graduate stage; the second to the graduate course and, perhaps, in lesser degree, to the final year of the school course; the first—by far the strongest and most important—to the greater part of the school course. Throughout the school, no matter how it comes about, the pupil must feel that the undertaking of any mathematical investigation is desirable, nay more, unavoidable. There must be a foundation of actual facts in the boy's personal experience and that experience must be ordinary, immediate and direct, if there is to be intellectual hunger, otherwise the symbols, the shorthand of Mathematics, become mere conventions, substitutes for realities. In an essay on "The Child and the Curriculum," Professor Dewey says: "If the subject-matter of the lessons be such as to have an appropriate place within the expanding consciousness of the child, if it grows out of his own past doings, thinkings and sufferings, and grows into application in further achievements and receptivities, then no device or trick of method has to be resorted to in order to enlist 'interest.'"[†] These quotations are most pertinent to the case under discussion. If these and similar pedagogic principles are kept in view, I do not think there is any insuperable difficulty in devising a mathematical course that should be satisfactory to the scientific mind (and necessarily acceptable to engineers), and yet not so highly specialised as to be unsuited to non-engineers. The engineer, at any rate the young engineer, does not need highly specialised courses. What he does need is that his Mathematics shall be part of his expanding consciousness, a real and familiar weapon; the elements of reality, of vividness, are derived from his science work, his design, his experimental work generally—in short, his whole environment. So long as we are agreed that it is desirable for boys to be trained in scientific methods, the elementary training of the average boy should not consciously be different from that of the boy who intends to specialise and may become an eminent engineer. If at 15 or 16 we make a slight difference in the form of training of the boy who wishes to go in for applied science, it is a difference in motive for study and so in subject-matter, not a difference in method. The embryo engineer has the advantage in that his outlook is

* Branford (Benchara). "A Study of Mathematical Education, including the Teaching of Arithmetic." (Oxford: Clarendon Press. 1908.)

† "The School and the Child," being selections from the educational essays of John Dewey. Edited by J. J. Findlay. p. 42. (London: Blackie and Son. 1906.)

broader, his interests are immediate and keener, so that he at once realises the necessity for, as well as the utility of, the advanced mathematical weapon: his outlook is not circumscribed by the four walls of his classroom, his interest needs no stimulation.

It is too often true that, under present conditions, the teacher of Mathematics must either neglect entirely certain aspects of his subject, or introduce them in a forced mechanical and uneducational manner, violating the doctrine that "every central truth should arise in response to some demand springing from a practical problem involving manual as well as mental dexterity." No matter how interesting, how appealing to the imagination, how logical in its development, how up to date a system may be, it is quite uneducational unless its genesis is natural and the necessity for each step is keenly felt. Just as our ancestors developed their simple concepts of number, space and quantity in the course of their struggles with their physical environment, so must the mathematical ideas of boys be self-developed, the external assistance being reduced to a minimum. It is only logical to treat the subject informally, for at the period when boys are ready for mathematical study—as distinct from arithmetic—the methods found suitable for younger pupils are no longer practicable: new appeals must be made to freedom and interest, for it is at this period that the power to appreciate most exceeds the power to express and understanding most outstrips ability to explain: hence we must be heretics in the matter of formal proofs.

Again, I think there is little doubt that the use of text-books is responsible for much that is deplorable in the present condition of mathematical study for boys. For graduates, text-books may—as has been suggested—be of greater importance than either teacher or lecturer, but I would go so far as to say *do without printed text-books*—except perhaps collections of examples as exercises in drill, in manipulation, to save time—in all but the highest forms. *Let every boy make his own text-book*; it will then be a record of his own experience as well as a spur to further effort. This may be a counsel of perfection, but I believe that much of the vast improvement in science teaching during the last eight or nine years is due to the adoption of this plan and there should be no great difficulty in applying the same principle to Mathematics, at any rate in the lower school and fourth forms. I know this is done in Geometry in some schools, but it should be extended to the whole science and it would then be found that the difficulty of "watertight compartments"—due to the claims to recognition of the different branches of mathematical study—would automatically vanish.

Apart from comparatively unimportant questions of detail, I do not think one could improve on the syllabus proposed by Professor Perry to the joint meeting of Sections A and L of

the British Association in 1901, if indeed a formal syllabus be wanted. The differences between this syllabus and the suggestions contained in the report on the teaching of Mathematics issued by the Committee of the Mathematical Association are small. But all schemes hitherto formulated seem to me to have overlooked or ignored the fact that "the necessity for every step must be vividly felt," and no syllabus, however excellent, can create the atmosphere requisite for this stimulus to self-activity. Again, most schemes are too systematic, the authors having apparently failed to note that the impulse to systematise is not strong in the average boy. The danger of any formal syllabus is that it tends to over-rigidity, it makes for the crystallization of a system which, while perhaps suitable for one group of boys, or for a particular district, may be most unsuitable for another set, or a different locality; the system may appeal to one teacher, while others may not be able to do it justice. Hence, I would beg that teachers be allowed greater scope for individuality in their choice of subject-matter as in their choice of presentation. It is only necessary to ensure that a proper regard be paid to those aspects which place Mathematics in front of any other branch of human study—the research aspect and the practical aspect. Boys must learn to regard Mathematics as a science and not as an examination subject, and I am afraid that, without the requisite motive, Mathematics as usually taught does not, nay cannot, encourage that power of visualisation, that sense of sympathy with the machine, so necessary to the engineer. The result of school Mathematics at present is that there is frequently no desire, even if there be the capacity, to continue the study.

It appears, then, that the chief source of weakness in boys' mathematical work is due to this want of real motive, to this lack of valid reasons for undertaking the investigation of any particular group of problems, methods, or theorems, to its want of appeal. To obtain the correct atmosphere we must alter the environment. It means emphatically the reorganisation of the whole curriculum and a much larger place must be taken by manual training, by handicraft, particularly in the lower school and fourth forms. By manual training I do not mean that fictitious form in which the mastery of tools, or the production of objects is made the sole end; wherein is allowed no intellectual responsibility, no freedom of choice of methods, tools, or materials; where the work degenerates into mere routine: but that in which the mental and moral states and the growth necessarily involved in every process are kept in view; that which is a real *occupation*, in that the maximum of consciousness must be put into all that is done. Practical geography on modern lines is almost as valuable in affording adequate grounds for mathematical investigation, and in many ways is practically a handicraft. The natural concomitants of such subjects as these, *provided sufficient time be allotted and they are approached in the right spirit*, are the atmosphere in which the necessity for formal

Mathematics is vividly felt, in which the advantage attaching to the use and power of manipulation of symbols is clearly evident, and that interest which may be utilised to the fullest extent without danger of over indulgence. All problems arise naturally and the treatment of the subjects of any syllabus is rationalised. I believe that if the time allotted to manual training in schools be judiciously—and largely—increased, and if care be taken to combine the mathematical, scientific, practical and literary work in a real and not imaginary or forced co-ordination, there will be no necessity to differentiate the work suitable for the embryo engineer from that suitable for the embryo clerk, at any rate until an age which experts agree is not over-early for specialisation. In connection with this question of correlation I should like to acknowledge my indebtedness to a short but suggestive and illuminating chapter on "Culture and Occupation" in Mr. Branford's "Study of Mathematical Education," mentioned above.

If a correlated system such as I have indicated be tried, it is clear that in the first place the purely arithmetical work will be much curtailed, as it will practically be limited to multiplication and division—of course, contracted to a definite number of significant figures—and to proportion; next, in Algebra, stress will be laid on the idea of functionality and graphical work will be used to illustrate this idea of continuously varying quantities, not, as it too often is, reduced to a valueless mechanical operation; next, Solid and Plane Geometry will be combined, the first necessarily requiring most attention; finally, the boys will learn to watch their units and the dimensions of the quantities they employ, while the importance of accuracy and speed in calculation will be realised when arithmetical errors mean much more than another "sum" wrong. Along these lines of development it will be found possible to introduce in schools many branches of mathematical study now perforce neglected for lack of opportunity or perfunctorily attempted owing to want of time. There seems to be no adequate reason why the general level of attainment in Mathematics should not be raised, why such subjects as the infinitesimal calculus, vector calculus, together with the extension of the methods of the infinitesimal calculus to vectors, harmonic analysis and Fourier's theorem, simple differential equations, &c. should not receive some attention in the upper forms of our schools. It is only a question of removing artificial restrictions and encouraging individuality and research methods.

T. S. USHERWOOD.

MATHEMATICAL EXAMINATIONS AT OXFORD.

I.—THE EXAMINATIONS FOR THE DEGREE OF BACHELOR OF ARTS.

“At the end of the eighteenth century the exercises required for the degree of Bachelor of Arts had become merely formal, and offered no opportunity to candidates of parts and learning of obtaining distinction at their examination. The commencement of the present system is to be found in a statute of the year 1800, which gave an option to each candidate to offer himself for the ordinary examination held in every term, or for a more strict examination in Easter term, at which honours were awarded according to merit.”*

Honours “in disciplinis mathematicis et physicis” as well as “in literis humanioribus” were first given in 1807, the names being arranged in two classes.

In 1852, the First Public Examination, or Moderations, was introduced as an intermediate examination between Responsions and the Public Examination for the degree: so that from 1852 onwards, in order to take a degree, it was necessary to take—

- (i) Responsions, a pass examination in Latin, Greek, Arithmetic, and either Euclid, Books I., II., or Algebra;
- (ii) Moderations, a pass or honours in Classics, with either Logic or Euclid, Books I., II., III., and Algebra; and
- (iii) A pass or honours in two final schools, of which one must be Literæ Humaniores, and the other might be either Mathematics or Natural Science, or Law and Modern History.

Honours in Moderations, in disciplinis mathematicis, were also given after an examination, held twice a year in pure Mathematics.

In the final honour school, the subjects of examination were “Mixed as well as Pure Mathematics.”

These regulations are all set out in a little book entitled “New Examination Statutes, 1852. Abstracts of their principal provisions, with a catalogue of books, either expressly mentioned or treating of the subjects required.”

The list of books in Mathematics, which contains some eighty or ninety titles, is both comprehensive and formidable, as will be seen from the following examples:—

Schemata Geometrica ex Euclide et aliis, Tabulis Æneis expressata in usum Tyronum.

Hymers’s Treatise on the Theory of Algebraical Equations.

* Quoted from the Oxford Historical Register, 1220–1900.

Serrets' Algèbre Supérieure.
 Babbage's Tables of Logarithms.
 Whewell's Conic Sections.
 De Morgan's Trigonometry.
 Salmon's Treatise on Conic Sections.
 Gregory's Solid Geometry.
 Chasles, Géométrie Supérieure.
 Price's Treatise on the Differential Calculus.
 Moigno, Leçons de Calcul Différentiel et de Calcul Intégral.
 Duhamel, Cours d'Analyse.
 Poinso't, Eléments de Statique.
 Poisson, Traité de Mécanique.
 Frost's Collection of Senate House Problems.
 Airy's Mathematical Tracts.
 Pontécoulant, Théorie Analytique du Système du Monde.

It is to be presumed that in those early days, even as now, candidates would find more guidance for their work in the tradition of the examination, and in the study of papers previously set, than in any syllabus or list of books.

The regulations requiring two final schools before the degree of B.A. could be taken were soon changed, but it was not until 1886 that mathematicians were relieved from the necessity of taking a pass in classics in Moderations.

Since 1886 candidates who have passed Responsions and wish to devote themselves to mathematical studies have two examinations before them: Mathematical Moderations to be taken after one or two years' residence, and the Final Honour School of Mathematics, after three or four years.

In 1877 we find the following regulations issued by the Board of Studies:—

“ MODERATIONS, 1877.

For those who seek honours in Mathematics:—

A.

List of Subjects of Examination.

1. Algebra and Theory of Equations.
2. Trigonometry, Plane and Spherical.
3. Plane Geometry, including the Conic Sections, treated both geometrically and analytically.
4. Geometry of Three Dimensions, including the straight line, plane, and sphere, treated both geometrically and analytically, and the surfaces of the second order referred to their principal axes.
5. The Differential Calculus, including its applications to plane geometry, and to the determination of tangents and normals to surfaces and lines in space.
6. The Integration of Differential Expressions, with geometrical applications.
7. The Elements of the Calculus of Finite Differences.

B.

Some misunderstanding having arisen with regard to the term 'Differential Expressions' in the list of subjects of Examination for Mathematical Honours in the First Public Examination, the Board of Studies give notice that under this term Differential Equations are included."

In the examination seven papers were set, according to the following scheme :—

- I. Algebra.
- II. Trigonometry and Theory of Equations.
- III. Pure Geometry.
- IV. Analytical Geometry.
- V. Differential Calculus.
- VI. Integral Calculus, &c.
- VII. Problems.

In the paper on Pure Geometry, analytical methods were not allowed, nor was the use of the calculus allowed in the first four papers.

In the same year, 1877, the regulations issued by the Board of Studies for the final school were :—

"HONOUR SCHOOL OF MATHEMATICS, 1877.

A.

The following is the syllabus of the subjects in which candidates shall be examined :—

Pure Mathematics.

1. Algebra.
2. Trigonometry, Plane and Spherical.
3. Geometry of two and three dimensions.
4. Differential Calculus.
5. Integral Calculus.
6. Calculus of Variations.
7. Calculus of Finite Differences.
8. Theory of Chances.

Mixed Mathematics.

1. Mechanics of Solid and Fluid Bodies.
2. Optics, Geometrical and Physical.
3. Newton's Principia, Sections I., II., III., and parts of IX. and XI.
4. Astronomy, including the more elementary parts of the Lunar and Planetary Theories.

B.

The Board of Studies for directing the Honour School of Mathematics give notice that for the present, and until further notice, the subjects of the Papers in the examination will be as follows :—

- I. and II.—Elementary Pure Mathematics.
- III.—Elementary Mechanics of Solid and Fluid Bodies.
- IV. to VII.—Pure Mathematics.
- VIII. to X.—Mechanics of Solid and Fluid Bodies.

XI.—Optics (Geometrical and Physical).

XII.—Newton's *Principia* and Astronomy.

The methods of the Differential and Integral Calculus will not be admitted in the Elementary Papers I, II, III."

The papers set were, of course, in the traditional English manner. Each paper, for which three hours were allowed, contained eleven or twelve questions, most of them a piece of book work followed by a problem or rider more or less connected with it.

The amount of knowledge required, as determined by tradition, was not very great in any one subject, and was, in fact, limited to the propositions to be found in the regular text books in vogue at the time. For the attainment of the highest honours it was not expected that candidates should have any deep knowledge of any one branch of Mathematics, or show any acquaintance with modern developments, but, on the other hand, it was essential to have great skill in the solution of problems in a wide range of subjects, and ability to work both accurately and quickly. Problems requiring a great deal of astuteness were set on subjects which were quite out of fashion among mathematicians; they were indeed

"Tricks to show the stretch of human brain,
Mere curious pleasure, or ingenious pain."

For this state of things there were, as it seems to me, two chief causes.

The first is, that we do not, at Oxford, have professors who first teach their students and then examine them themselves on what they have taught them. If things were arranged in this way it would be natural, and indeed certain, that the best students would follow the professor in obtaining a wide and deep knowledge of the particular subject at which he is working. But when we have, as we have in Oxford, the system of separate colleges, each with its tutor, who, indeed, takes his turn with the rest in examining, but who is usually employed in preparing his pupils to be examined by someone else, it is obvious that the system tends to make the course of study rather wide and superficial than deep and rigorous.

The other chief cause was that it was recognised, thirty years ago, that the chief aim of a university education, even in the honours courses, was to turn out men trained in quickness of thought and in nimbleness of wits, rather than men of deep and extensive knowledge.

The modern fashion is quite otherwise. We are asked "to encourage soundness of work and breadth of reading and to discourage waste of time in acquiring acquaintance with the bye-paths of a subject, and an unnecessary degree of skill in manipulation."*

* See page 221 below.

The changes that have been made from time to time, which are to show their perfect work in the Final Honour School in 1913, have been directed to the following ends:—

- (i) to introduce the study of applied mathematics at an earlier period in the course :
- (ii) to discourage excessive training in manipulation, by allowing freedom of method :
- (iii) to encourage candidates to become proficient and learned in some few advanced subjects, rather than to have a superficial knowledge of many, by allowing a certain amount of freedom of choice in the subjects offered for examination :
- (iv) to give the mathematical theory of electricity a prominent place in the final examination.

The detailed history of these changes is as follows:—

In 1884, in the Final School, an elementary paper on Geometrical Optics, Astronomy and Newton's Principia was introduced, and the scheme of the papers was:—

A.—*Elementary Papers*, in which the use of the Differential Calculus was not allowed.

1. Algebra and Trigonometry.
2. Geometry.
3. Mechanics and Hydrostatics.
4. Geometrical Optics, Astronomy and Newton's Principia, Book I., Sect. 1, 2, 3.

B.—*Advanced Papers*.

5. Algebra and Trigonometry.
6. Geometry.
7. Differential Calculus.
8. Integral Calculus.
9. Statics of Solids and Fluids.
10. Dynamics of a particle.
11. Dynamics.
12. Optics and Astronomy.
13. Problems.

This yielding to the forces of reaction was, however, not to last long, for, in 1888, Newton is omitted and the scheme of papers is:—

A.—*Elementary Papers*.

1. Algebra and Trigonometry.
2. Geometry.
3. Elementary Differential and Integral Calculus with Solid Geometry.
4. Elementary Mechanics and Hydrostatics.
5. Optics and Astronomy.

The use of the Differential and Integral Calculus to be allowed in Papers 4 and 5.

B.—Advanced Papers.

6, 7, 8. Pure Mathematics.

9, 10, 11, 12. Mixed Mathematics.

13. Problems in Pure and Mixed Mathematics.

In 1887, in Moderations the list of subjects was altered by the addition of—

“The Elements of Mechanics of Solids and Fluids, to be treated without the aid of the Differential and Integral Calculus,”

and the omission of Solid Geometry and Finite Differences.

This is by far the most important change so far; but the limitation “to be treated without the aid of the Differential and Integral Calculus” was found to work very badly.

In 1889 Solid Geometry was restored to the list of subjects of examination in Moderations.

In 1904 some far-reaching changes were made in Moderations. The limitation as to the use of the differential calculus in elementary mechanics was removed, and more important still, freedom of choice was to be allowed between the methods of Pure Geometry and those of Analytical Geometry.

Up till 1904, Paper III. had been headed Pure Geometry, and the questions had to be answered by the methods of Pure Geometry without the aid of analytical methods. This, of course, meant that the candidates had to spend much time and labour in acquiring skill in these methods, whilst, when once freedom of method became the rule, skill in Pure Geometry was of little use for success in the examination.

As the preparation for the Honour Mathematical examinations leaves but very little leisure for even the ablest men, the inevitable result has been that Pure Geometry which, according to some authorities, has “a special value as a training of the intellectual faculties”^{*} has ceased to be an object of special study.

The regulations embodying these changes were as follows:—

“MODERATIONS, 1904.

Candidates will be examined in the following subjects:—

1. Algebra and the Theory of Equations.
2. Trigonometry, Plane and Spherical.
3. Pure Geometry: Analytical Geometry of two dimensions, and of three dimensions as far as the simpler properties of Surfaces of the Second Order.
4. Differential and Integral Calculus, with applications to Plane Geometry, and to the Geometry of Solids and Surfaces; Differential Equations.
5. The Elements of Mechanics of Solids and Fluids.

^{*} See p. 221 below.

The following suggestions are made by the Board of the Faculty of Natural Science for the direction of the first three Examinations held under these Regulations:—

1. Eight papers should be set, viz. :—

- I. Algebra and Theory of Equations.
- II. Trigonometry (Plane and Spherical) and Geometry.
- III, IV. Geometry.
- V, VI. Differential and Integral Calculus.
- VII. Mechanics.
- VIII. Problems.

2. Subject to the discretion of the Moderators there should be no restriction in the method of treating questions except that the use of the Infinitesimal Calculus should not be permitted in Papers I, II, III, IV. For example, freedom of choice should in general be allowed between the methods of Pure Geometry and those of Analytical Geometry.

3. Questions on Multiple Integration may be set, but only the simplest definite integrals should be required.

4. The Examination may include questions on Mechanics involving the use of the Infinitesimal Calculus—

In Statics—on virtual work and centres of gravity ;

In Dynamics—on the composition and resolution of velocities and accelerations in one plane as expressed by means of rectangular, polar and intrinsic co-ordinates with fixed directions of reference ; on rectilinear motion and composition of simple harmonic motions ;

In Hydrostatics—on centres of pressure.

5. In Papers V, VI questions on Mechanics may be set by way of illustration.”

These regulations of 1904 continued in force till 1911 when a new scheme for Moderations was introduced, to be followed in 1913 by a new scheme for the Final Honour School ; these two reforms are, to some extent, complementary, and taken together must have a profound effect on the training of the candidates. The further changes in Moderations were in part due to the fact that nowadays very many candidates take Mathematical Moderations as an intermediate examination before taking their degree in Physics, or Engineering, or Chemistry, the proportion of those who go on to Mathematical Final Schools after Moderations being not more than about 40 per cent.

It was felt that for these candidates, and in particular, for those who afterwards took Engineering, it would be better to increase the amount of Mechanics, and to decrease the amount of what is known in England as Algebra, and further, in order to provide for candidates who might wish to neglect almost entirely some branches of Mathematics, and yet to attain the highest honours, the old arrangement by which each subject had its own paper or papers was done away with, and each paper set must now contain questions on a variety of subjects, whilst entire freedom of method was to be the rule.

This new scheme for Moderations, which is the one now in force, is as follows :—

“ MODERATIONS, 1911.

Candidates will be examined in the following four groups of subjects, of which Group 1 will be treated as the smallest, and Group 4 as not less than either Group 2 or Group 3 :—

1. Algebra: Theory of Equations: Plane and Spherical Trigonometry. (No questions will be set on the theory of Infinite Products, or on the theorems of Uniform Convergency; but simple questions involving the principles of the theory of Uniform Convergency may be set.)

2. Pure Geometry: Analytical Geometry of two dimensions, excluding the theory of Invariants and Covariants of two Conics: Analytical Geometry of three dimensions as far as the simpler properties of Surfaces of the Second Order, excluding the theory of Confocal Surfaces.

3. Differential and Integral Calculus, with simple applications to Plane and Solid Geometry: Differential Equations.

4. The Elements of the Statics of Solids and Fluids: the Elements of the Dynamics of Particles, and of Rigid Bodies moving in two dimensions.

Seven papers, of which one will be a problem paper, will be set in the Examination. Each paper will contain questions on subjects in at least three of the above four groups. At least three of the questions in every paper but the problem paper will be of a simple and elementary character.

Freedom of method will be allowed except when otherwise stated.”

By the regulation by which “ at least three of the questions “ in every paper . . . will be of a simple and elementary “ character,” it is intended to make provision for those candidates who, not being professed mathematicians and not aiming at high honours, yet wish to take the course for the sake of its training, and to be able to count mathematical moderations as an intermediate examination towards their degree.

FINAL SCHOOL.

The history of the changes since 1888 in the examination for the final school is not so long.

In 1894 the Planetary and Lunar Theories were omitted from the list of subjects, and in 1903, “ applications of the Theory of Potential to Electro-statics ” were added, physical optics being omitted. The great change is yet to come, as will be seen from the following regulations for the examination in 1913 and afterwards :—

“ HONOUR SCHOOL OF MATHEMATICS, 1913.

Candidates will be examined in the following subjects :—

Algebra, including the Elements of the Algebra of Quantics; Theory of Equations; Trigonometry, plane and spherical; Infinite series and infinite products.

Geometry, pure and analytical, of two and three dimensions.

Differential and Integral Calculus; Differential Equations.

The Elements of the Theory of Functions of a Complex Variable, with applications to the Elementary Functions and to Elliptic Functions.

The Elements of the Calculus of Finite Differences.

The Elements of the Calculus of Variations.

Statics and Dynamics of Particles, Rigid Bodies, and Strings; the Elements of Analytical Dynamics; Statics of rods slightly bent.

Hydrostatics; the Elements of Hydrodynamics; Waves on liquids.

Attractions; Theory of Potential.

Electrostatics; Magnetostatics; Steady Electric Currents (flow in linear circuits, laminæ, and solid bodies); Electromagnetism (magnetic force due to currents, induction); Electrodynamics (mechanical effects of currents); Dielectric currents (propagation of plane waves in a homogeneous dielectric).

Vibrations of strings; Propagation of sound; Vibrations of air in pipes.

The Elements of Geometrical Optics.

Spherical Astronomy.

Half the Examination will be in Pure and half in Applied Mathematics.

The papers will be set according to the following scheme:—

Papers I.—VI.—Three of these will be on Pure and three on Applied Mathematics and each paper will include at least three questions of a simple and elementary character. No questions will be set on Electricity or Magnetism in these papers.

Papers VII.—IX.—Half of each of these papers will be on Pure and half on Applied Mathematics, and a Candidate will not be required to show proficiency both in Pure and also in Applied Mathematics in these papers in order to obtain the highest honours.

Paper X.—This paper will consist of problems in Pure and Applied Mathematics.

The following suggestion is made by the Board of the Faculty of Natural Science for the guidance of Examiners in the first three Examinations held under these Regulations:—

That the seventh, eighth, and ninth papers contain respectively:—

VII. Algebra and Analysis; and Statics, Hydrostatics, Electricity and Magnetism.

VIII. Geometry; and Dynamics, Hydrodynamics, Vibrations, and Waves.

IX. Analysis; and Electricity and Magnetism."

The first examination under these regulations will doubtless be a thunderclap, but the storm has been long brewing. The tradition of our examinations has been gradually changing with the spirit of the times, the amount of detailed knowledge expected of the candidates has greatly increased, and some acquaintance with modern rigorous method is demanded.

From the papers that have been recently set under the old scheme, it is evident that candidates are expected to show their mettle rather by a thorough study of some few subjects than by

a superficial acquaintance with many, and the attempt to fit an examination of this sort into the old framework has shown us the need for a radical change,

II.—EXAMINATIONS FOR UNIVERSITY SCHOLARSHIPS.

There are at Oxford two University Scholarships in Mathematics, which are, in fact, prizes given after examination to the best men of the year. Candidates for the Senior Scholarship must have passed the examinations for the degree of Bachelor of Arts, but must not have completed seven years from their matriculation; candidates for the Junior Scholarship must be undergraduates of less than two years' standing.

The scholarships are in the hands of trustees who appoint the examiners. The examination is "left entirely at the discretion of the examiners."

The examiners elect the scholars.

As the object of the examination is merely to pick out the best men and not to class meritorious candidates, the questions set have always been severe tests of mathematical ability according to the fashion of the times.

Six papers have usually been set in each examination.

The Junior Scholarship Examination.—Thirty years ago the papers set in the Junior Scholarship were labelled Algebra; Trigonometry and Theory of Equations; Pure Geometry; Analytical Geometry; Differential Calculus; and Problems. No knowledge of the Integral Calculus was required. In 1903, questions on the Integral Calculus and on Differential Equations began to creep in, and the headings to the papers were changed, but there was still a paper on Pure Geometry, in which analytical methods were not to be employed; since 1904, when the new regulations for Moderations were introduced, there have been no headings to the papers and no restriction as to method.

The new ideas as to what is important in the study of Mathematics have, of course, had their effect in gradually changing the style of the papers set.

The examination is still entirely in Pure Mathematics, but the range of knowledge expected is ever widening, and questions requiring some knowledge of the rigorous investigations of the foundations of the infinitesimal calculus, and of the theory of infinite series, are now usual, as well as the older type of question requiring great dexterity and skill in manipulation, so that breadth of knowledge is now as important for success as skill in the rapid solution of problems.

The Senior Scholarship Examination.—The Senior Scholarship is a reward and encouragement to men who continue the study of Mathematics after taking their degree. The subjects of examination were the subjects of the Final School, with amplifications and additions, and so long as the old ideas con-

tinued to reign the examination fulfilled its purpose very well. Six papers were set:—

- I., II. Pure Mathematics.
- III. Problems in Pure Mathematics.
- IV., V. Applied Mathematics.
- VI. Problems in Applied Mathematics.

The candidate expected to find some questions on Elliptic Functions, on Spherical Harmonics, on the Theory of Plane Curves, on the Lunar and Planetary Theories, on Elasticity, on Hydrodynamics, and might also find himself confronted with Quaternions and Thermodynamics. The giving up of two whole papers to Problems shows that great skill in manipulation was of the first importance. This worked very well twenty years ago, but as time went on more and more questions were set on the Theory of Functions and on the Mathematical Theory of Electricity, and the range of possible subjects grew so vast that changes had to be made.

First, there was one problem paper instead of two, with three papers on Pure Mathematics and three on Applied Mathematics. Then the plan was tried of giving notice beforehand to candidates as to subjects of special study, which were to have great weight in the next examination; or notice was given that each paper would contain questions both in Pure and Applied Mathematics.

Finally, the examination was abolished altogether—the last examination was in January 1911—and the Senior Scholarship is now “awarded to the candidate who shall present the dissertation of greatest merit on any subject of Pure or Applied Mathematics, to be selected by the candidate himself.”

By the kindness of the Delegates of the University Press I am enabled to add, as an appendix, copies of the papers set in the years 1885 and 1911 in—

- Mathematical Moderations;
- The Final Honour School of Mathematics;
- The Junior University Scholarship;
- The Senior University Scholarship.

To any one who will diligently compare them, one with the other, these papers tell their own story; indeed, except for the history of intermediate changes, this report need hardly have been written.

A. L. DIXON.

APPENDIX.

FIRST PUBLIC EXAMINATION.

TRINITY TERM, 1885.

I.

Algebra.

1. Prove the rule for finding the highest common divisor of two expressions.

Find the condition that

$$ax^2 - (a^2 - b^2)x + b, \quad bx^2 - (a^2 - b^2)x + a,$$

may have a highest common factor of the second degree, a being prime to b .

2. Solve the equations:

$$(1) \frac{x^2 - a^2}{bx - a^2} + \frac{x^2 - b^2}{ax - b^2} = 1;$$

$$(2) \begin{cases} (7a^2 + 4ab + b^2)x^2 + (a^2 + 4ab + 7b^2)y^2 = 8(a^2 + ab + b^2), \\ x + y = 2; \end{cases}$$

$$(3) ays + by + cs = bsx + cs + ax = cxy + ax + by = a + b + c.$$

3. Find the sum of the cubes of n terms of an arithmetical progression, and show that if n is even and $a + \beta$ is the middle pair of terms the sum of the cubes of the terms is

$$n\alpha \{a^2 + (n^2 - 1)\beta^2\}.$$

4. If one quantity varies as each of a set of other quantities when the rest of the set are constant, show that when all vary the first quantity varies as the product of the other quantities.

The quantity of water discharged per minute by an intermittent stream during each hour is constant, but at the beginning of each hour varies as the number of hours from the previous midnight and also as the number of hours from the succeeding midnight; the maximum number of gallons discharged per minute is 144; find the whole quantity discharged per day.

5. Show how to find the number of combinations of $p + q$ things taken r together when p are of one sort and q of another.

If $r > p > q$, show that the number is $p + q - r + 1$.

6. Find the general term in the expansion of

$$(a_0 + a_1s + \dots + a_ns^n)^m$$

If C_r is the coefficient of s^r in the expansion of

$$(1 + s + \dots + s^n)^m,$$

show that

$$(1) C_r = C_{m-r};$$

$$(2) C_{m+1} = \frac{n+m+1}{n-1, m+2} m^2.$$

7. If the m^{th} differences of consecutive terms of a series are in geometrical progression, show how to find the n^{th} term of the series.

Sum each of the following series to n terms:

$$(1) 1 + 0 - 1 + 0 + 7 + 28 + 79 + \dots;$$

$$(2) -\frac{2 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1 \cdot 2^2}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{6 \cdot 2^3}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{13 \cdot 2^4}{4 \cdot 5 \cdot 6 \cdot 7} + \dots;$$

$$(3) 3 + x + 9x^2 + x^3 + 33x^4 + x^5 + 129x^6 + \dots$$

8. Apply the method of convergent fractions to the solution in integers of an equation of the form $ax + by = c$.

Show that p being a prime number, the numbers $2, 3, \dots, p-2$ may be arranged in pairs in one way only such that the product of each pair is of the form $mp + 1$, and find the pairs in the case of the number 17.

9. If x is positive, show that $\log_e(1+x)$ is less than x and greater than $\frac{x}{1+x}$.

10. Find the highest power of a prime number n which is contained in $\frac{m!}{n!}$, and hence prove that the product of p successive integers is divisible by $p!$.

Show that the highest power of 2 contained in $2^r - 1$ is $2^r - r - 1$.

11. The probability of an event happening in one trial being known, find the probability of its happening r times at least in n trials.

A bag contains three balls which may be black or white; a ball has been drawn and replaced three times, and it is known that twice at least the ball has been white; find the probability that all the balls are white.

12. Express as a determinant the product of two determinants of the third order; and show that

$$\begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ -bc + ca + ab & bc - ca + ab & bc + ca - ab \\ (c+a)(a+b) & (a+b)(b+c) & (b+c)(c+a) \end{vmatrix} \\ = 3(b-c)(c-a)(a-b)(a+b+c)(bc+ca+ab).$$

II.

Trigonometry and Theory of Equations.

1. Prove the identity

$$\sin B \cdot \sin C \cdot \sin(B-C) + \sin C \cdot \sin A \cdot \sin(C-A) \\ + \sin A \cdot \sin B \cdot \sin(A-B) + \sin(B-C) \cdot \sin(C-A) \cdot \sin(A-B) = 0.$$

2. In a plane triangle ABC prove that the expression

$$\cos A + \cos B + \cos C$$

is positive and lies in value between 1 and $\frac{3}{2}$.

3. If α, β be the roots of the equation

$$a \cos x + b \sin x = c,$$

show that

$$\sin \alpha + \sin \beta = \frac{2bc}{a^2 + b^2}$$

and
$$\tan 2\alpha + \tan 2\beta = \frac{4ab(a^2 - b^2)}{(2c^2 - a^2 - b^2)^2 - 4a^2b^2}$$

4. Having given that $\tan \frac{A}{2} \cdot \tan \frac{B}{2} \cdot \tan \frac{C}{2} - 1 = 0$,

prove that

$$\sec A \cdot \sec B \cdot \sec C + \sec A + \sec B + \sec C = \tan A \cdot \tan B \cdot \tan C.$$

5. In a plane triangle, using the usual notation, prove that

$$(1) a^2 \sin(B - C) + b^2 \sin(C - A) + c^2 \sin(A - B) = 0;$$

$$(2) \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3};$$

(3) $\{R^2 - 2Rr\}^{\frac{1}{2}}$ = distance between the centres of the inscribed and circumscribed circles.

6. Express $\cos n\theta$ in terms of a series of ascending powers of $\cos \theta$, n being an odd positive integer: and show that (with the same restriction in the value of n)

$$\begin{aligned} n^2 - \frac{n^2(n^2 - 1^2)}{3} + \frac{n^2(n^2 - 1^2)(n^2 - 3^2)}{5} - \dots \\ = 1 - \frac{n^2 - 1^2}{2} + \frac{(n^2 - 1^2)(n^2 - 3^2)}{4} - \dots \end{aligned}$$

each series being continued as far as possible.

7. Sum the series:

$$(1) x \sin \theta + 2x^2 \sin 2\theta + 3x^3 \sin 3\theta + \dots \text{ to infinity;}$$

$$(2) 2 \cot 2\theta \cdot \operatorname{cosec} 2\theta + 2^2 \cot 2^2\theta \cdot \operatorname{cosec} 2^2\theta + \dots \\ + 2^n \cdot \cot 2^n \theta \cdot \operatorname{cosec} 2^n \theta;$$

(3) $\sin^2 a + \sin 2a \cdot \sin(a + \beta) + \sin 3a \cdot \sin(a + 2\beta) + \dots$ to infinity.

8. In a spherical triangle, prove that

$$(1) \tan \frac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cdot \cot \frac{1}{2}C;$$

$$(2) \tan E = \frac{\sin(s - a) + \sin(s - b) + \sin(s - c) - \sin s}{2\sqrt{\sin s \cdot \sin(s - a) \cdot \sin(s - b) \cdot \sin(s - c)}};$$

$$(3) \cos a \cdot \cos b \cdot \cos c \cdot \sin^2 A + \cos A \cdot \cos B \cdot \cos C \cdot \sin^2 a \\ = \cos^2 a - \cos^2 A.$$

9. If A' , B' , C' be the middle points of the sides of the spherical triangle ABC , and if from A' two great circles be drawn cutting AB in C' , C'' , and AC in B'' , B' respectively; prove that

$$\pm \cos A = \cos B' C' \cdot \cos B'' C'' - \cos B' C'' \cdot \cos B'' C';$$

the joining lines being arcs of great circles.

10. Solve the equations:

$$(1) 6x^2 + x^2 - 6x - 8 = 0;$$

(2) $6x^4 + 17x^2 - 83x^2 + 82x - 24 = 0$, two of whose roots are respectively of the form $a, \frac{1}{a}$;

(3) $2x^4 - 25x^4 + 46x^2 + 259x^2 - 426x + 144 = 0$, which has commensurable roots.

11. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$,

find the value of $(\beta + \gamma - \alpha)^2 + (\gamma + \alpha - \beta)^2 + (\alpha + \beta - \gamma)^2$ in terms of the coefficients.

12. What are the roots of $x^3 - 1 = 0$? Distinguish its primitive roots.

Show that the roots of $x^3 - 7x^2 + 14x - 7 = 0$

are $4 \sin^2 \frac{\pi}{7}, 4 \sin^2 \frac{2\pi}{7}, 4 \sin^2 \frac{3\pi}{7}$.

III.

Pure Geometry.

1. The feet of the perpendiculars on the sides of a triangle from any point of the circumscribing circle are collinear.

From the angular points of the triangle ABC are drawn three chords AP, BQ, CE of a circumscribing conic, so as to intersect in the point O . BR, CP, AQ meet the polar of O in X, Y, Z respectively. Prove that, if S be any point on the conic, the lines SX, SY, SZ meet the sides BC, CA, AB respectively in collinear points.

2. Any circle which passes through the limiting points of a coaxial system of circles cuts every circle of the system orthogonally.

Find the envelope of a system of circles touching a given circle and cutting another given circle orthogonally.

3. The number of regular solids with plane faces is five.

4. Perpendicular tangents to an ellipse intersect in T . Find the locus of T , and prove that the sum of the squares of the chords intercepted on these tangents by the auxiliary circle is constant.

5. $\{ABCD\}$ is a harmonic range on a given line. O_1, O_2 are the middle points of AB, CD , and P is any point on the line. Prove that $PA \cdot PB + PC \cdot PD = 2 PO_1 \cdot PO_2$.

AC, BD are conjugate diameters of a central conic, and P is any point on the arc AB . PA, PB meet CD in Q, R respectively. Prove that the range $\{QDCR\}$ is harmonic.

6. Four fixed tangents to a conic determine on any fifth a range of points whose anharmonic ratio is constant.

$PG, P'G'$ are normals to a conic at the points P, P' respectively. Shew that these lines, the chord PP' , and the principal axes of the conic all touch a parabola.

7. Defining a conic by its focus and directrix property, prove that confocal conics are inscribed in the same quadrilateral.

S and H are the foci of a confocal system of conics. A parabola is drawn, with S as focus, so as to touch the minor axis of the system. Show that its directrix will pass through H . Prove also that the common tangents of the parabola and any conic of the system subtend right angles at S .

8. If two triangles circumscribe a conic, their vertices lie on another conic.

Hence, or otherwise, show that, if it is possible to inscribe two triangles in a conic S so that their sides envelope a second conic S' , then an infinite number of such triangles may be described.

9. A system of conics circumscribing a given plane quadrangle has a common self-conjugate triangle.

If two plane quadrangles have a common triangle of diagonals, their vertices lie either on a conic or on a pair of lines.

10. Give a general account of any method for the geometrical solution of problems concerning maxima and minima.

AB is a chord of a parabola. Determine the maximum inscribed polygon of n sides that can be constructed with AB as one of its sides.

11. "If a circle be drawn through the foci S, H of two confocal ellipses, cutting the ellipses in P and Q , the tangents to the ellipses at P and Q will intersect on the circumference of the same circle."

Assuming the truth of this theorem, generalise it (1) by projection, (2) by reciprocation with respect to the point S , (3) by reciprocation with respect to any point in the plane.

IV.

Analytical Geometry.

1. Find the ratio in which the connector of two given points is divided by a given straight line.

$ABCD$ is a parallelogram, EF a straight line parallel to BD meeting AB, BC, CA in E, F, G respectively; show that
 $EG : GF :: GA : GC$.

2. Find the condition that two given circles may cut each other orthogonally.

Show that the distances of a point from the centres of two such circles are in the same ratio as the distances of the centre of each circle from the polar of the point with respect to the other.

3. Find the coordinates of the centres of similitude of two given circles.

Show that $r_2^2 C_1 - r_1^2 C_2 = 0$ represents a circle which has the centres of similitude of the circles $C_1 = 0, C_2 = 0$ as extremities of a diameter; r_1, r_2 being the radii of the circles. Show also that the three circles have the same radical axis.

4. Show that $x \cos a + y \sin a = (a^2 \cos^2 a + b^2 \sin^2 a)^{1/2}$ is the equation of a tangent to the ellipse $a^{-2} x^2 + b^{-2} y^2 = 1$.

Q is a point on a tangent at a point P of the ellipse, and CY the perpendicular from the centre upon the tangent; show that if $QP \cdot CY = ab$ the locus of Q is $a^{-2} x^2 + b^{-2} y^2 = 2$.

5. Find the equation of the normal at a given point of the conic $tr^{-1} = 1 + e \cos \theta$.

If the normals at the points $\theta_1, \theta_2, \theta_3$ of the parabola $r = a \sec^2 \frac{\theta}{2}$ meet in a point whose angular coordinate is a , show that $\theta_1 + \theta_2 + \theta_3 - 2a$ is a multiple of four right angles.

6. Prove that the chords of intersection of a circle and a conic are equally inclined to an axis of the conic.

Show that the connector of the centre of an equilateral hyperbola with the point of contact of a circle of curvature is perpendicular to the chord of intersection of the curves.

7. Find the conditions that the general equation of the second degree in trilinear coordinates may represent a circle.

Show that the equation of the circle which has the vertices B, C of the triangle of reference as extremities of a diameter is

$$a\beta\gamma + b\gamma a + ca\beta = a \cos A (a\alpha + b\beta + c\gamma).$$

8. If the constants of the equation

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0$$

vary, due to any change of rectangular axes, show that

$$(a + b)^2 \Delta^{-1} \text{ and } (a + b)^2 (ab - h^2)^{-1}$$

remain invariable, and give a geometrical interpretation of the invariants.

Trace the loci

$$(1) 2x^2 + 5xy + 2y^2 + 3x + 3y + 1 = 0;$$

$$(2) x^2 + 4xy + y^2 + 2x + 2y + 1 = 0;$$

and find the coordinates of the foci of (2).

9. Show that if $a_1 b_2 + a_2 b_1 - 2h_1 h_2 = 0$, the lines

$$a_1 x^2 + 2h_1 xy + b_1 y^2 = 0, \quad a_2 x^2 + 2h_2 xy + b_2 y^2 = 0$$

form a harmonic pencil.

Show that a tangent to the conic

$$(b + b')^{-1} x^2 + (a + a')^{-1} y^2 = (ab' + a'b)^{-1}$$

is cut harmonically by the conics $ax^2 + by^2 = 1, a'x^2 + b'y^2 = 1$.

10. If $S = 0$ is a conic and $L = 0$ a straight line, what does $S - L^2 = 0$ represent?

A parabola has contact of the third order with the ellipse $a^{-2}x^2 + b^{-2}y^2 = 1$, and makes intercepts $x_1, x_2; y_1, y_2$ on the axes; show that $2a^2(x_1x_2)^{-1} + 2b^2(y_1y_2)^{-1} + 1 = 0$.

11. Find the angle between two straight lines whose direction cosines are given, and show that by suitable choice of axes the equations of any two straight lines may be put in the forms

$$\frac{x-a}{0} = -\frac{y}{\sin a} = \frac{z}{\cos a}; \quad \frac{x+a}{0} = \frac{y}{\sin a} = \frac{z}{\cos a}.$$

AB is the shortest distance between two straight lines AC, BD ; show that if $AC = BD$, CD is equally inclined to AC, BD .

12. Show that (f, g, h) is the centre of the section of the surface $ax^2 + by^2 + cz^2 = 1$ made by the plane

$$afx + bgy + chz = af^2 + bg^2 + ch^2.$$

Find the loci of the centres of circular sections of the surface.

V.

Differential Calculus.

1. Obtain from first principles the differential coefficient of $e^{ax} \cos bx$.

2. State and prove Leibnitz's theorem for the n th differential coefficient of a product of two functions of a variable.

If $f(\tan \theta) = \tan^n \left(\frac{\pi}{4} \pm \frac{\theta}{2} \right)$, prove that $f^{(n)}(0) = (n^2 - n^2)f^n(0)$.

3. Prove Taylor's theorem, and find an expression for the remainder after the first n terms.

Find the first five terms of the expansion of $\sin(e^x)$ in ascending powers of x by the method of Arbogast.

4. Investigate a method for evaluating a function when it becomes of the form $\frac{0}{0} \cdot 1^\infty$, or 0^∞ .

Evaluate $\frac{e - (1+x)^{\frac{1}{x}}}{(\operatorname{cosec} x - \cot x) \sin x^{\tan x}}$ when $x = 0$.

5. Eliminate the constants and transcendental functions from $y = e^{ax} \cos bx$; and the arbitrary functions from

$$z = \phi(x^2 + y^2) + (x^2 - y^2)\psi\left(\frac{y}{x}\right).$$

6. Show how to find and distinguish between the maxima and minima values of a function of one variable.

TP, TQ are the tangents at the extremities of the shortest circular arc which can be drawn touching two equal non-intersecting circles: prove that $TP + TQ$ is equal to the length of a concentric arc terminated by the centres of these two circles.

7. With the ordinary notation prove that

$$(1) \quad \rho = p + \frac{d^2p}{d\phi^2};$$

$$(2) \quad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2;$$

and show how to find the points of inflexion of a polar curve.

8. Show how to find the rectilinear asymptotes of a curve whose equation is given (1) in rectangular coordinates, (2) in polar coordinates; and distinguish the case of a multiple point at infinity.

Find the asymptotes of

$$(x + y - 2a)(x^2 - y^2)^2 - 2a^2(x^2 + y^2)(x + y) + 2a^3(x - y)(x - y + 2a) = 0.$$

9. Trace the curves

(1) $x^2y + ax(x^2 - y^2) = a^2y^2;$

(2) $(y + 3x)^2(y^2 - x^2) + 3a(y + 3x)x^2 - 5a^2x^2 = 0;$

(3) $r = \frac{3a}{4} \left[\tan \frac{3\theta}{4} - \cot \frac{3\theta}{4} \right].$

10. Prove that the line $x \cos \theta + y \sin \theta = a \sin m\theta$ envelopes an epi-cycloid or hypo-cycloid according as m is less or greater than unity.

Hence prove (1) that the radius of curvature of an epi-cycloid is in the constant ratio $1 - m^2$ to the perpendicular from the centre of the fixed circle on to the tangent; that the second evolute of an epi-cycloid is a similar and similarly placed epi-cycloid.

11. Find the equations of the tangent plane and normal at any point of the surface $F(x, y, z) = 0$.

Tangent planes are drawn to the surface $x^m + y^m + z^m = a \cdot m^{m-1}$ at points where it is met by the surface

$$a^2(x^{2m-2} + y^{2m-2} + z^{2m-2}) = (x^m + y^m + z^m)^2;$$

find their envelope, m being a variable parameter.

12. Find the equation of the osculating plane at any point of a tortuous curve.

Prove that the osculating planes of the curve whose equations are $y(z + x)^2 = 2ay^2 = 2a(x^2 - z^2)$ all touch the cone $3y^2 = 4(z^2 - x^2)$.

VI.

Integral Calculus, &c.

1. Define a definite integral. Obtain from the definition the values of the definite integrals

$$\int_0^{\pi} \sin 3x dx, \quad \int_0^{\infty} \frac{dx}{a - x^2},$$

explaining any difficulty that may present itself in the process.

2. Evaluate

(1) $\int \frac{dx}{5 + 4x - x^2};$

(2) $\int \frac{dx}{\sin x \cos x \log \tan x};$

(3) $\int \frac{dx}{a^2 + x^2};$

(4) $\int_0^{\pi} \cos^2 x dx.$

3. If the equation $\phi(x) = 0$ have m equal roots a , and if $\phi(x) = \phi_1(x) \cdot (x - a)^m$, prove that, when $\frac{\psi(x)}{\phi(x)}$ is converted into a series of partial fractions, the coefficient of $(x - a)^{n-m}$ ($n < m$) is that of h^n in the expansion of $\frac{\psi(a+h)}{\phi_1(a+h)}$ in ascending powers of h .

Prove that

$$\int_0^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)(x^2 + c^2)} = \frac{\pi}{2(b+a)(c+a)(a+b)}$$

4. Show that

$$(1) \int_a^c \int_a^x \phi(x) dx dx = \int_0^{c-a} x \phi(x+a) dx - \int_0^{c-b} x \phi(x+b) dx;$$

$$(2) \int_0^{\pi} \frac{x dx}{1 + \sin^2 x} = \pi \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin^2 x} = \frac{\pi^2}{2\sqrt{2}}.$$

5. Find the length of a curve in the form

$$s = \frac{dp}{d\phi} + \int p d\phi.$$

Prove that, in the case of any closed curve, $\int p d\phi$ has the same value, wherever the origin may be, if the integral be taken completely round the curve.

The equation of a curve is given in the form $p^2 = 2ar - a^2$; show that the length of the arc from

$$r = 2a \text{ to } r = 3a \text{ is } a(1 + \log 2).$$

6. If $y = zx$ and the equation of a closed curve be given in the form $f(x, z) = 0$, prove that the area may be obtained by evaluating $\frac{1}{2} \int x^2 dx$.

Obtain the area of the loop of the curve

$$x^3 + y^3 = 3axy.$$

7. Find the portion of the volume intercepted between the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ which is contained between the planes $y = mx, y = nx$.

8. Integrate the differential equations:

$$(1) dy [x \cos x - x^2(x-1) \sin y]$$

$= dx [yz \sin x + y \cos x - (1 + \cos y)x^2]$, there being an integrating factor which is a function of x only;

$$(2) \frac{d^2y}{dx^2} + 2a \frac{dy}{dx} + a^2y = \sin ax + e^{ax} + x;$$

$$(3) 2y \frac{dy}{dx} + y^2x = \frac{e^{-ix}}{1+x^2};$$

$$(4) y = x \frac{dy}{dx} + c \sqrt{1 + 2 \frac{dy}{dx} - \frac{dy^2}{dx^2}}.$$

9. Define a singular solution of an equation of the form

$$\phi \left(x, y, \frac{dy}{dx} \right) = 0.$$

How is such a solution obtained? Give a geometrical interpretation of the method, and of the exceptional cases that may arise.

Examine fully for singular solutions the equation

$$(x^2 - a^2)p^2 - 2xyp - x^2 = 0.$$

10. Prove that

$$\Sigma a^n \phi(x) = \frac{a^x}{a-1} \left[\phi(x) - \frac{a}{a-1} \Delta \phi(x) + \frac{a^2}{(a-1)^2} \Delta^2 \phi(x) - \dots \right];$$

and sum the series

$$1^3 \cdot 2^1 + 2^3 \cdot 2^2 + 3^3 \cdot 2^3 + \dots \text{ to } n \text{ terms.}$$

11. Solve the equations:

$$(1) u_{x+2} + 2u_{x+1} - 3u_x = x + (-3)^x;$$

$$(2) u_{x+1} u_x - 2u_x - 3u_{x+1} = 14.$$

VII.

Problems.

1. An arithmetic series of which the first term is

$$(p + 1)(3p - 2)a,$$

and the common difference $-2a$, consists of $3p^2 - 1$ terms; show that the sum of the products of every two terms vanishes.

2. Obtain in a rational form the eliminant of the equations

$$\begin{aligned} by^2 + 2fyx + cx^2 = 0, \quad cx^2 + 2gzx + ax^2 = 0. \\ ax^2 + 2hxy + by^2 = 0. \end{aligned}$$

3. Find all the values of the number n less than 14 which make $n^3 - 1$ a multiple of 240.

4. If $\tan \phi = \frac{\sin \theta \cdot \sin \alpha}{\cos \theta \pm \cos \alpha}$, then $\tan \theta = \frac{\sin \phi \cdot \sin \alpha}{\cos \phi \pm \cos \alpha}$.

5. Given three concentric circles, prove that another concentric circle can be drawn such that a triangle of tangents may be described about it having a vertex on each of the given circles.

Show further that three such circles can always be drawn, all real, and all different unless the three given circles are all coincident.

6. A, B, C are the centres of three given circles. O_1 is the centre of the circle bisecting their circumferences, O_2 the centre of the circle cutting them orthogonally, and O_3 the centre of the circle drawn through A, B, C . Prove that O_1, O_2, O_3 lie on a straight line, and that O_2 bisects $O_1 O_3$.

7. The common chord of the auxiliary circle and of the circle of curvature at the extremity of one of the equiconjugates of an ellipse is parallel to that diameter.

8. If P is the pole of a chord which subtends a constant angle at the focus S of a conic and SP intersects the chord in Q , show that the locus of the harmonic conjugate of S with respect to P, Q is a conic having the same focus and directrix as the given conic.

9. Show that the equation of the real pair of common tangents of the circles $x^2 + y^2 - ax = 0, \quad x^2 + y^2 - by = 0$, is

$$4a(2b - a)x^2 + 8abxy + 4b(2a - b)y^2 - 4a^2bx - 4ab^2y - a^2b^2 = 0.$$

10. The trilinear co-ordinates of a point P are f, g, h , the triangle of reference being ABC ; show that the equation of the conic which bisects BC, CA, AB, PA, PB, PC is

$$\begin{aligned} f^{-1}a(-a\alpha + b\beta + c\gamma) + g^{-1}\beta(a\alpha - b\beta + c\gamma) \\ + h^{-1}\gamma(a\alpha + b\beta - c\gamma) = 0. \end{aligned}$$

11. From a fixed point S in the plane of a plane curve a perpendicular SY is drawn to the normal at a point P of the curve, and a circle is drawn through P to touch SY at S : prove that when the triangle PSY is a maximum or minimum, this circle passes through the centre of curvature at P .

12. If r, r' be the radii vectores and ρ, ρ' be the radii of curvature at corresponding points of a curve and its reciprocal polar, prove that $\rho\rho'$ is proportional to $[rr']^2$.

13. The curve $x^3 + y^3 = a^3$ meets the axes of co-ordinates in the points $A, B; C, D$ respectively. Prove that the excess of its length over the sum of the distances AB and CD is

$$AB \times \sqrt[3]{2} \int_0^{\frac{\pi}{2}} \sqrt{\cos \phi} d\phi.$$

14. Solve the differential equation

$$\left(\frac{dy}{dx} - ax\right)^2 + x\left(\frac{dy}{dx} - ax\right) - (y - \frac{1}{2}ax^2) = a^2.$$

15. If $x + y = 1$ and p be integral, show that

$$\sum_{p=1}^{\infty} \frac{x^p + y^p}{p^2} = \frac{\pi^2}{6} - \log x \log y.$$

SECOND PUBLIC EXAMINATION.

TRINITY TERM, 1885.

I.

Algebra and Trigonometry.

1. Prove that

$$\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)} + \frac{(x-a)(x-c)(x-d)}{(b-a)(b-c)(b-d)} + \frac{(x-a)(x-b)(x-d)}{(c-a)(c-b)(c-d)} + \frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)} = 1.$$

2. A person borrows £ a ; find how much he must pay annually so that the whole debt may be discharged in n years, allowing compound interest at r per cent.

3. Solve the equations:

$$(1) \frac{x-b}{x-a} + \frac{x-a}{x-b} = \frac{x+b-2a}{x+a-2b} + \frac{x+a-2b}{x+b-2a};$$

$$(2) x + y + z = ab, \quad x^{-1} + y^{-1} + z^{-1} = a^{-1}b, \quad xyz = a^3.$$

4. In every geometric progression of an odd number of terms, the sum of the squares of the terms is equal to the product of the sum of all the terms by the excess of the sum of the odd terms above the sum of the even.

5. The number of solutions of $ax + by = c$ in positive integers can never exceed the greatest integer in $\frac{c}{ab} + 1$.

A farmer spends 184*l.* in buying sheep at the rate of 2 guineas a head and oxen at 12*l.* 10*s.* a head. How many of each did he buy?

6. Find a formula for the number of numbers prime to and less than a given number N .

If x be any one of these primes and n the number of them, prove that $x^n - 1$ is divisible by N .

7. If $\frac{p}{q}, \frac{p'}{q'}$ be two consecutive convergents to a continued fraction, shew that $pq' - p'q = \pm 1$.

Express each of the roots of $x^2 - 2ax + 1 = 0$ in the form of a continued fraction.

8. The arithmetic mean of any number of positive quantities is greater than their geometric mean.

If, x, y, z be three positive quantities in order of magnitude, prove that

$$\frac{x}{z} + \frac{y}{x} + \frac{z}{y} > \frac{x}{x} + \frac{x}{y} + \frac{y}{z}$$

9. If A, B, C be the angles of a plane triangle, shew that
 $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$.

In a triangle given $A = 72^\circ$ and $\sec B - \sec C = 2$, find B and C .

10. If O_1, O_2, O_3 are the centres of the escribed circles of a plane triangle ABC , shew that the area of the triangle $O_1 O_2 O_3 =$

$$\text{area of triangle } ABC \left\{ 1 + \frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c} \right\}.$$

11. If $i^2 + 1 = 0$, prove the formula $2i \sin \theta = e^{i\theta} - e^{-i\theta}$.

Given $\tan y = \sec a \tan x$, prove that

$$y = x + \tan^2 \frac{a}{2} \sin 2x + \frac{1}{2} \tan^4 \frac{a}{2} \sin 4x + \frac{1}{3} \tan^6 \frac{a}{2} \sin 6x + \dots$$

12. Give a method for solving a cubic equation.

Find the condition that the cubic $ax^3 + 3bx + c = 0$ shall have two equal roots.

13. Find by Sturm's theorem, or otherwise, the positions of the roots of the equation

$$x^3 - x^2 - 4x + 3 = 0.$$

II.

Geometry.

1. If a straight line touch a circle and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

Two circles touch one another internally at the point A , and a chord APQ is drawn cutting them in P and Q . If the tangent at P meets the outer circle in H and K , prove that HQ is equal to KQ .

2. If the internal bisector of the angle A of a triangle ABC meet the circumscribed circle in D , and DG be drawn perpendicular to AB meeting it in G , show that AG is equal to half the sum of AB and AC .

Construct a triangle of which the sum of two sides, the included angle, and the radius of the circumscribed circle are given.

3. The opposite sides of a quadrilateral $ABCD$ are produced to meet in R, S ; prove that the line which joins the middle points of the diagonals of the quadrilateral will pass through the middle point of RS .

4. If QV be an ordinate of a diameter PV of a parabola, prove that QV is a mean proportional between PV and the parameter of P .

If S be the focus of the parabola, and SP be produced to Q' so that $PQ' = SP$, then an ellipse which has S as focus, and V as centre, and which passes through Q , will pass also through Q' , and will be such that VQ, VQ' are a pair of its conjugate semi-diameters.

5. Prove that the rectangles contained by the segments of any two chords of an ellipse which intersect one another are in the ratio of the squares of the parallel diameters.

Show geometrically that the square of a diameter of an ellipse varies directly as the distance of the middle point of the parallel focal chord from the directrix.

6. ABC is a given triangle and through B any line is drawn meeting AC in K . Through C a parallel line is drawn meeting AB produced in L . Prove geometrically that the envelope of KL is a hyperbola, and find its foci.

7. Find the length of the tangent drawn from (h, k) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Show that if from any point P tangents PT, PT' are drawn to two circles, meeting them in T, T' , then $PT^2 \sim PT'^2 = 2PN \cdot AB$, where AB is the line joining the centres A, B of the circles, and PN is the perpendicular from P on their radical axis.

8. Prove that the length of a focal chord of the parabola $y^2 = mx$ making an angle θ with the axis is $m \operatorname{cosec}^2 \theta$.

If two focal chords are at right angles, show that the sum of their reciprocals is constant.

9. Find the equation of the tangent at any point of the conic $lr^{-1} = 1 + e \cos \theta$.

The centre of a circle is the focus of an ellipse of eccentricity e . The two curves intersect in P , and one of their common tangents touches the ellipse in Q . Prove that

$$2(\cos \alpha - \cos \beta) = e \sin^2 \alpha,$$

where α, β are the vectorial angles of P and Q respectively, the focus being the origin and the major axis of the ellipse the initial line.

10. Determine the condition that the two lines $y = mx, y = m'x$, may be conjugate diameters of the conic $ax^2 + 2hxy + by^2 = 1$.

Show that the circle $2(ab - h^2)(x^2 + y^2) = a + b$ meets this conic at the extremities of its equal conjugate diameters.

11. Obtain the equation of a parabola in the form

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1.$$

If the axes are rectangular, find the equation to the directrix.

12. Find the equation of the rectangular hyperbola which circumscribes a given parallelogram, two adjacent sides of which are the axes of coordinates; and show that its asymptotes cut off from the axes intercepts equal to the sides of the parallelogram.

III.

Mechanics and Hydrostatics.

1. Define acceleration, and explain how it is measured.

If the acceleration caused by gravity be the unit of acceleration, and the velocity of a mile in five minutes the unit of velocity, find the unit of length.

2. Prove the proposition known as the parallelogram of velocities.

If a body start from rest and move in such a manner that the velocity at any instant varies as the square of the time of motion, show that the space described in three minutes is nine times the space that would be described in one minute by a second body moving uniformly with the velocity which the first body had at the end of the first minute.

3. Show that the times of sliding down all chords from the highest point of a vertical circle are equal.

4. Equal spherical inelastic bodies are placed at short distances apart in a smooth horizontal straight groove. The first is projected from one end along the groove with velocity V . Find the velocity after n impacts.

5. Enunciate the proposition known as the parallelogram of forces, and show how it depends on the Second Law of Motion.

If the direction and magnitude of the resultant of two forces, the directions of which are inclined at a given obtuse angle, be known, prove that one of the forces is greatest when the direction of the other is perpendicular to the direction of the resultant.

6. Find the centre of gravity of a plane triangular lamina.

If the corners of the triangle be cut off by three straight lines parallel respectively to the opposite sides, the three areas cut off being equal, show that the centre of gravity of the remainder coincides with the centre of gravity of the original triangle.

7. Two partially elastic spheres whose centres are moving in the same straight line impinge directly upon one another. Find the velocities of the spheres after impact, and the kinetic energy lost by impact.

A bucket and a counterpoise, connected by a string passing over a pulley, just balance one another, and an elastic ball is dropped into the centre of the bucket from a distance h above it; shew that the time that elapses before the ball ceases to rebound is $\frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$, and the loss of kinetic energy by the impacts is $\frac{2Mmgh}{2M+m}$, where m , M are the masses of the ball and the bucket, and e is the coefficient of restitution.

8. A particle is projected with a given velocity in a given direction, find the range on the horizontal plane and the time of flight.

A string which is 3 feet long and is just able to sustain a weight of 83 lbs. is attached to a stone weighing 8 lbs., and is whirled in a vertical plane round a fixed point 19 feet from the ground. Show that the string will break if the velocity of the stone is 30 feet per second, and find when and where the stone will strike the ground.

9. Explain what is meant by "whole pressure." Find an expression for it; and determine the centre of pressure of a plane area.

One side of a hollow cube whose base is horizontal is kept in position by four equally strong joints at the four corners, and the cube may just be filled with water without the supports giving way. A heavier fluid is poured into the cube and the supports give way when the cube is half full, show that the specific gravity of the fluid is 3.2.

10. Describe the Hydrostatic Balance, and show how it is used to compare the specific gravities of two liquids.

When a vessel is filled by means of equal volumes of two fluids the specific gravity of the compound is one-third greater than what it would have been if the vessel had been filled by means of equal weights of the fluids. Compare the specific gravities of the two fluids.

IV.

Geometrical Optics, Astronomy, and Newton's Principia, Book I.
Sects. 1, 2, 3.

1. State the laws for the reflexion and refraction of a ray of light at a plane surface.

Show that a person 6 feet high standing at a distance of 21 inches from the foot of a plane mirror 5 inches long placed on the ground

at an inclination $\sin^{-1} \frac{1}{2}$ to the horizon, can see very nearly his whole length in the mirror.

2. Find the geometrical focus of a pencil of rays after direct refraction at a spherical surface, and trace the relative changes of position of the conjugate foci.

A hollow globe of glass has a small object midway between the centre and the interior surface. If μ be the refractive index and r, r' the radii of the interior and exterior surfaces of the glass, prove that the object when observed from a point outside the globe, on the opposite side of the centre, will appear nearer than it really is by a distance

$$\frac{\mu - 1}{4\mu} r \left(1 - \frac{r}{r'}\right) \div \left\{ 1 + \frac{\mu - 1}{2\mu} \left(1 - \frac{r}{r'}\right) \right\}.$$

3. A compound microscope is composed of two convex lenses of focal lengths 1 and 3 inches, separated by a distance of 2 inches; find the position of a small object when distinctly seen through the microscope by an eye which requires pencils to diverge from a distance of 12 inches to produce distinct vision.

4. Describe Newton's telescope, and find an expression for the magnifying power.

5. Describe an astronomical method for determining the velocity of light.

6. Describe the Transit Instrument; and find an expression for the uncorrected level error.

7. Find the parallax in hour angle of a heavenly body in terms of the latitude of the place of observation and the declination of the body, assuming the distance of the body from the earth to be so great that the sine and circular measure of the parallax may be considered equal.

Find the locus of all the bodies which on this assumption have their parallaxes in hour angle for a given place and time equal to a given quantity.

8. Find the time at which the sun rises at a given place on a given day.

Supposing the obliquity of the ecliptic = $23^\circ 28'$ and $\log 2 = \cdot 3010300$,

$$L \tan 23^\circ 28' = 9 \cdot 6376106,$$

$$L \tan 49^\circ 2' = 10 \cdot 0613473,$$

find approximately the latitude of the place at which the sun sets at 4 o'clock on the shortest day.

9. State and prove Newton's Lemma I.

Prove that the quadrilateral of maximum area that can be formed with four given straight lines is such that a circle can be described about it.

10. Every body, which moves in a curve line described in a plane, and describes areas proportional to the times of describing them about a point either fixed or moving uniformly in a straight line, by radii drawn to that point, is acted on by a centripetal force tending to the same point.

A heavy particle falls from the cusp to the vertex of a cycloid whose axis is vertical; show that a particle could describe the cycloid in the same manner under the action of a constant force tending to a certain moving point.

11. An ellipse is described under the action of a force tending to the focus; find the law of force and the velocity at any point.

If, without changing the velocity, the direction of motion of the body receive a very slight alteration, show that the position of the major axis will be altered, unless the body be at an extremity of the latus rectum through the focus to which the force does not tend.

12. State Kepler's Laws of Planetary Motion, and show they are only approximately true.

V.

Algebra and Trigonometry.

1. Prove that the expression $(x + y)^n - x^n - y^n$ is divisible by $(x^2 + xy + y^2)^2$ if n be a positive integer of the form $6n + 1$.

If α, β are the imaginary cube roots of unity, show that

$$(1 - \alpha)^{2n} + (1 - \beta)^{2n} = 3^{2n} - 3^{2n-1} 2n + 3^{2n-2} \frac{2n(2n-3)}{2} - 3^{2n-3} \frac{2n(2n-4)(2n-5)}{3} + \dots \text{ to } n+1 \text{ terms.}$$

2. The quantity e , the base of Napierian logarithms, cannot be the root of a quadratic equation with rational coefficients.

Prove this; and show that if n be positive, then

$$e^n |n > (1+n)^n.$$

3. If in an infinite continued fraction $\frac{b_1}{a_1} - \frac{b_2}{a_2} - \frac{b_3}{a_3} - \dots$ every component has its numerator not less than unity and its denominator greater than its numerator by unity, the value of the continued fraction is unity.

Find the n^{th} convergent of

$$\frac{(a-b)^2}{b} - \frac{a^2}{b} - \frac{(a-b)^2}{b} - \frac{a^2}{b} - \dots$$

4. From a bag containing n balls, each equally likely to be white or black, a ball is drawn which turns out to be white; this is replaced, and another ball is drawn which also turns out to be white. If the second ball is replaced, prove that the chance of the next draw giving a black ball is

$$\frac{1}{2}(n-1)(2n+1)^{-1}.$$

5. Find the sum of the sines of a series of angles which are in arithmetical progression; and show that if m and n be positive integers, m being less than n , and $\beta = \frac{2\pi}{n}$, then the sum of the series

$$\sin^{m-1} \alpha + \sin^m (\alpha + \beta) + \sin^{m+1} (\alpha + 2\beta + \dots + \sin^m (\alpha + n-1\beta)$$

is independent of α and β .

$$\text{If } u_r = (2\cos\theta)^r - \frac{r-1}{1} (2\cos\theta)^{r-1}$$

$$+ \frac{(r-2)(r-3)}{1 \cdot 2} (2\cos\theta)^{r-2} - \frac{(r-3)(r-4)(r-5)}{3} (2\cos\theta)^{r-3} + \dots$$

prove that u_0, u_1, \dots, u_r form a recurring series, the sum of which to infinity is $\frac{1}{4} \operatorname{cosec}^2 \frac{\theta}{2}$.

6. In a spherical triangle, prove that

$$(1) \cot a \sin b = \cot A \sin C + \cos b \cos C.$$

$$(2) \tan \frac{E}{2} \left(\sec C + \tan \frac{a}{2} \tan \frac{b}{2} \right) = \tan \frac{a}{2} \tan \frac{b}{2} \tan C,$$

where E is the spherical excess.

Also from the known value of the area of a spherical triangle in terms of the spherical excess, deduce the area of a plane triangle.

7. Obtain values for the angular radii R, r of the circumscribed and inscribed circles respectively of a spherical triangle.

Prove that if $\tan R = 2 \tan r$, the triangle must be equilateral.

8. Prove that the equation whose roots are the cubes of the roots of the equation

$$f(x^3) + x\phi(x^3) + x^2\psi(x^3) = 0,$$

the functions being rational and integral, is

$$\{f(x)\}^3 + x \{ \phi(x) \}^3 - 3f(x)\phi(x)\psi(x) + x^2 \{ \psi(x) \}^3 = 0.$$

9. Show how to determine the roots of a biquadratic in terms of the roots of its reducing cubic; and prove that if the biquadratic $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ has three equal roots, then its invariants $ae - 4bd + 3c^2$, $ace + 2bcd - ad^2 - be^2 - c^3$ are both zero.

Can the equation $x^4 + px^2 + q = 0$ have three equal roots?

10. Define the discriminant of a quantic; and show that the discriminant of a quantic of the n^{th} degree in m variables is of order $m(n-1)^{m-1}$ and of weight $n(n-1)^{m-1}$.

Show that the discriminant of $(x-a)(x-\beta)\phi(x)$ is equal to the discriminant of $\phi(x)$ multiplied by

$$\{ \phi(a) \times \phi(\beta) \}^2 \{ a - \beta \}^2.$$

11. Prove that the determinant whose constituents are the first minors of a determinant Δ of the n^{th} order is equal to Δ^{n-1} .

Hence or otherwise find the value of

$$\begin{vmatrix} b+c-a-d, & bc-ad, & bc(a+d)-ad(b+c) \\ c+a-b-d, & ca-bd, & ca(b+d)-bd(c+a) \\ a+b-c-d, & ab-cd, & ab(c+d)-cd(a+b) \end{vmatrix}.$$

VI.

Geometry.

1. If the general equation of the second degree represents two straight lines, find the angle between them.

2. If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of four points on an ellipse, find the condition that these points should all lie on a circle.

Show that the locus of the middle points of the common chords of the ellipse and its circles of curvature is given by the equation

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2 = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^2.$$

3. Investigate an expression for the distance between two points whose areal coordinates are given.

Hence show that the expression $a^2yz + b^2zx + c^2xy$ is equal to the product of the intercepts made on any line through the point (x, y, z) by the circle circumscribing the triangle of reference.

4. Find the polar reciprocal of one circle with respect to another, and find expressions for the latus rectum and eccentricity of the resulting conic.

5. If the circle whose centre is at the point (a, b) and whose radius is c , be inverted with regard to the origin, the result is a circle whose radius is $ck^2(a^2 + b^2 - c^2)^{-1}$ when k is the constant of inversion.

If three circles whose radii are a, b, c touch each other, the radius of the circle touching all three is given by the equation

$$\frac{1}{R^2} - \frac{2}{R} \sum \left(\frac{1}{a} \right) + \sum \left(\frac{1}{a^2} \right) - 2 \sum \frac{1}{ab} = 0.$$

6. State and prove Pascal's Theorem.

If A, B, C, D, E, F be six points on a conic, show that sixty different Pascal lines can be obtained by joining these points in different ways. Show also that the Pascal lines for the four hexagons $ABCDEF, ABFDEC, ABCEDF, ABFEDC$ pass through a point.

7. Show that the equations to two straight lines which do not intersect may always be put in the form

$$y = mx, z = c \text{ and } y = -mx, z = -c.$$

If through these straight lines a pair of planes be drawn at right angles to each other, show that the straight line through the origin parallel to the line of intersection of these planes lies on the cone

$$y^2 - m^2x^2 + (1 - m^2)z^2 = 0.$$

8. Find an expression for the volume of a tetrahedron in terms of the coordinates of its angular points.

Show that the volume of the tetrahedron formed by the four planes $my + nz = 0, ns + lz = 0, lx + my = 0,$ and $lx + my + ns = a$ is $\frac{2a^3}{3lmn}$.

9. Show that two series of parallel planes can be drawn each of which cuts a given ellipsoid in circles.

If two planes be drawn through the point (p, q, r) to cut the ellipsoid $(a^2 + b^2)x^2 + a^2y^2 + (a^2 - c^2)z^2 = a^4$ in circles, find the equation to the sphere which passes through these two circles.

10. Show how to determine the general character of the section of a surface of the second order by a plane.

If e be the eccentricity of the section of the surface

$$x^2 + 2y^2 + 3z^2 = 1.$$

by the plane $x + y + z = 0$, show that $(2\sqrt{3} + 1)e^2 = 2$.

11. Prove that the three conicoids which can be drawn through a given point P confocal with a given ellipsoid are respectively an ellipsoid, a hyperboloid of one sheet, and one of two sheets.

If a_1, a_2, a_3 be the semi-axes of the confocals corresponding to the semi-axis a of the original ellipsoid, and if

$$a_1^2 + a_2^2 + a_3^2 = 3a^2,$$

show that the locus of P is a sphere.

VII.

Differential Calculus.

1. If $y = \frac{1}{1 + x^2}$, find $\frac{d^2y}{dx^2}$

2. Explain what is meant by partial differentiation.

If $U = f(x, y, z)$, show that in a transformation from one rectangular system of axes in space to another the function

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2}$$

is unaltered.

3. State and prove Maclaurin's theorem.

Show that the expression for $\sin x$ in ascending powers of x is convergent.

4. Show how to find the maxima and minima values of a function of three variables x, y, z connected by the equation

$$f(x, y, z) = 0.$$

Find the distance from the origin $(0, 0, 0)$ of the vertex of the parabolic section of $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane

$$\frac{\sqrt{b^2 - c^2}}{a} x + \frac{\sqrt{a^2 - c^2}}{b} y + \frac{\sqrt{a^2 - b^2}}{c} z = 0.$$

5. If θ and ϕ be variables, find the envelope of the plane

$$x \sin \theta \cos \phi + y \sin \theta \sin \phi + z \cos \theta = a \sin \theta + b.$$

6. Trace the curve

$$(x^2 - y^2)^2 - 6ax(x^2 - y^2)^2 + 32a^2x^2 + 25a^4x^2 = 0.$$

7. Form the equation to the Hessian of a plane curve

$$f(x, y) = 0.$$

Find the number of points of inflexion on the curve

$$x^3 + y^3 = 3axy.$$

8. Find the equation to the tangent plane at any point of a surface.

Express in the form of determinants the conditions that the general equation of the second degree in x, y, z shall represent a pair of planes.

9. Find the equations to the normal at any point of the surface

$$f(x, y, z) = 0.$$

Show that

$$\frac{(b^2 - c^2)x}{\frac{df}{dx}} + \frac{(c^2 - a^2)y}{\frac{df}{dy}} + \frac{(a^2 - b^2)z}{\frac{df}{dz}} = 0$$

is the differential equation to the orthogonal trajectory

$$f(x, y, z) = 0$$

of the system of lines represented by

$$\frac{a(x - a \cos \theta \cos \phi)}{\cos \theta \cos \phi} = \frac{b(y - b \cos \theta \sin \phi)}{\cos \theta \sin \phi} = \frac{c(z - c \sin \theta)}{\sin \theta};$$

a, b, c being constants and θ, ϕ variables.

10. Investigate the differential equation to developable surfaces.

If the cylinder $x^2 + y^2 = a^2$ has the conic

$$x = a \cos \theta, \quad y = a \sin \theta, \quad z = b \cos \theta,$$

described on its surface and be set rolling on the plane $y = a$, in the direction of the x -axis, show that the trace of the conic on the plane is the curve given by the differential equation

$$\frac{dx}{dz} = \frac{a}{\sqrt{b^2 - x^2}}$$

11. Find an expression for the radius of absolute curvature at any point of a tortuous curve; and show that in the curve of intersection of the surfaces

$$x^2 + y^2 = a^2, \quad z = b \tan^{-1} \frac{y}{x}$$

the radius of absolute curvature is constant.

12. Find the equation for determining the principal radii of curvature at a given point of a surface.

Prove that the principal radii of curvature at a point (x, y, z) of the surface $(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$

$$\text{are } b \text{ and } \left(\frac{x^2 + y^2 + z^2 + a^2 - b^2}{x^2 + y^2 + z^2 - a^2 - b^2} \right) b.$$

VIII.

Integral Calculus.

1. Define a definite integral; and prove the formula

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx.$$

Prove that $\int_0^{\pi} \frac{x dx}{1 + \sin^2 x} = \frac{\pi^2}{2\sqrt{2}}.$

2. Show how to find the area of a pedal of a closed curve; and prove that it is equal to the sum of the areas of the curve and of the pedal of its evolute.

Prove that the whole area of the pedal of the lemniscate $r^2 = a^2 \cos 2\theta$ with respect to the pole is $2a^2$.

3. Show that the area of a surface $z = f(x, y)$ is given by the integration between proper limits of

$$\iint \left\{ 1 + \left(\frac{dx}{dx} \right)^2 + \left(\frac{dx}{dy} \right)^2 \right\} dx dy.$$

If dS be an element of surface of an ellipsoid at a point P , and A the area of the section of the ellipsoid by a plane through its centre parallel to the tangent plane at P , then $\iint \frac{dS}{A} = 4$, the integral being extended over the whole surface.

4. Show how to transform $\iiint \phi(x, y, z) dx dy dz$ into a triple integral with respect to three new variables u, v, w , which are given by the equations

$$x = f_1(u, v, w), y = f_2(u, v, w), z = f_3(u, v, w).$$

If $u_1 x = u_2 u_3, u_2 y = u_3 u_1, u_3 z = u_1 u_2$, prove that

$$\iiint V dx dy dz$$

is transformed into $4 \iiint V_1 du_1 du_2 du_3.$

5. Determine the value of $\int_0^{\infty} \frac{x^{2m} dx}{1 + x^{2n}}$, when m and n are positive integers, and $n > m$.

Prove that the value of $\iint x^{n-1} y^{-1} e^{x+y} dx dy$ is

$$\frac{\pi}{\sin \frac{\pi}{n}} (e^n - 1),$$

the integral being so taken as to give to the variables all positive values consistent with the condition that $x + y$ is not greater than c .

6. Prove by means of the Calculus of Variations that the minimum value of

$$\int_{x_0}^{x_1} (a-x)^2 \left(\frac{dy}{dx} \right)^2 dx \text{ is } \frac{(y_1 - y_0)^2 (a - x_1)(a - x_0)}{x_1 - x_0}$$

where y_0, y_1 are the values of y corresponding respectively to the initial and final values of x .

7. Solve the equations :

$$(1) \left(\frac{d}{dx} - a \right)^4 u = e^{ax} \cos bx;$$

$$(2) x(a - y) \left(\frac{dx}{dy} \right) - 2a y \left(\frac{dx}{dy} \right) = x(a + y).$$

8. Prove that a partial differential equation of the first order $u = f(v)$ can only lead to a partial differential equation of the second order of the form $Rr + Ss + Tt = V$, when u and v are so related as to satisfy identically the condition

$$\frac{du}{dp} \frac{dv}{dq} - \frac{du}{dq} \frac{dv}{dp} = 0.$$

Obtain the two first integrals of the equation $xy(r - t) = x^2 - y^2$.

9. Solve the equations

$$(1) \begin{cases} u_x + 1 = 3u_x + 2v_x \\ v_x + 1 = 4u_x - 3v_x \end{cases};$$

$$(2) 4u_x^2(u_x^2 - 1) = u_{xx}^2.$$

10. If Bernouilli's numbers are given by the identity

$$(e^x - 1)^{-1} = \frac{1}{x} - \frac{1}{2} + \frac{B_1}{2}x - \frac{B_2}{4}x^2 + \dots$$

show that

$$B_{2n-1} = (-1)^{n+1} \left\{ 1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \dots + \frac{\Delta^{2n}}{2n+1} \right\} 0^{2n}.$$

11. Prove that the mean distance of a point within a given rectangle whose sides are $2a$, $2b$, from the centre of gravity of the rectangle is

$$\frac{d}{3} + \frac{a^2}{6b} \log \frac{b+d}{a} + \frac{b^2}{6a} \log \frac{a+d}{b},$$

$2d$ being the length of the diagonal.

IX.

Statics of Solids and Fluids.

1. Show that any number of forces acting on a rigid body may be replaced by a single force and a couple whose plane is perpendicular to the direction of the single force.

2. Three uniform rods whose weights are proportional to their lengths a , b , c are jointed together so as to form a triangle which is placed on a smooth horizontal plane on its three sides successively, its plane being vertical; prove that the stresses along the sides a , b , c when horizontal are proportional to

$$(b+c) \operatorname{cosec} 2A, (c+a) \operatorname{cosec} 2B, (a+b) \operatorname{cosec} 2C \text{ respectively.}$$

3. State clearly the laws of friction and of limiting friction.

The centres of two spherical surfaces, radii $a \sin \alpha$, and $a \cos \alpha$, are placed in the same horizontal line at distance a from each other; if the coefficients of friction for the two surfaces be μ and μ' respectively, prove that the length of that part of the curve of intersection on which a heavy particle could rest is

$$a \sin 2\alpha \tan^{-1} (\mu \cos \alpha + \mu' \sin \alpha).$$

4. Define the Centre of Mass of a body.

Find the Centre of Mass of a thin spherical shell whose density at any point varies inversely as the cube of its distance from a given external point.

5. If the mass of a string be so distributed that when the string is suspended by its extremities from two points, the tension varies as the density, prove that the form assumed by the string will be represented by the equation $y = c \log \sec \frac{z}{c}$.

Prove that the radius of curvature will vary as the density, and that its projection on the vertical will be of constant length.

6. By Newton's Universal Law of Attraction the stress between two particles of masses M and M' when at a distance D from each other is $\mu MM' D^{-2}$, where μ is some constant. Find the dimensions of μ in terms of the usual dynamical units, viz., mass, length, and time.

Find also in terms of the same units the dimensions of the Potential at any point due to any attracting mass.

7. Find the attraction of a thin spherical shell of uniform density on an external particle.

If the density at any point P inside a sphere depends only on its distance OP from the centre, and if the attraction of the sphere on any internal point Q varies as OQ^4 , show that the density at P varies as OP^2 .

8. Prove the formula $\nabla^2 V + 4\pi\rho = 0$.

If two bodies of equal mass have the same external equipotential surfaces, show that their attractions on all external points are the same in magnitude and direction.

9. If p be the pressure and ρ the density at any point (x, y, z) of a fluid, and if X, Y, Z be the resolved parts of the force acting at that point, show that $dp = \rho(Xdx + Ydy + Zdz)$.

A weightless cone is nearly filled with fluid and inverted on a horizontal table. The fluid is made to rotate with angular velocity ω , and the pressure required to keep the cone in contact with the table is three times the weight of the fluid; show that $3h\omega^2 \tan^2 a = 4g$, where h is the height of the cone, and a its semivertical angle.

10. Define the Metacentre, and investigate a formula for finding its position.

A uniform solid cone whose height is h , semivertical angle a and specific gravity σ , has its vertex fixed at a depth z below the surface of water. Show that the equilibrium is stable if $z^4 \sec^2 a > h^4 \sigma$.

11. A membrane in the form of a surface of revolution is partly filled with fluid, and t and t' are the tensions along and across a meridian at a point distant y from the axis, show that $t' - t = y \frac{dt}{ay}$.

X.

Dynamics of a Particle.

1. Find expressions for the accelerations of a particle moving in a curve, estimated along and perpendicular to the tangent.

The velocity of a particle describing a plane curve under the action of a constant force varies as the square root of the particle's distance from a given right line in the plane; find the direction of the force.

2. Show that in planetary motion the velocity v in terms of the time t is approximately expressed by the formula

$$v^2 = \frac{\mu}{a} (1 + 2e \cos nt + 3e^2 \cos 2nt).$$

3. Every apsidal line divides a central orbit into two parts which are equal and similar.

A particle is acted on by a central attraction $\mu \left(\frac{u^2}{a} + 3u^3 \right)$ and is projected from an apse at a distance $\frac{a}{1+e}$ with a velocity $\frac{2\sqrt{\mu}}{a} (1+e)$, e being < 1 ; find the orbit and the periodic time.

4. Two particles, projected in any manner, are acted on solely by their mutual attraction, which varies inversely as the square of the distance; show that the relative orbits of the particles about each other will be conic sections about a focus.

5. A particle moves on a smooth plane curve under given forces: show how to determine the motion and to find the pressure on the curve.

A small ring, starting from the point $\left(a, \frac{\pi}{2} \right)$, slides on a smooth wire $r = a(1 + \cos \theta)$ which has its axis vertical and pole upwards; find the pressure on the wire when the ring has reached the lowest point $(2a, 0)$.

6. If the height of a projectile, moving in a resisting medium, above the horizontal line through the point of projection, be connected with the time by the equation

$$y = \frac{g}{6} t (3 - t^2),$$

prove that the trajectory is determined by this and

$$x = \frac{g}{6} \cot \alpha \{ (1 + t^2) - 1 \};$$

α being the angle which the direction of projection makes with the horizon.

7. A particle is constrained to move, under the action of given forces, inside a fine smooth tube in the form of a plane curve which revolves in a given manner about an axis in its plane; determine the motion.

A smooth straight tube OA , of length $3a$, is made to revolve in a horizontal plane about a vertical axis through O with uniform angular velocity ω , and a particle P attached to an elastic string OP of natural length a is placed in the tube with an extremity fixed at O . Supposing the string to be initially just stretched and its coefficient of elasticity $= 2\omega^2 a$, show that (1) the particle will reach the end A of the tube in the time $t = \frac{\pi}{\omega}$; (2) its path is a limaçon,

8. A heavy particle is projected obliquely on the inside of a smooth cylinder $y^2 + z^2 = 2ax$, having its axis horizontal, from a point in the lowest generating line of the cylinder; show that the equations to the particle's path will be

$$x = V \cos \alpha t, \quad y = a \sin 2\theta, \quad z = a(1 - \cos 2\theta),$$

if $2a \frac{d\theta}{dt} = \sqrt{V^2 \sin^2 \alpha - 4ga \sin^2 \theta}$, and V be the initial velocity of the particle, α the angle which the direction of projection makes with the lowest generating line, and $V \sin \alpha > 2\sqrt{ga}$.

9. Determine the general equation to tautochronous curves; and prove that the cycloid is a tautochrone in the case of a heavy particle moving in a medium whose resistance varies as the velocity.

10. The tangent to the hodograph at any instant is parallel to the resultant acceleration of the particle at the corresponding point of its path and the velocity in it is equal to the acceleration of the particle.

Show that the hodograph of an ellipse described under a central attraction varying as the distance is a similar ellipse.

11. Two small masses attracting each other according to the law of the inverse square of the distance are attached by two strings each of length l to points in a smooth horizontal plane at a distance a apart. If the particles be slightly displaced from their positions of equilibrium and a be very great compared with l , show that the motion of each mass will be compounded of two harmonic motions whose periods are approximately as $1 + \frac{l}{a} : 1$.

12. Explain the principle of least action; and apply it to find the path of a particle moving freely under the attraction of a central force which varies inversely as the cube of the distance.

XI

Dynamics.

1. Given the principal axes at the centre of gravity of a plane lamina, show how to find the principal axes at any other point of the lamina.

A plane lamina is bounded by the ellipse $a^{-2}x^2 + b^{-2}y^2 = 1$; prove that the principal axes at a point h, k in the plane of the lamina are the line perpendicular to the lamina at that point and the lines given by the equation

$$4hk \{(x-h)^2 - (y-k)^2\} = \{a^2 - b^2 + 4(h^2 - k^2)\} (x-h)(y-k).$$

2. A plane triangle is equi-momental with a system of three particles each of one-third its mass placed at the middle points of its sides.

Prove this; and show how to construct a momental ellipse at the centre of gravity of the triangle.

3. A heavy body rotates about a horizontal axis under the action of gravity; determine the motion and find the pressures on the axis.

A lamina in the form of a square (of side $2a$) is capable of rotating about one of its sides which is fixed horizontally. The lamina, which is hanging vertically at rest, is struck at its centre of inertia by a particle of one-third its mass. The particle before impact was moving with a velocity $2\sqrt{10}ag$ in a direction perpendicular to the plane of the lamina, and after impact adheres to it. Prove that the lamina will just rise through an angle 90° .

4. If u, v, w are the linear velocities of a body in the direction of three co-ordinate axes through the centre of inertia, and $\omega_x, \omega_y, \omega_z$ are the angular velocities of the body round these axes, prove that the motion may be exhibited by rotation round a single axis if $\omega_x u + \omega_y v + \omega_z w = 0$.

A solid homogeneous sphere (of radius a) under the action of no forces is struck by a given blow P which is not normal to the surface. Show that the initial motion is one of pure rotation, and that the initial angular velocity is $\frac{1}{2} Pa^{-1} m^{-1} \sin \theta$, where θ is the angle between the tangent plane to the sphere at the point of application of the blow and the plane to which the direction of the blow is normal, and m the mass of the sphere.

5. A disc turns about a fixed point O round an instantaneous axis OI in its own plane. Suddenly an axis OI' also in the plane of the disc becomes fixed. Show that the kinetic energy is diminished in the ratio $1 : \sec^2 \theta$,

where θ is the angle between the tangents to the momental ellipse at O at the points where it is met by the axes OI, OI' .

6. A circular hoop rests upon a smooth horizontal plane with a particle at its lowest point, and is projected with a horizontal velocity V in its own plane. Find the value of V in order that the particle may just rise to the height of the centre of the hoop. If V be very small, prove that the particle will describe complete oscillations in the time $2\pi \left\{ \frac{ma}{(m+m')g} \right\}^{\frac{1}{2}}$, a being the radius of the hoop, and m, m' the masses of the hoop and particle respectively.

7. Obtain Euler's equations. $A \frac{d\omega_1}{dt} + (C - B) \omega_2 \omega_3 = L$, &c. for the motion of a body about a fixed point under the action of given forces.

If the body be such that two of its principal moments at the fixed point are equal, and if no forces act upon it except at the fixed point, show that the instantaneous axis traces out in the body a right circular cone, and that this axis is always in the same plane with the invariable axis and the axis of unequal moment of the body.

8. The motion of a body in space of two dimensions is given by the coordinates x, y of its centre of gravity, and the angle θ which any fixed line in the body makes with a line fixed in space. The body being in equilibrium under the action of gravity, find the time of a small oscillation.

An equilateral triangle of side a resting in a smooth hemispherical bowl of radius $2a$ is made to oscillate under the action of gravity about its position of equilibrium in such a way that one of its sides always remains parallel to itself. Find the length of the simple equivalent pendulum.

9. A vessel in the form of a surface of revolution with its axis vertical and having a finite horizontal aperture in its base is kept constantly full of liquid. Determine the rate at which liquid must be poured in, on the hypothesis that the motions of all particles in the same horizontal plane are uniform and vertical.

10. Prove that the velocity potential ϕ of a homogeneous incompressible fluid moving with irrotational motion in three dimensions satisfies the differential equation

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$$

If in a case of irrotational motion in two dimensions

$$\phi = Ae^{mz} \sin(mx - nt),$$

prove that each fluid particle describes a circle round its mean position with uniform angular velocity.

XII.

Optics and Astronomy.

1. Determine the positions of the primary and secondary foci of a small pencil of light obliquely reflected at a spherical surface.

2. The origin of the curve $r = a(1 + \cos \theta)$ is a source of light which is reflected by the curve. Prove that the radius of curvature of the caustic at the point corresponding to θ is $\frac{2}{3} a \sin \theta$.

3. A succession of waves in which the displacement is represented by $c \sin \frac{2\pi}{\lambda}(vt - z)$ is moving directly towards a screen in which there is a slit of breadth $2b$. Find the magnitude of the vibration at a point on the other side of the screen.

Interpret your result when λ is very much greater or very much less than b .

4. A wave of light is incident directly on a plate of glass of thickness T and refractive index μ , and a second wave is incident on a similar plate of glass at an angle β . Show that the difference of retardation of the waves is $2T \left(\sin^2 \frac{\beta}{2} - \mu \sin^2 \frac{\gamma}{2} \right)$ where $\sin \beta = \mu \sin \gamma$.

5. Find the length of day at any place.

The number of seconds occupied by the sun in rising on a given day at a given place may be nearly expressed by $\frac{132}{\sqrt{\cos(l + \delta) \cos(l - \delta)}}$, where l is the latitude of the place and δ the sun's declination.

6. Define Aberration, and prove the formula
aberration = $\kappa \sin$ (earth's way).

Determine the aberration of a star in longitude.

7. Account for the stationary positions and retrograde motions of the planets.

Find the angle subtended by two planets at the sun when they are in a stationary position.

8. In the Lunar Theory obtain the equation

$$P = \mu u^2 \left(1 - \frac{3}{2} s^2 \right) - \frac{1}{2} m' u^2 u^{-1} \{ 1 + 3 \cos 2(\theta - \theta') \}.$$

9. Assuming the expression for the longitude of the Moon

$$\theta = pt + 2s \sin(cpt - a) + \dots + \frac{1}{4} me \sin \{ (2 - 2m - c)pt - 2\beta + a \}$$

determine the change in the position of the apse and in the eccentricity of the Moon's orbit produced by the last term (called the Evection).

10. Prove (by numerical calculation) that the moon is kept in her orbit by the earth's attraction.

11. In the Planetary Theory prove that $\frac{da}{dt} = \frac{2na^2}{\mu} \frac{dR}{d\epsilon}$.

12. Integrate the equations for the eccentricity and the longitude of perihelion, given

$$\begin{aligned} \frac{de}{dt} &= -aD_2 e' \sin(\varpi - \varpi'); \\ e \frac{d\varpi}{dt} &= a \{ D_1 e - D_2 e' \cos(\varpi - \varpi') \}. \end{aligned}$$

Hence show that when the eccentricity is a maximum or minimum the apsidal line is either moving most rapidly or most slowly.

XIII.

Problems.

1. Show that $\sec \frac{\pi}{7} + \sec \frac{3\pi}{7} + \sec \frac{5\pi}{7} = 4$.

2. If $r = e^{-\frac{\pi x}{\beta}}$ and $\cosh x \equiv \frac{1}{2}(e^x + e^{-x})$, prove that

$$f(x) = 2 \left\{ \sqrt{r} \cosh \frac{\pi x}{2\beta} + \sqrt{r^3} \cosh \frac{3\pi x}{2\beta} + \sqrt{r^5} \cosh \frac{5\pi x}{2\beta} + \dots \text{ad infinitum} \right\}$$

is a solution of the functional equation

$$f(x + 2a) = \frac{1}{r} e^{\frac{\pi x}{\beta}} f(x)$$

3. If a, b, c are the lengths of the sides of a triangle circumscribing the parabola $y^2 = mx$, prove that its area is equal to $2abc \sqrt{\frac{m}{f_1 f_2 f_3}}$, where f_1, f_2, f_3 are the lengths of focal chords of the parabola parallel to the sides of the triangle.

4. Find the relation between the axes of two confocal ellipses when it is possible to inscribe a triangle in one which shall be circumscribed to the other.

5. Show that

$$(b^2 - y^2) \frac{dy^2}{dx^2} - 2xy \frac{dy}{dx} + a^2 - x^2 = 0$$

is the differential equation to the involutes of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

6. The tangent at a point P of the cissoid $y^2(a-x) = x^2$ meets the curve again in Q , and the tangent at Q meets the curve again in R . If O be the cusp of the cissoid, prove that

$$\cot EOQ - \cot POQ = \frac{1}{2} \cot POR.$$

7. If ρ, ρ' are the principal radii of curvature of a paraboloid of revolution at a point P , ρ being $> \rho'$, prove that the distance of P from the focus is $\frac{1}{2} \sqrt{\rho\rho'}$, and the angle between the tangent plane at P and the focal

distance of P is $\sin^{-1} \sqrt{\frac{\rho'}{\rho}}$.

8. A curve is drawn on a sphere cutting all the meridians at a constant angle a . If θ and ϕ be the latitude and longitude of a point P , the longitude being measured from the point where the curve cuts the equator, show that

$$\phi = \tan a \log \tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right).$$

Also if ψ be the angle between the osculating plane to the curve and the tangent plane to the sphere at P , show that

$$\cot \psi = \sin a \tan \theta.$$

9. Prove that

$$y = \frac{x}{1.3.5} - \frac{x^3}{3.5.7} + \frac{x^5}{5.7.9} - \frac{x^7}{7.9.11} + \dots \text{ad infinitum}$$

is a particular integral of the differential equation

$$x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 8y = \tan^{-1} x;$$

and hence or otherwise find the sum of the series.

10. A uniform heavy string of length l rests on a rough cycloid whose cusps are vertical and vertex upwards, one end of the string being at a cusp and the other end hanging freely. Show that the string is just on the point of slipping if

$$l = 8a + 12a (\sigma^2 - 1) (\mu^2 + 4)^{-1},$$

where a is the radius of the generating circle of the cycloid.

11. An elliptic lamina is partly immersed in a homogeneous fluid with a diameter in the surface; find the locus of the centre of pressure, supposing the lamina to be moveable about a fixed horizontal axis through its centre.

12. Incompressible fluid is contained in a cone of height a , and semi-vertical angle $\frac{\pi}{4}$, and the density of a depth x is λx . If when the fluid is poured into a surface of revolution it is found that the density at depth x is then $\lambda x^2 a^{-1}$, show that the equation to the generating curve is

$$a^2 y^2 = 2x(a^2 - x^2)^2.$$

13. A fine inextensible string OP has one of its extremities attached to the highest point O of the circumference of a fixed vertical circle, and the other to a heavy particle P . Initially the string is stretched in the direction of the tangent to the circle at O , and P is projected vertically downwards; find the velocity of projection so that the string may be completely wound round the circle, supposing its length to be equal to the circumference.

14. Two smooth planes are each inclined to the vertical at an angle $\frac{\pi}{4}$ and intersect in a horizontal line. A uniform circular disc of radius a is making small oscillations in such a manner that the line joining the points of contact of the disc and the planes is always horizontal. Show that the length of the simple equivalent pendulum is $\frac{a}{4}$.

15. A smooth fine tube of length $4a$ is bent into the form of part of the curve $r = a(1 + \cos \theta)$, the tube being symmetrical about the initial line which is vertically upwards. The tube is fixed in a vertical plane, and the tangents at its extremities are vertical. A uniform chain of length $8a$ passes through the tube and rests in the position of equilibrium. If the chain be slightly displaced, show that its velocity V at the moment when the end of the chain leaves the tube is given by the equation

$$5V^2 = 39ag.$$

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MATHEMATICS. I

[See Note after Question 12.]

1. Explain Cardan's method of solving a cubic equation which has only one real root. How would you solve a cubic equation known to have three real roots?

Prove that, if the difference between two roots of a biquadratic equation is equal to the difference between the other two roots, the equation is of the form $(x^2 + px + q)^2 = r$; and hence, or otherwise, solve the equation

$$x^4 - 2x^3 - 33x^2 + 34x + 44 = 0.$$

2. Resolve into real factors $x^{2n} - 2x^n \cos \theta + 1$, where n is a positive integer.

Show that
$$\frac{\sin(2n+1)\theta}{(2n+1)\sin\theta} = \prod_{r=1}^{r=n} \left(1 - \frac{\sin^2\theta}{\sin^2\frac{r\pi}{2n+1}}\right).$$

3. Prove that in any spherical triangle $\cot a \sin b = \cot A \sin C + \cos b \cos C$.

If D is taken on BC beyond C so that $BD = s$, show that

$$\cot ADC = \sin(s-c) \sin(b+c) \operatorname{cosec} a \operatorname{cosec} b \operatorname{cosec} C.$$

4. Show that the locus of a point, the lengths of the tangents from which to two circles are in a given ratio, is a circle coaxial with the two circles.

From two points which are harmonic conjugates with respect to the centres of two circles perpendiculars are drawn to their common tangents. Show that the eight feet of these perpendiculars lie on a circle coaxial with the two circles.

5. If $a = 0$ is the equation of the tangent at the point P to the conic $S = 0$, interpret the equation $S + \lambda a^2 = 0$.

CP, CQ are a pair of conjugate semi-diameters of an ellipse, and a parabola has four-point contact with the ellipse at P . Show that its directrix cuts CP at right angles at a distance $CP + CQ^2/2 CP$ from C .

6. Find the equation of the plane section of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ which has the point (x', y', z') as its centre.

Show that the locus of the centres of the sections of a conicoid whose planes pass through a given line is a plane conic.

7. If a curve has an algebraic equation which is arranged in homogeneous sets in the form $u_n + u_{n-1} + \dots + u_m = 0$, u_n being the terms of highest degree and u_m those of lowest degree, show that there are in general m branches at the origin, the equations of the tangents to which are the factors of the equation $u_m = 0$, and n linear asymptotes parallel to the straight lines whose equations are the factors of $u_n = 0$. State without proof the modifications that generally result from the presence of a repeated factor in u_m or u_n .

Trace the curve $(x^2 - y^2)x^2 + y^2(y + 2x) = 0$.

8. Prove from first principles that

$$\int_a^b \sin x dx = \cos a - \cos b.$$

Prove that the limit, when the positive integer n is indefinitely increased, of

$$\left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \left(1 + \frac{3}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right]^{\frac{1}{n}}$$

is $4/e$.

9. If $s = f(\psi)$ is the intrinsic equation of a curve, show how to express the Cartesian coordinates of any point on it in terms of ψ .

Show that the coordinates of any point on the curve whose intrinsic equation is $s = c(\sec^n \psi = 1)$, where n is an odd integer greater than unity, can be expressed rationally in terms of $\tan \psi$; and show that, when $n = 3$, the curve is a cubic with a cusp.

10. Find from mechanical considerations the intrinsic equation of the common catenary.

The ends of a uniform heavy chain are attached to two fixed points A, B . When the chain is hanging in equilibrium the tangents to it at its extremities make angles $\tan^{-1}(3/4)$, $\tan^{-1}(5/12)$ with the horizontal in opposite senses. Prove that the inclination of the straight line AB to the horizontal is $\cot^{-1}(6 \log 3)$.

11. Explain what is meant by the velocity of a point A relative to a point B .

Two heavy particles start simultaneously from a point on a smooth plane, which is inclined to the horizontal at an angle of 30° . One of them is projected vertically upwards with velocity V , and the other slides from

rest down the plane. Find the time at which the magnitude of the relative velocity of the particles is a minimum, and their distance apart at this time.

12. A uniform cylinder is placed with its axis horizontal against an imperfectly rough fixed plane inclined to the horizontal at an angle α and then released. Show that the subsequent motion of the cylinder will be pure rolling, or rolling and sliding combined, according as the coefficient of friction between the cylinder and the plane is greater than or less than $\frac{1}{2} \tan \alpha$. How would this result be modified if the plane had the same mass as the cylinder, and was free to move on a smooth fixed horizontal plane?

(Assume that the centres of gravity of the cylinder and inclined plane lie in a vertical plane perpendicular to the inclined plane).

[Any number of the following questions may be substituted for an equal number of the preceding questions, but less weight will be assigned to them.]

A. Prove the formula $\frac{1}{2} bc \sin A$ for the area of a triangle.

The diagonals AC and BD of a plane convex quadrilateral $ABCD$ intersect in O . If a triangle is constructed with AB and CD as sides and the angle between AB and CD as the included angle, show that its area will be equal to the difference of the areas of the triangles AOD and BOC .

B. Differentiate $\sin x$ by first principles.

If $y = x^2 \tan^{-1} x$, prove that $\frac{dy}{dx} = \frac{x^3 + 2x^2y + 2y}{x^3 + x}$.

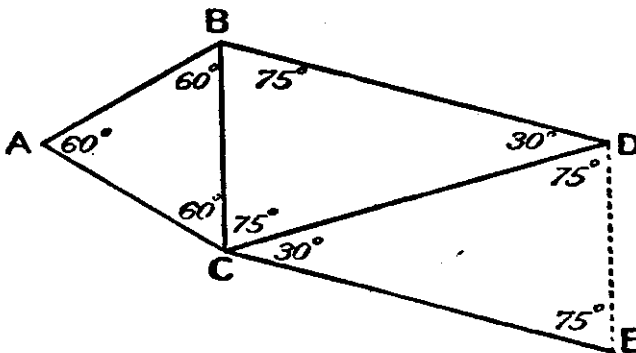
C. State Newton's second law of motion, and show how it leads to a measure of force.

Give reasons for the assumption that, immediately after a cannon is fired horizontally, the magnitudes of the velocities of the cannon and the ball are inversely as their masses. Is the assumption quite correct?

MATHEMATICS II.

[See Note after Question 11.]

1. Six light rods AB, BC, CA, BD, CD, CE are freely jointed at their extremities so as to form the framework shown in the figure, the points D, E being freely hinged to two fixed points in a vertical line. Determine the stress in each rod and the reactions at D and E when a weight W is attached to the joint A , assuming that the whole system is in a single vertical plane.



2. Show that the rate of change of moment of momentum of a system of particles about a fixed straight line is equal to the moment of the external forces about that line.

One end A of a light inextensible string is attached to a fixed point of a smooth horizontal plane. To the other end B of the string and to its mid-point equal particles are attached. The particle at B is held on the plane so that the string is straight and taut and is then projected along the plane in a direction at right angles to the string. Show that, when the two parts of the string are again in a straight line, their angular velocities will be in the ratio 4 : - 1.

3. Prove that the thrust exerted by a fluid on a body immersed in it is equal to the weight of the displaced fluid and acts vertically upwards through the centre of gravity of the latter.

A is a solid cylinder of radius a , length l , and density σ , resting in a cylindrical cup B of radius $b (> a)$. The centre of one end of A is connected to the centre of the bottom of B by a fine elastic string, whose natural length is l and whose modulus is equal to the weight of A in air. Water of density $\rho (> \sigma)$ is then poured slowly into the cup. Prove that, when A is just immersed, the volume of the water is

$$\pi l \left[b^2 \left(\frac{\rho}{\sigma} + 1 \right) - a^2 \right].$$

4. Assuming that $f(x)$ can be expanded in a series of ascending powers of x , show how to determine the coefficients.

Expand $\sin^{-1} x$ in powers of x .

If $y = \int_0^x e^{\sin x} dx$, expand y in powers of x as far as x^4 .

5. Interpret geometrically the integral $\int_x^{2x} f(x) dx$, where $y = f(x)$ is the equation of a plane curve.

Find the sum of the areas included between the axis of x and the arc of the curve $y = x \sin(x/a)$ from the ordinate $x = 0$ to the ordinate $x = n\pi a$; n being any positive integer, odd or even.

Show that the whole length and area of a single loop of the curve

$$r = 2a \cos n\theta \quad (n > 1)$$

are equal respectively to the whole perimeter and area of the ellipse $x^2 + n^2 y^2 = a^2$.

6. Show that the solution of the equation

$$\frac{d^2 y}{dx^2} + \phi(y) \frac{dy}{dx} + \psi(y) = 0$$

can be reduced to that of an equation of the form

$$u \frac{du}{dv} + u = f(v),$$

and solve the latter equation in the case $f(v) = 2v$.

7. Prove that, if a and b are real unequal positive quantities,

$$a^a b^b > \left(\frac{a+b}{2} \right)^{a+b}.$$

Show also that, if a, b, c are positive integers, which are not all equal,

$$\left(\frac{a^a + b^b + c^c}{a + b + c} \right)^{a+b+c} > a^a b^b c^c;$$

and extend the result so as to include all real rational positive values of a, b, c .

8. Evaluate the determinant

$$\begin{vmatrix} 1^2 + 2^2 & 2^2 + 3^2 & 3^2 + 4^2 & 4^2 + 5^2 \\ 2^2 + 3^2 & 3^2 + 4^2 & 4^2 + 5^2 & 5^2 + 6^2 \\ 3^2 + 4^2 & 4^2 + 5^2 & 5^2 + 6^2 & 6^2 + 7^2 \\ 4^2 + 5^2 & 5^2 + 6^2 & 6^2 + 7^2 & 7^2 + 8^2 \end{vmatrix};$$

and express as a product of linear factors the determinant

$$\begin{vmatrix} a^2 & a^2 + ab - b^2 & a^2 + ac - c^2 \\ -a^2 + ab + b^2 & b^2 & b^2 + bc - c^2 \\ -a^2 + ac + c^2 & -b^2 + bc + c^2 & c^2 \end{vmatrix}.$$

9. If P is any point on the circumscribing circle of the triangle ABC , show that

$$PA \sin A + PB \sin B + PC \sin C = 0.$$

O is a point in the plane of a triangle ABC at which the sides BC , CA , AB subtend angles α , β , γ respectively (measured so that $\alpha + \beta + \gamma = 2\pi$), and AO meets the circumscribing circle of the triangle OBC at D . Show that

$$DA \sin \alpha = OA \sin \alpha + OB \sin \beta + OC \sin \gamma,$$

and that

$$OA \cdot DA \sin \alpha = bc \sin (\alpha - A).$$

Hence obtain OA , OB , OC in terms of a , b , c , A , B , C , α , β , γ .

10. If the perpendiculars drawn from the vertices A and B of the tetrahedron $ABCD$ to the opposite faces intersect, show that the edges AB and CD are at right angles, and that the perpendiculars from C and D to the opposite faces intersect.

11. Find the equation of the tangent at the point (x', y') on the conic

$$ax^2 + 2hxy + by^2 + 2gx + fy + c = 0.$$

Prove that the conics

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and

$$\frac{x^2}{a^2} + \frac{y^2}{2a^2 - b^2} + 2\lambda x + 1 = 0$$

cut each other at right angles.

[Any number of the following questions may be substituted for an equal number of the preceding questions, but less weight will be assigned to them.]

A. Define a harmonic range.

Give, with proof, a method of constructing with the ruler only the harmonic conjugate of a point C with respect to two points A and B lying on a straight line through C .

B. Find the equations of the tangent and normal at any point of the curve

$$y = f(x).$$

Show that the point of contact of any tangent to the curve $x^2y = c^2$ is one of the points of trisection of the part of the tangent intercepted between the axes.

C. Determine the position of the centre of gravity of a uniform triangular plate.

A uniform plate $ABCD$ has the form of an equilateral triangle ABD and an isosceles triangle CBD , whose angle BCD is 120° , on opposite sides of the same base BD . A heavy particle, whose mass is two-thirds that of the plate, is attached to the plate at C . Show that the centre of gravity of the system lies on BD .

MATHEMATICS III.

See Notes after Question 11.

1. Show how to find (when possible) the general solution of the equation $ax - by = c$ in positive integers, where a, b, c are positive integers.

A man has 121 franc pieces and 70 half-crowns. Reckoning 25 francs to be the equivalent of £1 find the number of ways in which the man can pay away one-third of the money in his possession. What is the smallest number of coins with which he can make the payment?

2. Draw the graphs of $\tanh x$ and $\sin x$.

Show that the equation $\sin x = \tanh mx$, where m is positive, has an infinite number of real roots; that the large positive roots occur in pairs, one a little less than and one a little greater than $(2n + \frac{1}{2})\pi$, where n is a large positive integer; and that closer approximations to the pair of roots near $(2n + \frac{1}{2})\pi$ are

$$(2n + \frac{1}{2})\pi \pm a - ma^2,$$

where a denotes $\operatorname{sech} (2n + \frac{1}{2})m\pi$.

3. Show that in any spherical triangle

$$\sin \frac{1}{2}E \cos \frac{1}{2}a = \sin A \sin \frac{1}{2}b \sin \frac{1}{2}c,$$

where E is the spherical excess.

If D is the middle point of the side BC of the spherical triangle ABC , and E_1 and E_2 are the spherical excesses of the two triangles ABD and ACD , show that

$$\sin \frac{1}{2}(E_1 - E_2) = \frac{\sin \frac{1}{2}E \sin \frac{1}{2}(c - b) \sin \frac{1}{2}(c + b)}{\cos^2 \frac{1}{2}b + \cos^2 \frac{1}{2}c + 2 \cos \frac{1}{2}b \cos \frac{1}{2}c \cos \frac{1}{2}E}$$

4. State the theorem obtained by reciprocating the theorem "The three pairs of straight lines joining a point in the plane of a complete quadrilateral to its pairs of opposite vertices are in involution"; and give a proof of either theorem.

Given five collinear points, A, A', B, B', C , show how to construct, using the ruler only, the conjugate of C in the involution determined by the two pairs A, A' and B, B' .

5. Find the condition that the general equation of the second degree should represent two straight lines.

A complete quadrilateral is formed by the straight lines

$$ax^2 + 2hxy + by^2 = 0 \quad \text{and} \quad a'(x - \alpha)^2 + 2h'(x - \alpha)(y - \beta) + b'(y - \beta)^2 = 0.$$

Find the equation of those two of its diagonals which do not pass through the origin or the point (α, β) .

6. If $u = 0, v = 0$ are the equations of two lines, prove that one of the anharmonic ratios of the pencil formed by the lines

$$u + \lambda_1 v = 0, \quad u + \lambda_2 v = 0, \quad u + \lambda_3 v = 0, \quad u + \lambda_4 v = 0$$

$$\text{is} \quad \frac{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2)}.$$

Prove that one of the anharmonic ratios subtended at any point of an ellipse by the points whose eccentric angles are a_1, a_2, a_3, a_4 is

$$\frac{\sin \frac{1}{2}(a_1 - a_2) \sin \frac{1}{2}(a_3 - a_4)}{\sin \frac{1}{2}(a_1 - a_4) \sin \frac{1}{2}(a_3 - a_2)}.$$

7. (1) If $y = u_1 u_2 \dots u_n$ prove that

$$\frac{dy}{dx} = u_2 \dots u_n \frac{du_1}{dx} + u_1 u_3 \dots u_n \frac{du_2}{dx} + \dots + u_1 \dots u_{n-1} \frac{du_n}{dx};$$

(2) If $y = (x^2 + 2x \cos a + 1)^{-1}$, prove that

$$\frac{1}{y^{n+1}} \frac{d^n y}{dx^n} = (-1)^n n! \left[(n+1)x^n + \frac{(n+1)n}{2!} x^{n-1} \frac{\sin 2a}{\sin a} + \dots + \frac{\sin (n+1)a}{\sin a} \right].$$

Prove also that $\frac{d^3}{dx^3} (x^2 + x\sqrt{3} + 1)^{-1}$ vanishes when $x = 0$.

8. Trace the curves:

(1) $(y - ax)(y - bx)(y - cx) = x^3$, ($a > b > c > 0$);

(2) $r = a \cos \frac{1}{2} \theta$.

9. Prove that, if three coplanar forces keep a body in equilibrium, their lines of action are either concurrent or parallel.

Two planes at right angles to one another slope upwards in opposite directions from their line of intersection, which is horizontal. The steeper plane is rough and the other plane smooth. A uniform plank rests in a horizontal position perpendicular to the intersection of the planes with an end on each plane, and is in limiting equilibrium. When a weight equal to that of the plank is placed upon it close to that end which rests on the smooth plane the equilibrium is again limiting. Determine the angle of friction and the inclination of the rough plane to the horizontal.

10. Two imperfectly elastic spheres impinge obliquely. Show how to determine their velocities after impact.

A fine smooth endless tube, in the form of a circle whose plane is horizontal, contains three small equal spheres A, B, C . Initially A and B are at rest at the extremities of a diameter of the tube and C is in motion. Show that, if the coefficient of restitution is $\sqrt{2} - 1$, after the occurrence of two collisions, two more will occur simultaneously.

11. A uniform rod of length $2a$ is falling freely under gravity, so that its centre of mass moves in a vertical line, and is, at the same time rotating with angular velocity $3g/a$ in a vertical plane. At an instant, when the rod is inclined at an angle of 45° to the vertical and is approaching a vertical position its lower end is suddenly fixed, but it remains free to rotate about that end. Show that, if h be the distance the centre of mass of the rod has descended from rest, the sense of the rotation after the fixture will be the same as, or opposite to, what it was previously, according as $a >$ or $< h$.

Show also that, after the fixture, the rod will execute complete revolutions about its fixed end, unless the value of h lies between those of

$$\frac{a}{3} \{ 11 - 4\sqrt{2} \pm 4(6 - 3\sqrt{2}) \}.$$

[Any number of the following questions may be substituted for an equal number of the preceding questions, but less weight will be assigned to them.]

A. Sum to n terms the series:

(1) $\frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + \frac{1}{8.10} \dots$;

(2) $\sin a + \sin (a + \beta) + \sin (a + 2\beta) + \sin (a + 3\beta) + \dots$

B. Find the equation of the conic whose centre is the origin and which passes through the points $(0, 2)$, $(1, 0)$ and $(3, 3)$.

Is the conic an ellipse, parabola, or hyperbola?

C. Prove that the pressure at depth h below the surface of a liquid of density ρ is $g\rho h$, the pressure at the surface being zero.

A cubical box of edge 10 centimetres is half full of mercury (density 13.5) and half full of water (density 1). Taking g as 980, find the total thrust on the base and on each side.

MATHEMATICS. IV.

See Note after Question 11.

1. Prove that the sum of the moments of two coplanar forces about any point in their plane is equal to the moment of their resultant about the point.

Four equal forces act in order along the sides AB, BC, CD, DA of a quadrilateral inscribed in a circle of which AC is a diameter. Prove that the line of action of their resultant cuts CD at a distance $(CD + DA)(\cot \frac{1}{2} BCD - 1)^{-1}$ from D .

2. Prove the formula $h^2 u^2 \left(u + \frac{d^2 u}{d\theta^2} \right) = P$ for the orbit described by a particle under the action of a central force.

Show that, when the orbit is an ellipse with the centre of force at a focus, the law of force is the inverse square, and the rate at which the direction of motion at the point P of the orbit is changing is proportional to SP^{-2}, HP^{-1} , where S is the centre of force and H the other focus.

3. Find the radii of gyration of :

- (1) A uniform thin spherical shell, radius a , about a diameter ;
- (2) A uniform right circular cone, height h , radius of base r , about a straight line through its vertex perpendicular to its axis.

4. Define "pressure at a point," and show that, at any point of a fluid at rest, the pressure is the same in all directions.

An endless tube of small uniform bore in the form of a circle whose plane is vertical is filled with three liquids of densities 1, 2, and 3, which do not mix and are present in equal volumes. Prove that the liquids can remain at rest in the position in which the point of separation of the lightest and heaviest liquid is on a level with the centre of the circle.

5. The perpendicular drawn through a fixed point O to OP , where P is any point on a curve, meets the normal at P at the point G and the tangent at P at the point T . Express as differential coefficients the tangent of the angle OPT and the lengths OG and OT , taking O as origin of polar coordinates.

If Q is taken on OP between O and P so that OQ is equal to the length of the perpendicular drawn from O to the tangent at P , prove the following construction for the normal at Q to its locus.

"From C , the centre of curvature at P , draw a perpendicular to OP and draw GK at right angles to this perpendicular and meeting it at K . Then the normal at Q to the locus of Q is parallel to PK ."

6. Evaluate the integrals ;

$$\int e^{ax} \cos bx \, dx, \quad \int x(x^2 + x + 1)^{-1} \, dx, \quad \int x(\tan^{-1} x)^2 \, dx.$$

Obtain a reduction formula for the integral $\int \sin^m x \cos^n x \, dx$, and prove that

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^4 x \, dx = \frac{3\pi}{512}$$

7. Prove the formula $\frac{1}{3} \pi \int r^3 \sin \theta \, d\theta$ for the volume of the surface formed by the revolution of a closed plane curve about the initial line.

The outer loop of the curve $r = a \cos^2 \frac{1}{2} \theta$ revolves about the initial line. Show that the volume of the surface generated is $\frac{1}{15} \pi a^3 \left(7 + \frac{1}{2^n} \right)$.

8. If a conic is its own homologue, show that the centre and axis of homology are pole and polar with respect to the conic; and, conversely, that any conic is its own homologue, any pole and polar being centre and axis of homology.

Two chords of a conic, AB and CD , intersect at O . The parallel through O to AD meets BC at P and the parallel through O to BC meets AD at Q . Show that the straight line PQ will meet the conic in real, coincident, or imaginary points according as the conic is an hyperbola, parabola, or ellipse.

9. If the tangential equation of a conic S is $al^2 + bm^2 + cn^2 + 2fmn + 2gml + 2hln = 0$, prove that the coordinates of the pole of the line $lx + my + nz = 0$, with respect to S are

$$(al + hm + gn, hl + bm + fn, gl + fm + on).$$

P, Q, R are the poles of the sides of the triangle of reference ABC with respect to S . Prove that AP, BQ, CR meet in a point, and that the point of concurrence will lie on S if

$$\frac{ba}{f^2} + \frac{ca}{g^2} + \frac{ab}{h^2} - 2 \left(\frac{af}{gh} + \frac{bg}{hf} + \frac{ch}{fg} \right) + 3 = 0.$$

10. Find an expression for the sine of the angle between two lines whose direction cosines are given.

If O is the origin, P the point (x_1, y_1, z_1) , Q the point (x_2, y_2, z_2) , and R the point

$$(y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1),$$

prove that OR is perpendicular to the plane POQ and of length $OP \cdot OQ \sin(POQ)$.

11. (1) Find the sum of the infinite series $\sin \theta + \frac{1}{2} \sin 2\theta + \frac{1}{4} \sin 3\theta + \dots$

(i) When θ lies between 0 and 2π , and (ii) when θ lies between $2n\pi$ and $(2n+2)\pi$.

(2) If a value of $\sin^{-1}(\cos \alpha + i \sin \alpha)$ is $u + iv$, where u and v are real and v is positive, show that, when

$$\pi > \alpha > 0, u = \sin^{-1} \frac{\cos \alpha}{\sqrt{1 + \sin \alpha}} \text{ and } v = \log (\sqrt{\sin \alpha} + \sqrt{1 + \sin \alpha});$$

and that, when $2\pi > \alpha > \pi, u = \sin^{-1} \frac{\cos \alpha}{\sqrt{1 - \sin \alpha}}$

$$\text{and } v = \log (\sqrt{-\sin \alpha} + \sqrt{1 - \sin \alpha}),$$

all the surds being taken positively.

[Any number of the following questions may be substituted for an equal number of the preceding questions, but less weight will be assigned to them.]

A. Prove that, in order to transform logarithms from one given base to another given base, they must be multiplied by a constant quantity.

Write down the expansions of e^{-x} and $\log(1+2x)$ in ascending powers of x , and state in each case for what values of x the expansion is convergent.

B. Show that (1) $4 (\cos 2\theta + \cos 6\theta) (\cos 6\theta + \cos 8\theta) = 1 + \sin 15\theta \operatorname{cosec} \theta$;

$$(2) 1 - 2 \cos A \cos B \cos C - \cos^2 A - \cos^2 B - \cos^2 C = 0,$$

if $A + B + C = (2n + 1)\pi$.

C. Prove that the equation $ax^2 + 2hxy + by^2 = 0$ represents two straight lines.

Find the equation of the straight lines joining the origin to the points of intersection of the line $x + y = 1$ with the conic $2x^2 + y^2 + x + 3y = 1$.

MATHEMATICS. V.

See Note after Question 11.

1. To a mass m , placed on a smooth plane inclined at an angle α to the vertical, a light inextensible string is attached, which passes over the top of the plane, supposed smooth, and supports a second mass m , hanging freely. Determine the acceleration of the system and the tension of the string.

Show that the direction of motion of the upper mass immediately after it has left the plane is inclined to the horizontal at an angle $\tan^{-1}(\frac{1}{2} \tan \frac{1}{2} \alpha)$.

2. Prove the rule for finding the moment of inertia of a body about an axis when the moment of inertia about a parallel axis through the centre of mass is known.

A uniform solid hemisphere of radius a can turn freely about a horizontal axis which coincides with a tangent to its curved surface at the point most remote from its base, and executes small oscillations under the action of gravity. Determine the length of the simple pendulum which has the same period of oscillation.

3. Define the centre of pressure of a plane lamina immersed in a homogeneous liquid, and show that, if there is no pressure on the upper surface of the liquid, its position is unchanged when the lamina is rotated about the line in which its plane meets the surface.

A rectangle of breadth a and depth b lies in a homogeneous liquid, not exposed to atmospheric pressure, with its upper edge in the surface, and is perforated by a circular hole of radius r with its centre at depth d . Find the centre of pressure, and show that, when the perforation is small, it lies below or above that of the unperforated lamina according as d is less than or greater than $\frac{2b}{3}$.

4. Show that the locus of the poles of a given straight line with respect to a system of conics passing through four given points is a conic which passes through the vertices of the common self-conjugate triangle of the system.

Show that the pencil joining any point of this locus to the various poles is homographic with the pencil formed by the corresponding polars of the system of conics with respect to any point.

5. Show that a pair of coplanar conics can in general be simultaneously projected into circles, and give the details of the necessary construction. Under what circumstances is the construction real?

Two conics intersect in the four points A, B, C, D and a pair of common tangents meet on CD . If one of these tangents touches one conic at P and the other at Q , show that C and D are conjugate points with respect to any conic which passes through the four points A, B, P, Q .

6. Find the equation of the normal at the point $(am^2, 2am)$ on the parabola $y^2 = 4ax$.

The normal at $P (am^2, 2am)$ meets the curve again in Q , and AQ meets the ordinate of P in R , A being the vertex. Show that the area of the triangle APR is $2a^2m^3 (m^2 + 1) / (m^2 + 2)$.

7. Show that the equation of the chord joining the points (x_1, y_1, z_1) (x_2, y_2, z_2) on the conic $u \equiv fyx + gzx + hxy = 0$ is

$$fx / x_1x_2 + gy / y_1y_2 + hx / z_1z_2 = 0.$$

PQ is a chord of u touching the conic v whose equation is $v^2 = 2yz$. The other tangents to v from P, Q , meet u again in S, R , respectively. Show that RS touches v .

8. Show that the intercepts made on the axes of coordinates, supposed rectangular, by the tangent and normal at any point of a curve are

$r^2 \frac{d\theta}{dy}$ and $r \frac{dr}{dx}$ on the axis of x , and $-r^2 \frac{d\theta}{dx}$ and $r \frac{dr}{dy}$ on the axis of y ; where

r and θ are the polar coordinates of the point referred to the same origin.

The normal at a point P of a curve meets a fixed straight line through the origin O at G , and the projection of PG on OP is of constant length. Determine the nature of the locus of P .

9. Prove that $f(D) e^{ax} X = e^a f(D + a) X$, explaining the sense in which the symbols are to be interpreted.

Solve the equations:

$$(1) \frac{dy}{dx} + y \cot x = x; \quad (2) \frac{d^2y}{dx^2} + y = x^2 + \sin x;$$

$$(3) (5x^2y + 8xy^2 + 6y^3) dx + (x^3 + 4x^2y + 6xy^2) dy = 0,$$

given that it has an integrating factor of the form x^n .

10. Show that:

- (1) every prime number greater than 3 is of the form $6m \pm 1$;
- (2) if n be a prime number and N prime to n , $N^{n-1} - 1$ is divisible by n .

Prove that, if q is an odd prime number, and $2q + 1$ is also a prime number, then $q^{2q+1} - (2q + 1)^q + q + 1$ is divisible by $2q(q + 1)(2q + 1)$.

11. Prove that, if $f(x)$ is a rational integral function of x and $f'(x)$ is its first derived function, a real root of $f'(x) = 0$ lies between every pair of adjacent real roots of $f(x) = 0$.

If $f(x) = 0$ is a cubic equation, one of whose roots is the harmonic mean of the roots of $f'(x) = 0$, prove that this root is the geometric mean of the other two roots of $f(x) = 0$.

[Any number of the following questions may be substituted for an equal number of the preceding questions, but less weight will be assigned to them.]

A. Express $\sin 4\theta$ and $\sin 5\theta$ in terms of $\sin \theta$.

Give a geometrical explanation of the fact that, if we are given $\sin 4\theta$, there are 8 possible values of $\sin \theta$, but that, if we are given $\sin 5\theta$, there are 5 possible values.

B. Show that the sum of the focal distances of a point on an ellipse is equal to the major axis.

Through one of the foci of a given ellipse a variable ellipse is drawn to touch the given ellipse and to have a focus at the other focus of

the given ellipse. Show that the length of the major axis of the variable ellipse is constant, and that the locus of its other focus is a circle.

C. Evaluate $\int \frac{dx}{x}$, $\int \frac{dx}{\sqrt{1-x^2}}$, $\int x^2 \sin ax$.

Solve the equation $\frac{d^2y}{dx^2} + a^2y = 0$.

D. Find the magnitude of the horizontal force which will keep a body of weight W at rest on a smooth plane, which is inclined to the horizontal at an angle α .

Show that, if the body were kept at rest on the plane by a tight string, attached to a point above the plane and making an angle β with the upper part of the plane, the reaction between the body and the plane would be $W \cos(\alpha + \beta) \sec \beta$.

MATHEMATICS. VI.

See Note after Question 11.

1. Examine the convergency of the series

$$\frac{x}{1-x^2} - \frac{x}{1-x^4} + \frac{x^3}{1-x^6} - \frac{x^7}{1-x^8} + \frac{x^{15}}{1-x^{16}} - \dots$$

for real values of x , and find, when possible, its sum to infinity.

Show (1) that the series whose n th term is $\frac{1}{n} - \frac{1}{2n+1} + \frac{1}{2n+2}$ is convergent, (2) that its sum to n terms differs from its sum to infinity by less than $\frac{3}{4n}$, (3) that its sum to infinity is $\frac{3}{2} - \log 2$.

2. Prove that any periodic continued fraction is a root of a quadratic equation with rational coefficients.

$$\text{If } x = \frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{a + \dots}}}}}}}$$

$$y = \frac{1}{c + \frac{1}{b + \frac{1}{a + \frac{1}{c + \frac{1}{b + \frac{1}{a + \frac{1}{c + \dots}}}}}}}$$

show that $xy = \frac{Q}{P^2 + 2Q} - \frac{Q^2}{P^2 + 2Q} + \frac{Q^3}{P^2 + 2Q} - \dots$

where $P = abc + a + c - b$, $Q = (ab + 1)(bc + 1)$.

3. If A' and B' are the inverses of the points A and B , the origin of inversion being O and the radius of inversion being k , show that $A'B' = k^2 AB/OA \cdot OB$.

A, B, C, D are the vertices of a tetrahedron of volume V inscribed in a sphere of radius E . The rectangles $BC \cdot AD, CA \cdot BD, AB \cdot CD$ are denoted by α, β, γ , and $2\sigma = \alpha + \beta + \gamma$. Show, by inversion with respect to D or in any other manner, that

(1) Any two of α, β, γ are together greater than the third.

(2) $36V^2E^2 = \sigma(\sigma - \alpha)(\sigma - \beta)(\sigma - \gamma)$.

4. Find the co-ordinates of the point dividing the line joining the points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ in the ratio $\lambda : \mu$.

Prove that two cones can be drawn to pass through the circle $x^2 + y^2 = 1, z = 0$ and the conic $25x^2 - 30xz - 23z^2 + 60z - 25 = 0, y = 0$, and find their equations.

5. Show that the co-ordinates of any point on the hyperboloid

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$$

can be represented parametrically by the formulae

$$\frac{x}{a} = \frac{1 + \lambda\mu}{\lambda + \mu}, \quad \frac{y}{b} = \frac{\lambda - \mu}{\lambda + \mu}, \quad \frac{z}{c} = \frac{1 - \lambda\mu}{\lambda + \mu},$$

and that the generators through it are

$$\frac{x}{a} - \frac{z}{c} = \lambda \left(1 - \frac{y}{b}\right), \quad \frac{x}{a} + \frac{z}{c} = \frac{1}{\lambda} \left(1 + \frac{y}{b}\right),$$

and
$$\frac{x}{a} - \frac{z}{c} = \mu \left(1 + \frac{y}{b}\right), \quad \frac{x}{a} + \frac{z}{c} = \frac{1}{\mu} \left(1 - \frac{y}{b}\right).$$

Hence, or otherwise, show that four generators of one system, are cut by a variable generator of the opposite system in a constant anharmonic ratio.

6. If θ is a function of x and y , and $x = f(u, v)$, $y = \psi(u, v)$, express $\partial\theta/\partial x$, $\partial\theta/\partial y$ in terms of $\partial\theta/\partial u$, $\partial\theta/\partial v$.

If u, v are the squares of the distances of the point (x, y) from the points $(-a, 0)$, $(a, 0)$ respectively, show that

$$\frac{1}{4} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = u \frac{\partial^2 \phi}{\partial u^2} + (u + v - 4a^2) \frac{\partial^2 \phi}{\partial u \partial v} + v \frac{\partial^2 \phi}{\partial v^2} + \frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial v}.$$

7. Discriminate between a maximum of a function and its greatest value, and prove the rule for finding maxima and minima.

Find the maxima and minima of the function $\frac{4}{x} - \frac{1}{x-1}$, and determine which is which.

Show also that the function $\frac{a \cos x + b \sin x}{c \cos x + d \sin x}$ has no maxima or minima.

8. Show that in polar co-ordinates the element of area $dA = r dr d\theta$.

Show that $\iint \frac{dA}{r}$, taken over the area cut off from a parabola of latus-rectum $4a$ by an ordinate distant c from the vertex ($c < a$), where r denotes the distance from the focal, is equal to

$$4\sqrt{ac} - 2(a - c) \log \frac{\sqrt{a} + \sqrt{c}}{\sqrt{a} - \sqrt{c}}$$

9. Obtain formulae for finding the position of the centre of gravity of a uniform plane lamina whose bounding curve is given.

A uniform lamina has the form of the area bounded by the curve $ay^2 = x^2$ and the straight line $x = y$. If it is freely suspended from the point corresponding to the origin, find the angle at which the straight part of its boundary is inclined to the vertical.

10. X, Y, Z are the components parallel to rectangular axes of a system of forces; L, M, N the moments of the system about the axes. Prove that, if $LX + MY + NZ = 0$, the system reduces to either a single force or a single couple.

$ABCD$ is a face of a cube; A', B', C', D' the other extremities of the diagonals of the cube which pass through A, B, C, D respectively. Forces of magnitudes 1, 2, 2, 7, 4 act along $BC, CD, DB', B'C, C'D$ respectively; show that they are equivalent to a single force.

11. Find expressions for the tangential and normal accelerations of a particle moving in a plane curve.

A particle describes the curve $s = c \tan \psi$ under the action of a force whose normal component is constant. Prove that the force acts in a fixed direction.

[Any number of the following questions may be substituted for an equal number of the preceding questions, but less weight will be assigned to them.]

A. (1) Find the whole area of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

(2) Find the volume of the portion cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the plane $z = a \cos \alpha$.

B. Obtain the equations of a straight line in space in the form

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}.$$

Find the equations of the straight line through the point (2, 1, 1) perpendicular to the lines

$$\frac{x}{3} = \frac{y - 2}{1} = \frac{z - 1}{2} \quad \text{and} \quad \frac{x + 1}{8} = \frac{y}{1} = \frac{z - 2}{3}.$$

C. A body of mass m is acted upon by a force F in the direction of its motion, and has its velocity increased from v_0 to v in traversing a distance s . Prove that $m(v^2 - v_0^2) = 2Fs$, without assuming the theory of energy.

A railway truck initially travelling at the rate of 20 feet per second is acted upon by a retarding force equal to one-eightieth of its weight. How far will it travel before it comes to rest, and for how long a time? (Assume that $g = 32$ foot-second units of acceleration.)

MATHEMATICS. VII

A.

1. Prove that, if x, y are integers prime to one another, $x^2 + y^2$ and $x^4 + y^4$ can have no common factor with the possible exception of 2.

2. AB is a uniform light elastic string of natural length $4a$ and modulus λ , C, D points on the string such that, when it is straight and unstretched, $2AC = CD = 2DB$. The extremities A, B of the string are attached to two points in the same horizontal plane whose distance apart is $4a$, and equal weights W are attached to the string at C and D . Show that, when the system is hanging in equilibrium, the total extension of the string is $a(\sec \theta - \cos \theta)$, where θ is the inclination of the end portions of the string to the horizontal.

Also find the ratio $W : \lambda$ in the case where $\theta = 60^\circ$.

3. A light straight uniform rod AB has its end A fixed so that the axis of the rod at A is horizontal, and has a weight W attached to it at a point P . Find the position of the point P for which, when the system is in equilibrium, the vertical height of P above B is greatest. (Assume the bending of the rod to be slight.)

4. A motor car of mass M is urged forwards by a force $MA \cos^2$ ($sk \cos \alpha$) and is opposed by a force $Mv^2k \sin \alpha$, where s is the distance travelled from rest, v the velocity, and A, k, α constants. Show that, when the car has run over a great distance, its velocity is oscillating between the values $\left\{ \frac{A}{2k} (\operatorname{cosec} \alpha \pm 1) \right\}^{\frac{1}{2}}$.

5. One end of a light inextensible string of length $2a$ is attached to a point on a smooth straight fixed wire, which is inclined at an angle of 45° to the horizontal. The other end of the string is attached to a ring of mass m , which slides on the wire. A particle of mass $2m$ is attached to the middle point of the string. Show that the inclination of either portion of the string to the wire when the system is in equilibrium is $\cot^{-1} 2$, and find the period of a small oscillation about this position.

B.

6. A variable triangle ABC is inscribed in a fixed conic so that the side AB always touches a certain fixed curve, the point of contact being E , and the side AC always touches a second fixed curve, the point of contact being Q . If P is the point of contact of the third side BC with its envelope, show that AP , BQ , and CR meet at a point.

7. A, B, C are the vertices of a plane triangle and D, E, F are the feet of the perpendiculars from A, B, C to BC, CA, AB respectively. P is any point associated with the triangle ABC , and P_1, P_2, P_3 are the points similarly associated with the similar triangles AEF, BFD, CDE . Show that the triangle $P_1P_2P_3$ is equal in all respects to the triangle formed by the feet of the perpendiculars drawn from P to BC, CA, AB , and that the corresponding sides of the two triangles are parallel.

8. P and P' are corresponding points on two curves which are reciprocals with respect to a parabola; ρ and ρ' are the radii of curvature at P and P' of their respective loci, and ρ_1 and ρ_1' are the radii of curvature of the parabola at the points where the tangents are parallel to the tangents at P and P' . Show that $\rho\rho' = \rho_1\rho_1'$.

9. Show that the tangents to the parabola $y^2 = 4a(x + a)$ at the points $(a \sinh^2(u \pm v) - a, 2a \sinh(u \pm v))$,

where u is variable but v is constant, intersect on a confocal parabola; and that the sum of the lengths of the tangents drawn from any point on the second parabola to the first exceeds the length of the arc between the points of contact by the constant quantity

$$a(\sinh 2v - 2v).$$

10. Show that it is possible to find a function of $\cosh x \operatorname{cosec} y - \mu \cot y$ which is a solution of the differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$; and find it.

C.

11. P is one of the four intersections of the two conics

$$lx^2 + my^2 + nz^2 = 0, \quad l'x^2 + m'y^2 + n'z^2 = 0.$$

Show that, if $(mn' + m'n)(n'l' + n'l)(lm' + l'm) = 0$, the tangents to the conics at P are harmonic conjugates with respect to two of the three common chords of the conics which pass through P .

12. If P is the point (x, y, z) , P' the point (x', y', z') , then the pencil $P(P_1P_2, P_3P_4)$ is harmonic if $(12)(34) + (14)(32) = 0$, (rs) standing for the determinant

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x & y & z \end{vmatrix}$$

13. From a fixed point perpendiculars are let fall on the generators of a cone of the second degree. Prove that they all lie on a cone of the fourth degree.

14. Show that if $n > 1$, $\int_0^1 \tanh \frac{1}{n^2} dx < \frac{1}{n}(1 + \log n)$.

15. A thin spherical shell of radius a is filled with air at atmospheric pressure, and held with its highest point at distance d below the surface, exposed to atmospheric pressure, of a liquid whose density is such that the height of a barometer filled with it is $3a$. The shell is then perforated by a small hole at its lowest point. Assuming Boyle's law, prove that when $d=0$ the fluid rises to a distance b from the top of the shell, where $\frac{b^3}{a^3} = \frac{1}{2}(9 - \sqrt{33})$. Prove also that when d is large the volume of the air becomes $4\pi a^3/(d + 3a)$ approximately.

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I

Elementary Algebra and Trigonometry.

1. Solve the equations :

$$(1) \quad \frac{x}{a + \lambda} + \frac{y}{b + \lambda} + \frac{z}{c + \lambda} = 1,$$

$$\frac{x}{a + \mu} + \frac{y}{b + \mu} + \frac{z}{c + \mu} = 1,$$

$$\frac{x}{a + \nu} + \frac{y}{b + \nu} + \frac{z}{c + \nu} = 1,$$

$$(2) \quad x + y + z = 0,$$

$$x^2 + y^2 + z^2 = 50,$$

$$x^3 + y^3 + z^3 = -50.$$

2. Show that the number of permutations n together of $p + q + r + \dots$ things, of which p are alike of one kind, q alike of another, r alike of a third kind and so on, is the coefficient of x^n in

$$\left\{ x \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^p}{p} \right) \right. \\ \left. \left(1 + x + \frac{x^2}{2} + \dots + \frac{x^q}{q} \right) \left(1 + x + \dots + \frac{x^r}{r} \right) \dots \right.$$

Find the number of ways in which 25 letters can be placed in 10 pigeon-holes, so that there may be at least one in each of three definite pigeon-holes.

3. If $f(x) \equiv u_0 + u_1x + u_2x^2 + \dots + u_nx^n$, n and r being integers, find the value of

$$u_0 + u_1x^r + u_2x^{2r} + \dots + u_nx^{nr}.$$

Discuss the convergency of

$$x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \dots$$

for different values of x , and prove that the sum to infinity when $x = 1$ is

$$\frac{1}{2} \left(\log 2 + \frac{\pi}{\sqrt{3}} \right)$$

4. Prove *ab initio* that the Arithmetic mean of three unequal positive quantities is greater than the Geometric mean.

By considering the series

$$1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{3} + \dots + 1,$$

(n integral), or otherwise, prove that if n is even

$$(n+1)^{n+1} n^{n-1} (n-1)^{n-3} \dots \left(\frac{n}{2} + 1\right) < 2^{n^2+2n-3} \cdot 3^{n-5} \cdot 4^{n-7} \dots \left(\frac{n}{2}\right),$$

and obtain the corresponding theorem when n is odd.

Prove that the product of any n consecutive integers is divisible by $\lfloor \frac{n}{2} \rfloor$.

If p is a prime number greater than 2, show that

$$2^{\frac{p-1}{2}} (2p-1)(2p-3) \dots (p+2) \text{ is divisible by } \left\lfloor \frac{p+1}{2} \right\rfloor.$$

6. If r is the radius of the incircle of a triangle and r_1, r_2, r_3 the radii of the circles lying entirely within the triangle and touching the incircle and two sides, prove that

$$r = \sqrt{r_2 r_3} + \sqrt{r_3 r_1} + \sqrt{r_1 r_2}.$$

7. Find the quadratic factors of $x^n - 1$, when n is an even integer.

$$\text{Prove that } 8 \sin \frac{\pi}{7} \sin \frac{2\pi}{7} \sin \frac{3\pi}{7} = \sqrt{7}.$$

8. ABC is a right-angled spherical triangle, in which C is the right angle. State Napier's rules, and prove that

$$\cos c = \cot A \cot B.$$

ABC is a spherical triangle such that its inscribed circle has a radius $\pi/4$, and D, E, F are the points of contact of the inscribed circle. Prove that the triangle DEF is equal in all respects to the polar triangle of ABC .

9. Show that if $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + px^2 + q = 0$, the equation whose roots are $\beta\gamma + \alpha\delta, \gamma\alpha + \beta\delta, \alpha\beta + \gamma\delta$ is

$$x^3 - 4qx - p^2q = 0.$$

Hence, or otherwise, find the value of

$$(\beta\gamma + \alpha\delta)^2 + (\gamma\alpha + \beta\delta)^2 + (\alpha\beta + \gamma\delta)^2.$$

10. Explain and justify Horner's method of approximating to a root of an equation with integral coefficients.

Find correct to three decimal places the root of the equation

$$2x^5 - 4x^3 - 2x^2 + 2x + 1 = 0$$

which lies between 1.4 and 1.5.

II.

Elementary Geometry.

1. The bisector of the angle C of a triangle ABC meets the side AB in F and passes through the in-centre I and the ex-centre I_2 ; and G is the middle point of II_2 . Show (1) that CF is divided externally and internally in I_2 and I in the ratio $BC + CA : AB$; (2) that G is the circumcentre of the triangle AIB .

Find a geometrical construction for the vertex C of a triangle, when $AB, BC + CA$ and the length of the perpendicular from C on AB are given.

2. O_1, O_2, O_3 are the centres of three circles. Prove that the radical axis of the three circles of similitude of these circles taken in pairs passes through the circumcentre of the circle O_1, O_2, O_3 .

3. Prove that a focal chord of a conic is divided harmonically by the focus and the corresponding directrix.

Show how to construct a conic which has a given line as directrix and passes through three given points; and find the number of solutions.

4. Prove that the tangents drawn from any point to a central conic are equally inclined to the focal distances of the point.

Two conics have one focus S in common, and the corresponding directrices intersect in D . Show that the tangents common to the two conics pass through the intersection of the line drawn at S perpendicular to DS with the connector of the other foci of the conics.

5. The normal at a point P of an ellipse meets the axes in G, g . S, H are the foci and the circle gGS meets SP in Q . Prove that $PQ = HP$ and that the diameter conjugate to CP bisects SQ .

6. Prove that the polar reciprocal of a conic with regard to any finite point on it is a parabola.

7. The coordinates of a point P are

$$\left(\frac{a\lambda^2 + b\lambda + c}{a''\lambda^2 + b''\lambda + c''}, \frac{a'\lambda^2 + b'\lambda + c'}{a''\lambda^2 + b''\lambda + c''} \right),$$

and those of Q, Q', Q'' are

$$\left(\frac{a}{a''}, \frac{a'}{a''} \right), \quad \left(\frac{b}{b''}, \frac{b'}{b''} \right), \quad \left(\frac{c}{c''}, \frac{c'}{c''} \right).$$

Prove that the locus of P , as λ varies, is a conic which touches QQ', QQ'' ; and that, when $b'' = 0$, the centre of the locus is the middle point of QQ'' .

Show that $(\frac{1}{2}, \frac{1}{2})$ is the centre of the conic given by

$$x : y : 1 = \lambda^2 + 2\lambda - 1 : 2\lambda^2 + 4\lambda - 4 : \lambda^2 - 2\lambda + 5.$$

8. Find the coordinates of the centre of curvature of the parabola $y^2 = 4ax$ at the point $(af^2, 2af)$.

Q is a variable point on the normal at a fixed point of a parabola, and R_1, R_2 are the feet of the two other normals which pass through Q . Prove that the connector of the centres of curvature at R_1, R_2 envelopes a parabola.

9. Find the ellipse of maximum area which touches the sides of a given triangle, and show that the circum-radius of the triangle is double the geometric mean of the radii of curvature of the ellipse at the points of contact.

10. P is any point on an ellipse of eccentricity $\frac{1}{2}$, and SH are the foci. Show that the connector of the in-centre and the centroid of the triangle SHP is parallel to the major axis.

11. Find explicitly the trilinear coordinates of the centre of the conic

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

Obtain the length of the radius of the circle

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = ka (a\alpha + b\beta + c\gamma).$$

12. Find the equation of the director circle of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Prove that an equilateral hyperbola intercepts equal segments on two perpendicular tangents to a confocal ellipse.

III.

Elementary Differential and Integral Calculus, and Solid Geometry.

1. Express $\frac{d^n}{dx^n}(uv)$ in terms of u , v and their successive differential coefficients.

Having given $(1-x^2) \log y = 1$, find a relation connecting the values when $x = 0$ of

$$\frac{d^{2n}y}{dx^{2n}}, \frac{d^{2n+2}y}{dx^{2n+2}}, \frac{d^{2n+4}y}{dx^{2n+4}}.$$

2. Find the asymptotes of the cubic

$$xy(x+y-a-b) + c(x+y-a)(x+y-b) - cxy = 0;$$

and show that none of them meets the curve except at infinity.

Also find a cubic through the origin which has the same lines for asymptotes and does not meet them except at infinity.

3. If p is the perpendicular from the origin on a tangent to a curve, and ϕ the angle between the axis of x and this perpendicular, interpret

$$\frac{dp}{d\phi} \text{ and } p + \frac{d^2p}{d\phi^2}.$$

Find the equation of the circle of curvature of the envelope of the line

$$x \cos \phi + y \sin \phi = f(\phi)$$

at the point of contact with the line.

4. Define a definite integral as the limit of a sum.

Prove by mathematical induction or otherwise that

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{2n-1};$$

and hence find a convergent series equal to $\int_1^2 \frac{dx}{x}$.

5. Express in forms free from $\sqrt{-1}$ the results of integrating

$$\int \frac{x^2+2}{x^2+4} dx \text{ and } \int \frac{x^4+4}{(x^2+2)^2} dx.$$

6. If $a_1 + a_2 + \dots + a_n = -2$, and u_1, u_2, \dots, u_n are all of the form $ax + b$, transform

$$\int u_1^{a_1} u_2^{a_2} \dots u_n^{a_n} dx$$

into an integral of the form

$$\int v_1^{a_1} v_2^{a_2} \dots v_{n-1}^{a_{n-1}} dy,$$

where v_1, v_2, \dots, v_{n-1} are all of the form $a'y + b'$.

Integrate $\int x^m(x+1)^\mu(x-1)^{\mu-m-2} dx$,

where m is a positive integer, and μ any constant.

7. A circle of radius a rolls completely round a fixed equal circle. Find the length of the path of a point of its circumference.

8. Prove that the result of eliminating p from the equations

$$y = px + f(p) \text{ and } 0 = x + f'(p)$$

is a solution of the equation

$$y = x \frac{dy}{dx} + f\left(\frac{dy}{dx}\right).$$

Integrate the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos x.$$

9. Prove that the locus of a point of which the tangential distances from three given spheres have given ratios is in general a circle.

Also prove that as a rule there is one sphere which cuts four given spheres orthogonally. Are there any exceptions to the rule?

10. Find, either by proving and using a general theorem involving differential coefficients or independently, the equation of the tangent plane at (x', y', z') to the conicoid $ax^2 + by^2 + cz^2 = 1$.

Obtain the condition that the normal at

$$\left(\frac{l}{\sqrt{a}}, \frac{m}{\sqrt{b}}, \frac{n}{\sqrt{c}} \right)$$

to the above conicoid, and that at

$$\left(\frac{l}{\sqrt{a'}}, \frac{m}{\sqrt{b'}}, \frac{n}{\sqrt{c'}} \right)$$

to the conicoid $a'x^2 + b'y^2 + c'z^2 = 1$, may intersect; and show that, if it is satisfied, the intersection lies on a conicoid $Ax^2 + By^2 + Cz^2 = 1$, where A, B, C are independent of l, m, n .

11. Prove that all the conicoids

$$(a+k)x^2 + (b+k)y^2 + (c+k)z^2 = k,$$

where a, b, c are given, and k, k' are arbitrary, have the same planes of circular section.

If $a < b < c$, find a conicoid of the system which has an umbilic at the point $(1, 0, 1)$.

IV.

Elementary Mechanics.

[The use of the Calculus is allowed.]

1. Find the centre of gravity of a uniform circular sector.

A uniform circular sector $OABC$ of weight w (OA, OC being the bounding radii and B the middle point of the arc AC) can turn freely about a horizontal axis through O perpendicular to the plane of the sector, and therefore hangs initially with OB vertical. A weight W is now attached to A . Show that in the new position of equilibrium the line OB produced will meet a line through the initial position of B parallel to the initial direction of OC at a point whose distance from the initial position of B is directly proportional to W .

Find the distance when $W = w$.

2. A rod of length l can turn freely about one end, O , which is fixed, and the other end, P , which is higher than O , rests against a rough vertical wall distant a from O . Prove that in the position of limiting equilibrium the rod is inclined to the vertical at an angle

$$\sin^{-1} \left\{ \frac{a}{l} \sqrt{\frac{P - a^2 \cos^2 \epsilon}{P^2 \cos^2 \epsilon - a^2 \cos 2\epsilon}} \right\}$$

where ϵ is the angle of friction.

3. Find the dimensions of force and of horse-power.

An engine works at constant horse-power P , and starts a train from rest, the resistance B at first being constant. Find the relation between the velocity of the train and the distance traversed during the time when the resistance remains unchanged. Verify the dimensions in your result.

4. Given that the tension of an elastic string is equal to the modulus of elasticity multiplied by the ratio of the extension to the original length, find the work done in stretching the string through a given length.

A ring slides on a smooth circular wire, whose plane is vertical, and is fastened to the highest point of the wire by a fine elastic string, whose modulus of elasticity is equal to the weight of the ring. Prove that the ring, if placed initially at the lowest point of the wire and slightly displaced, will move continuously round the wire provided the natural length of the string is not greater than four fifteenths of the diameter of the wire.

5. The centre of gravity of an engine of mass m , which is going round a curve, moves along a horizontal circle of radius r with uniform velocity v , the plane of the rails being inclined to the horizontal at angle α . If all the wheels remain in contact with the rails, and if the normal force exerted by the inner rail has no component in the plane of the rails, find the normal forces exerted by each rail. Prove also that $\tan \alpha$ must be not less than $\frac{hv^2 - agr}{av^2 + hgr}$, where h is the distance of the centre of gravity of the engine from the plane of the rails, and $2a$ is the distance between the rails.

6. Three uniform rods, AB, BC, CD , each of weight w , are connected by smooth joints at B and C and supported by strings at A and D . Draw a force diagram; and determine the magnitudes and directions of the reactions at the joints in terms of w and the inclinations of the strings to the vertical. Show also that the reactions can only be equal if the strings are equally inclined to the vertical.

7. Two imperfectly elastic balls impinge obliquely. Find their velocities after impact.

A particle is projected against a cushion of a billiard table. Show that it will return to its original position after impinging on the four cushions, provided the direction of projection passed through a point whose distances from the first and second cushions are $b/(1-e)$, $a/(1-e)$ respectively, a, b being the lengths of the cushions and e the coefficient of elasticity between the particle and the cushions.

8. A cylindrical hydrometer graduated in the ordinary way floats in a liquid whose specific gravity at depth x is $a + bx$, a and b being constants. Find the apparent specific gravity of the liquid as shown by the hydrometer in terms of a, b , and d , the depth to which the hydrometer is immersed in water.

9. The water on one side of a lock gate of breadth d is a feet deep, on the other side b feet deep ($a > b$), and the gate turns about two hinges at depths c_1, c_2 below the surface on the deeper side. The outer end of the gate rests against a similar gate on the other side of the river, the two meeting in mid-stream at an angle to 2α . Determine completely the thrust on each hinge.

10. Explain the action of a syphon.

A closed cylindrical vessel $h + d$ feet deep contains d feet of water and air at atmospheric pressure. Show that a syphon whose lower end is at the level of the bottom of the vessel cannot draw off as large a fraction as $\frac{h}{H + h = d}$ of the water in the vessel, if the air remains at the same temperature throughout, H being the height of the water barometer.

V.

Elementary Optics and Astronomy.

1. A small portion, near the origin, of the sphere formed by revolution about the axis of x of the circle $x^2 + y^2 = 2rx$, separates two media of absolute refractive indices μ, μ' . A ray, refracted from the first medium into

the second, passes through $(u, 0)$ before and $(u', 0)$ after refraction. Prove the equation of first approximation

$$\frac{\mu}{u} - \frac{\mu'}{u'} = \frac{\mu - \mu'}{r}.$$

Also prove that, if y is small compared with x and $x - r$, a pencil from (x, y) becomes after refraction approximately one from (x', y') , where

$$\frac{y}{y'} = \frac{x - r}{x' - r} = \frac{\mu x}{\mu x'}.$$

2. A ray passes in a principal plane through a prism of angle i . Prove that

$$(\mu^2 - 1) \sin^2 i = \{\cos i + \cos(\phi - \psi)\} \{\cos i - \cos(\phi + \psi)\},$$

where ϕ, ψ are the angles of incidence and emergence.

If D is the minimum deviation by a prism of angle i for light of index μ , prove that light of index $\mu + \delta\mu$ incident in the same direction will suffer deviation

$$D + 2 \sin \frac{i}{2} \left\{ 1 - \mu^2 \sin^2 \frac{i}{2} \right\}^{-\frac{1}{2}} \delta\mu.$$

3. Along and immediately round the axis of x , there are $n + 1$ media separated by portions of spherical refracting surfaces with centres on the axis. Assuming the correctness of Question 1, prove that if (x_n, y_n, z_n) in the last medium is the image of (x, y, z) in the first, both being near the axis, then these coordinates are connected by relations of the form

$$\frac{y}{y_n} = \frac{z}{z_n} = \frac{x}{a_n x_n + b_n} = \frac{1}{c_n x_n + d_n},$$

where a_n, b_n, c_n, d_n (which need not be found) depend on the refractive indices, the radii and relative positions of the surfaces, and the position of the origin on the axis.

A telescope is in such adjustment that an object at infinity on the axis has an image at infinity. Prove that small equidistant objects on the axis have equidistant images.

4. Explain the formation of focal lines when a small pencil is obliquely reflected at a spherical surface; and prove the formula

$$\frac{1}{v_1} + \frac{1}{u} = \frac{2}{r \cos \phi}.$$

A concave mirror consists of an annulus, one-tenth of an inch wide, of a spherical surface of radius 10 inches, the radius of the sphere to a point of the annulus being inclined at about 45° to the axis. On the axis there is an infinitely distant source of light. Prove that after reflection the light passes accurately through a short straight line, and also passes close to the circumference of a circle (a locus of primary foci) of which the radius is about 50 times the length of that straight line.

5. A bright element of area dS illuminates an element of area dS' at a distance r from it, the angles of emergence and incidence being ϕ, ϕ' respectively. Find the measure of the whole illumination, stating clearly the assumptions made, and experiments or observations which justify the assumptions.

Assuming that the diameters of the sun, earth, and moon are as 1000 : 9 : 2½, and that the ratio of the diameter of the sun to its distance from the earth is small, find the ratio of the light received by the earth and that received by the moon from a central element of the sun's surface; and

using rough approximations, show that not more than $\frac{1}{12}$ of the total earth light is lost to an observer on the moon at a moment when the sun is just totally eclipsed on the earth, and the centres of sun, moon, and earth are collinear.

6. The transit instrument of an observatory has an azimuthal (or deviation) error of x'' . Find in seconds of time the consequent correction of the observed time of transit of a known star.

The clock times of the transits of two stars, of declinations δ_1, δ_2 respectively, across the meridian south of the pole, are observed and corrected for collimation and level errors, and are then found to agree with the actual times of transit shown in the Nautical Almanac. Prove that the clock is right if $\delta_1 \neq \delta_2$ or if $\delta_1 = \delta_2 =$ latitude, but not if $\delta_1 = \delta_2 \neq$ latitude unless the instrument is free from azimuthal error.

7. What is the declination of the sun on a day when it sets due north-west at a place in north latitude λ ?

Prove that in latitude 60° on a day when the sun sets north-west it shines

$$\frac{24}{\pi} \left\{ \cos^{-1} \frac{1}{7} - \cos^{-1} \frac{1}{\sqrt{21}} \right\}$$

hours longer on a wall facing south-west than on one facing south. (The angles are in circular measure.)

8. If a is the radius of the earth, D the distance of its centre from a heavenly body, and s the zenith distance of the body at a point of the earth's surface, prove that the parallax is

$$\frac{a}{D} \sin s, \text{ to the second order in } \frac{a}{D}.$$

Find to the first order the parallax in hour angle, to an observer in latitude ϕ , of a body whose declination and hour angle are, according to his observations, δ' and h' respectively; and show that the expression for the parallax in declination is

$$\frac{a}{D} \left\{ \sin \phi \cos \delta' - \cos \phi \sin \delta' \cos h' \right\},$$

correct to the second order in $\frac{a}{D}$.

9. State Kepler's three laws of planetary motion.

Prove the two formulæ

$$nt = u - e \sin u, \quad \tan^2 \frac{\theta}{2} = \frac{1+e}{1-e} \tan^2 \frac{u}{2}.$$

10. Find in tenths of an inch the length of a second of arc on a sphere of which the radius is a mile.

In a plate on which a part of the sky is photographed the central point, taken as origin, is a star of Declination δ_0 and Right Ascension α_0 , the axis of y is the line of constant Right Ascension, and the unit of length is a length at the origin depicting a minute of arc of the meridian. Prove that the coordinates in the plate of a star of Declination δ and Right Ascension α are

$$x = \frac{\operatorname{cosec} 1' \cos \delta \sin (\alpha - \alpha_0)}{\sin \delta \sin \delta_0 + \cos \delta \cos \delta_0 \cos (\alpha - \alpha_0)},$$

$$y = \frac{\operatorname{cosec} 1' \{ \sin \delta \cos \delta_0 - \cos \delta \sin \delta_0 \cos (\alpha - \alpha_0) \}}{\sin \delta \sin \delta_0 + \cos \delta \cos \delta_0 \cos (\alpha - \alpha_0)}.$$

VI.

Pure Mathematics.

[Full marks on this Paper can be obtained by complete answers to about TEN questions.]

1. (1) Find two factors of $4x^3 + 2a^2x + a^3$;

(2) Solve the equation

$$a_1 - \frac{1}{a_2} - \frac{1}{a_3} - \dots - \frac{1}{a_n} + \frac{1}{x} - \frac{1}{a_n} - \dots - \frac{1}{a_1} = 0.$$

2. Show that five substitutions of the form $y = lx^2 + mx + p$ (other than $y = x$) can be found, which will transform the equation $x^3 + ax^2 + bx + c = 0$ into itself.

3. Prove that, if $f(x) / [F(x)]^2$ is expressed as a series of partial fractions, and if $f'(x)F'(x) - f(x)F''(x)$ is divisible by $F(x)$, then the denominator of every term of the series will be a perfect square, it being assumed that the roots of the equation $F(x) = 0$ are all different.

4. Assuming any necessary facts as to convergency, prove that, if

$$\Theta(u, a) \equiv \sum_{n=-\infty}^{n=+\infty} e^{un + \frac{1}{2}na} \text{ and } \Theta_1(u + \frac{1}{2}a, a) \equiv e^{-\frac{1}{2}a} \Theta(u, a)$$

then

$$\Theta(u, a) \Theta(v, a) \equiv \Theta(u + v, 2a) \Theta(u - v, 2a) + \Theta_1(u + v, 2a) \Theta_1(u - v, 2a).$$

5. (1) Sum the series

$$\sin \theta + x \sin 2\theta + \dots + x^{n-1} \sin n\theta + x^n \sin(n-1)\theta + \dots + x^{2n-2} \sin \theta.$$

(2) Prove that, if

$$4(\rho^2 - \rho + 1)^2 \sin^2 3\theta = (\rho + 1)^2 (\rho - 2)^2 (2\rho - 1)^2,$$

then the six values of ρ are

$$\cos(a \pm \theta) \sec \theta, \cos(2a \pm \theta) \sec(a \pm \theta), \cos(3a \pm \theta) \sec(2a \pm \theta)$$

where $\pi = 3a$.

6. If a polynomial with integral coefficients is the product of two factors of which one has integral coefficients, then the coefficients of the other must be integral.

Show that $x^{p-1} + x^{p-2} + \dots + x + 1$, where p is prime, cannot be expressed as the product of two rational integral functions of x with integral coefficients.

7. Define $\sin z$ and $\tan z$, where z is a complex quantity, and from your definition prove the periodicity of these functions.

8. Prove that, if $a^2 < a_i - 1, a_i + 1$ for any one value of i satisfying the relations $1 \leq i \leq n-1$, then the equation

$$(a_0, a_1, \dots, a_n)(x, 1)^n = 0$$

has at least one pair of imaginary roots.

Hence (or otherwise) show that, if m_1, m_2, \dots, m_n are real positive quantities, which are not all equal, then

$$\frac{\sum m_i}{n} > \left(\frac{\sum m_1 m_2}{n_1} \right)^{\frac{1}{2}} > \left(\frac{\sum m_1 m_2 m_3}{n_2} \right)^{\frac{1}{3}} \dots > (m_1 m_2 \dots m_n)^{\frac{1}{n}},$$

where $n_i! (n-i)! = n!$

9. State and prove Sturm's Theorem.

Show that the equation $\tan 2mx = a$ gives $2m$ real values of $\tan x$.

10. Find the number of the real roots of the equation

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} = 0.$$

11. If all the elements of the determinant

$$| a_{ik} | \quad (i, k = 1, 2, \dots, m)$$

are real, all except those in the principal diagonal being negative, and if the sum of the elements in each row is positive, prove (by induction or otherwise) that the value of the determinant is a positive quantity.

12. If the elements a_{ik} of the determinant

$$\Delta \equiv | a_{ik} | \quad (i, k = 1, 2, \dots, n)$$

are real and satisfy the conditions

$$a_{11}^2 + a_{22}^2 + \dots + a_{nn}^2 = 1 \quad (i = 1, 2, \dots, n)$$

prove that $\Delta^2 > 1$.

13. Show that the weight of a covariant of a binary quantic of degree n is $\frac{1}{2}(n\theta + p)$, where p is the order of the covariant in the variables and θ is its degree in the coefficients.

Find that covariant of $(a_0, a_1, a_2, a_3, a_4, a_5)$ $(x, y)^5$ in which the term $a_0 a_4$ occurs in the coefficient of the highest power of x .

14. Show that, by means of a linear substitution, it is possible to transform

$$f(x_1, x_2) \equiv ax_1^4 + 4bx_1^3x_2 + 6cx_1^2x_2^2 + 4dx_1x_2^3 + ex_2^4$$

into the form $y_1^4 + 6\mu y_1^2y_2^2 + y_2^4$, and prove that μ is a root of a sextic equation, one root of which is $a / (\beta - \gamma)$, where α, β, γ , are the roots of the equation $\theta^3 - I\theta + 2J = 0$; I, J being the invariants of f . Find the other values of μ in terms of α, β, γ .

VII.

Pure Mathematics.

1. Prove that the equation of any straight line through the point whose trilinear coordinates are α, β, γ can be written

$$\frac{\alpha - \alpha'}{l} = \frac{\beta - \beta'}{m} = \frac{\gamma - \gamma'}{n} = r,$$

where r is the distance of (α, β, γ) from $(\alpha', \beta', \gamma')$, and give a geometrical interpretation of l, m, n .

Prove that the area of the conic whose areal equation is $\sqrt{ux} + \sqrt{vy} + \sqrt{wz} = 0$ is $\pi \Delta \sqrt{uvw} (u + v + w)^{-1}$, where Δ is the area of the triangle of reference.

2. Prove that tangents drawn from any point x', y', z' on the conic, $S, ax^2 + by^2 + cz^2 = 0$ to the conic $S', px^2 + qy^2 + rz^2 = 0$, meet S again in points the line joining which is

$$axx'(-ap^{-1} + bq^{-1} + cr^{-1}) + byy'(ap^{-1} - bq^{-1} + cr^{-1}) + czz'(ap^{-1} + bq^{-1} - cr^{-1}) = 0.$$

Hence or otherwise show that the condition that it should be possible to inscribe quadrilaterals in S which are circumscribed to S' is

$$e^2 - 4\Delta'\Theta\Theta' + 8\Delta\Delta'^2 = 0.$$

3. Form the equation of the cubic which has a cusp at $a, 0$, the cuspidal tangent being the axis of x , and the asymptotes being the axis of y and lines parallel to $x^2 - y^2 = 0$.

Prove that this cubic is of the third class, and that if A, B, C are the points of contact of the tangents from any point P , and ϕ_1, ϕ_2, ϕ_3 the angles at which OA, OB, OC (where O is the origin) are inclined to the axis Ox , then

$$\sqrt[3]{\tan \phi_1} + \sqrt[3]{\tan \phi_2} + \sqrt[3]{\tan \phi_3} = 0,$$

$$\tan \phi_1 \tan \phi_2 \tan \phi_3 = -8 \tan^3 POx.$$

4. Show that if a curve has its maximum number of double points the coordinates of any point on it can be expressed as rational algebraic functions of a variable parameter.

Find the areal equation of a tricuspidal quartic referred to the triangle formed by joining the cusps as triangle of reference, and express the coordinates of any point on the curve in terms of a single variable parameter.

5. Write down Plücker's equations, and prove *one* of them.

Show that a unicursal curve of degree n cannot have fewer than $n - 3$ bitangents.

6. Find the lengths and positions of the axes of the section of the ellipsoid $a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2 = 1$ by the plane

$$lx + my + nz = 0.$$

If the plane passes through a fixed diameter of the ellipsoid, prove that the axes of the sections in which it cuts the ellipsoid all lie on a cone of the third degree which passes through the principal axes of the ellipsoid and through the projections of the diameter on the principal planes.

7. Find the equation of an enveloping cone of a conicoid referred to its principal axes.

Show that in general four straight lines can be drawn through a given point P to touch two confocal conicoids: and prove that, if P moves on a conicoid of the confocal system, the lines will be inclined at a constant angle to the normal at P to this conicoid, provided the tangent plane at P is at a constant distance from the centre of the conicoids.

8. Find the general equation in tetrahedral coordinates of a conicoid circumscribed about the tetrahedron of reference.

A conicoid is circumscribed about a tetrahedron $ABCD$. Prove that if the tangent plane to the conicoid at D cuts the plane ABC in the Pascal line of the hexagon $AABBCC$ with respect to the conic in which the plane ABC cuts the conicoid, then a similar property will be true for each of the other vertices and the opposite faces, and show that the four Pascal lines are coplanar.

Reciprocate the theorem.

9. Obtain the direction cosines of the binormal of a curve.

Show that the direction cosines of the binormal at any point x, y, z of the spheroconic in which the ellipsoid $a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2 = 1$ is cut by the sphere $x^2 + y^2 + z^2 = r^2$, are proportional to

$$\frac{(r^2 - a^2)x}{a^4(b^2 - c^2)}, \quad \frac{(r^2 - b^2)y}{b^4(c^2 - a^2)}, \quad \frac{(r^2 - c^2)z}{c^4(a^2 - b^2)}.$$

VIII.

Pure Mathematics.

[Full marks on this paper can be obtained by complete answers to about NINE questions.]

1. If $f(x), f'(x), f''(x), \dots, f^{(n)}(x)$ are continuous throughout the interval from $x = a$ to $x = a + h$, and if $f^{(n+1)}(x)$ exists throughout that interval, find the differential coefficient of

$$f(a+h) - f(x) - \frac{a+h-x}{1} f'(x) - \dots - \frac{(a+h-x)}{1.2.3\dots n} f^{(n)}(x) - \frac{(a+h-x)^p}{1.2.3\dots n.p} P,$$

where P is independent of x , which lies in the interval; and deduce the remainder after $n+1$ terms in the expansion of $f(a+h)$ in powers of h in Cauchy's form.

$$\frac{h^{n+1}(1-\theta)^n}{1.2.3\dots n} f^{(n+1)}(a+\theta h).$$

Expand $(1+x)^m$ in powers of x , when m is real, and $-1 < x < 1$.

2. Evaluate $\int_0^{\infty} \frac{x^k-1}{1+x} dx$, where k is a proper fraction with an even denominator and an odd numerator.

Prove that, if c is positive,

$$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} \frac{2-x}{1-x+cx^2} dx = \pi \sqrt{2} c^{-\frac{1}{2}}.$$

3. Prove that between corresponding limits

$$\iint du dv = \pm \iint \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) dx dy.$$

Evaluate $\int_0^1 \int_0^1 (3x-xy)^m (3y-2xy)^n (3-y-2x) dx dy$,

where m and n are positive integers.

4. Find, as a series proceeding by powers of x , a solution of the equation $\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = 0$, such that $y = 1, \frac{dy}{dx} = 0$, when $x = 0$.

If this series is called $J_0(x)$, prove, by mathematical induction or otherwise, that

$$s = \left(\frac{d}{dx} - \frac{n-1}{x} \right) \left(\frac{d}{dx} - \frac{n-2}{x} \right) \dots \frac{d}{dx} J_0(x)$$

satisfies the equation

$$\frac{d^2s}{dx^2} + \frac{1}{x} \frac{ds}{dx} + \left(1 - \frac{n^2}{x^2} \right) s = 0.$$

5. Find a relation between x, y and the constants, a, b, c, u , having given that always

$$\frac{d^2x}{dt^2} + (a+b) \frac{dy}{dt} - abx = 0, \quad \frac{d^2y}{dt^2} - (a+b) \frac{dx}{dt} - aby = 0,$$

and that

$$x = c, y = 0, \frac{dx}{dt} = 0, \frac{dy}{dt} = u \text{ when } t = 0.$$

6. Transform $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$, taking for independent variables $x = \frac{x^2 + y^2}{t}$, $\theta = \tan^{-1} \frac{y}{x}$ and t .

Prove that every solution of the equation which is of the form $f(t) \phi(\theta) \psi(x)$ is included in the form

$$t^k (A \cos m\theta + B \sin m\theta) \psi(x),$$

where $\psi(x)$ satisfies

$$4x^2 \frac{d^2 \psi}{dx^2} + x(x+4) \frac{d\psi}{dx} = (kx + m^2) \psi,$$

and k, m are numerical, real or imaginary.

7. Show how to find the centre and radius of spherical curvature at any point of a curve in space.

$$x = f(t), y = \phi(t), z = \psi(t).$$

A particle is projected from a point, and is acted on by forces which depend only on its position and not on its motion or on the time. Prove that the centre of spherical curvature of its path at the point of projection is the intersection of a plane, which is the same whatever be the direction of projection, and a straight line, which is the same whatever be the speed.

8. Prove that along a line of curvature of the surface $z = f(x, y)$ the differential equation

$$(1 + p^2) s - pqr + \{(1 + p^2) t - (1 + q^2) r\} \frac{dy}{dx} + \{pqt - (1 + p^2) s\} \left(\frac{dy}{dx}\right)^2 = 0$$

is satisfied.

Spheres with their centres at the origin touch lines of curvature of the paraboloid $z = kxy$. Prove that the points of contact lie on the locus

$$(x^2 - y^2) \{2k^2xyz + k^2(x^2 - y^2 - z^2) + 1\} = 0.$$

9. Express $\int_a^x \frac{dx}{\sqrt{(x^2 - a^2)(x^2 + b^2)}}$ as a multiple of an elliptic integral of the first kind in standard form.

Hence or otherwise evaluate the complete elliptic integral of modulus $\frac{1}{\sqrt{2}}$ in terms of gamma functions.

10. If

$$\int_0^\theta \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} + \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int_0^\mu \frac{d\mu}{\sqrt{1 - k^2 \sin^2 \mu}},$$

prove that $\mu = x + y$, where

$$\begin{aligned} \frac{\cos x}{\cos \theta} &= \frac{\cos y}{\cos \phi} = \frac{\sin x}{\sin \theta \sqrt{1 - k^2 \sin^2 \phi}} = \frac{\sin y}{\sin \phi \sqrt{1 - k^2 \sin^2 \theta}} \\ &= \frac{1}{\sqrt{1 - k^2 \sin^2 \theta \sin^2 \phi}} \end{aligned}$$

Show that

$$\operatorname{cn}(u + v) + i \sin(u + v) = \frac{\operatorname{cn} u + i \sin u \operatorname{dn} v}{\operatorname{cn} v - i \sin v \operatorname{dn} u}$$

11. What is meant by a multiform function of z , and by following a branch of a multiform function?

The point $z = x + iy$ traverses a closed circuit in the plane of (x, y) , starting from and ending at a point z_0 . The circuit does not pass through either of the points $z = a, z = b, z = c$. If one branch of the function

$$\{(s-a)(s-b)(s-c)\}^{\frac{1}{2}}$$

is followed round the circuit, how can it be decided whether the original value of the function at z_0 is resumed when z_0 is reached again?

Evaluate $\int \{(a+s)^{\frac{1}{2}} - (a-s)^{\frac{1}{2}}\} ds$ round a circle about $s = a$ from the origin to the origin again.

12. Justify the assumption of the Calculus of Variations that

$$\delta \frac{d^2y}{dx^2} = \frac{d^2}{dx^2} \delta y.$$

If \dot{x}, \dot{y} denote $\frac{dx}{dt}, \frac{dy}{dt}$, and if $T + V = \text{constant}$, where V is a given function of x, y , and T a given function of x, y, \dot{x}, \dot{y} which is homogeneous and of degree n in the two last, prove that along a route from one fixed point (x_0, y_0) to another fixed point (x_1, y_1) , for which $\int T dt$ is a minimum, we must have

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + (n-1) \frac{\partial V}{\partial x} = 0$$

and
$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{y}} \right) - \frac{\partial T}{\partial y} + (n-1) \frac{\partial V}{\partial y} = 0.$$

IX.

Mixed Mathematics.

1. A closed polygon of jointed rods is in equilibrium on a smooth horizontal plane under forces, one at each joint. If the directions of all but one of these forces is given, exhibit graphically the direction of the remaining one, and the relative magnitudes of all.

2. Forces proportional to the sides of a plane polygon act normally to those sides respectively at their middle points, all inwards or all outwards, Prove that they are in equilibrium.

A quadrilateral of freely jointed rods is in equilibrium under forces proportional to the lengths of the rods acting normally outwards at their middle points. Prove that a circle can be circumscribed to the quadrilateral, and that the stresses at the joints act along tangents to the circle.

3. Find the condition that given forces acting on a rigid body at points with given rectangular coordinates may have a single resultant force.

The coordinate planes are smooth rigid boundaries, and the direction cosines of the vertical are l, m, n . Prove that if a rigid body rests in contact with the planes respectively at

$$(0, y_1, z_1), (x_2, 0, z_2), (x_3, y_3, 0),$$

then
$$mx(x_2 - x_3) + nl(y_2 - y_1) + lz(x_1 - x_2) = 0;$$

and find the equations of the vertical through the centre of gravity.

If the body is a homogeneous ellipsoid of semi-axes a, b, c , prove that the vertical height of the centre above the origin is

$$\{a^2 + b^2 + c^2 - (mx_2 - ny_2)^2 - (nx_3 - lz_3)^2 - (ly_1 - mx_1)^2\}^{\frac{1}{2}}.$$

4. A uniform rigid circular-table top of weight W stands on four equal and like slightly compressible legs, which are at equal distances from the centre measured along two diameters at right angles to one another, and a weight W' is placed on the table immediately above a leg. Assuming that compression varies as pressure, and that the legs are not fastened to the top, find the four pressures (1) when $W' < W$, and (2) when $W > W'$.

5. The cross-section of a cylindrical body is the curve $\rho = f(s)$, where s, ρ are arc and radius of curvature, and the centre of gravity is on the normal at (ρ, s) at a distance h from its foot. Prove that, if the cylinder is placed on a rough horizontal plane with this normal vertical, it will not be in stable equilibrium (1) if $f'(s) < h$, or (2) if $f'(s) = h$ and $f''(s) \neq 0$; and examine cases when $f'(s) = h$, $f''(s) = 0$, and $f'''(s) \neq 0$.

6. Find differential equations for determining the form, and the tension at any point, of a uniform inelastic string under given forces.

A uniform inelastic string with fixed ends is repelled from a given plane by a force everywhere proportional to the mass acted upon and the distance from the plane. Prove that it assumes the form of a plane curve, and that the tension along it varies as the product of the distances from two planes parallel to the given plane.

7. A heavy uniform elastic string rests on a smooth curve in a vertical plane. Prove that the tensions at two points of it on the same level are equal.

A very slightly elastic band of mass m per unit unstretched length and modulus of elasticity λ just surrounds a sphere of radius a in a vertical diametral plane without pressure at the lowest point. Prove that to the first order in λ^{-1} the tension at a height y above the lowest point is

$$mg(a + y) - \frac{1}{2} m^2 g^2 \{a^2 + (a + y)^2\} \lambda^{-1};$$

and find the pressure per stretched unit length at the highest point.

8. Find the attraction of a uniform solid ellipsoid of revolution on a particle at an end of its axis.

The density at any point of a solid ellipsoid of revolution about the major axis is μ times the harmonic mean between the distances of the point from the foci. Prove that the potential at a focus is μ times the volume.

9. What is the metacentre of a solid of revolution floating partially immersed with its axis vertical? Find its position.

A homogeneous right circular cylinder of height $2h$ and radius a floats in a liquid of double its density with its axis vertical. Find limits between which the ratio of h to a must lie that the equilibrium may be unstable when it floats freely, but stable when it is only free to turn about a tangent in the plane of flotation.

10. A flexible weightless vessel in the form of a surface of revolution about the axis of y , which is vertical, is filled with homogeneous heavy liquid. If the origin is in the free surface of the liquid, and the downward direction is positive, prove that the principal tensions at any point in the meridian section by the plane of xy are given by

$$\frac{d}{ds} \left(xt \frac{dy}{ds} \right) + g\rho xy \frac{dx}{ds} = 0, \quad \ell = \frac{d}{ds} (xt).$$

If the vessel is in the form of a hemisphere with a fixed rim, what result may be expected?

X.

Mixed Mathematics.

1. A particle of mass m moves under the action of a varying force, the components of which at the point (x, y, z) are $m(x^2 + 2yz)$, $m(x^2 + 2xy)$, $m(y^2 + 2xz)$ parallel to the axes of coordinates. Initially it is at rest at the point $(a, 0, 0)$. Prove that at time t its x -coordinate is

$$\frac{1}{3}[f(t, a) + f(t, \beta) + f(t, \gamma)],$$

where $f(t, h)$ is a value of u satisfying the equation

$$\frac{1}{3}u^3 = u^2 - a^2,$$

and a, β, γ are three particular values of h .

2. Particles P, Q , each of unit mass, are constrained to describe the circles $x^2 + y^2 = 4a^2$ and $x^2 + y^2 = a^2$ under their mutual attractions which are $\mu^2 . PQ$. If initially both particles are at rest, find the motion, and prove that, if the oscillations are small, their period is $\frac{2\pi}{\mu} \sqrt{\frac{1}{3}}$.

3. Show that, if a particle of unit mass moves on a sphere of unit radius under the action of a force tending to the pole, and if $f(\rho)$ is the component of the force towards the pole along the meridian, the differential equation of the path will be

$$\frac{d^2u}{d\theta^2} + u = \frac{f(\rho)}{h^2(u^2 + 1)},$$

where ρ is the polar distance (measured along the sphere), θ the longitude, $h = \text{const.} = \theta \sin^2 \rho$, and $u = \cot \rho$.

If the force is $ka^{-1}(4a^2 - c^2)^{-1}$, where a is the radius of the sphere and c is the chordal distance of the particle from the pole, show that the path is a spheroconic, having the pole as focus.

4. The motion of a particle is defined by the equations

$$x^2\ddot{x} + A = y^2\ddot{y} + B = 0,$$

where A and B are constants. Prove that the orbit is given by

$$x^2 = a(t + a')^2 - Aa^{-1}, \quad y^2 = b(t + b')^2 - Bb^{-1},$$

where a, b, a', b' are constants.

Show also that, if the initial conditions are chosen so that the product ab is negative, the orbit is part, but not the whole, of a loop of a curve which does not extend to infinity, and explain how this can occur in a case of non-periodic motion.

5. A particle of mass m moves under gravity on the cycloid $x = a(u - \sin u)$, $y = a(1 - \cos u)$, the y -axis being vertically downwards, under a resistance $m\lambda v$, where v is the velocity. Initially $v = 0$ and s (measured from the vertex) $= c\lambda$.

Prove that the particle will oscillate, the areal distances (from the vertex) of its points of instantaneous rest being obtained by giving s in succession the values 1, 2, 3 ... in

$$\lambda c \exp. (-k\pi/\lambda) \text{ where } \lambda = \{g/a - k^2\}^{\frac{1}{2}}.$$

6. A particle describes the parabola $y^2 = 4ax$ in such a manner that at every point of its path $\ddot{y} = k^2x$. Initially the particle is at the origin, moving along Oy . Prove that the hodograph is a cubic of the form

$$x^2 = py^2 + qy^3.$$

7. A particle moves upon a smooth surface under any forces. Find equations determining the motion.

If the surface is that formed by rotating the catenary $y = c \cosh(x/c)$ about the axis of y , and if the particle is subject to an attractive force $h^2(r^2 + 2a^2)/r^3$ perpendicular to the axis, find the projection of the path on the plane of zx . [$r^2 \equiv x^2 + y^2$; initially, $r = b$ and the velocity is $h(a^2 + b^2)^{1/2}/b^2$, the component perpendicular to Oy being h/b].

8. A particle can describe an orbit $r = f(\theta)$ under the action of a centre of force situated at the pole. Show that, if the orbit is rotated in its own plane about the pole with angular velocity ω , accelerations $(2\omega\dot{\theta} + \omega^2)r$ towards the pole and $\frac{1}{r} \frac{d}{dt}(r^2\omega)$ transversely have to be added to that which is due to the central force.

If the central force is $F(r) + \mu r^{-2}$, show that the path of the particle can be realised by giving to the orbit corresponding to the force $F(r)$ an angular velocity about the pole.

9. Obtain the equation of continuity in the form

$$\rho \frac{\partial(x, y, z)}{\partial(a, b, c)} = \text{constant},$$

and deduce from it the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0.$$

10. Investigate the method of conjugate functions as applied to two-dimensional fluid motion, and illustrate your answer by reference to any suitable problem arising in connection with an elliptic cylinder.

XI.

Mixed Mathematics.

[Full marks can be obtained on this paper by complete answers to about NINE questions.]

1. Explain d'Alembert's principle and apply it to obtain the equations of motion of a rigid body.

A rigid body is oscillating about a horizontal axis under the action of its weight only. Determine the resultant reaction between the axis and the body at any instant in terms of the weight of the body, its moment of inertia about the axis, the distance of the centre of gravity from the axis and the angles at which this line of shortest distance is inclined to the vertical at the instant considered and at the end of a swing.

2. A napkin-ring in the form of a thin hollow cylinder of radius a is projected forward along an imperfectly rough table with velocity u and an angular velocity ω about the axis of the cylinder tending to make the ring return to the point of projection. Determine the whole motion, finding how far the ring will advance, and the condition that it should return to the point of projection.

3. Find the moment of inertia of a right-circular cone about a generator.

A right-circular cone of mass M rests with a generator in contact with a perfectly rough horizontal plane, and is struck at the highest point horizontally and in a direction perpendicular to the axis, so that the cone

begins to roll about the generator in contact with the plane. If I is the impulse of the blow, prove that the centre of gravity of the cone will describe a horizontal circle with uniform velocity in time

$$\frac{3\pi Mh}{20I} (\sec a + 5 \cos a),$$

where $2a$ is the vertical angle of the cone, and h its height.

4. A disc of mass M is rotating about its axis, which is at rest, with uniform angular velocity on a smooth horizontal table. From a point vertically above the circumference of the disc a particle of mass m is dropped and adheres to the disc after impact. Find the impulsive blow, and show that a fraction $2m/(M + 3m)$ of the kinetic energy of rotation is lost by the impact.

5. A rod BC of weight W rests in a horizontal position supported by two strings AB, CD fastened to its ends, and AB is equal to BC and is inclined at an angle θ to the horizon. If CD is cut, find the initial tension of the string AB . Determine also, when $\theta = \pi/2$, the initial radius of curvature of the path described by C .

6. An elliptic cylinder is placed on a perfectly rough horizontal plane, with the major axis of a cross-section vertical, and is slightly displaced so that rolling takes place. If the cylinder does not leave the table, obtain in the form of a definite integral the time that elapses before the major axis is again vertical; and show that the time of a small oscillation about the position of stable equilibrium is

$$\pi \sqrt{\frac{b(a^2 + 5b^2)}{(a^2 - b^2)g}}.$$

7. A rod of length $2l$ and weight w can move freely about one end, and when horizontal is made to rotate with angular velocity $\sqrt{\frac{kg}{l}}$ about a vertical axis. Show that during the subsequent motion the inclination of the rod to the vertical lies between $\pi/2$ and $\cos^{-1} \frac{1}{2} \{(k^2 + 9)l - k\}$. Prove also that the reaction at the fixed end always lies in the vertical plane through the rod, and that the vertical component of the reaction is never less than $\frac{w}{4}$.

8. Obtain Euler's geometrical equations connecting the motion of a body about a point in it with the angular velocities of the body about three mutually perpendicular axes moving with it.

A body is moving about a fixed point under the action of no forces, in such a way that one of its principal axes at the fixed point, about which the moment of inertia is C , describes a cone in space of constant vertical angle 2θ with constant angular-velocity Ω . Prove that the other two principal moments of inertia must be equal, and find the angular velocity of the body about the principal axis in question.

9. If T is the kinetic energy of a system of bodies in motion, prove completely that in general

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = \Sigma m \left(\ddot{x} \frac{\partial x}{\partial \theta} + \ddot{y} \frac{\partial y}{\partial \theta} + \ddot{z} \frac{\partial z}{\partial \theta} \right),$$

and similar equation in $\phi, \psi \dots$, where $\theta, \phi, \psi \dots$ are the general coordinates of the system. Point out any restrictions as to these coordinates required by your proof, explaining where the proof would fail, were the restrictions not satisfied.

Show that, if the geometrical equations do not contain the time explicitly, the equation of energy can be derived from Lagrange's equation by multiplying the $\theta, \phi \dots$ equations by $\dot{\theta} \dot{\phi} \dots$, adding the results and integrating.

10. A body moves about a fixed point under the action of no forces. Prove that its motion may be represented by supposing the momental ellipsoid at the fixed point to roll on a fixed plane with its centre fixed.

An ellipsoid of revolution is set revolving with angular velocity ω about an axis inclined at an angle a to the axis of symmetry. Prove that the axis of symmetry will describe a cone in space, in the same sense as the body revolves, in time.

$$\frac{2\pi(1+k^2)}{\omega \sqrt{(1+k^2)^2 \sin^2 a + 4 \cos^2 a}}$$

where k is the ratio of the axis of symmetry to the equatorial axis.

11. A billiard ball resting on an imperfectly rough horizontal table is struck in such a way that it does not leave the table. Show that the centre of the ball will not in general begin to move parallel to the vertical plane through the blow, unless either the plane is diametral or the blow is horizontal. Show also that sliding will continue during a time $\frac{1}{2}S/\mu g$ where μ is the coefficient of friction and S the initial velocity of the point of contact, and that the centre of the ball will describe a parabola during this time.

XII.

Mixed Mathematics.

[Full marks may be obtained for complete answers to about NINE questions.]

1. Prove that any system of rays which are normals to a surface will be normals to a surface after any number of refractions.

2. A ray, proceeding from an origin of light at the point $(c, 0)$, impinges at the point $(a \cos a, a \sin a)$ upon the inner reflecting surface of the circle $x^2 + y^2 = a^2$ ($a > c$): prove (1) that the equation of the reflected ray is

$$x(c \sin 2a - a \sin a) + y(a \cos a - c \cos 2a) = ac \sin a,$$

(2) that the caustic is the sextic curve

$$\{ (4c^2 - a^2)(x^2 + y^2) - 2a^2cx - a^2c^2 \}^2 = 27 a^4 c^2 y^2 (x^2 + y^2 - c^2)^2.$$

3. How is a sun-dial graduated?

Prove that the boundary of the solar shadow of the cylindroid $z(x^2 + y^2) = 2mxy$ on the plane of xy is a three-cusped hypocycloid.

4. Prove for the orbit of an undisturbed planet P that, if $|e| < 1$, $ASP = v$, $nt = m$, then

$$v = m + 2(1 - e^2)^{\frac{1}{2}} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{2}e\right)^k J_k^2(ie) \frac{1}{i} \sin im,$$

where

$${}_n J_k^2(s) = \int_0^\pi (2 \cos x)^k \cos(ix - s \sin x) dx.$$

5. Show that, if the orbit of a planet is assumed to be circular, two observations of the planet will suffice for the determination of its orbit.

6. Define the "equation of time" and show why it vanishes four times a year.

7. Define a "source" in fluid motion.

A, B, C are three collinear points at which are sources of strengths $m, m, -m$, respectively. If $AB = 3a, BC = a$, show that one stream-line consists partially of a circle of centre C and radius $2a$.

8. Determine the motion of any point of a fluid in consequence of the presence of a single vortex line of strength m .

Three parallel vortices have strengths $m, m, -2m$, respectively. Prove that the consequent stream-lines are bicircular quartics.

9. Show that the attraction of a thin homogeneous shell bounded by two similar, coaxial ellipsoids on an external particle is in the direction of the axis of the cone which has its vertex at the particle and envelops the shell.

10. Prove that the potential of any body of mass M at a point P , whose coordinates referred to principal axes at the centroid G , are x, y, z , is approximately

$$M/r + \frac{1}{2} \frac{\Sigma(B + C - 2A)x^2}{r^3},$$

where $r \equiv \sqrt{\Sigma x^2}$ is large, and A, B, C are the principal moments of inertia at G .

The body being a homogeneous tetrahedron $OABC$, of which the edges OA, OB, OC are mutually perpendicular and of lengths $a\sqrt{2}, a, a\sqrt{3}$, prove that $C + A = 2B$, and show that, if P lies in either of the central cyclic planes of the momental ellipsoid at the centroid G , a particle placed at P and constrained to move along the straight line PG will describe this line as though it were moving freely under the attraction of a mass M situated at G . (It is assumed that a^4/GP^4 is a negligible quantity.)

11. A is the centre of a small fixed insulated sphere, and B that of a movable one of equal radius and of mass m which is supported by a light insulating thread of length l cm. attached to a point C vertically above A . At first the spheres are in contact. Each then receives a charge q and equilibrium is established when the angle $ACB = \theta$. Find a relation between q, m, θ .

Also prove that the diminution in the potential of the two spheres, if they are kept in electrical communication throughout, cannot be less than $2mg/q^{-1} \sin^2 \frac{1}{2} \theta$.

12. Prove that the potential at an external point P due to a conductor, which is insulated and charged to potential V , is fV/ϵ , where $-f$ is the charge which would be induced on the same conductor, if uninsulated, by a charge ϵ at P .

13. The ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ ($a > b > c$) is charged and insulated. Assuming that the equipotential surfaces are confocal ellipsoids, prove that the capacity of the body is twice the reciprocal of

$$\int_0^{\infty} \frac{d\lambda}{\sqrt{a^2 + \lambda} \cdot \sqrt{b^2 + \lambda} \cdot \sqrt{c^2 + \lambda}}$$

and verify that it may be expressed in the form $ae/sm^{-1}\epsilon$, the modulus of the elliptic function being e'/e , and e, e' being the eccentricities of the ellipses

$$x^2/a^2 + z^2/c^2 = 1, \quad x^2/a^2 + y^2/b^2 = 1.$$

14. A particle charged with q units of electricity is at a distance f ($< a$) from the centre of a hollow spherical conductor of radius a which is put to earth: prove that the work done in moving the particle to the centre is

$$\frac{1}{2} f^2 q^2 / a (a^2 - f^2).$$

XIII

Problems.

1. If $dd' = a^2 + ab + b^2$,
 and $Pd = (a + \lambda d)^2 + (a + \lambda d)(b + \mu d) + (b + \mu d)^2$,
 prove that $Pd' = a'^2 + a'b' + b'^2$,
 where $a' = d' + a(\lambda + \mu) + b\mu$, $b' = b\lambda - a\mu$.

Solve in positive integers one or other of the equations

$$7(x^2 \pm xy + y^2) = 1003009.$$

2. The coordinates (x, y) , (x', y') of corresponding points in two configurations are connected by the relations

$$\frac{y'}{y} = \frac{x'}{ax + b} = \frac{1}{cx + d}$$

Prove that, if the second configuration is shifted parallel to the axis of x through a distance

$$b - \frac{(1-a)(1-d)}{c},$$

it will be in perspective with the first; and find the axis and centre of perspective.

3. Show that the mean value of the moment of inertia of a body about a line through a chosen point of the body is the arithmetic mean of the three principal moments at that point.

Find in terms of integrals the probability that the moment of inertia about a line chosen at random through the point may exceed the mean; and show that if two of the principal moments are equal and greater than the third the probability is $\frac{1}{\sqrt{3}}$.

4. If s, c, d are the real parts of

$$m(u + iv), \quad cn(u + iv), \quad dn(u + iv),$$

when u, v are real, prove that

$$k^2 c^2 \left(1 - \frac{1}{c^2 + s^2}\right) = d^2 \left(1 - \frac{1}{d^2 + k^2 s^2}\right)$$

and
$$\frac{\partial s}{\partial u} = cd - \frac{s^2 d}{c} \left(1 - \frac{1}{d^2 + k^2 s^2}\right).$$

5. A particle describes a plane orbit under a force $\frac{\mu}{r^2}$ to the origin, μ being variable. Prove that if $\frac{c}{r} = 1 + \epsilon \cos(\theta - \alpha)$ is the instantaneous ellipse at any time, i.e., the ellipse which the particle would proceed to describe if μ were to retain the value which it has at that time, then

$$\left(c \frac{dc}{dt} - \epsilon \frac{d\epsilon}{dt}\right)^2 + c^2 \epsilon^2 \left(\frac{d\alpha}{dt}\right)^2 = \left(\frac{dc}{dt}\right)^2.$$

6. If the minors A, B, C, F, G, H of the discriminant of

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

are proportional to the corresponding constituents a, b, c, f, g, h , find f, g, h in terms of a, b, c .

7. A circle cuts a three-cusped hypocycloid in six points. Prove that, if the tangents to the hypocycloid at three of these points are concurrent, those at the other three will also be concurrent.

8. A, B, C are any three points on the curve

$$x = at, \quad y = bt^2, \quad z = ct^3.$$

Show that the osculating planes at the points intersect the plane ABC in three concurrent lines.

9. Two particles, P, P' , of masses m and m' respectively, move without friction on two horizontal lines which are in perpendicular directions but do not intersect. P, P' attract each other according to the law of the direct distance. Find the ratio of m to m' in order that the motion of their centre of inertia may be periodic, and show that in that case it describes an algebraic curve.

10. A solid is free to move about a vertical axis Oz , its moment of inertia about which is $2mk^2$. In the solid is a canal of infinitesimal section in the form of a vertical circle of radius h about a point C as centre, the plane of the canal being perpendicular to the plane COz and the distance of C from Oz being h .

A particle of mass m is in the canal at one end of the horizontal diameter of the circle and the system is at rest. If the particle falls under gravity, find the motion and show that the angle between the extreme positions of OC is about 34° .

11. Show that, if a geodesic on a surface of revolution cuts a meridian curve at an angle α at a point distant ρ from the axis of revolution, then $\rho \sin \alpha$ is constant.

Any two points are taken on a geodesic on the surface formed by the revolution of an ellipse about the minor axis.

Express the distance between them in the form

$$4 \int_0^\theta \frac{d\theta}{(1 - k^2 \sin^2 \theta)^{3/2}},$$

where

$$\sqrt{a^2 - c^2} \sin \theta = \sqrt{a^2 - c^2 e^2} \sin \phi,$$

ϕ being the angle the normal at a point makes with the equatorial plane, a the equatorial semi-axis, e the eccentricity of a meridian curve, and c the constant value of $\rho \sin \alpha$.

12. Two smooth planes each inclined at an angle α to the vertical intersect in a horizontal straight line. A smooth rod of length $2l$ is placed horizontally with its ends resting on the planes and its direction at right angles to their line of intersection. It is then slightly displaced about a vertical axis through its middle point. Prove that it will reach its lowest position after a time.

$$\frac{1}{2} \sqrt{\frac{l \tan \alpha}{3g}} \int_0^\pi \sqrt{\operatorname{cosec}^2 \frac{\theta}{2} + 12 \cot^2 \alpha \cos^2 \frac{\theta}{2}} d\theta.$$

13. A heavy uniform string of weight $2w$ and length $2l$ hangs freely from two fixed points B, C in the same horizontal plane distant $2a$ from each other. A second similar string of length l and weight w has its lower end fastened to the middle point D of BC , and the upper end A can move freely along a horizontal line bisecting BC at right angles. If A is moved slowly along this line from the position of equilibrium, prove that the tangent to the second string at D will be horizontal when and only when ABC are the vertices of an equilateral triangle; and find an expression for the work done in bringing A to this position.

14. A liquid moves irrotationally between two fixed infinite circular cylinders, one entirely within the other, whose axes are parallel and a distance c apart. Determine the points of maximum and minimum velocity, and prove that the ratio of these critical velocities is $\frac{a(a-a'+c)}{a'(a-a'-c)}$, where a is the radius of the larger and a' that of the smaller cylinder.

15. Two charges, one positive the other negative, are placed outside an insulating conducting sphere, at points the line joining which subtends a right angle at the centre of the sphere. Prove that if the sphere is at zero potential the charges are directly as their distances from the centre of the sphere. Find an expression for the ratio of the areas of the positively and negatively charged portions of the sphere in this case:

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I.

1. (1) If $x - \frac{1}{1+x} - \frac{1+x}{x} = 3p$, and $\omega^2 + \omega + 1 = 0$,

show that

$$\left[x - \frac{\omega}{1+x} - \frac{\omega^2(1+x)}{x} \right]^2 = 27(p - \omega)(p - \omega^2)^2.$$

(2) Eliminate x, y, z from the following equations:

$$x + y + z = a, \quad yz + zx + xy = 0, \quad xyz = b^2, \quad (y - z)(z - x)(x - y) = c^2.$$

2. Solve the equations:

$$(1) \frac{80}{x-5} + \frac{15}{x-10} = \frac{64}{x-4} + \frac{3}{x-2} + 20;$$

$$(2) u + v = 7, \quad ux + vy = 12, \quad ux^2 + vy^2 = 22, \quad ux^3 + vy^3 = 42;$$

$$(3) a(x + yz) = b(y + zx) = c(z + xy), \quad x^2 + y^2 + z^2 + 2xyz = 1.$$

3. If $\frac{x}{a+\alpha} + \frac{y}{b+\beta} + \frac{z}{c+\gamma} + \frac{u}{d+\delta}$

$$= \frac{x}{a+\beta} + \frac{y}{\beta+b} + \frac{z}{c+\beta} + \frac{u}{d+\beta}$$

$$= \frac{x}{a+\gamma} + \frac{y}{b+\gamma} + \frac{z}{c+\gamma} + \frac{u}{d+\gamma} = 0,$$

and

$$\frac{1}{a+p} + \frac{y}{b+p} + \frac{z}{c+p} + \frac{u}{d+p} = \frac{x}{p},$$

express x, y, z, u in their simplest forms.

4. If $\frac{p}{q}, \frac{p'}{q'}$ be successive convergents to a rational fraction x , prove that $pq' - p'q, pp' - qq', x^2, q^2x^2 - p'^2$ have all the same sign.

$\frac{a}{b}$ is a fraction in its lowest terms, and $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ are the nearest fractions,

below and above, to $\frac{a}{b}$, that have denominators less than b . Prove that $p_1 + p_2 = a, q_1 + q_2 = b$.

5. Sum the following series (in each case to infinity):—

$$(1) 1 + a \cdot \frac{x}{1} + a(a + \kappa) \frac{x^2}{1 \cdot 2} + \dots$$

$$\dots + a(a + \kappa) \dots (a + \overline{n-1} \kappa) \frac{x^n}{\underline{n}} + \dots$$

$$(2) \frac{1}{1+x} + \frac{1}{2} \cdot \frac{1}{1+x^2} + \frac{1}{4} \cdot \frac{1}{1+x^4} + \frac{1}{8} \cdot \frac{1}{1+x^8} + \dots;$$

$$(3) \frac{3}{2} - \frac{1}{2} \cdot \frac{7}{4} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{11}{6} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{15}{8} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{19}{10} - \dots$$

6. Prove the Exponential Theorem, and show that

$$n^4 + n(n-2)^4 + \frac{n(n-1)}{2} (n-4)^4 + \dots = 2^{n-1} n (3n-2).$$

7. If $n, p, q, r, s \dots$ are positive integers and

$$n = p + q + r + s + \dots,$$

prove that $\frac{n!}{\underline{p} \underline{q} \underline{r} \underline{s} \dots}$ is an integer.

If x be a number which cannot be expressed in either of the forms ap^r or $ap^r + \beta p^s$ where p is a prime and a, β, r, s are positive integers of which the two former are less than p , show that $\underline{nx} \div \{\underline{x}\}^n$ is divisible by $\{\underline{n}\}^2$, x and n being positive integers.

8. Every integer which divides the sum of the squares of two relatively prime numbers is itself the sum of two squares.

Every prime number of the form $4n + 1$ is the sum of two squares.

9. If a, b, c are three positive quantities, prove that

$$\frac{a^m + b^m + c^m}{3} > \left(\frac{a+b+c}{3} \right)^m \text{ if } m < 0 \text{ or } > 1,$$

and $\frac{(b^m + c^m)(c^m + a^m)(a^m + b^m)}{8} < \left(\frac{a+b+c}{3} \right)^{3m} \text{ if } 0 < m < 1.$

10. Find the condition that the infinite product

$$(1 + a_1)(1 + a_2)(1 + a_3) \dots,$$

where a_1, a_2, \dots are all positive, should converge to a finite limit.

A_n, G_n are respectively the A.M. and G.M. of the series $a, a+b, a+2b, \dots, a + \overline{n-1}b$, prove that the limit of $\frac{A_n}{G_n}$ is $\frac{e}{2}$, a and b being assumed to be positive quantities.

11. An event has happened, which must have arisen from some one of a given number of causes. Find the probability of the existence of each of the causes.

If a mass of homogeneous fluid be thoroughly shaken and allowed to come to rest again, prove that the chance that no particle of the fluid now occupies its original position is $\frac{1}{e}$.

12. If Δ be the value of a determinant of the n^{th} order, prove that the reciprocal determinant is Δ^{-1} .

If (ab) denote the determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$, prove the identity

$$(ad)^4 (be)^4 + (bd)^4 (ca)^4 + (cd)^4 (ab)^4$$

$$\equiv 2 \{ (bd)^2 (cd)^2 (ab)^2 (ac)^2 + (cd)^2 (ad)^2 (bc)^2 (ba)^2 + (ad)^2 (bd)^2 (ca)^2 (cb)^2 \}.$$

II.

Trigonometry and Theory of Equations.

1. Prove that, if x be any angle,

$$(\sin^2 x)^{\sin^2 x} (\cos^2 x)^{\cos^2 x} < \frac{1}{2};$$

and that, if A, B, C be the angles of an acute angled plane triangle,

$$\tan^4 A \tan^4 B \tan^4 C < 432.$$

2. If $\cosh x$ denote $\frac{1}{2}(\epsilon^x + \epsilon^{-x})$, and if $\cosh \theta \sin 2\phi = 1$, prove that

$$\cosh n \theta \sin^2 2\phi = 2^{n-1} (\sin^{2n} \phi + \cos^{2n} \phi).$$

3. Prove that the distance between the centres of the circumscribed and one of the escribed circles of a plane triangle is $\sqrt{R^2 + 2Rr_1}$; and, assuming any known geometrical fact which may be convenient, show that the distance between the centre of the former circle and the centre of the circle which is inscribed in the triangle whose vertices are the centres of the three escribed circles is

$$R - \frac{a(r + r_1) + b(r + r_2) + c(r + r_3)}{\sqrt{ar_1r_2} + \sqrt{br_2r_1} + \sqrt{cr_1r_2}} \cdot \sqrt{\frac{rr_1r_2r_3}{abc}}.$$

4. Establish the identity, in which n is any positive integer,

$$\frac{\sin n \phi}{\sin \phi} = 2^{n-1} \left(\cos \phi - \cos \frac{\pi}{n} \right) \left(\cos \phi - \cos \frac{2\pi}{n} \right) \dots \\ \dots \left(\cos \phi - \cos \frac{(n-1)\pi}{n} \right).$$

5. Express the series

$$\cos n \theta + \frac{1 \cdot n}{1 \cdot (2n-1)} \cos (n-2) \theta \\ + \frac{1 \cdot 3 \cdot n(n-1)}{1 \cdot 2 \cdot (2n-1)(2n-3)} \cos (n-4) \theta + \dots,$$

which continues to $n+1$ terms, as a coefficient in an expansion; and prove that it is equal to

$$2^n \left\{ \cos^n \theta - \frac{n(n-1)}{2 \cdot (2n-1)} \cos^{n-2} \theta \right. \\ \left. + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} \cos^{n-4} \theta + \dots \right\},$$

carried to $\frac{n}{2} + 1$ or $\frac{n+1}{2}$ terms according as n is even or odd.

6 Having given that

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \text{ ad inf. } = f(x),$$

obtain the sum of the series

$$a_0 + a \cos n \theta + a_n \cos 2n \theta + \dots$$

Sum to infinity $1 + \frac{\cos \theta}{4} + \frac{\cos 2 \theta}{8} + \dots$, expressing the result in a real form.

7. Assuming the expression for $\sin x$ as a product express $\frac{x}{\sin x}$ as a sum of partial fractions.

Prove that

$$\frac{x}{216} = \frac{1}{5 \cdot 7} - \frac{2^2}{11 \cdot 13} + \frac{3^2}{17 \cdot 19} - \frac{4^2}{23 \cdot 25} + \dots$$

8. In a spherical triangle c , $a + b$ and A are given. Find a and b .

In the tetrahedron $VABCD$ it is known that $VA = 2\sqrt{3}$, $VB = AB = 2$, $AC + CB = 4$, that the angles VCA and VCB are supplementary, and that the faces VAB and VAC are at right angles. Find the lengths of the edges AC , CB and VB .

9. ABC is a spherical triangle, D the point of contact with BC of that escribed circle of the triangle which touches it externally, r_1 the spherical radius of that circle, and p_1 the spherical perpendicular from its centre upon AD produced. Prove that, if $\angle BAD > \angle DAC$,

$$\cot \frac{DAC}{2} \tan \frac{BAD}{2} = \frac{\sin r_1 + \sin p_1}{\sin r_1 - \sin p_1} = \tan^2 \frac{ADC}{2}.$$

10. If $p_0x^n + p_1x^{n-1} + \dots + p_n = 0$, have a commensurable root which when expressed in its lowest terms is $\frac{a}{b}$, prove that a must be a factor of p_n and b of p_0 .

Prove that $\sqrt[3]{\sqrt{2} + 1} - \sqrt[3]{\sqrt{2} - 1}$ is incommensurable, but that it cannot be written as the sum of a rational quantity and a quadratic surd.

11. If the cubic equation $ax^3 + bx^2 + cx + d = 0$ have an imaginary root $r(\cos \theta + \sqrt{-1} \sin \theta)$, prove that $(bc \sin^2 \theta - ad \sin^2 3\theta)^2 = \sin^2 2\theta (bd \sin 3\theta - c^2 \sin \theta) (ac \sin 3\theta - b^2 \sin \theta)$; and deduce the condition for a pair of equal roots.

Also, denoting by α, β, γ the roots of the above equation, find the cubic whose roots are $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 1$ and the two allied functions of α, β, γ .

12. Explain, and prove the principle of, Horner's method of approximation to a real root of a rational integral equation.

Find, by Horner's or any other method, the real root of the equation

$$x^5 - 3x^3 + 6x - 6 = 0$$

to four places of decimals.

III.

Pure Geometry.

1. Two equal tangents TP and TQ of a parabola are cut in M and N by a third tangent, show that $TM = QN$.

2. Through a point O is drawn a straight line cutting a conic in AB , and on AB are taken points OD , such that

$$\frac{1}{OC} + \frac{1}{OD} = \frac{1}{OA} + \frac{1}{OB}.$$

Then if MN be the points of contact of tangents from D , and LE those of tangents from C , show that either LM and EN or LN and EM meet in O .

2. If four circles are mutually orthogonal, show that each of the centres is the orthocentre of the triangle formed by the other three, and that all the circles cannot be real.

4. Give a geometrical construction for the points in which any straight line cuts a conic determined by four given points and the tangent at one of them.

Construct the conic which passes through the four points $ABCD$ and is such that AB and CD are conjugate lines with regard to it.

5. Describe a method of projecting orthogonally a given angle into an angle of any size.

A pair of conjugate hyperbolas is described with a given pair of conjugate diameters of an ellipse as asymptotes; show that, if one of the hyperbolas touch the ellipse, so will the other, and that the lines joining the centre to the points of contact will be conjugate diameters of all the conics.

6. The vertices of a triangle ABC are joined to the point O , and ad' are taken on BC so that Oa, Od' bisect the angle BOC ; similarly $\beta\beta', \gamma\gamma'$ are constructed on CA, AB respectively. Show that the points $ad', \beta\beta', \gamma\gamma'$ lie three by three on four straight lines.

Deduce a method of describing a conic, with a given focus, through three given points; and show that there cannot be more than one ellipse satisfying the conditions.

7. Two of the vertices of a parallelogram circumscribing a central conic lie on the directrices, show that the other two are at a constant distance apart.

8. AOB and COD are two diameters of a circle and QE is a chord parallel to AB ; if P be the intersection of CQ and DE or of DQ and CE and if from P be drawn PM parallel to AB to meet CD in M , then $OM^2 = OD^2 + PM^2$.

9. A point P is taken such that the square of the tangent from P to a given circle is equal to the rectangle contained by a given length and by the perpendicular distance from P to a given straight line, show that every position of P is on one or other of two circles.

The circle of similitude of two circles is co-axial with them.

10. Give a geometrical construction for the vertex and picture-plane that two imaginary points (given as the intersections of a straight line and a conic) may be projected into the circular points.

If $ABC \dots$ and $A'B'C' \dots$ are two homographic ranges on a conic show that AA', BB', CC', \dots all touch a conic; and determine the common points of the conics.

11. Find the rectangles of greatest and least perimeter which can be circumscribed to a given ellipse.

12. A circle can be drawn touching the four lines $a\beta, \beta\gamma, \gamma\delta, \delta a$ in $DCBA$ and a circle can also be described through the points $a\beta\gamma\delta$: show that AC and BD are perpendicular.

Obtain the formula connecting the distance between the centres of two such circles and the radii.

IV.

1. Determine the condition that the pair of lines

$$Ax^2 + 2Hxy + By^2 = 0,$$

should be harmonic conjugates with regard to the pair

$$A'x^2 + 2H'xy + B'y^2 = 0.$$

If $u \equiv ax^2 + 2hxy + by^2$, $v \equiv a'x^2 + 2h'xy + b'y^2$, show that the locus of a point, the tangents from which to the conics $u = 1$, $v = 1$ form a harmonic pencil, is given by $cu + dv = 1$, where c and d are constants.

2. Find the equation of the tangent at any point of the conic $l = r(1 + e \cos \theta)$, and deduce the equations of the real or imaginary asymptotes.

3. Show that, in general, from a point P in the plane of a parabola three lines can be drawn normal to the curve. Determine some geometrical criterion for distinguishing between the cases in which these lines are all real from those in which two are imaginary.

The feet of the normals from P are A, B, C . On the sides of the triangle ABC as diameters are drawn three circles. The radical axes of these circles taken in pairs meet in Q . Prove that when the locus of P is a right line, the locus of Q is also a right line.

4. $OP (=k)$ and $OQ (=k)$ are two axes inclined at an angle ω . Find the coordinates of the focus of the parabola which touches OP and OQ at their extremities.

If P and Q vary in position, while PQ remains constant, and $\omega = \frac{\pi}{2}$, show that the locus of the vertex is

$$(x^2 + y)^2 = a^2 x^2 y^2.$$

5. CP, CD are any conjugate semi-diameters of an ellipse. Find the equation of the circle of curvature at P , and show that the length of the common chord of this circle and the ellipse is $2CD \left[1 - \frac{w^2}{p^2} \right]^{\frac{1}{2}}$, where p, w are the perpendiculars from the centre of the ellipse on the tangent at P and the chord.

If a circle be drawn through P, D touching the ellipse at Q , show that the chords of curvature at P, D and the diameter conjugate to CQ meet in a point R , and that the locus of R as P varies is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2 = 8 \left(\frac{xy}{ab} \right)^2.$$

6. The four directrices of a conic are two of the three pairs of chords of intersection of the conic and its director circle. Prove this, and deduce the equation of the directrices of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

7. Find the equation of a conic having double contact with two given conics S and S' .

A parabola Σ has double contact with a given conic S , one point of contact being the fixed point P . Show that the directrix of Σ always touches a parabola which has double contact with the director circle of S , the chord of contact being the tangent at P .

8. A conic touches the sides of a triangle ABC at the points L_1, M_1, N_1 . Show that AL_1, BM_1, CN_1 meet in a point, and find the tangent at the point where AL_1 meets the curve again.

To B, L_1, C a fourth harmonic L_2 is found, and points M_2, N_2 are similarly taken on the other sides. Three other conics S_2, S_3, S_4 are inscribed in the triangle, their points of contact with the sides being $L_2, M_2, N_2; L_3, M_3, N_3; L_4, M_4, N_4$. ρ_1, ρ_2, ρ_3 are the radii of curvature of S_2, S_3, S_4 at the points L_2, M_2, N_2 respectively. Show that

$$a\rho_1^{-1} + b\rho_2^{-1} + c\rho_3^{-1} = 0,$$

where a, b, c are the sides of the triangle.

9. Find the tangential equation of the nine-points circle of a triangle ABC , and deduce the property that it touches the escribed circles of the triangles.

If P, Q, R be the points of contact with the escribed circles, then the corresponding sides of PQR, ABC intersect in points which lie on a straight line.

10. Find the invariant-condition that a conic S should be inscribed in a triangle which is self-conjugate with respect to a second conic S' .

A triangle ABC is self-conjugate with respect to a rectangular hyperbola whose centre is O . Prove that a parabola can be described with focus at O so as to touch the sides of the triangle.

V.

Differential Calculus.

1. Show that

$$\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \dots = \frac{\pi}{2} \left\{ \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}} \right\} - \frac{1}{2}.$$

2. Prove that

$$\theta \cot \theta = 1 - \frac{1}{3} \sin^2 \theta - \frac{2}{3 \cdot 5} \sin^4 \theta - \frac{2 \cdot 4}{3 \cdot 5 \cdot 7} \sin^6 \theta - \dots$$

3. If $x (e^x - 1)^{-1} = B_0 + B_1 x + B_2 \frac{x^2}{2} + \dots$, show that

$$\begin{aligned} \frac{r}{(r+1)(2r+1)} &= B_r + \frac{2r(2r-1)}{3 \cdot 4} B_{r-1} \\ &+ \frac{2r(2r-1)(2r-2)(2r-3)}{3 \cdot 4 \cdot 5 \cdot 6} B_{r-2} + \dots + \frac{2r(2r-1)}{3 \cdot 4} B_1 + B_0 \end{aligned}$$

Expand $x \operatorname{cosec} x$ in terms of the coefficients B_0, B_1, B_2 &c.

4. If u and v be functions of x and y defined by the equations

$$x = u + e^{-v} \sin u \text{ and } y = v + e^{-v} \cos u, \text{ then}$$

$$\frac{du}{dy} = \frac{dv}{dx}.$$

Obtain a partial differential equation of the third order by eliminating the arbitrary functions from

$$z = (x+y)f(x) + (x-y)F(y).$$

5. If u_n be a homogeneous function of the n th order and π denote the operative symbol $x \frac{d}{dx} + y \frac{d}{dy}$, show that

$$\pi(\pi-1)(\pi-2)\dots(\pi-r+1)u_n = \pi(\pi-1)(\pi-2)\dots(\pi-r+1)u_n.$$

6. Investigate the rule for applying Undetermined Multipliers to problems of maxima and minima.

If $\frac{a^2}{x^2} + \frac{b^2}{y^2} + \frac{c^2}{z^2} = 1$, then $x^2 + y^2 + z^2$ has only one singular value, and this is a minimum.

7. Obtain the polar equation of the normal at any point of the curve $r = f(\theta)$.

On the radius vector OP of an equiangular spiral is taken a point Q such that PQ is constant; show that the normal at Q to the locus of Q passes through the centre of curvature of the spiral at P .

8. The origin is a conjugate point on the curve

$$(y^2 - x)^2 (x + y - 2) = x^4.$$

Prove this and trace the curve.

9. Obtain the envelope of the straight line

$$x \cos a + y \sin a = 4a \sin 2a;$$

and show that the evolute of the envelope is the envelope turned through half a right angle and doubled.

10. If PM be the tangent at an ordinary cusp P , and QM be the perpendicular from a neighbouring point Q on PM , find the limit of the ratio of $QM \tan QPM$ to the difference between the arc PQ and the chord PQ , when Q moves up to P .

11. Trace the curve $x^2 + y^2 = a^2$.

Show that the curve has no nodes or points of inflection; and find the maximum and minimum values of the radius of curvature.

12. Obtain the asymptotes of the curve $r \cos^2 2\theta = a (\pi - 4\theta)$, and trace the curve from $\theta = -\pi$ to $\theta = \pi$.

VI.

Problems.

1. Prove that

$$(1) \tan^{-1} \left(a \frac{d}{dx} \right) \sin mx = \frac{1}{2} \log \left(\frac{1 + am}{1 - am} \right) \cos mx;$$

$$(2) \lambda^x \mu^y \nu^z \frac{d}{dx} f(x, y, z) = f(x\lambda, y\mu, z\nu).$$

2. Prove that it is possible so to determine A, B, C that

$$A \{ \sin(\theta - a) - \sin \beta \} \{ \sin(\theta + a) + \sin \beta \} \\ + B \{ \sin(\theta - a) - \sin \beta \} \cos(\theta + a) \\ + C \{ \sin(\theta + a) + \sin \beta \} \cos(\theta - a) + \cos(\theta + a) \cos(\theta - a)$$

may be independent of θ , and that the value of the expression so given will be zero if, and only if, β is an odd multiple of $\frac{\pi}{2}$.

3. If in a given triangle the maximum ellipse be inscribed, and in the ellipse the greatest triangle, and again within the triangle the greatest ellipse, and so on ad infinitum, show that the sum of the areas of the series of triangles = $\frac{1}{3}$ of the area of the original triangle.

4. The tangent to an ellipse at the point P meets the axes in L and M , and the perpendicular from the centre in Y ; if $LY = PM$, show that LM is equal to the sum of the semi-axes.

5. Find values of a, b, c , the sides and hypotenuse respectively of a right-angled triangle, such that $a, c + b, c - b$ may be integral cubes.

6. ABC is a spherical triangle all of whose angles are right angles, and O is any point on the sphere inside the triangle; show that a spherical triangle can be formed by great circles twice as long as the great circles OA, OB, OC , and that the sum of the angles of this triangle will be four right angles.

7. The envelope of a straight line, the intercept on which between the axes is divided in any constant ratio by the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, is a curve of the system

$$(\mu x)^2 + (\nu y)^2 = 1.$$

8. Show that six common tangents can be drawn to the curve $r \cos \theta = a \sin^3 \theta$ and any conic having the axes of coordinates as principal axes. Also, if $\theta_1, \theta_2, \dots$ be the angular coordinates of the points of contact with the curve, prove that

$$\Sigma \tan \theta_1 \tan \theta_2 \tan \theta_3 = 0.$$

9. If, for all values of $a, b, c,$ and $x,$

$$F(ax, bx, cx) = \psi(x) F(a, b, c),$$

prove that $\psi(x)$ is a power of $x,$ and $F(a, b, c)$ a homogeneous function.

10. The tangents from a given point to two given conics are pairs of harmonic conjugates, and an intersection of two common tangents of the conics lies on the polar of the given point with regard to the first of them. Prove that the intersection of their other two common tangents is on the polar of the given point with regard to the second.

11. Two circles, with centres at A and $B,$ touch at $O,$ and a third circle with centre at C is drawn to touch the two given circles; if OC cut the third circle in D and $E,$ show that

$$BA : 2 \cdot DE :: OA \cdot OB : OD \cdot OE.$$

12. Six balls, three of which are red, two white, and one blue, are placed in a row, every arrangement being equally likely. Show that the odds are 4 to 1 against a red, a white, and a blue occurring successively, and 9 to 1 in favour of two balls of the same colour being together.

13. A variable line cuts a nodal cubic in the three points $A, B, C.$ The segment AB subtends a right angle at the node. Prove that, if the nodal tangents are perpendicular, the point C is fixed.

14. Given the centre of an ellipse, two tangents, and the point of contact of one of them, find geometrically the positions and lengths of the major and minor axes.

15. If X_n is the coefficient of x^n in the expansion of $(1 - 2xx + x^2)^{-1},$ prove that

$$\frac{dX_{n+1}}{dx} = \Sigma (X_\alpha X_\beta X_\gamma),$$

the summation including all permutations of the suffixes $\alpha, \beta, \gamma,$ which satisfy the condition $\alpha + \beta + \gamma = n.$

SENIOR MATHEMATICAL SCHOLARSHIP, 1885.

I

Pure Mathematics.

1. Find the equation of the four foci of the conic given by the general equation of the second degree in tangential coordinates.

2. Show how to determine a congruence whose roots are those common to two given congruences to the same prime modulus.

The values of x which are not divisible by 11 but which make

$$x^{10} + x^9 - 3x^8 + 3x^7 + 8x^6 + 3x^5 + 9x^4 - 1$$

divisible by 11 are congruous to primitive roots of 11.

3. Prove that every rational integral equation with real coefficients has a root of the form $a + b \sqrt{-1}$, where a and b are real, or may either of them vanish.

Is it true that a real root must necessarily be either commensurable or the sum of a commensurable quantity and a quadratic surd? Justify your answer.

4. If $P = 0$ be the condition in the coefficients which expresses that a a root of $a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$ and β a root of $b_0 x^n + b_1 x^{n-1} + \dots + b_n = 0$ are connected by a given relation $a = f(\beta)$, then

$$\frac{dP}{da_0} = a \frac{dP}{da_1} = a^2 \frac{dP}{da_2} = \dots = a^n \frac{dP}{da_n}$$

and
$$\frac{dP}{db_0} = \beta \frac{dP}{db_1} = \beta^2 \frac{dP}{db_2} = \dots = \beta^n \frac{dP}{db_n}.$$

Again, if $Q = 0$ express that a, a' , two roots of the first equation, are connected by a given relation $a = \phi(a')$, then $\frac{dQ}{da_0}, \frac{dQ}{da_1}, \dots, \frac{dQ}{da_n}$ form a recurring series whose scale of relation is $aa' - (a + a') + 1$.

5. A quantic $(a_0, a_1, \dots) (x_1, x_2, \dots)^n$ becomes by linear transformation $(A_0, A_1, \dots) (X_1, X_2, \dots)^n$, where a_1, a_2, \dots and X_1, X_2, \dots are equally numerous sets of variables; and there is a function F of the coefficients such that

$$F(A_0, A_1, \dots) = \phi(\lambda, \mu, \dots) F(a_0, a_1, \dots),$$

where ϕ depends only on the constants of the transformation, which may be perfectly general. Prove, without making any assumption as to the form of ϕ , that $F(a_0, a_1, \dots)$ is homogeneous.

6. Any invariant of a binary cubic is a power of its discriminant.

7. If α, β, γ are vectors whose tensors are OA, OB, OC , and if W is the volume of the tetrahedron $OABC$, show that

$$S.\alpha\beta\gamma = -6W,$$

and that $V.\alpha\beta\gamma + V.\beta\gamma\alpha + V.\gamma\alpha\beta = \alpha S.\beta\gamma + \beta S.\gamma\alpha + \gamma S.\alpha\beta$.

Prove also that the tensor of this last sum of vectors is

$$-OA.OB.OC \{ \cos^2 BOC + \cos^2 COA + \cos^2 AOB + 6 \cos BOC \cos COA \cos AOB \}^{\frac{1}{2}}.$$

8. If a curve have its maximum number of double points, the coordinates of any point on it can be expressed as rational algebraic functions of a variable parameter.

How many double points has the curve

$$x = a \cos 3t + b \cos t, \quad y = a' \sin 3t + b' \sin t?$$

9. Prove that the anharmonic ratios of the pencil of four tangents to a cubic from any point upon it are constant.

In the case of the cubic

$$x^3 + y^3 + z^3 + 6xyz = 0$$

every such anharmonic ratio is equal to one of the imaginary cube roots of -1 .

10. If the four straight lines whose coordinates are obtained by giving r successively the values 1, 2, 3, 4 in

$$a_r x + b_r y + c_r z + f_r = 0,$$

be generators of one system of a ruled quadric, prove that all the determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 & f_1 & g_1 & h_1 \\ a_2 & b_2 & c_2 & f_2 & g_2 & h_2 \\ a_3 & b_3 & c_3 & f_3 & g_3 & h_3 \\ a_4 & b_4 & c_4 & f_4 & g_4 & h_4 \end{vmatrix} \text{ vanish.}$$

11. Define elliptic coordinates of a point on an ellipsoid; and show how any small area on the ellipsoid can be expressed by means of them.

Prove that, if through all the points of the perimeter of a small area surrounding a point on the section of the ellipsoid

$$a^{-2} x^2 + (a^2 - h^2)^{-1} y^2 + (a^2 - k^2)^{-1} z^2 = 1$$

by the plane of ax the lines of curvature of the confocal hyperboloids be drawn, the area they cut from the plane of xy shall be in the limit of similar form to the chosen small area.

12. Prove that every twisted cubic curve is a partial intersection of two quadrics; and that the developable of which it is the edge of regression is of class 3, and has the correlative property; which state.

In general, seven of the intercepts on tangents to a twisted cubic between the curve itself and a fixed plane subtend right angles at any fixed point.

II.

Pure Mathematics.

1. Find the area of the pedal (with respect to the centre) of the evolute of an ellipse.

2. If $x_1 = r \cos \theta_1, x_2 = r \sin \theta_1 \cos \theta_2, x_3 = r \sin \theta_1 \sin \theta_2 \cos \theta_3, \dots$

$$x_{n-1} = r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-2} \cos \theta_{n-1},$$

$$x_n = r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-2} \sin \theta_{n-1},$$

show that

$$\begin{aligned} & \iint \dots dx_1 dx_2 \dots dx_n \\ &= \iint \dots r^{n-1} (\sin \theta_1)^{n-2} (\sin \theta_2)^{n-3} \dots \sin \theta_{n-1} dr d\theta_1 d\theta_2 \dots d\theta_{n-1}. \end{aligned}$$

3. Prove that

$$(1 + cn \ 2u) dn^2 u = (1 + dn \ 2u) cn^2 u.$$

If $y = \int_1^x \frac{dx}{\sqrt{2(x-1)(x-2)(x-3)}}$ where $x < 2$ and > 1 ,

express x in terms of $sn \ y$ to a proper modulus.

4. Show that

$$\int_0^{\frac{\pi}{2}} \log(a + \sin^2 x) dx = \pi \log(\sqrt{a} + \sqrt{a+1}) - \pi \log 2,$$

and that $\int_0^{\frac{\pi}{2}} \sin x \log \sin x dx = \log 2 - 1.$

5. Obtain the value of the integral

$$\iiint x^{a-1} y^{b-1} z^{c-1} f(x+y+z) dx dy dz,$$

the integration to include all elements in which $x, y,$ and z are positive and such that $x + y + z < 1$.

The value of

$$\iiint \left(\frac{1-u}{1+u} \right)^k dx dy dz$$

where

$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

the integration extending throughout an octant of the ellipsoid $u = 1$, is

$$\frac{\pi abc}{4} \int_0^1 \left(\frac{1-s^2}{1+s} \right)^k ds.$$

6. The ratio of $\Gamma(x+1)$ to $x^x e^{-x}$ is finite when $x = \infty$.

Prove this, and discuss the convergency of the series

$$1 + \frac{ab}{cd} + \frac{a(a+1)b(b+1)}{c(c+1)d(d+1)} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)d(d+1)(d+2)} + \dots$$

when

$$c + d = a + b + 1.$$

7. Solve the equations

$$(1) \frac{d^2 y}{dx^2} - \frac{2}{x^3} y = a^2 y;$$

$$(2) x^2 \frac{d^2 z}{dx^2} - y^2 \frac{d^2 z}{dy^2} = x^2 - y^2;$$

$$(3) \phi(4x) - 3\phi(2x) + 2\phi(x) = x;$$

and show that if

$$f(y)dx = f(x)dy$$

where

$$f(x) = \sqrt{a + bx + cx^2 + dx^3 + ex^4},$$

then

$$f(x) + f(y) = (x-y) \sqrt{c_1 + e(x+y) + f(x+y)^2}.$$

8. If $u_1 = c_1, u_2 = c_2, \dots, u_n = c_n$ is the complete solution of the auxiliary equations employed in Lagrange's process of solving the partial equation

$$P_1 \frac{dx}{ds_1} + P_2 \frac{dx}{ds_2} + \dots + P_n \frac{dx}{ds_n} = Q,$$

show that every solution of this partial equation is of the form

$$f(u_1, u_2, \dots, u_n) = 0,$$

where f is an arbitrary function.

Solve the equation $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$; and deduce a complete primitive obtainable by Charpit's process.

$$9. \text{ If } x(e^x - 1)^{-1} = 1 - \frac{x}{2} + B_1 \frac{x^3}{2} - B_2 \frac{x^5}{4} + \dots,$$

show that

$$2|2\pi \div (2\pi)^{2n} B_{2n-1} = \left(1 - \frac{1}{2^{2n}}\right) \left(1 - \frac{1}{3^{2n}}\right) \left(1 - \frac{1}{5^{2n}}\right) \dots,$$

where 2, 3, 5, ... are the prime numbers.

10. If $(1 - 2hx + h^2)^{-1} = P_0 + P_1h + P_2h^2 + \dots$,

show that
$$\int_{-1}^{+1} P_m P_n dx = 0$$

and
$$\int_{-1}^{+1} P_n^2 dx = \frac{2}{2n+1},$$

and that
$$\frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2} \left\{ 1 + 5 \left(\frac{1}{2}\right)^2 P_2 + 9 \left(\frac{1.3}{2.4}\right)^2 P_4 + \dots \right\}.$$

11. A periodic function can be expressed as the sum of a series of sines and cosines. Prove this, and obtain the values of the coefficients in terms of the given function.

A periodic zigzag locus consists of the equal sides of an indefinite number of right-angled isosceles triangles ... ABC, CDE, EFG, \dots with their bases ... AC, CE, EG, \dots in the axis of x and the vertices ... BDF, \dots alternately above and below that axis. Find the equation of the locus, taking the axis of y through one of the vertices.

III.

Problems in Pure Mathematics.

1. Tangents are drawn from the vertices of reference to the nodal cubic

$$x(y^2 + z^2) + y(x^2 + z^2) + z(x^2 + y^2) = 6xyz;$$

show that they all touch also the conic

$$4(x^2 + y^2 + z^2) = 7(xy + yz + zx).$$

2. Prove that an infinite number of triangles can be inscribed in the cubic $y^2 - kz^2 x = 0$ and circumscribed to the cubic $y^2 - x^2 z = 0$; their points of contact being in a right line, which envelopes the cubic $ky^2 - x^2 z = 0$.

3. The value of $\iint \{(x^2 + y^2 + z^2)^2 - 4c^2 z^2\}^{-1/2} y dx dy$, the integration in which extends over the area between the two curves

$$(x^2 + y^2 + z^2)^2 - 4c^2 z^2 = a^4, \quad (x^2 + y^2 + z^2)^2 - 4c^2 z^2 = b^4,$$

where $a > b > c$, is

$$\left[\frac{a^2}{c} \log \frac{\sqrt{a^2 + c^2} + c}{\sqrt{a^2 + c^2} - c} + 2\sqrt{a^2 + c^2} \right]_c^a$$

4. The points, which are cusps of cardioids having their vertices at the origin and passing through the point $(0, c\sqrt{2}, 0)$, and which are also centres of curves of the type

$$r^2 = a^2(4 \cos^2 \theta + 3 \sin^2 \theta)$$

that pass through the point $(c\sqrt{3}, 0, 0)$ and have the origin at the vertex $\theta = 0$, have for the equations of their locus

$$\frac{45x^2 - 54c^2}{8x\sqrt{3} - 15c} = \frac{x^2 + y^2 + z^2 + 6c^2}{c} = \frac{9y^2 - 10c^2}{2y\sqrt{2} - 3c}.$$

5. If $\lambda^3 + \mu^3 = a(\lambda^3 - 3\lambda\mu^2)$ be the Boothian equation of a curve, show that the sum of the inclinations to the axis of x of the normals which can be drawn to the curve from any point is a multiple of two right angles.

6. The radii of gyration of a uniform solid about any three rectangular axes through its centre of gravity are k_1, k_2, k_3 . Show that $k_1^2 + k_2^2 + k_3^2$ is the mean square of the distance of two points taken at random within its bounding surface.

7. Denoting $\frac{n(n+1)(n+2)\dots(n+r-1)}{r!}$ by $\phi(n, r)$, show that, if n be a positive even integer,

$$\begin{aligned} \phi(n, r) + \phi(n-2, r) + \phi(n-4, r) + \dots + \phi(2, r) \\ = \frac{1}{2} \phi(n, r+1) + \frac{1}{2^2} \phi(n, r) + \frac{1}{2^3} \phi(n, r-1) + \dots \\ \dots + \frac{1}{2^r} \phi(n, 2) + \frac{1}{2^{r+1}} n \end{aligned}$$

8. A, B, C, \dots are variable magnitudes so associated that when all but one of them, A , are kept constant and equal to B', C', \dots the greatest value which A can have is A' ; and A', B', C', \dots are connected by a linear relation

$$aA' + bB' + cC' + \dots = 0,$$

whose coefficients are independent of the particular values B', C', \dots . Prove that, if A be kept constant and equal to A' , and all the rest except B constant as before, then B' is the greatest or least value which B can assume, according as ab is positive or negative.

9. Those lines whose equations are of the type

$$z = my = nx + p,$$

the intercepts on which between the coordinate planes are cut in any given constant ratio by the ellipsoid

$$ax^2 + by^2 + cz^2 = 1,$$

are tangents to a surface whose equation is of the form

$$Aa^2x^2 + B(b^2y^2 + c^2z^2) = 1.$$

10. If one of two equal small circles of a sphere roll completely round the other, and if $(A), (B), (C)$ be the spherical areas of the roulettes traced on the sphere by A, B, C , three points on the same great circle fixed with regard to the rolling circle, then shall

$$(A) \sin BC + (B) \sin CA + (C) \sin AB = 4\pi (\sin BC + \sin CA + \sin AB).$$

11. If $\psi(x) = \int \frac{dx}{\phi(x)}$ and λ be a constant determined from the equation $\psi(\lambda) = 1 + \psi(h)$, prove that

$$e^{x\psi\left(\frac{d}{dx}\right)} \cdot e^{\lambda x} = e^{\lambda x}.$$

Show also that $e^{kx^{-1} \frac{d^2}{dx^2}} \cdot e^{\lambda x^2} = \frac{1}{(1-2kk^2)^{1/2}} e^{\frac{\lambda x^2}{1-2kk^2}}$.

12. A conic rolls on a straight line, and a surface is generated by rotating the path of a focus about this straight line; show that the sum of the reciprocals of the principal radii of curvature is the same at all points.

13. If C be Euler's constant, and $\omega_1, \omega_2, \dots, \omega_n$ the n^{th} roots of unity, prove that

$$(1) \quad \Gamma(\omega) = \frac{e^{-C\omega}}{\prod_{m=1}^{\infty} \left(1 + \frac{\omega}{m}\right) e^{-\frac{\omega}{m}}};$$

$$(2) \quad [\Gamma(\omega)]^n = \prod_{r=1}^{r=n} \Gamma(\omega - \omega_r) \cdot \prod_{m=0}^{m=\infty} \left[1 - \frac{\omega^n}{(s+m)^n}\right],$$

provided s be a positive quantity.

14. If X^n be the n^{th} Legendre's coefficient expressed as a function of x , prove

(1) that the series $X_1^2 + X_3^2 + \dots + X_n^2$ is divergent;

(2) that $\frac{d X_{\alpha+1}}{dx} = \Sigma (X_{\alpha} X_{\beta} X_{\gamma})$,

where the summation extends over all values of $\alpha\beta\gamma$, which satisfy the relation $\alpha + \beta + \gamma = n$.

15. If $f(x)$ decrease continually between $x = 0$ and $x = a$ show that

$a \int_0^a f(x) \frac{dx}{1-x} + \log(1-a) \int_0^a f(x) dx$ is negative, a being positive and less than unity.

IV.

1. Of all the points within a triangle and in its plane that at which the potential of the perimeter is least is the centre of the inscribed circle.

2. Define a centrobaric body, and prove that if a body is centrobaric relatively to any one attracting mass, it is centrobaric relatively to every other.

Is the earth centrobaric? Discuss this question in connection with the phenomena of Precession and Nutation.

A solid sphere has at any point P a density which varies as OP^{-2} , where O is an external point. Prove that the centre of mass is at the centre of the section by the polar plane of O .

3. A heavy uniform elastic string hangs in equilibrium over two smooth pegs in a horizontal plane. Prove that, if the tangent at any point at a distance x from the line of symmetry of the catenary make an angle ψ with the horizon, then

$$\log \frac{1 + \sin \psi}{\cos \psi} + k \tan \psi = \frac{x}{c}$$

(where c and k are constants), and that the free ends descend to a depth

$c \left(1 + \frac{k}{2} \right)$ below the vertex of the catenary.

An elastic string is formed of such material that the density ρ is $\mu x^{\frac{2}{a}}$, and the modulus of elasticity $\lambda \rho \frac{a-x}{x}$, at the point whose distance from one end of the string when unstretched is x , a being the natural length of the string. Prove that if the string be hung up by this end it will be uniformly stretched.

4. The canonical form of the displacement of a rigid body is a twist about a screw, and is unique.

If p , q are the pitches of two screws, a the angle and d the shortest distance between their axes, prove that, if $p + q = d \tan a$, the screws are reciprocal, i.e., that a wrench on either will not disturb a body free to twist about the other.

5. A heavy particle is projected horizontally along the inside of a surface of revolution whose axis is vertical. Find the initial curvature of its path.

The surface being a prolate spheroid, of semi-axes a and b , and the point of projection any point in the equator, prove that, if the particle just reach the level of the focus before returning, the initial radius of curvature of the

path is $\frac{2b^2}{\sqrt{a^2 + 3b^2}}$.

6. A particle is describing an ellipse about a centre of force in the focus, and the absolute force is diminished by $\frac{b^3}{2a^3}$ times its amount at the time when the true anomaly is θ . Show that it is an even chance that the particle's new orbit is a hyperbola, if all values of θ are supposed equally likely.

7. What is the principle of "varying action"? Give a general account of the way in which it may be applied to questions of stability and instability of motion.

8. Find the amount of work done by an impulse applied to a particle moving in any manner in space.

If a system be at rest and be set in motion by impulses at given points so that the motions of these points are prescribed, then the vis viva of the subsequent motion is less than that of any other hypothetical motion the system could take, in which these points have the prescribed motion.

Two uniform rods AB, BC , of equal density and section, are freely jointed at B , and lie at rest on a smooth table. The point A is suddenly moved in a direction perpendicular to AB with velocity V ; prove that, if $AB = 2a$, $BC = a$, and the angle ABC is right, then the initial angular velocity of AB is $\frac{3V}{5a}$.

9. A sphere of radius a moves on a perfectly rough surface under the action of forces whose resultant passes through its centre. Write down all the equations of motion.

If the surface be a right cylinder, the intrinsic equation of whose cross section is $\frac{ds}{d\phi} = b \sin \sqrt{\frac{2}{7}\phi} + a$, and there be no external forces, show that

the path traced out by the point of contact becomes a parabola when the cylinder is developed on a plane.

10. The motion of a body about a fixed point O under the action of no forces being represented by the rolling of Poincot's ellipsoid on an invariable plane, a parallel to the invariable line is drawn at P the point of contact, so as to meet the principal planes in the points L, M, N respectively. Prove that, for all positions of P , LP, MP, NP have constant values, which are proportional respectively to the squares of the semi-axes of the ellipsoid.

Hence, or otherwise, prove that the motion of the body may be represented by the rolling of an hyperboloid fixed in the body on a right circular cylinder, whose axis is the invariable line, the squares of the axes of the hyperboloid being inversely proportional to $G^2 \sim AT, G^2 \sim BT, G^2 \sim CT$.

V

Applied Mathematics.

1. Define principal points and nodal points of a system of media separated by spherical refracting surfaces with a common axis; and prove that the distance of the first nodal point from the first focus is equal to the distance of the second principal point from the second focus.

Find also, by geometrical construction, the principal points of the system formed by the combination of two coaxial systems with given foci, principal points, and nodal points.

2. Light polarized in the plane of incidence falls on a refracting surface of glass. Find the intensities of the refracted and reflected streams.

A beam of circularly polarized light is used to produce Newton's rings by incidence on one lens which is in contact with another; and the angle of incidence is the polarizing angle. Prove that the maximum intensity of light in the innermost reflexion ring is to that of the incident light as

$$8\mu^4(1-\mu^2)^2:(1+\mu^2)^2(1+\mu^4)^2$$

approximately, where μ is the refractive index of glass.

3. If ω, ω' be the angles which any direction makes with the optic axes of a crystal, and if u, v be the wave velocities in that direction, then

$$\sin \omega \sin \omega' = \frac{u^2 \sim v^2}{a^2 - c^2}.$$

Also prove that if one of the planes of polarization of the wave front which is at right angles to the given direction meets the plane of the optic axes in a line dividing the acute angle between them into two parts a, a' , then

$$\sin^2 \omega + \sin^2 \omega' = \frac{u^2 \sim v^2}{a^2 - c^2} \cdot \frac{\sin^2 a + \sin^2 a'}{\sin a \sin a'}.$$

4. The potential of a solid homogeneous ellipsoid of small ellipticity at an external point is $\frac{M}{\rho} + \frac{A+B+C-3I}{2\rho^3}$ approximately, the error being of the order of the square of the eccentricity of the section of greatest and least axes.

A particle under the attraction of such an ellipsoid is constrained to describe a circle concentric and coplanar with a principal section of that ellipsoid; and initial circumstances of its motion are given. Express the period of its revolution as a multiple of a complete elliptic integral.

5. A harmonic distribution of density on a spherical surface produces potential which is the allied spherical solid harmonic of positive degree at any point within, and that of negative degree at any point without, the sphere.

A given superficial distribution of density on a sphere of radius a is such that at any point of its surface the potential due to it is $4\pi k \cdot \mu^n$, where μ is the sine of the latitude of the point and n is a positive integer. Find the surface density at any point of the sphere and the potentials at points interior and exterior to it.

6. Determine the aberration of a star in declination and right ascension.

There is a small error in the used value k of the constant of aberration, which should be k' . Two stars are observed, and the values of their aberrations in declination as calculated are β_1 and β_2 . Also the observed difference of their declinations is d , and the real difference d' . Prove that

$$\frac{d' \sim d}{\beta_1 \sim \beta_2} = \frac{k'}{k}.$$

7. Assuming the differential equation of the moon's latitude, and that to the second order

$$T = -\frac{1}{2} k^2 \omega^2 \cdot m^2 \sin \psi \text{ and } P - Ss^{-1} = -\frac{1}{2} k^2 \omega^2 \cdot m^2 (1 + \cos \psi),$$

where $\psi = (2 - 2m)\theta - 2\beta$, calculate s to the second order.

Prove, also, that the excess of the moon's longitude in the plane of its orbit over θ , which is that in the ecliptic, is, to the third order,

$$\frac{1}{2} k^2 \sin 2\phi + \frac{1}{2} m k^2 \cos \phi \sin(\psi - \phi),$$

in which $\phi = g\theta - \gamma$ and ψ has the same meaning as before.

8. Prove the formulæ of the planetary theory

$$h \sin i \frac{d\Omega}{dt} = Zr \sin(\theta - \Omega) = \frac{dB}{dt},$$

where Z is the component of disturbing force at right angles to the plane of the instantaneous ellipse, h is twice the rate of description of sectorial area in that ellipse, and B, r, θ, i, Ω have their usual meaning.

A particle, originally describing an ellipse about a focus, is disturbed by a small constant force acting always at right angles to the plane of that ellipse. Prove that the motion of the plane of its instantaneous ellipse at any subsequent time is such as to satisfy

$$\sin(\theta - \Omega) \frac{di}{dt} = \sin i \cos(\theta - \Omega) \frac{d\Omega}{dt}.$$

9. Integrate the equations determining secular variations of eccentricity and perihelion longitude of two mutually disturbing planets, viz.

$$\frac{de}{dt} = -\frac{m'na^2a'}{4\mu} D_2 e' \sin(\varpi - \varpi'),$$

$$e \frac{d\varpi}{dt} = \frac{m'na^2a'}{4\mu} \{D_1 e - D_2 e' \cos(\varpi - \varpi')\},$$

and two similar equations for the second planet.

The instantaneous ellipse of the motion of a planet disturbed by a single other planet was at a certain epoch circular. Prove that the perihelion progredes or regredes at a constant angular rate.

10. Prove that an ellipsoid of three unequal axes is a possible form of equilibrium of a mass of gravitating fluid which rotates with angular velocity ω about a fixed axis; and show that, if $a, a \frac{dn}{du}, a \frac{cn}{u'}$, be the semi-axes of the ellipsoid, and M the mass in terms of the astronomical unit, then

$$a^3 n^2 u' \cdot \omega^2 = 3M \int_0^u (cn^2 u - cn^2 u') \frac{sn^2 u}{cn^2 u} du$$

$$= 3M \int_0^u \left(\frac{cn^2 u}{dn^2 u} - \frac{cn^2 u'}{dn^2 u'} \right) \frac{sn^2 u}{cn^2 u} du,$$

and
$$\int_0^u u \frac{d}{du} \left(\frac{sn^2 u cn^2 u}{dn^2 u} \right) du = 0.$$

11. Investigate the motion of any point of a fluid due to a given single vertex of strength m .

Three parallel vortices have strengths $m, m, -2m$ respectively. Prove that the consequent stream lines are bicircular quartics.

VI.

Problems in Applied Mathematics.

1. A sphere of radius a is placed upon an imperfectly rough horizontal plane, and is started in such a manner that initially the centre of mass and the point of contact are moving at right angles with velocities in the ratio 2:7. Show that the centre of the sphere will describe the parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum, before proceeding to describe a straight line, if the velocity of projection be properly chosen.

2. A particle, describing the lemniscate $r^2 = a^2 \cos 2(\theta - \omega)$ under the attraction of a central force μ (distance) $^{-7}$ at the pole, is disturbed by a small force f tending to the same point. Prove that

$$\frac{1}{a} \frac{da}{dt} \cot 2(\theta - \omega) = - \frac{d\omega}{dt} = f \frac{a^2}{2} \sqrt{\frac{3}{\mu}} \{\cos 2(\theta - \omega)\}^{\frac{1}{2}}.$$

3. If $2T = f(x)x^2 = 2\phi(x) + C$ be the equation of energy for a system having a single degree of freedom, show that, for the position of equilibrium $x = a$, given by $\phi'(a) = 0$, to be thoroughly stable, it is necessary and sufficient that the earliest derived function $\phi^{(n)}(x)$ which assumes a non-vanishing value $\phi^{(n)}(a)$ when $x = a$ be one of even order, and that $\phi^{(n)}(a)$ and $f(a)$ have different signs; and prove that the time of fall from the position $x = a + h$, where h is small, to the position of equilibrium in question is

$$\frac{\Gamma\left(\frac{1}{n}\right) \sqrt{\pi}}{h^{\frac{n}{2}-1} \Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} \cdot \sqrt{-\frac{\Gamma(n)}{2n} \cdot \frac{f(a)}{\phi^{(n)}(a)}}.$$

4. A bar of finite length l is heated so that its two ends are at temperature zero, and is placed in a medium whose temperature is constantly zero. If initially the temperature is $v = c \sin \frac{\pi x}{l}$, prove that

after a time t it is $ce^{-kx - \frac{k\pi^2 t}{l^2}} \cdot \sin \frac{\pi x}{l}$, h, k being the coefficients of the equation of motion of heat

$$\frac{dv}{dt} = k \frac{d^2v}{dx^2} - hv.$$

5. An infinite volume of an elastic fluid is arranged in spherical layers of equal density about a point. The particles of the fluid attract each other with a force varying as the distance. If p be the pressure per unit area at the centre, ρ the density at the centre, m the mass of the gas, and f the force m would exert on unit mass at unit distance, then $m^2 p^2 = 32\pi^2 \rho^2$.

6. A smooth circular tube of radius a is caused to slide uniformly on a smooth horizontal table, in such a manner that its centre is always at a fixed point. A particle is in motion in the tube, and T is the time of a small oscillation about a position of equilibrium. Show that T has two values, according as the angular velocity of the plane of the tube about the

vertical is $\geq \sqrt{\frac{g}{h}}$, where h is the height of O above the table; and prove

that, in the former case, $ghT^2 \sin^2 \beta = 4\pi^2 a^2 \cos \beta$, if β be the angular distance of the point of equilibrium from the point of contact with the table.

7. A hollow sphere of external and internal radii a and b and smooth internal surface is very nearly full of water, and rolls under gravity down an inclined plane. Find the pressure of the water at any point; and prove that the values of the whole fluid pressure on the containing surface, for different inclinations of the plane, are inversely proportional to the various diameters of a certain ellipse.

8. A particle moves on the surface of a sphere of unit radius under the action of a force tending to the pole. If the co-ordinates of its position at any time be the co-latitude ρ and longitude θ , prove that, if the particle be projected perpendicularly to the meridian in latitude 45° with velocity $\sqrt{2\mu}$, it will describe the curve $\cos \theta = \tan \rho$, if $2\mu \operatorname{cosec}^2 \rho \cot^2 \rho$ be the component of the force resolved along the tangent to the meridian.

9. An inelastic string of length a is at rest in the form of the plane curve whose intrinsic equation is $s = a(1 - \epsilon^{-\psi})$, one end being at the origin and both ends being free. If the curve receive an impulse along the tangent at the origin, prove that each element of the string will begin to move at the same angle to its tangent.

10. A string rests on the upper side (which is entirely concave) of a smooth curve in a vertical plane. The mass m per unit length of the string varies from point to point, the law of variation and the form of the curve being such that the pressure varies directly and the tension inversely as m . Prove that the length and the mass of the portion of the string between the lowest point and the point where the tangent makes an angle ϕ with the horizon are respectively constant multiples of

$$(1 - \cos a)^2 \left(\frac{1}{\sin a} \frac{d}{da} \right)^2 \left\{ \frac{1}{\sin a} \log \frac{\sin \frac{1}{2}(a + \phi)}{\sin \frac{1}{2}(a - \phi)} \right\}$$

and $(1 - \cos a) \left(\frac{1}{\sin a} \right) \frac{d}{da} \left\{ \frac{1}{\sin a} \log \frac{\sin \frac{1}{2}(a + \phi)}{\sin \frac{1}{2}(a - \phi)} \right\},$

where a is the inclination to the horizon of the curve's asymptote.

11. A regular solid, entirely immersed in a liquid whose density varies as the depth, is freely moveable about its centre of mass which is fixed. Show that it will be in equilibrium in all positions.

12. In a given crystal it is known that the ratio of the greatest to the least wave-velocity is λ . Prove that the angle between the optic and ray axes in the same octant of the wave-surface cannot exceed $\sin^{-1} \frac{\lambda - 1}{\lambda + 1}$

13. A small conical pencil is refracted into a uniaxial crystal through a face cut at right angles to the axis. Prove that the connectors of the two primary and the two secondary foci of the pencils into which it is separated are parallel.

14. A sphere, whose centre of mass and centre of figure do not coincide is moving on a perfectly smooth horizontal plane. If at any instant the motion of the centre of mass is vertical, with a velocity due to falling to the centre of figure, then the sphere will not remain on the plane during the whole of the subsequent motion.

15. A uniform incompressible fluid is in steady motion in two dimensions under the action of conservative forces. Prove that the velocities u, v at a point (x, y) parallel to the axes are connected by the relation

$$v \nabla^2 u = u \nabla^2 v.$$

JUNIOR MATHEMATICAL SCHOLARSHIP, 1911.

I.

1. Prove that

$$1 + 2m + 2^2 \frac{m(m+1)}{2} + \dots + 2^n \frac{m(m+1) \dots (m+n-1)}{n} \\ = \frac{m(m+1) \dots (m+n)}{n} \left[\frac{1}{m} + {}^nC_1 \cdot \frac{1}{m+1} + \dots + {}^nC_n \cdot \frac{1}{m+n} \right],$$

for any value of m and for positive integral values of n .

$$\begin{aligned}
 2. \text{ If } & x_n(u_n + u_{n-1}) + x_{n-1}u_nu_{n-2} = 0 \\
 & x_n + x_{n-1}(u_{n-1} + u_{n-2}) + x_{n-2}u_{n-1}u_{n-3} = 0 \\
 & x_{n-1} + x_{n-2}(u_{n-2} + u_{n-3}) + x_{n-3}u_{n-2}u_{n-4} = 0 \\
 & \dots \\
 & x_2 + x_3(u_2 + u_1) + x_1u_3 = 0 \\
 & x_2 + x_1 = u_1,
 \end{aligned}$$

prove by induction, or otherwise, that $x_1 = u_1 + u_2 + u_3 + \dots + u_n$.

3. Show that any continued fraction, the constituents of which are all integers, may be written either with an even or an odd number of quotients.

Prove that the necessary and sufficient condition that a continued fraction of n quotients should be reciprocal (i.e., of the form

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3} + \dots + \frac{1}{a_2} + \frac{1}{a_1}} \text{ is that } P_{n-1} = Q_n.$$

Prove that if P, Q and $\frac{Q^2 + 1}{P}$ are integers and P is greater than Q , then $\frac{P}{Q}$ is equal to a reciprocal continued fraction with an even number of quotients.

4. If u_n is the n^{th} term of the recurring series

$$1, 7, 41, 239, \dots,$$

prove that (1) $u_{2n} = M(3) + (-1)^n 2^{2n-1}$
 $u_{2n+1} = M(3) + (-1)^n 2^{2n};$

(2) $u_n = M(2n - 1) + 1$, when $(2n - 1)$ is a prime number,

5. If x, p, q be all positive and $x < 1$, then

$$\frac{1 - x^p}{p} \text{ is } > \text{ or } < \frac{1 - x^q}{q} \text{ according as } p \text{ is } < \text{ or } > q,$$

p and q being commensurable.

Each of a group of men of the same age is entitled to an annuity of λa payable at the end of each year which he survives. Prove that the present value of any man's expectation is less than the present value of an annuity of $\lambda a \cdot \frac{m}{n}$ for n years, where m is the average of complete years subsequently survived by the group, p is the probability of surviving the first year, and n is any integer less than $\frac{m}{p}$.

6. If two solutions of the equation

$$ax^2y^2 + hxy(x + y) + g(x^2 + y^2) + bxy + f(x + y) + c = 0$$

are $x = y_1, y = z_2$ and $x = z_1, y = z_2$,

prove that $x = z_2, y = z_1$ is also a solution provided that

$$a - g^2 = fh - bg.$$

Prove also that with this condition

$$\Pi(ax^2 + 2hx^2 + (2g + b)x^2 + 2fx + c)$$

$$= (abc + 2fgh - af^2 - bg^2 - ch^2)(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2,$$

the product being taken for $x = x_1, x_2$ and x_3 .

7. Explain what is meant by uniform convergence.

Prove that if z lies between $-\pi$ and π the series

$$\frac{z}{2m^2} - \frac{\sin z}{m^2 + 1^2} + \frac{\sin 2z}{2(m^2 + 2^2)} - \frac{\sin 3z}{3(m^2 + 3^2)} + \dots$$

and those obtained by differentiating this series term by term, once and twice respectively, are uniformly convergent. Hence, or otherwise, show that the sum of the series within these limits is $\frac{\pi \sinh mz}{2m \sinh m\pi}$.

8. Prove that the series

$$1 + nx + \frac{n^{(2)}}{2} \cdot \frac{x^2}{1+x} + \frac{(n+1)^{(3)}}{3} \cdot \frac{x^3}{1+x} + \frac{(n+1)^{(4)}}{4} \cdot \frac{x^4}{(1+x)^2} + \frac{(n+2)^{(5)}}{5} \cdot \frac{x^5}{(1+x)^2} + \dots,$$

where $m^{(r)} = m(m-1)(m-2)\dots(m-r+1)$, is absolutely convergent if $\left| \frac{x^2}{1+x} \right| < 4$, and prove also that for all values of x for which $|x| < 1$ the sum is equal to $(1+x)^n$.

9. Prove that the positive root of the equation $ax^{n+1} - (a+1)x^n + 1 = 0$, other than unity, lies between unity and $\frac{a+1}{a}$ if n is greater than a , and between $\sqrt[n]{\frac{1}{a+1}}$ and unity if n is less than a , a being any positive quantity and n a positive integer.

Prove also that the root is equal to $\frac{a+1}{a} \sin^2 \theta$ where θ satisfies the equation

$$n \log \sin \theta + \log \cos \theta = \frac{n}{2} \log a - \frac{n+1}{2} \log (a+1).$$

10. Prove that every rational symmetric function of the roots of an algebraic equation can be expressed rationally in terms of the coefficients.

If the roots of $x^4 + qx^3 - rx + s = 0$ are $\alpha, \beta, \gamma, \delta$, prove that

$$\Sigma \alpha^2 (\beta - \gamma) (\gamma - \delta) (\delta - \beta) = 2qr \Sigma \beta\gamma\delta (\beta - \gamma) (\gamma - \delta) (\delta - \beta).$$

II.

1. Three fixed circles pass through a point O and meet again in pairs in A, B, C . Variable points P, Q, R are taken one on each circle, so that QE, RP, PQ pass through A, B, C respectively. Prove that the circle through P, Q, R cuts orthogonally a fixed circle through O .

2. Prove that any three coplanar circles can be stereographically projected into three great circles on a sphere, and state the condition that the projection may be real.

C_1, C_2, C_3 are three coplanar circles. A circle S_1 is drawn so as to be orthogonal to C_1 and coaxial with C_2 and C_3 , and S_2 and S_3 are similarly obtained. Prove that S_1, S_2 and S_3 are coaxial.

3. The bisectors of the angles of a triangle ABC meet the opposite sides in D, E, F . The middle points of AD, BE, CF are K, L, M respectively. Prove that the area of the triangle KLM is

$$\frac{r^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{2 \cos \frac{B-C}{2} \cos \frac{C-A}{2} \cos \frac{A-B}{2}}$$

4. O is the orthocentre and DEF the pedal triangle of a spherical triangle ABC . Prove that

$$\frac{\sin EF}{\tan a \tan OA} = \frac{\sin FD}{\tan b \tan OB} = \frac{\sin DE}{\tan c \tan OC}$$

5. P is a point within a spherical triangle ABC , and x, y, z are the perpendicular arcs from P to the sides. Prove that if $\cos AP + \cos BP + \cos CP$ has a maximum value m , then $\sin a \sin x = \sin b \sin y = \sin c \sin z$ and $m^2 = 3 + 2z \cos a$.

6. Prove that the envelope of the asymptotes of the rectangular hyperbolas which have three-point contact with a given curve at a given point is a three-cusped hypocycloid, and that the envelope of the axes is obtained by turning this hypocycloid through two right angles.

7. A, B, C are three conics, and P a fixed point. The envelope of the polars of P with respect to the conics which touch the common tangents of A and B is denoted by (A, B) . Prove that (A, B) is a conic, and that (A, B) , (B, C) , and (C, A) are inscribed in a triangle.

8. A variable conic through four fixed points $ABCP$ meets a fixed straight line l in X and Y . (PQ, XY) is a harmonic range on the conic. Prove that the locus of Q is a conic through A, B, C .

If $ABCP$ are kept fixed while l always passes through a fixed point, prove that the loci of Q pass through another fixed point.

9. Prove that the necessary and sufficient condition that a given homology is harmonic is that the vanishing lines coincide.

If two triangles are in harmonic homology, prove that the meets of their sides with the conic with respect to which they are reciprocal from six pairs of points of an involution range on the conic.

10. If two coplanar figures are correlative (that is, if to a point in one figure corresponds a unique line in the other, and conversely), prove that the locus of a point of one figure which lies on its correlative line in the other figure is a conic.

If the point P describes a conic which is met by the correlative line p in Q_1, Q_2 , prove that the envelope of PQ_1, PQ_2 is a curve of the fourth class, which degenerates into two coincident conics when the figures are reciprocal with respect to a conic.

III

1. If u, v, w be three functions of x, y , and z such that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

prove that there are also three functions of x, y , and z , U, V , and W , such that

$$u = \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}, \quad v = \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x}, \quad w = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$$

and

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0.$$

2. If a determinant is such that the sum of the constituents of each row and column is zero, prove that all its first minors are equal.

3. If P is any point on the conic $yz + zx + xy = 0$, and D a point on the side BC of the triangle of reference such that PD and BC are conjugate with respect to the conic

$$x^2 + y^2 + z^2 - 4yz - 4zx - 4xy = 0,$$

and E and F are points with similar properties with respect to CA and AB respectively, prove that D , E , and F are collinear.

4. $ABCD$ and $A'B'C'D'$ are two quadrangles; show that there are three points at which the corresponding vertices of these quadrangles subtend involution pencils.

5. ABC is the triangle of reference and D a point whose homogeneous coordinates are 1, 1, 1. Write down three linear transformations of the form

$$\begin{aligned} x' &= l_1x + m_1y + n_1z \\ y' &= l_2x + m_2y + n_2z \\ z' &= l_3x + m_3y + n_3z \end{aligned}$$

which have respectively the properties (1) of transforming A to B , B to C , C to A and leaving D unaltered; (2) of transforming C to D and D to C whilst A and B remain unaltered; (3) of transforming A to B , B to C , C to D , and D to A .

6. Prove that the system of cubic curves drawn through the node, and seven fixed points on a nodal cubic all touch one another at the node.

7. The side and angle of an equilateral spherical triangle are connected by the relation

$$\tan^2 A \left(1 + \sec^2 \frac{a}{2} + \sec^4 \frac{a}{2} \right) = \sec^2 \frac{a}{2};$$

prove that, if small circles are described on the sides of this triangle as diameters, an infinite number of spherical triangles can be described so as to have their vertices on these circles and their sides passing through the vertices of the equilateral triangle; and prove that the area of any one of these together with that of the equilateral triangle is one-fourth that of the sphere.

8. Through three points on a circle taken in pairs, circles are described to cut orthogonally a circle which touches the first circle; prove by inversion (or otherwise) that if the three circles intersect at angles A , B and C then $\tan S$ is equal to the sum of two of the quantities $\tan(S - A)$, $\tan(S - B)$, $\tan(S - C)$, where $2S = A + B + C$.

9. If X , Y , Z , and W are four homogeneous quadratics in x , y , z , the trilinear coordinates of a point, and if there is an identity connecting them of the form

$$aX^2 + bY^2 + \dots + 2hXY + \dots \equiv 0$$

but no linear identity connecting them, prove that the conics

$$X = 0, \quad Y = 0, \quad Z = 0, \quad W = 0$$

pass through two common points.

10. If u , y , and z are functions of x and v such that

$$\begin{vmatrix} \frac{\partial^2 u}{\partial x \partial v} & \frac{\partial^2 y}{\partial x \partial v} & \frac{\partial^2 z}{\partial x \partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = 0$$

and

$$x^2 + y^2 + z^2 = 1,$$

prove that in general

$$\frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} = 0.$$

11. Write down two polynomials, of the n^{th} and $(n-1)^{\text{th}}$ degrees respectively in x , which have no common factor, and are such that the sum of their squares is the n^{th} power of $x^2 + 1$.

12. Prove that $(by - 1)(c - a)t = (a - b)\{(x - c)\}$ is a Pascal line for the points on the hyperbola $x = t, y = t^{-1}$ whose parameters are $0, a, \infty, b, t, c$; and find the coordinates of a point on this line, through which there pass two other Pascal lines of the hexagon.

IV.

1. Find the most general form of the equations $ax^2 + bx^2y + dxy^2 + ey^3 = 0$ and $a'x^2 + b'x^2y + d'xy^2 + e'y^3 = 0$ in order that the lines which they represent should have a common pair which are the double lines of the involution formed by the remaining pairs.

2. Show that the area of the triangle formed by three points whose areal coordinates are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is

$$\Delta \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

where Δ is the area of the triangle of reference.

Triangles of constant area are circumscribed to a conic S such that two of the vertices lie on a fixed tangent L ; prove that the locus of the third vertex is a quartic curve having four-point contact with S at the point of contact of the tangent to S parallel to L . Prove also that two of the asymptotes of the quartic are those of S and that the quartic has a cusp at infinity on L .

3. A system of conics touches the lines AB and AC at B and C respectively; prove that for each conic there can be found one pair of lines through A such that the normals at their points of intersection with the conic are concurrent, and that the locus of the point of concurrence is a hyperbola one of whose asymptotes is perpendicular to BC .

4. Prove that all conics which have the same director-circle and a given pair of conjugate lines can be inscribed in a rectangle whose sides are parallel to the bisectors of the angle between the given pair of conjugate lines.

5. Explain how the analytical solution of the problem: "To describe a conic to pass through three given points and to touch two given straight lines" may be obtained by finding the common tangents of a certain pair of conics through the given points. Four conics are drawn to pass through three points $A, B,$ and C , and to touch two straight lines P and Q ; prove that P and Q are conjugate lines for all conics which touch the four chords of contact of the four conics with P and Q .

6. Prove that the locus of the centre of the circle which touches the parabola $y^2 = 4ax$ and passes through the focus, is the cubic

$$2(x + a)^2 = 27a \{(a - x)^2 + y^2\}.$$

Two circles of radii r_1 and r_2 touch a parabola and touch each other at the focus; prove that, if l is the latus-rectum, $r_1^{-1} - (r_1 r_2)^{-1} + r_2^{-1} = 3l^{-1}$.

7. Find the invariant relation between two conics S and S' when triangles inscribed in S are self-conjugate with respect to S' , and show that the centre of S' must lie on S when the former has a pair of conjugate diameters parallel to the asymptotes of the latter.

Prove that in general the locus of the centroids of triangles inscribed in S , self-conjugate with respect to S' , is a conic similar and similarly situated to S , but that if the centre of S' is on S the locus reduces to a straight line.

8. The surface $\frac{x^2}{a^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$ is cut by any tangent plane of the cone

$$\frac{x^2}{a^2 - \delta^2} + \frac{y^2}{\beta^2 - \delta^2} + \frac{z^2}{\gamma^2 - \delta^2} = 0,$$

and the equation of the section referred to any rectangular axes in its plane is

$$ax^2 + 2hxy + by^2 + 2gz + 2fy + c = 0.$$

Prove that

$$\frac{(ab - h^2)^2}{(abc + 2fgh - af^2 - bg^2 - ck^2)^2} = \frac{\delta^2}{a^2\beta^2\gamma^2}$$

9. If the normals at the two points in which a straight line meets a quadric intersect, prove that the normals at the two points in which the line meets any confocal also intersect. Hence show that the line of intersection of two confocal quadrics is a line of curvature on each.

10. The polar planes of a point P with respect to two conicoids S and S' meet in a line L ; prove that, if P moves on a straight line, the lines L are, in general, generators of the same system of a hyperboloid of which the generators of the opposite system are the polar lines of the locus of P with respect to the conicoids which pass through the curve of intersection of S and S' .

V.

1. If $f(x)$ and $F(x)$ are finite, one-valued and continuous, and $f'(x)$ and $F'(x)$ exist, at every point of the interval $a \leq x \leq b$, then

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(\xi)}{F'(\xi)} \text{ where } a < \xi < b,$$

provided that $f'(x)$ and $F'(x)$ are not infinite, and $F'(x)$ not zero, at any point in the interval.

If also $\frac{f'(x)}{F'(x)}$ is finite, positive, and increasing throughout the interval,

prove (by consideration of the function $\frac{f(x) - f(a)}{F(b) - F(a)} - \frac{f(a)}{F(a)}$, or otherwise) that

$$\left| \frac{f(\eta)}{F(b)} - \frac{f(a)}{F(a)} \right| = \left| \frac{f(a)}{F(a)} - \frac{f(a)}{F(a)} \right| \text{ where } a < \eta < b.$$

2. If $y = (x + \sqrt{1+x^2})^n$ prove that $(1+x^2)y_{n+1} + (2n+1)xy_{n+1} = 0$, where $y_n = \frac{d^n y}{dx^n}$, and deduce that $y_{n+1} = \frac{n.1.3.5 \dots (2n-1)}{(1+x^2)^{n+1}}$.

Evaluate $\left(\frac{1}{\cosh x} \frac{d}{dx}\right)^{n+1} \cosh nx$ and $\left(\frac{1}{\cosh x} \frac{d}{dx}\right)^{n+1} \sinh nx$.

3. Prove that

$$\begin{vmatrix} 1, & 1, & 1, & \dots & 1 \\ \frac{\partial}{\partial x_1}, & \frac{\partial}{\partial x_2}, & \frac{\partial}{\partial x_3}, & \dots & \frac{\partial}{\partial x_n} \\ \frac{\partial^2}{\partial x_1^2}, & \frac{\partial^2}{\partial x_2^2}, & \frac{\partial^2}{\partial x_3^2}, & \dots & \frac{\partial^2}{\partial x_n^2} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial^{n-1}}{\partial x_1^{n-1}}, & \frac{\partial^{n-1}}{\partial x_2^{n-1}}, & \frac{\partial^{n-1}}{\partial x_3^{n-1}}, & \dots & \frac{\partial^{n-1}}{\partial x_n^{n-1}} \end{vmatrix} = \begin{vmatrix} 1, & 1, & 1, & \dots & 1, \\ x_1, & x_2, & x_3, & \dots & x_n \\ x_1^2, & x_2^2, & x_3^2, & \dots & x_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{n-1}, & x_2^{n-1}, & x_3^{n-1}, & \dots & x_n^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & 4 & \dots & n \end{vmatrix}$$

4. *OA, OB, OC* are three fixed straight lines in space and *OP* is a straight line through *O*. If the sum of the angles *AOP, BOP, COP* has a critical value, prove that the planes *AOP, BOP, COP* are equally inclined to one another.

Hence or otherwise prove that, if the sum of the solid angles of a tetrahedron has a critical value, the tetrahedron is regular.

[The solid angle at any vertex is measured by the area intercepted by the planes through that vertex on the sphere of unit radius, whose centre is at the vertex.]

5. Prove that the differential equation $u^{2n} dv + (\cos \alpha + i \sin \alpha) v^{2n} du = 0$, where $u = x + iy, v = x - iy$, represents two mutually orthogonal systems of cardioids.

Interpret the equation $\frac{du}{dv} = \left(\frac{u}{v}\right)^{2n}$ as a geometrical property of the tangent to the cardioid.

6. If $f(x)$ be positive when $x \geq 1$, and continually diminish to zero as x increases without limit, prove that $\sum_{n=1}^{\infty} f(x)$ and $\int_1^{\infty} f(x) dx$ are both convergent or both divergent.

Prove that $\frac{1}{1^r} + \frac{1}{2^r} + \dots + \frac{1}{n^r}$ lies between $1 + \int_1^n \frac{dx}{x^r}$ and $\frac{1}{n^r} + \int_1^n \frac{dx}{x^r}$, and deduce that if $S_r = \frac{1}{1^r} + \frac{1}{2^r} + \dots + \frac{1}{n^r} + \dots$ to infinity ($r > 1$) then $\lim_{r \rightarrow 1} (r-1) S_r = 1$.

7. A point *A* is taken on the side of a square and points *B* and *C*, one on each of the adjacent sides. Show that the chance that the angle *BAC* is $\frac{\pi}{4}$ is $\frac{1}{2}$.

5. Explain how $f(x)$ is a polynomial of degree n in x such that the roots of $y^2(x^2 - 1)$ be expanded in a series of

“conic to pass through the form of lines” may be obtained of conics through the three points $A, B,$ and C that P and Q are conjugate of contact of the four circles $\alpha, \beta, \gamma,$ and δ .

6. Prove that the locus of the focus of a parabola $y^2 = 4ax$ and $\int \left(1 - \frac{1}{y^2}\right) \frac{dx}{x^2 - 1}$ is a rational function

Solution of the equation

Two circles of radii a, b touch at the focus; prove that, if $\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$.

9. Integrate the equation

$$(1 + y + y^2) \left(1 + y + y^2 + \frac{d^2y}{dx^2} \right) = 2y \left(\frac{dy}{dx} \right)^2$$

by aid of the substitution $\frac{dy}{dx} = 1 + y + y^2$ or otherwise, and the equation

$$(y - 1) \frac{d^2y}{dx^2} = \left(\frac{dy}{dx} \right)^2 \left(x \frac{dy}{dx} - y \right)$$

by changing the independent variable from x to y , or otherwise.

10. Given that $y = \cot 2x$ is a particular integral of the equation.

$$\frac{dy}{dx} - 2y^2 \sec 2x + \operatorname{cosec}^3 x = 0,$$

obtain the complete primitive.

VI.

1. If $f(x + iy) = u + iv$ where u and v are real, prove that the curves
 $u = \text{constant}, v = \text{constant}$
 cut orthogonally.

Find the orthogonal trajectories of the curves $r_1, r_2, r_3 = \text{constant}$, where r_1, r_2, r_3 are the distances of a point from three fixed points and l, m, n are constants.

2. A parabola is drawn so as to have four-point contact with an equiangular spiral: prove that the locus of its focus is another equiangular spiral.

3. Prove that if $u = 0$ is any algebraic curve and (x', y') any fixed point the curve

$$(x - x') \frac{\partial u}{\partial y} + (y' - y) \frac{\partial u}{\partial x} = 0$$

passes through all its double points and touches it at all its cusps.

Hence, or otherwise, show that the number of normals which can be drawn to a curve of degree m is $m^2 - 2\delta - 3\kappa$ where δ is the number of double points and κ the number of cusps.

4. Prove that a, b, c, d, e being any constants, the line

$$x \cos \theta + y \sin \theta = (a\theta + b) \cos \theta + (c\theta + d) \sin \theta + e$$

touches the involute of a cycloid.

A triangle whose sides are given in magnitude moves so that two of the sides touch involutes of cycloids; prove that the third also touches the involute of a cycloid.

5. Find the volume generated by the complete revolution about the axis of x of the area in the first quadrant bounded by that axis and the three conics whose equations are

$$\frac{x^2}{2} - \frac{y^2}{6} = 1, \quad \frac{x^2}{9} + \frac{y^2}{1} = 1, \quad \frac{x^2}{16} + \frac{y^2}{8} = 1.$$

6. A circle is drawn to pass through the points whose coordinates are (x_1, y_1) and (x_2, y_2) and to have its centre on the axis of x . If σ denotes the value of the integral $\int \frac{ds}{y}$ taken along the arc of the circle between these two points, prove that

$$(x_1 - x_2)^2 + y_1^2 + y_2^2 = 2y_1y_2 \cosh \sigma.$$

7. Prove that the radius of curvature of the envelope of the line

$$\lambda x + \mu y + 1 = 0$$

$$\text{is } (\lambda^2 + \mu^2)^{\frac{3}{2}} (\lambda f_1 + \mu f_2)^{-2} (f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2)$$

where λ and μ are connected by the equation $f(\lambda, \mu) = 0$, and the suffixes 1 and 2 denote differentiation with respect to λ and μ respectively.

8. Find (to two terms in each case) the expansion (y in powers of x , or x in powers of y as may be suitable) of the real branches of the curve

$$xy^2 + x^2 - 2y^2x^2 = y^4$$

at the origin and at infinity.

9. If l, m, n are the direction cosines of a line fixed in space, with reference to the moving axes, formed by the tangent, principal normal and binormal of a curve, whose radii of curvature and torsion are ρ and σ , prove that

$$\frac{dl}{ds} - \frac{m}{\rho} = 0, \quad \frac{dm}{ds} + \frac{n}{\sigma} + \frac{l}{\rho} = 0, \quad \frac{dn}{ds} - \frac{m}{\sigma} = 0.$$

If $\left(\sigma \rho^2 \frac{d\sigma}{ds} - \rho \sigma^2 \frac{d\rho}{ds} \right)^2$ bears a constant ratio to $(\rho^2 + \sigma^2)^2$, prove that the principal normal makes a constant angle with a line fixed in space.

10. The curve whose equation is $y = e^s$ (s being the arc measured from a certain point on the curve) revolves about the axis of x and thus generates a surface of revolution, the coordinates of any point of which may be taken to be s and v , the angle between the meridian through the point and a fixed plane through the axis. On this surface a curve is drawn crossing the meridians at an angle whose cosecant varies as e^s ; prove that the length σ of the arc of such a curve is given by

$$(v_1 - v_2)^2 + e^{-2s_1} + e^{-2s_2} = 2e^{-s_1 - s_2} \cosh \sigma$$

where s_1, v_1 and s_2, v_2 are the coordinates of its extremities.

SENIOR MATHEMATICAL SCHOLARSHIP, 1911.

I.

1. Prove that δ , the length of the chord intercepted on the line $a + m\beta + n\gamma = 0$ by the conic $S(a, \beta, \gamma) = 0$, is given by the equation

$$4a^2b^2c^2 \Sigma(l, m, n) U(l, m, n) + \delta^2 [S(a', \beta', \gamma')]^2 = 0,$$

where $\Sigma = 0$ is the tangential equation of the conic; $U = 0$ is the tangential equation of the circular points at infinity; a', β', γ' are the respective

minors of the set $\begin{vmatrix} l, m, n \\ a, b, c \end{vmatrix}$; and a, b, c are the sides of the triangle of reference.

If $S = 0$ is a circle, obtain the equation in the form

$$4a^2b^2c^2 \Sigma + \delta^2 U[S(0, -c, b)]^2 = 0.$$

2. Show that the locus of a point at which two sets of four points subtend equianharmonic pencils is, in general, a quartic curve passing through the eight points.

Examine specially the case in which one pair of the first set of points coincides with the corresponding pair of the second set.

3. The coordinates of a line in rectangular Cartesian coordinates are l, m, n, f, g, h ; prove that:

(1) the normals to the conicoid $ax^2 + by^2 + cz^2 = 1$, at all the points in which it cuts the conicoid

$ax(ny - mx + f) + by(lz - nx + g) + cz(mx - ly + h) = 0$,
meet the line;

(2) the line is a generator of $ax^2 + by^2 + cz^2 = 1$, if

$$\frac{al}{f} - \frac{bm}{g} = \frac{cn}{h} = + \sqrt{-abc}.$$

4. If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the direction cosines of the tangent, principal normal and binormal respectively of a tortuous curve; prove the relations

$$\frac{dl_1}{ds} = \frac{l_2}{\rho}, \quad \frac{dl_2}{ds} = -\frac{l_3}{\sigma}, \quad \frac{dl_3}{ds} = \frac{l_3}{\sigma} - \frac{l_1}{\rho}.$$

The osculating planes of one curve are the reciprocals of the points of another curve with regard to the conicoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. If σ, σ' are the radii of torsion at corresponding points of the two curves, and p, p' are the distances of the corresponding osculating planes from the centre of the conicoid, prove that

$$\sigma^2 \sigma'^2 p^4 p'^4 = a^4 b^4 c^4.$$

5. Evaluate, when m, n, a, b are positive,

$$(1) \int_0^{\infty} \frac{\sin mx \sin nx}{x^2} dx;$$

$$(2) \frac{2}{\pi} \int_0^{\infty} dx \int_a^b \frac{\sin mx \sin yz}{x^2} f'(y) dy.$$

Prove that, if m is a positive proper fraction,

$$\sin m\pi > m(-m)\pi.$$

6. Investigate the conditions under which

$$\frac{\partial}{\partial c} \int_a^b \phi(x, c) dx = \int_a^b \frac{\partial}{\partial c} \phi(x, c) dx,$$

paying special regard to the case in which one or both limits are infinite.

If $X \equiv (x-a)(x-b) \frac{d}{dx}$, $Y \equiv (y-a)(y-b) \frac{d}{dy}$, $X^0 = 1$, $Y^0 = 1$;

prove that, if all the integrals involved are finite,

$$X^r \int_a^b \frac{f(y)}{a-y} dy = - \int_a^b Y^{r-1} f(y) dy + \int_a^b \frac{Y^r f(y)}{a-y} dy,$$

and

$$\frac{d}{dx} X^r \int_a^b \frac{f(y)}{a-y} dy = \int_a^b \frac{d}{dy} \frac{Y^r f(y)}{a-y} dy,$$

where r is any positive integer.

7. An engine of mass M , whose maximum power is ML^2T^{-3} , runs on the level against a constant resistance $2MLT^{-2}$. From the time the engine starts working it increases its rate of working uniformly from zero to the maximum and it attains this maximum rate after a time T . If the engine is started (by another engine) with velocity u and then begins working, prove that at any time t ($> T$) its velocity v and distance s from the starting point are given by the equations

$$(v^2 - u^2) + 4stT^{-2} = L^2tT^{-4},$$

$$(v - pt)^2 (v + pt)^2 = u^{2+q},$$

where p and q are certain constants.

If the engine starts from rest, prove that the limit towards which its velocity tends is $V(S/l + 1)$, where V is its velocity and S its distance from the starting point at time T .

8. A homogeneous right circular cone rolls on a perfectly rough plane, whose inclination to the horizon is γ , starting from rest in the position of unstable equilibrium. Show that, if the vertex always remains in contact with the plane, $\cot \gamma \leq 5 \cot \beta$; where β is the greatest inclination of the plane for which the cone can rest on its side, vertex upwards, without toppling over.

Show also that the resultant friction acts at a distance from the vertex which is constant and depends only on the size and shape of the cone.

9. Assuming that Legendre's equation

$$(1 - z^2) \frac{d^2 y}{dz^2} - 2z \frac{dy}{dz} + n(n+1)y = 0$$

has a solution of the form $\int \frac{T dt}{(t - z)^{n+1}}$, where T is a function of t only, obtain, when n is a positive integer, two independent integrals of the forms $P_n(z)$ and $\int_{-1}^z \frac{P_n(t)}{z-t} dt$, where $P_n(z)$ is the Legendre polynomial of order n .

If $P_n(z)$ is for shortness written P_n , prove that

$$(1) \quad \{n(n+1) - n(n+1)\} \left\{ (x^2 - 1)^n \frac{d^r P_n}{dx^r} \frac{d^r P_n}{dx^r} \right. \\ \left. = (x^2 - 1)^{r-1} \left\{ \frac{d^{r+1} P_n}{dx^{r+1}} \frac{d^r P_n}{dx^r} - \frac{d^r P_n}{dx^r} \frac{d^{r+1} P_n}{dx^{r+1}} \right\} \right\};$$

$$(2) \quad \int (x^2 - 1)^{r+1} \left[\frac{d^{r+1} P_n}{dx^{r+1}} \right]^2 dx = (x^2 - 1)^{r+1} \frac{d^r P_n}{dx^r} \frac{d^{r+1} P_n}{dx^{r+1}} \\ - \{n(n+1) - r(r+1)\} \int (x^2 - 1)^r \left(\frac{d^r P_n}{dx^r} \right)^2 dx.$$

Find the value of $\int_{-1}^1 (1 - x^2)^r \left(\frac{d^r P_n}{dx^r} \right)^2 dx$.

II

1. An elastic string of length $2l$ hangs in the form of a "catenary of uniform strength" so that the tension at any point is wa , where w is the weight per unit length at the point in the stretched state and a is a constant. If the unstretched length $2l_0$ is such that $l_0 = a \tanh l/2a$, prove that the weight per unit length in the unstretched state is proportional to $(a^2 + a_0^2)(a^2 - a_0^2)^{-2}$, where a_0 is the length of the arc measured from the strongest point of the string.

2. Each force of a system is turned about an axis perpendicular to the Poinsot axis through a given angle, the points of application remaining fixed. Prove that the locus of new Poinsot axes for different values of the angle of rotation can have its equation put in the form $s^2(x^2 + y^2) = k^2 z^2$, where k is a constant.

3. A funicular polygon is in equilibrium under the action of forces acting at the corners, the components of the force acting at the r^{th} corner (x_r, y_r, z_r) being X_r, Y_r, Z_r . If a new polygon be constructed such that its sides are conjugate to those of the former with respect to $x^2 + y^2 + z^2 + 1 = 0$, prove that it will be in equilibrium under the action of forces such that the force at the r^{th} corner is (L_r, M_r, N_r) , where

$$L_r = y_r Z_r - z_r Y_r; \quad M_r = z_r X_r - x_r Z_r; \quad N_r = x_r Y_r - y_r X_r.$$

4. Prove that the potential of a uniform circular ring of mass m at a point P is equal to m/r , where r is the arithmetico-geometrical mean between the greatest and least distances of P from the ring. (The arithmetico-geometrical mean between two positive quantities p and q is the common value of

$$\text{Lt}_{n \rightarrow \infty} p_n \quad \text{or} \quad \text{Lt}_{n \rightarrow \infty} q_n$$

where p_n and q_n are formed as follows:

$$p_1 = \frac{1}{2}(p + q); \quad q_1 = \sqrt{pq}; \quad \dots \dots$$

$$p_{n+1} = \frac{1}{2}(p_n + q_n); \quad q_{n+1} = \sqrt{p_n q_n}.$$

If the point P is very close to the ring, prove that the value of the potential is approximately $(m/\pi a) \log(8a/p)$, where a is the radius of the ring and p is the shortest distance of the point from the circumference.

5. Prove Green's theorem that

$$\iint (lu + mv + nw) dS = \iiint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dx dy dz,$$

stating the limitations of the functions u, v, w involved.

If V_0 be the potential of a solid sphere of radius a_0 for any law of attraction which depends only on the relative positions of the attracting points, prove that

$$\frac{1}{a} \left(2 \frac{\partial^3}{\partial x^3} - \frac{\partial^3}{\partial x^2} - \frac{\partial^3}{\partial y^2} \right) \int_0^a V_0 a_0 da_0$$

is the potential of a spherical shell of radius a and of surface density $(3 \cos^2 \theta - 1)$, where θ is the colatitude measured from the x -axis, the point at which the potential is to be found being external to the sphere of radius a .

If the law of force is the inverse n^{th} power of the distance, prove that the formula holds also for internal points if $n < 3$.

6. If the index of refraction (μ) at a point of the earth's atmosphere distant r from the centre is given by the equation

$$\mu = \sqrt{(\mu_0^2 - 1)R/r + 1},$$

where R is the radius of the earth, prove that, when the sun is apparently on the point of rising, the zenith distance of his upper limb is $90^\circ + \theta$, θ being given by

$$\tan \theta = (\mu_0^2 - 1)/2\mu_0.$$

7. A uniformly bright spherical cap, of height h and radius a , lies with its base in contact with a plane; prove that the intensity of illumination at any point of the plane is proportional to

$$\cos^{-1}(\cos a \sec \theta) - \sin^2 a \sin \theta \cos^{-1}(\cot a \tan \theta)$$

$$- \cos a \cos \theta \sqrt{1 - \cos^2 a \sec^2 \theta},$$

where a and θ are determined by the relations

$$\theta + \phi = a, \quad a \sin \theta = (a - h) \sin a;$$

ϕ being the angular elevation of the cap as seen from the point in the plane.

8. Prove that the equation of time has a stationary value when

$$(1 - e^2)^{\frac{1}{2}} (\cos^2 \lambda + \cos^2 \omega \sin^2 \lambda) = \cos \omega \{1 + e \cos (\lambda - \omega)\}^2,$$

where λ is the sun's true longitude, e the eccentricity of the earth's orbit, ω the angle between the plane of the equator and the plane of the earth's orbit, and ϖ the longitude of perihelion.

9. Prove that the earth's surface can be conformally represented on a plane by the transformation

$$x + iy = c \left[l + i \left\{ \phi + \frac{1}{2} e \log \frac{1 - e \sin \mu}{1 + e \sin \mu} \right\} \right],$$

where c is a constant, l the longitude, μ the geographical latitude, e the eccentricity of the earth's surface supposed a spheroid, and ϕ is given by $\operatorname{sech} \phi = \cos \mu$.

Show that loxodromes on the sphere transform into straight lines.

III.

1. If $z \equiv x + yi = \operatorname{sn}^2 \frac{1}{2} w \equiv \operatorname{sn}^2 \frac{1}{2} (u + v)$, show that the curves $u = a$, $v = b$, belong to a system of confocal, orthogonal Cartesian ovals, for all values of the constants a and b .

If a conformal representation of the w plane be made on the z plane, show that the ratio of corresponding infinitesimal areas around two corresponding points P and Q is inversely proportional to the product of the distances of Q from the three foci of the system of Cartesian ovals.

If A be the intersection of the curves $u = u_1$, $v = v_1$, B that of $u = u_2$, $v = v_2$, and O be the origin, show that

$$\frac{AB^2}{(1 - k^2 OA^2)(1 - k^2 OB^2)} = \gamma \frac{1 - k^2 \operatorname{sn} u_1 \operatorname{sn} u_2 \operatorname{sn} v_1 \operatorname{sn} v_2}{1 - k^2 \gamma},$$

where $\gamma \equiv \operatorname{sn} \frac{1}{2} (u_1 + u_2 + v_1 + v_2) \operatorname{sn} \frac{1}{2} (u_1 + u_2 - v_1 - v_2) \times \operatorname{sn} \frac{1}{2} (u_1 - u_2 + v_1 - v_2) \operatorname{sn} \frac{1}{2} (u_1 - u_2 - v_1 + v_2)$

and deduce a theorem corresponding to Ivory's theorem for confocal quadrics.

$$\left[\operatorname{cn} (a + \beta) = \operatorname{cn} a \operatorname{cn} \beta - \operatorname{sn} a \operatorname{sn} \beta \operatorname{dn} (a + \beta); \right. \\ \left. \operatorname{sn} (a + \beta) \operatorname{sn} (a - \beta) = \frac{\operatorname{sn}^2 a - \operatorname{sn}^2 \beta}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 \beta} \right]$$

2. Prove that, if s be any point in a ring-shaped space, within which $f(s)$ is regular, which is bounded by two concentric circles C and C' of centre c , $f(s)$ can be expanded at the point s in the form

$$f(s) = a_0 + a_1 (s - c) + a_2 (s - c)^2 + \dots + \frac{b_1}{s - c} + \frac{b_2}{(s - c)^2} + \dots$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(t) dt}{(t - c)^{n+1}}, \text{ and } b_n = \frac{1}{2\pi i} \int_{C'} (t - c)^{n-1} f(t) dt.$$

Show that in the ring defined by $|a| < |z| < |b|$,

$$\Gamma(a)\Gamma(\beta) \frac{b^\beta z^\alpha}{(z-a)^\alpha (b-z)^\beta}$$

can be expanded in the form

$$\frac{1}{2}(A_0 + B_0) + \sum_{n=1}^{\infty} \left(A_n \frac{a^n}{z^n} + B_n \frac{z^n}{b^n} \right),$$

where

$$A_n = \sum_{s=0}^{\infty} \frac{\Gamma(a+n+s)\Gamma(\beta+s)}{\Gamma(n+s+1)\Gamma(s+1)} \frac{a^s}{b^s},$$

$$B_n = \sum_{s=0}^{\infty} \frac{\Gamma(\beta+n+s)\Gamma(a+s)}{\Gamma(n+s+1)\Gamma(s+1)} \frac{a^s}{b^s}.$$

3. Starting from the relation

$$\begin{aligned} [\wp'(u)]^2 &= 4[\wp(u)]^3 - g_2\wp(u) - g_3 \\ &\equiv 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3), \end{aligned}$$

show, by making the substitution

$$x - e_1 = \frac{(e_1 - e_2)(e_1 - e_3)}{\wp(u) - e_1},$$

that $\wp(u)$ is a periodic function of u .

Show also that the three periods got in this way are not independent.

4. Prove the formula

$$\wp(u) + \wp(v) + \wp(u+v) = \left\{ \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right\}^2.$$

If $3a$ is a period of $\wp(u)$ prove that the sum of the nine distinct values of $\wp(u+a) - \wp(3u)$ is zero.

5. Obtain an expression for the energy of a system of currents flowing in linear circuits, and give dynamical reasons for calling this energy kinetic.

Compare the expressions $\frac{1}{2}(Fu + Gv + Hw)$ and $\frac{1}{8\pi} (aa + b\beta + c\gamma)$ given by Maxwell for the kinetic energy per unit volume, and mention any case in which these expressions are not equivalent.

6. Find the most general quadratic function ϕ of the six quantities $X, Y, Z, \alpha, \beta, \gamma$, which satisfies the equation

$$\iiint \frac{d\phi}{dt} dxdydz = \iint (lu + mv + nw) dS,$$

where u, v, w , are also quadratic functions of the same six quantities (l, m, n) the direction cosines of an element of a surface, and the volume- and surface-integrations refer to any region in which the differential equations

$$\begin{aligned} \sigma^{-1}(\dot{X}, \dot{Y}, \dot{Z}) &= \text{curl}(\alpha, \beta, \gamma), \\ -(\dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \text{curl} X, Y, Z \end{aligned}$$

hold.

Interpret the four linearly independent forms which result and the corresponding surface integrals.

7. Prove that the frequencies $p/2\pi$ of the principal oscillations of simplest type which can take place inside a hollow perfectly conducting sphere of radius a are given by the equation

$$(1 - p^2 a^2) \tan pa = pa.$$

8. Explain the application of the method of the Least Action to the deduction of the circuital relations.

9. Obtain solutions of the differential equations in Question 6 which are infinite only at the point $(0, 0, vt)$, where v is a constant.

Prove that these solutions represent the field due to a conducting spheroid moving with a velocity v .

IV.

1. (1) Prove that $\text{Lt}_{n=\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right)$ is finite.

(2) If $s > 1$, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\prod \left(1 - \frac{1}{p^s} \right)}$$

where p stands for the succession of prime numbers 2, 3, 5, 7

(3) Show that $\prod \left(1 - \frac{1}{p} \right)$ is divergent.

(4) Show that $\frac{1}{1^c} \log 2 + \frac{1}{2^c} \log 3 + \frac{1}{3^c} \log 5$ is the sum of the series $\sum_1^{\infty} \frac{c_n}{n^c}$ where c has the following values, according to the remainder when n is divided by 30,

$$c_n = 1, \text{ Rem. } 1, 7, 11, 13, 17, 19, 23, 29;$$

$$c_n = 0, \text{ Rem. } 2, 3, 4, 5, 8, 9, 14, 16, 21, 22, 25, 26, 27, 28;$$

$$c_n = -1, \text{ Rem. } 0, 6, 10, 12, 15, 18, 20, 24.$$

2. Discuss the validity of the binomial theorem

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2} x^2 + \dots$$

for all real commensurable values of m and for all complex values of x .

3. If $\phi(x), \psi(x)$ be two polynomials of degree n , and $f(x) \equiv \prod_{s=1}^{s=n} (x - r_s)$,

find the expansion of $\frac{\phi(x)\psi(a) - \phi(a)\psi(x)}{(x-a)f(x)}$ in partial fractions, and show that the eliminant of $\phi(x)$ and $\psi(x)$ is given by a determinant of the n th order whose constituents are

$$\frac{\phi(r_s)\psi(a) - \phi(a)\psi(r_s)}{r_s - a_s}$$

s being the row number and t the column number, the sets of quantities $a_1, a_2, \dots, a_n, r_1, r_2, \dots, r_n$ being arbitrary.

Apply this method to find the eliminant of $\phi(x)$ and $x^{2m}\phi\left(\frac{1}{x}\right)$, where $\phi(x)$ is of degree $2m$; and by putting $r_s = a_s$ for all values of s , and then

$$a_{n+s} = \frac{1}{a_s},$$

resolve the resulting determinant into two factors of the same degree.

4. Prove that, in general, a binary quantic of order $2n - 1$ can be expressed as the sum of the $(2n - 1)$ th powers of n linear forms.

Show that, in general, a quintic can be expressed in the form

$$(a, b, 0, 0, e, f) (x, y)^5.$$

5. P is a point on a hyperboloid of one sheet whose equation is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

and $a^2 + \mu$, $a^2 - \nu$ are the squares of the primary semi-axes of the two conicoids through P confocal with the given hyperboloid. Show that the semi-axes of the central section, which is parallel to the tangent plane at P , are $\sqrt{\mu}$ and $\sqrt{\nu}$.

Show also that the generating lines of the hyperboloid are given by the equations

$$\frac{d\mu}{\{(a^2 + \mu)(b^2 + \mu)(\mu - c^2)\}} \pm \frac{d\nu}{\{(a^2 - \nu)(\nu - b^2)(c^2 + \nu)\}} = 0.$$

Deduce from this an integral of the equation $u_1 + u_2 + u_3 + u_4 = 0$, where u is a given elliptic integral of the first kind; also find an expression for an element of length along a generator, and deduce a formula for

$$E(u_1) + E(u_2) + E(u_3) - E(u_1 + u_2 + u_3).$$

6. Prove that the condition that a surface, whose equation in rectangular Cartesian coordinates is $\phi(x) + \psi(y) + \chi(z) = 0$, may have its principal radii of curvature equal and of opposite sign, is

$$\frac{\partial^2 \phi}{\partial x^2} \left\{ \left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \chi}{\partial z} \right)^2 \right\} + \frac{\partial^2 \psi}{\partial y^2} \left\{ \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \chi}{\partial z} \right)^2 \right\} + \frac{\partial^2 \chi}{\partial z^2} \left\{ \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right\} = 0.$$

Show that

$$\frac{1}{\text{cn } \sqrt{2}z} = \frac{1}{2} \left(\frac{\text{cn } x}{\text{cn } y} + \frac{\text{cn } y}{\text{cn } x} \right)$$

is such a surface if the modulus of the elliptic functions is $\frac{1}{2}$.

7. A flexible surface is subject to fluid pressure. Prove that, if at any point p be the pressure, t and t' the principal tensions, r and r' the radii of curvature of the normal sections along which t and t' act, ρ and ρ' the principal radii of curvature, and τ and τ' the tensions along the sections of which ρ and ρ' are the radii of curvature,

$$p = \frac{t}{r} + \frac{t'}{r'} = \frac{\tau}{\rho} + \frac{\tau'}{\rho'}$$

A rectangular box with vertical sides which contains a liquid, the density of which varies as the depth, is closed at the bottom by a rectangular piece of thin flexible cloth the two ends of which are fastened at the same horizontal level, while the liquid is prevented from escaping at the sides. Show that the intrinsic equation of the cross section of the cylindrical surface formed by the cloth is of the form

$$s = \int^{\phi} \frac{d\phi}{(A + B \cos \phi)^{\frac{3}{2}}}$$

8. A long circular cylinder of radius a is contained within a coaxial cylinder of radius b , the space between them being filled with liquid. If the inner cylinder be given a velocity u in a given direction perpendicular to the axis, show that the velocity potential for the initial motion is

$$\phi = -\frac{ua^2}{b^2 - a^2} \left(r + \frac{b^2}{r} \right) \cos \theta$$

where r is the distance from the axis of the cylinders, and θ is measured from the direction of u .

7. Prove that the radius of curvature of the envelope of the line

$$\lambda x + \mu y + 1 = 0$$

$$\text{is } (\lambda^2 + \mu^2)^3 (\lambda f_1 + \mu f_2)^{-3} (f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2)$$

where λ and μ are connected by the equation $f(\lambda, \mu) = 0$, and the suffixes 1 and 2 denote differentiation with respect to λ and μ respectively.

8. Find (to two terms in each case) the expansion (y in powers of x , or x in powers of y as may be suitable) of the real branches of the curve

$$xy^3 + x^3 - 2y^2x^2 = y^6$$

at the origin and at infinity.

9. If l, m, n are the direction cosines of a line fixed in space, with reference to the moving axes, formed by the tangent, principal normal and binormal of a curve, whose radii of curvature and torsion are ρ and σ , prove that

$$\frac{dl}{ds} - \frac{m}{\rho} = 0, \quad \frac{dm}{ds} + \frac{n}{\sigma} + \frac{l}{\rho} = 0, \quad \frac{dn}{ds} - \frac{m}{\sigma} = 0.$$

If $\left(\sigma \rho^2 \frac{d\sigma}{ds} - \rho \sigma^2 \frac{d\rho}{ds} \right)^2$ bears a constant ratio to $(\rho^2 + \sigma^2)^2$, prove that the principal normal makes a constant angle with a line fixed in space.

10. The curve whose equation is $y = e^s$ (s being the arc measured from a certain point on the curve) revolves about the axis of x and thus generates a surface of revolution, the coordinates of any point of which may be taken to be s and v , the angle between the meridian through the point and a fixed plane through the axis. On this surface a curve is drawn crossing the meridians at an angle whose cosecant varies as e^s ; prove that the length σ of the arc of such a curve is given by

$$(v_1 - v_2)^2 + e^{-2s_1} + e^{-2s_2} = 2e^{-s_1 - s_2} \cosh \sigma$$

where s_1, v_1 and s_2, v_2 are the coordinates of its extremities.

SENIOR MATHEMATICAL SCHOLARSHIP, 1911.

I.

1. Prove that δ , the length of the chord intercepted on the line $a + m\beta + n\gamma = 0$ by the conic $S(a, \beta, \gamma) = 0$, is given by the equation

$$4a^2b^2c^2 \Sigma(l, m, n) U(l, m, n) + \delta^2 [S(a', \beta', \gamma')]^2 = 0,$$

where $\Sigma = 0$ is the tangential equation of the conic; $U = 0$ is the tangential equation of the circular points at infinity; a', β', γ' are the respective minors of the set $\begin{vmatrix} l, m, n \\ a, b, c \end{vmatrix}$; and a, b, c are the sides of the triangle of reference.

If $S = 0$ is a circle, obtain the equation in the form

$$4a^2b^2c^2 \Sigma + \delta^2 U [S(0, -c, b)]^2 = 0.$$

2. Show that the locus of a point at which two sets of four points subtend equianharmonic pencils is, in general, a quartic curve passing through the eight points.

Examine specially the case in which one pair of the first set of points coincides with the corresponding pair of the second set.

3. The coordinates of a line in rectangular Cartesian coordinates are l, m, n, f, g, h ; prove that:

(1) the normals to the conicoid $ax^2 + by^2 + cz^2 = 1$, at all the points in which it cuts the conicoid

$ax(ny - mx + f) + by(lx - nz + g) + cz(mx - ly + h) = 0$,
meet the line;

(2) the line is a generator of $ax^2 + by^2 + cz^2 = 1$, if

$$\frac{al}{f} - \frac{bm}{g} = \frac{cn}{h} = + \sqrt{-abc}.$$

4. If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the direction cosines of the tangent, principal normal and binormal respectively of a tortuous curve; prove the relations

$$\frac{dl_1}{ds} = \frac{l_2}{\rho}, \quad \frac{dl_2}{ds} = -\frac{l_3}{\sigma}, \quad \frac{dl_3}{ds} = \frac{l_3}{\sigma} - \frac{l_1}{\rho}.$$

The osculating planes of one curve are the reciprocals of the points of another curve with regard to the conicoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. If σ, σ' are the radii of torsion at corresponding points of the two curves, and p, p' are the distances of the corresponding osculating planes from the centre of the conicoid, prove that

$$\sigma^2 \sigma'^2 p^4 p'^4 = a^4 b^4 c^4.$$

5. Evaluate, when m, n, a, b are positive,

$$(1) \int_0^{\infty} \frac{\sin mx \sin nx}{x^2} dx;$$

$$(2) \frac{2}{\pi} \int_0^{\infty} dx \int_a^b \frac{\sin mx \sin yx}{x^2} f'(y) dy.$$

Prove that, if m is a positive proper fraction,

$$\sin m\pi > m(-m)\pi.$$

6. Investigate the conditions under which

$$\frac{\partial}{\partial c} \int_a^b \phi(x, c) dx = \int_a^b \frac{\partial}{\partial c} \phi(x, c) dx,$$

paying special regard to the case in which one or both limits are infinite.

If $X \equiv (x-a)(x-b) \frac{d}{dx}$, $Y \equiv (y-a)(y-b) \frac{d}{dy}$, $X^0 = 1$, $Y^0 = 1$;

prove that, if all the integrals involved are finite,

$$X^r \int_a^b \frac{f(y)}{x-y} dy = - \int_a^b Y^{r-1} f(y) dy + \int_a^b \frac{Y^r f(y)}{x-y} dy,$$

and

$$\frac{d}{dx} X^r \int_a^b \frac{f(y)}{x-y} dy = \int_a^b \frac{\frac{d}{dy} Y^r f(y)}{x-y} dy,$$

where r is any positive integer.

7. An engine of mass M , whose maximum power is ML^2T^{-3} , runs on the level against a constant resistance $2MLT^{-2}$. From the time the engine starts working it increases its rate of working uniformly from zero to the maximum and it attains this maximum rate after a time T . If the engine is started (by another engine) with velocity u and then begins working, prove that at any time t ($> T$) its velocity v and distance s from the starting point are given by the equations

$$(v^2 - u^2) + 4tsT^{-2} = L^2FT^{-4},$$

$$(v - pt)^2 (v + pt)^2 = u^{2+q},$$

where p and q are certain constants.

If the engine starts from rest, prove that the limit towards which its velocity tends is $V(S/l + 1)$, where V is its velocity and S its distance from the starting point at time T .

8. A homogeneous right circular cone rolls on a perfectly rough plane, whose inclination to the horizon is γ , starting from rest in the position of unstable equilibrium. Show that, if the vertex always remains in contact with the plane, $\cot \gamma \leq 5 \cot \beta$; where β is the greatest inclination of the plane for which the cone can rest on its side, vertex upwards, without toppling over.

Show also that the resultant friction acts at a distance from the vertex which is constant and depends only on the size and shape of the cone.

9. Assuming that Legendre's equation

$$(1 - s^2) \frac{d^2 y}{ds^2} - 2s \frac{dy}{ds} + n(n+1)y = 0$$

has a solution of the form $\int \frac{T dt}{(t-s)^{n+1}}$, where T is a function of t only, obtain, when n is a positive integer, two independent integrals of the forms

$P_n(s)$ and $\int_{-1}^s \frac{P_n(t)}{s-t} dt$, where $P_n(s)$ is the Legendre polynomial of order n .

If $P_n(x)$ is for shortness written P_n , prove that

$$(1) \quad \{m(m+1) - n(n+1)\} \int (x^2-1)^r \frac{d^r P_m}{dx^r} \frac{d^r P_n}{dx^r} dx \\ = (x^2-1)^{r+1} \left\{ \frac{d^{r+1} P_m}{dx^{r+1}} \frac{d^r P_n}{dx^r} - \frac{d^r P_m}{dx^r} \frac{d^{r+1} P_n}{dx^{r+1}} \right\};$$

$$(2) \quad \int (x^2-1)^{r+1} \left[\frac{d^{r+1} P_n}{dx^{r+1}} \right]^2 dx = (x^2-1)^{r+1} \frac{d^r P_n}{dx^r} \frac{d^{r+1} P_n}{dx^{r+1}} \\ - \{n(n+1) - r(r+1)\} \int (x^2-1)^r \left(\frac{d^r P_n}{dx^r} \right)^2 dx.$$

Find the value of $\int_{-1}^1 (1-x^2)^r \left(\frac{d^r P_n}{dx^r} \right)^2 dx$.

II

1. An elastic string of length $2l$ hangs in the form of a "catenary of uniform strength" so that the tension at any point is $w\alpha$, where w is the weight per unit length at the point in the stretched state and α is a constant. If the unstretched length $2l_0$ is such that $l_0 = a \tanh l/2a$, prove that the weight per unit length in the unstretched state is proportional to $(\alpha^2 + s_0^2)(\alpha^2 - s_0^2)^{-2}$, where s_0 is the length of the arc measured from the strongest point of the string.

2. Each force of a system is turned about an axis perpendicular to the Poinsot axis through a given angle, the points of application remaining fixed. Prove that the locus of new Poinsot axes for different values of the angle of rotation can have its equation put in the form $x^2 + y^2 = k^2 z^2$, where k is a constant.

3. A funicular polygon is in equilibrium under the action of forces acting at the corners, the components of the force acting at the r^{th} corner (x_r, y_r, z_r) being X_r, Y_r, Z_r . If a new polygon be constructed such that its sides are conjugate to those of the former with respect to $x^2 + y^2 + z^2 + 1 = 0$, prove that it will be in equilibrium under the action of forces such that the force at the r^{th} corner is (L_r, M_r, N_r) , where

$$L_r = y_r Z_r - z_r Y_r; \quad M_r = z_r X_r - x_r Z_r; \quad N_r = x_r Y_r - y_r X_r.$$

4. Prove that the potential of a uniform circular ring of mass m at a point P is equal to m/r , where r is the arithmetico-geometrical mean between the greatest and least distances of P from the ring. (The arithmetico-geometrical mean between two positive quantities p and q is the common value of

$$\text{Lt}_{n \rightarrow \infty} p_n \quad \text{or} \quad \text{Lt}_{n \rightarrow \infty} q_n,$$

where p_n and q_n are formed as follows:

$$p_1 = \frac{1}{2}(p + q); \quad q_1 = \sqrt{pq}; \quad \dots$$

$$p_{n+1} = \frac{1}{2}(p_n + q_n); \quad q_{n+1} = \sqrt{p_n q_n}.$$

If the point P is very close to the ring, prove that the value of the potential is approximately $(m/\pi a) \log(8a/p)$, where a is the radius of the ring and p is the shortest distance of the point from the circumference.

5. Prove Green's theorem that

$$\iint (lu + mv + nw) dS = \iiint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dx dy dz,$$

stating the limitations of the functions u, v, w involved.

If V_0 be the potential of a solid sphere of radius a_0 for any law of attraction which depends only on the relative positions of the attracting points, prove that

$$\frac{1}{a} \left(2 \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \int_0^a V_{\rho_0} d\rho_0$$

is the potential of a spherical shell of radius a and of surface density $(3 \cos^2 \theta - 1)$, where θ is the colatitude measured from the z -axis, the point at which the potential is to be found being external to the sphere of radius a .

If the law of force is the inverse n^{th} power of the distance, prove that the formula holds also for internal points if $n < 3$.

6. If the index of refraction (μ) at a point of the earth's atmosphere distant r from the centre is given by the equation

$$\mu = \sqrt{(\mu_0^2 - 1)E/r + 1},$$

where E is the radius of the earth, prove that, when the sun is apparently on the point of rising, the zenith distance of his upper limb is $90^\circ + \theta$, θ being given by

$$\tan \theta = (\mu_0^2 - 1)/2\mu_0.$$

7. A uniformly bright spherical cap, of height h and radius a , lies with its base in contact with a plane; prove that the intensity of illumination at any point of the plane is proportional to

$$\cos^{-1}(\cos a \sec \theta) - \sin^2 a \sin \theta \cos^{-1}(\cot a \tan \theta)$$

$$- \cos a \cos \theta \sqrt{1 - \cos^2 a \sec^2 \theta},$$

where a and θ are determined by the relations

$$\theta + \phi = a, \quad a \sin \theta = (a - h) \sin a;$$

ϕ being the angular elevation of the cap as seen from the point in the plane.

8. Prove that the equation of time has a stationary value when

$$(1 - e^2)^{\frac{1}{2}} (\cos^2 \lambda + \cos^2 \omega \sin^2 \lambda) = \cos \omega \{1 + e \cos (\lambda - \varpi)\}^2,$$

where λ is the sun's true longitude, e the eccentricity of the earth's orbit, ω the angle between the plane of the equator and the plane of the earth's orbit, and ϖ the longitude of perihelion.

9. Prove that the earth's surface can be conformally represented on a plane by the transformation

$$x + iy = c \left[l + i \left\{ \phi + \frac{1}{2} e \log \frac{1 - e \sin \mu}{1 + e \sin \mu} \right\} \right],$$

where c is a constant, l the longitude, μ the geographical latitude, e the eccentricity of the earth's surface supposed a spheroid, and ϕ is given by $\operatorname{sech} \phi = \cos \mu$.

Show that loxodromes on the sphere transform into straight lines.

III.

1. If $z \equiv x + yi = \operatorname{sn}^2 \frac{1}{2} w \equiv \operatorname{sn}^2 \frac{1}{2} (u + vi)$, show that the curves $u = a$, $v = b$, belong to a system of confocal, orthogonal Cartesian ovals, for all values of the constants a and b .

If a conformal representation of the w plane be made on the z plane, show that the ratio of corresponding infinitesimal areas around two corresponding points P and Q is inversely proportional to the product of the distances of Q from the three foci of the system of Cartesian ovals.

If A be the intersection of the curves $u = u_1$, $v = v_1$, B that of $u = u_2$, $v = v_2$, and O be the origin, show that

$$\frac{AB^2}{(1 - k^2 OA^2)(1 - k^2 OB^2)} = \gamma \frac{1 - k^2 \operatorname{sn} u_1 \operatorname{sn} u_2 \operatorname{sn} v_1 \operatorname{sn} v_2}{1 - k^2 \gamma},$$

where $\gamma \equiv \operatorname{sn} \frac{1}{2} (u_1 + u_2 + v_1 + v_2) \operatorname{sn} \frac{1}{2} (u_1 + u_2 - v_1 - v_2) \times \operatorname{sn} \frac{1}{2} (u_1 - u_2 + v_1 - v_2) \operatorname{sn} \frac{1}{2} (u_1 - u_2 - v_1 + v_2)$

and deduce a theorem corresponding to Ivory's theorem for confocal quadrics.

$$\left[\operatorname{cn} (a + \beta) = \operatorname{cn} a \operatorname{cn} \beta - \operatorname{sn} a \operatorname{sn} \beta \operatorname{dn} (a + \beta); \right.$$

$$\left. \operatorname{sn} (a + \beta) \operatorname{sn} (a - \beta) = \frac{\operatorname{sn}^2 a - \operatorname{sn}^2 \beta}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 \beta} \right]$$

2. Prove that, if s be any point in a ring-shaped space, within which $f(s)$ is regular, which is bounded by two concentric circles C and C' of centre c , $f(s)$ can be expanded at the point s in the form

$$f(s) = a_0 + a_1 (s - c) + a_2 (s - c)^2 + \dots + \frac{b_1}{s - c} + \frac{b_2}{(s - c)^2} + \dots$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(t) dt}{(t - c)^{n+1}}, \text{ and } b_n = \frac{1}{2\pi i} \int_{C'} (t - c)^{n-1} f(t) dt.$$

Show that in the ring defined by $|a| < |s| < |b|$,

$$\Gamma(a)\Gamma(\beta) \frac{b^\beta x^\alpha}{(x-a)^\alpha (b-x)^\beta}$$

can be expanded in the form

$$\frac{1}{2}(A_0 + B_0) + \sum_{n=1}^{\infty} \left(A_n \frac{a^n}{x^n} + B_n \frac{x^n}{b^n} \right),$$

where

$$A_n = \sum_{s=0}^{\infty} \frac{\Gamma(a+n+s)\Gamma(\beta+s)}{\Gamma(n+s+1)\Gamma(s+1)} \frac{a^s}{b^s},$$

$$B_n = \sum_{s=0}^{\infty} \frac{\Gamma(\beta+n+s)\Gamma(a+s)}{\Gamma(n+s+1)\Gamma(s+1)} \frac{a^s}{b^s}.$$

3. Starting from the relation

$$\begin{aligned} [\wp'(u)]^2 &= 4[\wp(u)]^3 - g_2\wp(u) - g_3 \\ &\equiv 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3), \end{aligned}$$

show, by making the substitution

$$x - e_1 = \frac{(e_1 - e_2)(e_1 - e_3)}{\wp(u) - e_1},$$

that $\wp(u)$ is a periodic function of u .

Show also that the three periods got in this way are not independent.

4. Prove the formula

$$\wp(u) + \wp(v) + \wp(u+v) = \left\{ \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right\}^2.$$

If $3a$ is a period of $\wp(u)$ prove that the sum of the nine distinct values of $\wp(u+a) - \wp(3u)$ is zero.

5. Obtain an expression for the energy of a system of currents flowing in linear circuits, and give dynamical reasons for calling this energy kinetic.

Compare the expressions $\frac{1}{2}(Fu + Gv + Hw)$ and $\frac{1}{8\pi} (aa + b\beta + c\gamma)$ given by Maxwell for the kinetic energy per unit volume, and mention any case in which these expressions are not equivalent.

6. Find the most general quadratic function ϕ of the six quantities $X, Y, Z, \alpha, \beta, \gamma$, which satisfies the equation

$$\iiint \frac{d\phi}{dt} dx dy dz = \iiint (lu + mv + nw) dS,$$

where u, v, w , are also quadratic functions of the same six quantities (l, m, n) the direction cosines of an element of a surface, and the volume- and surface-integrations refer to any region in which the differential equations

$$\sigma^{-1}(\dot{X}, \dot{Y}, \dot{Z}) = \text{curl}(\alpha, \beta, \gamma),$$

$$-(\dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \text{curl} X, Y, Z$$

hold.

Interpret the four linearly independent forms which result and the corresponding surface integrals.

7. Prove that the frequencies $p/2\pi$ of the principal oscillations of simplest type which can take place inside a hollow perfectly conducting sphere of radius a are given by the equation

$$(1 - p^2 a^2) \tan pa = pa.$$

8. Explain the application of the method of the Least Action to the deduction of the circuital relations.

9. Obtain solutions of the differential equations in Question 6 which are infinite only at the point $(0, 0, vt)$, where v is a constant.

Prove that these solutions represent the field due to a conducting spheroid moving with a velocity v .

IV.

1. (1) Prove that $\text{Lt}_{n=\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right)$ is finite.

(2) If $s > 1$, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Pi \left(1 - \frac{1}{p^s} \right)}$$

where p stands for the succession of prime numbers 2, 3, 5, 7, ...

(3) Show that $\Pi \left(1 - \frac{1}{p} \right)$ is divergent.

(4) Show that $\frac{1}{15} \log 2 + \frac{1}{15} \log 3 + \frac{1}{15} \log 5$ is the sum of the series $\sum_{n=1}^{\infty} \frac{c_n}{n^2}$ where c has the following values, according to the remainder when n is divided by 30,

$$c_n = 1, \text{ Rem. } 1, 7, 11, 13, 17, 19, 23, 29;$$

$$c_n = 0, \text{ Rem. } 2, 3, 4, 5, 8, 9, 14, 16, 21, 22, 25, 26, 27, 28;$$

$$c_n = -1, \text{ Rem. } 0, 6, 10, 12, 15, 18, 20, 24.$$

2. Discuss the validity of the binomial theorem

$$(1+z)^m = 1 + ms + \frac{m(m-1)}{1 \cdot 2} z^2 + \dots$$

for all real commensurable values of m and for all complex values of z .

3. If $\phi(x), \psi(x)$ be two polynomials of degree n , and $f(x) \equiv \prod_{s=1}^{s=n} (x - r_s)$,

find the expansion of $\frac{\phi(x)\psi(a) - \phi(a)\psi(x)}{(x-a)f(x)}$ in partial fractions, and show that the eliminant of $\phi(x)$ and $\psi(x)$ is given by a determinant of the n th order whose constituents are

$$\frac{\phi(r_s)\psi(a_t) - \phi(a_t)\psi(r_s)}{r_s - a_t},$$

s being the row number and t the column number, the sets of quantities $a_1, a_2, \dots, a_n, r_1, r_2, \dots, r_n$ being arbitrary.

Apply this method to find the eliminant of $\phi(x)$ and $x^{2m} \phi\left(\frac{1}{x}\right)$, where $\phi(x)$ is of degree $2m$; and by putting $r_s = a_s$ for all values of s , and then

$$a_{s+t} = \frac{1}{a_s},$$

resolve the resulting determinant into two factors of the same degree.

4. Prove that, in general, a binary quantic of order $2n-1$ can be expressed as the sum of the $(2n-1)$ th powers of n linear forms.

Show that, in general, a quintic can be expressed in the form

$$(a, b, 0, 0, c, f)(x, y)^5.$$

5. P is a point on a hyperboloid of one sheet whose equation is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

and $a^2 + \mu$, $a^2 - \nu$ are the squares of the primary semi-axes of the two conicoids through P confocal with the given hyperboloid. Show that the semi-axes of the central section, which is parallel to the tangent plane at P , are $\sqrt{\mu}$ and $\sqrt{\nu}$.

Show also that the generating lines of the hyperboloid are given by the equations

$$\frac{d\mu}{\{(a^2 + \mu)(b^2 + \mu)(\mu - c^2)\}^{\frac{1}{2}}} \pm \frac{d\nu}{\{(a^2 - \nu)(\nu - b^2)(c^2 + \nu)\}^{\frac{1}{2}}} = 0.$$

Deduce from this an integral of the equation $u_1 + u_2 + u_3 + u_4 = 0$, where u is a given elliptic integral of the first kind; also find an expression for an element of length along a generator, and deduce a formula for

$$E(u_1) + E(u_2) + E(u_3) - E(u_1 + u_2 + u_3).$$

6. Prove that the condition that a surface, whose equation in rectangular Cartesian coordinates is $\phi(x) + \psi(y) + \chi(z) = 0$, may have its principal radii of curvature equal and of opposite sign, is

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2} \left\{ \left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \chi}{\partial z} \right)^2 \right\} + \frac{\partial^2 \psi}{\partial y^2} \left\{ \left(\frac{\partial \chi}{\partial z} \right)^2 + \left(\frac{\partial \phi}{\partial x} \right)^2 \right\} \\ + \frac{\partial^2 \chi}{\partial z^2} \left\{ \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right\} = 0. \end{aligned}$$

Show that

$$\frac{1}{\text{cn } \sqrt{2}x} = \frac{1}{2} \left(\frac{\text{cn } x}{\text{cn } y} + \frac{\text{cn } y}{\text{cn } x} \right)$$

is such a surface if the modulus of the elliptic functions is $\frac{1}{2}$.

7. A flexible surface is subject to fluid pressure. Prove that, if at any point p be the pressure, t and t' the principal tensions, r and r' the radii of curvature of the normal sections along which t and t' act, ρ and ρ' the principal radii of curvature, and τ and τ' the tensions along the sections of which ρ and ρ' are the radii of curvature,

$$p = \frac{t}{r} + \frac{t'}{r'} = \frac{\tau}{\rho} + \frac{\tau'}{\rho'}.$$

A rectangular box with vertical sides which contains a liquid, the density of which varies as the depth, is closed at the bottom by a rectangular piece of thin flexible cloth the two ends of which are fastened at the same horizontal level, while the liquid is prevented from escaping at the sides. Show that the intrinsic equation of the cross section of the cylindrical surface formed by the cloth is of the form

$$s = \int^{\phi} \frac{d\phi}{(A + B \cos \phi)^{\frac{1}{2}}}$$

8. A long circular cylinder of radius a is contained within a coaxial cylinder of radius b , the space between them being filled with liquid. If the inner cylinder be given a velocity u in a given direction perpendicular to the axis, show that the velocity potential for the initial motion is

$$\phi = -\frac{ua^2}{b^2 - a^2} \left(r + \frac{b^2}{r} \right) \cos \theta$$

where r is the distance from the axis of the cylinders, and θ is measured from the direction of u .

There are four coaxial cylinders, $r = c$, $r = 2c$, $r = 3c$, $r = 4c$. The spaces between $r = c$ and $r = 2c$, and between $r = 3c$ and $r = 4c$ are filled with liquid, whilst $r = 2c$ and $r = 3c$ are the surface of a solid of the same density as the liquid. The cylinders $r = c$ and $r = 4c$ are suddenly moved at right angles to their axes with equal and opposite linear velocities, u , $-u$, the same in magnitude and direction, but opposite in sense.

Show that the velocity of the solid cylinder will be $-\frac{101}{115}u$.

V.

1. Prove that a uniform function of z , which has its sole essential singularity at a finite point b , and has no accidental singularities, can be represented in the form of an infinite product of factors of the form

$$\left(\frac{k}{z-b} + l\right) e^{g\left(\frac{1}{z-b}\right)},$$

which converges everywhere except at $z = b$.

Find in a form free from imaginaries, an expression for a function with an essential singularity at infinity, which vanishes at all points r^3 , $r^3\omega$, $r^3\omega^2$ where r is a positive integer, and 1 , ω , ω^2 are the cube roots of unity.

2. Prove that an analytical function of a variable u , which possesses an algebraical addition theorem, is either:

(1) an algebraical function of u , or

(2) an algebraical function of e^{u^2} , or

(3) an algebraical function of $\wp(u)$;

the constants ω , g_2 , g_3 being suitably chosen.

3. If
$$\Phi(a, b) \equiv e^{-\pi(a+b)} \int_0^1 x^{a-1} (1-x)^{b-1} dx,$$

the integral being taken around a contour which begins from some point on the real axis lying between 0 and 1, describes a circuit round the point $z=1$ in the positive direction, then one round $z=0$ in the positive direction, then one round $z=1$ in the negative direction, and finally one round $z=0$ in the negative direction, and returns to the starting point; prove that

(1) $\Phi(a, b) = \Phi(b, a)$;

(2) $\Phi(a+1, b) = -\frac{a}{a+b} \Phi(a, b)$;

(3) when the real parts of a and of b are positive

$$\Phi(a, b) = -4 \sin(\pi a) \sin(\pi b) B(a, b)$$

$B(a, b)$ being the Eulerian integral of the first kind; and

(4)
$$\Phi(a, b) = -4\pi^2(a+b) \prod_{x=1}^{\infty} \left(1 - \frac{a}{x}\right) \left(1 - \frac{b}{x}\right) \left(1 + \frac{a+b}{x}\right).$$

4. If $\sigma(u)$ be a function of u which satisfies the relation

$$\begin{aligned} & \sigma(a-b) \sigma(a+b) \sigma(c-d) \sigma(c+d) \\ & + \sigma(b-c) \sigma(b+c) \sigma(a-d) \sigma(a+d) \\ & + \sigma(c-a) \sigma(c+a) \sigma(b-d) \sigma(b+d) = 0 \end{aligned}$$

for all values of a, b, c, d , and also the condition

$$\text{Lt}_{u \rightarrow 0} \left(\frac{\sigma(u)}{u}\right) = 1,$$

deduce that $\sigma(u)$ is an odd function of u , which satisfies the relation

$$\frac{\sigma(a+b) \sigma(a-b)}{\sigma^2(a) \sigma^2(b)} = \frac{d^2}{da^2} \log \sigma(a) - \frac{d^2}{db^2} \log \sigma(b).$$

Assuming that the expansion of $\sigma(u)$ in ascending powers of u is

$$\sigma(u) = u + A \frac{u^5}{5} + B \frac{u^7}{7} + C \frac{u^9}{9} + D \frac{u^{11}}{11} + \dots,$$

find C and D in terms of A and B .

5. Show that an elliptic function of u , whose zeros and poles are known, can be expressed in the form

$$c \Pi \sigma(u - a_r) / \Pi \sigma(u - b_s).$$

Find this expression for the determinant whose r th row is

$$| 1, \wp(u_r), \wp_1(u_r), \wp_2(u_r), \dots, \wp_{n-1}(u_r) |,$$

where $\wp_r(u_r)$ denotes $\frac{d^r}{du_r^r} \wp(u_r)$.

Show that

$$(-1)^{n-1} [2! 3! 4! \dots (n-1)!]^2 \frac{\sigma(nu)}{[\sigma(u)]^{n^2}}$$

$$= \begin{vmatrix} \wp_1 u & \wp_2 u & \wp_3 u & \dots & \wp_{n-1} u \\ \wp_2 u & \wp_3 u & \wp_4 u & \dots & \wp_n u \\ \dots & \dots & \dots & \dots & \dots \\ \wp_{n-1} u & \wp_n u & \wp_{n+1} u & \dots & \wp_{2n-1} u \end{vmatrix}.$$

6. A point charge of strength e is placed at a point P outside a dielectric sphere of centre C . Find an expression for the electrical potential outside the sphere in spherical harmonics.

Show that the field outside the sphere is the same as that due to the original charge together with a point charge $-e\beta a/b$ at the inverse point Q and to a line charge of density

$$e\beta(1-\beta)(x/c) \frac{-(1+\beta)}{2} / 2a$$

extending from C to Q , where x is measured from C , a is the radius of the sphere, b is the distance PC , $(1+\beta)/(1-\beta)$ is the dielectric constant, and c is equal to QC .

7. A conductor is placed in a field of force, the resolved part of which, due to external sources, along the outward drawn normal at a point P_0 is N_0 . If σ_0 be the electrical density at P_0 and σ the density at any other point P on the surface, prove that

$$N_0 = 2\pi\sigma_0 - \iint \frac{\sigma dS}{r^2} \cos \theta,$$

where θ is the angle between PP_0 and N_0 and r is the distance PP_0 .

Prove that, if the surface of the conductor is spherical, the homogeneous integral equation

$$0 = 2\pi\sigma_0 - \frac{1}{2n+1} \iint \frac{\sigma dS}{r^2} \cos \theta$$

is satisfied if σ is a surface harmonic of the n^{th} order.

8. A thin plane sheet of metal, of infinite length and of uniform width, is electrified. Prove that the density at any point is inversely proportional to the square root of the product of the distances of the point from the two edges.

Transform this theorem by inversion.

9. Prove that the mutual potential energy of two permanently magnetized spheres, of radii a and of intensities of magnetization I and I' , is

$$\frac{16 \pi^2 a^3}{9r^3} II' (\cos \epsilon - 3 \cos \theta \cos \theta'),$$

where ϵ is the angle between the directions of I and I' , θ and θ' the angles which these directions make with the line joining the centres and r is the distance between the centres.

Prove that, if the distance apart of the spheres is great compared with their radii and if the second sphere is of permeability μ and of no permanent magnetism, then the mutual potential energy is

$$-\frac{16\pi^2 a^3}{9r^3} \frac{\mu - 1}{\mu + 2} I^2 (1 + 3 \cos^2 \theta).$$

VI.

1. Show that if the coefficients p_1, p_2, \dots, p_n in the expression

$$\phi(x) \text{ where } \phi(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n,$$

are so determined that the numerically greatest value of $\phi(x)$ between the limits -1 and $+1$ for x is as small as possible, $\phi(x) \equiv \frac{1}{2^{n-1}} \cos n(\cos^{-1} x)$.

2. A curve of degree m and class n has δ double points, κ cusps, ι inflexions, τ double tangents; it passes f times through a circular point and touches the line at infinity g times. Find Plücker's numbers for its evolute, and show that the deficiency of both curves is the same.

Verify your results in the case of (1) a central conic, (2) a parabola.

$$[n = m(m-1) - 2\delta - 3\kappa, \quad \iota = 3m(m-2) - 6\delta - 8\kappa].$$

3. (1) Show how, by a real transformation of the form

$$x = (py + q) / (y + 1), \text{ the integral } \int \frac{dx}{(Ax^2 + 2Bx + C) \sqrt{(ax^2 + 2bx + c)}}$$

may be made to depend on integrals of the form $\int \frac{dx}{(A'x + C') \sqrt{(a'x + c')}}.$

(2) Prove that the solution of the equation

$$\frac{dy}{dx} = \frac{cx + fxy + gy^2}{ax^2 + bxy + cy^2}$$

$$\text{is } \begin{matrix} \log(x + \lambda_1 y), & a + e\lambda_1, & g + e\lambda_1^{-1} \\ \log(x + \lambda_2 y), & a + e\lambda_2, & g + e\lambda_2^{-1} \\ \log(x + \lambda_3 y), & a + e\lambda_3, & g + e\lambda_3^{-1} \end{matrix} = \text{constant},$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation

$$e\lambda^3 + (a - f)\lambda^2 + (g - b)\lambda + c = 0.$$

How must this be modified if (i) two, (ii) three roots of this equation are equal?

4. Prove by contour integration (or otherwise) that, if $\pi > a > -\pi$,

$$\int_0^{\infty} \frac{\cosh ax}{\cosh \pi x} \cos bx dx = \frac{\cosh \frac{b}{2} \cos \frac{a}{2}}{\cosh b + \cos a}.$$

5. Show that the equation of the locus of the foci of all central sections of the conicoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

is
$$\Sigma x^2 \cdot \Sigma \frac{x^2}{a^2} \cdot \left[\Sigma \frac{x^2}{a^2} \cdot \Sigma a^2 x^2 - (\Sigma x^2)^2 \right] = \Pi \left(a^2 \Sigma \frac{x^2}{a^2} - \Sigma x^2 \right).$$

6. A surface is the envelope of a sphere, whose centre moves on a fixed guiding curve whilst its radius varies in some known manner. Prove that one set of the lines of curvature of the surface are circles, and find expressions for the principal radii of curvature at any point.

A tubular surface is traced out by a sphere of constant radius b , whose centre describes a helix; O is any point on the helix, C is the centre of curvature at O . Prove that the principal radii of curvature of the surface at any point in the section by the normal plane of the helix at O are b , and $b \cdot CM/OM$, where M is the projection of the point on CO .

A certain curve is the path of a particle, if properly projected in each case, when acted on by each of two systems of forces; find the conditions of projection that the curve may be the path when the particle is acted on by both systems of forces.

A particle of unit mass is acted on by two forces, one, $= \nu OP^{-2}$, to the origin, another, $= \mu PC$, to a fixed point $C(-c, 0)$. Prove that there is a possible orbit of the form $Au = B + \cos \theta$, and that the corresponding value of h^2 is $F + A\mu \nu c^{-2}$, where A, B, F are constants, and h is the angular momentum about the origin.

7. Show that in any case the orbit may be represented by the same equations, where A, B, F are functions of θ satisfying the equations

$$-\frac{u^3}{\mu c} \frac{dF}{d\theta} = u \frac{dA}{d\theta} = \frac{dB}{d\theta},$$

and
$$F \left(\frac{B^2 - 1}{A} - \frac{1}{c} \right) = K + EA + 2\nu B - \mu c A^2$$

where K and E are constants.

8. H is the angular momentum of a material system about an axis whose direction, but not necessarily its position, is fixed in space. If U is the velocity of the axis perpendicular to itself, V the component perpendicular to the axis of the velocity of the centroid of the system, and θ the angle between the directions of U and V , prove that the moment of the effective forces of the system about the axis is (with a suitable convention as to the sign of θ) $\frac{dH}{dt} - MUV \sin \theta$, M being the mass of the system.

A homogeneous cubical block, of side $4a$, rests in equilibrium at the vertex of a rough paraboloid of revolution, of latus rectum $4a$, whose axis is vertical and vertex upwards. Show that, if the block receives a slight displacement by rolling on the paraboloid so that its motion is uniplanar and the plane of motion is parallel to two opposite faces of the block, the equilibrium is stable; and, if a is the extreme angular deflection of the block from the vertical, show that the time of a complete oscillation is

$$\sqrt{\frac{20a}{\pi g}} \left\{ \Gamma \left(\frac{1}{2} \right) \right\}^2 a^{-1/2}$$

9. (1) If u_n is any solution of Bessel's equation of order n ,

$$\left\{ \left(x \frac{d}{dx} \right)^2 + x^2 - n^2 \right\} y = 0,$$

and if u_{n-1} , u_{n+1} are quantities defined by the equations

$$u_{n-1} \equiv \left(\frac{d}{dx} + \frac{n}{x} \right) u_n, \quad u_{n+1} \equiv - \left(\frac{d}{dx} - \frac{n}{x} \right) u_n,$$

prove that u_{n-1} , u_{n+1} are solutions of Bessel's equations of orders $n-1$ and $n+1$ respectively.

Show also that $\left(\frac{d}{dx} + \frac{n+1}{x} \right) u_{n+1} = u_n$,

$$- \left(\frac{d}{dx} - \frac{n-1}{x} \right) u_{n-1} = u_n.$$

(2) Show that, when n is an integer,

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin \theta) d\theta.$$

RECENT CHANGES IN THE MATHEMATICAL TRIPOS AT CAMBRIDGE.

As I am writing for an International Commission it may be well to explain certain features in English education which, though familiar to Englishmen, may not be equally well known to others.

English university education may be said broadly to be dominated by examinations. The teacher is subordinate to the examiner and the course of study is determined, not by the tastes of either teacher or student, but by the schedule of the examination which the student is aiming at taking, whereas in some other countries at least the course of study is primarily settled between teacher and student and the student's proficiency is finally tested by his teacher, whether by means of a dissertation or by an examination. Many of the modern English Universities have adopted what may be regarded as a compromise between the two systems, as the examinations are conducted jointly by internal examiners who are the students' teachers and by external examiners from other Universities, and though the outlines of the ground covered are laid down in printed schedules, the details are to a great extent determined by the internal examiners' knowledge of what has actually been taught. In Cambridge, on the other hand, the papers are set in accordance with tolerably full schedules, the detailed interpretation being supplied largely by tradition, partly, perhaps, by reference to what is supposed to be generally taught throughout the university. In Mathematics at least, owing to the number of teachers, only a small number of candidates will, in general, have received any substantial amount of teaching from the actual examiners of any given year, and to secure as far as possible equal opportunities for all candidates, custom prohibits an examiner from giving any individual instruction to any candidate during a period of some months prior to the examination, a period during which the examiner may be supposed to have some idea of what questions will be set. Most of the examiners are usually chosen from the resident body of University or College teachers; such non-residents as are appointed are old Cambridge men usually fairly conversant with the current mathematical teaching.

As a consequence of this system any attempt to change the course of study takes the form of an alteration in the regulations for the examinations.

Students at the University take either pass examinations or an Honours course; the latter alone can be regarded as serious students, and I confine myself to them.

Candidates for Honours have to pass an elementary examination in a variety of subjects, known as the Previous Examination. It is a University examination, but is usually taken at school or quite early in the University course; and various other examinations, primarily designed for schools, may be substituted for it. Substantially the whole of the University course, which normally lasts over three years, is then devoted to work for one or more Tripos Examinations,* which are specialised examinations, each dealing with one subject, such as Mathematics, Classics, Natural Sciences, History, Mechanical Sciences (Engineering), &c.

Up to about 1882 there was a single tripos examination in each subject, usually taken about the middle of the fourth year. Then concurrently with other substantial changes most of the triposes were divided into two parts, of which Part I. normally taken at the end of the third year, carried a degree (Bachelor of Arts) with it; while Part II. was a more advanced examination normally taken at the end of the fourth year. Part II. of the Mathematical Tripos was for a short time held in January, but was soon moved to the end of the fourth year.

At the same time an important alteration was made in the form of the class-list in the Classical Tripos and some others by the abolition of the "order of merit." This curious institution which is, I believe, peculiar to Cambridge deserves a little explanation. The successful candidates in a Tripos are divided into three classes, according to the merit of their performances, numerical marks being assigned to each question, the marks obtained by each candidate are added up, and an order thus obtained. In those Triposes in which the order of merit prevailed, the candidates' names were published in this order in each class; so that any successful candidate was stated to rank 1st, 2nd, . . . 50th, . . . in merit among the competitors. The first in the whole list sometimes received a special honorific title, such as Senior Wrangler (in the case of Mathematics) or Senior Classic. The system obviously involved great confidence in the accuracy of marking attainable; and its prevalence and popularity might be ascribed in part to the general British interest in sport. University studies were reduced as far as possible to a semblance of a race; the candidates were the horses, their teachers the trainers, the examiners the judges. In Cambridge great interest was always felt in these competitions, especially in the Mathematical Tripos, large crowds always attending when the list was read out.† Even the general public, usually little interested in University matters, were involved; the lists were telegraphed to the newspapers, short biographies

* For an explanation of the meaning of this curious name, and an historical account of the growth of the mathematical tripos, see W. W. R. Ball, *History of Mathematics at Cambridge*.

† An entertaining account of the attitude of the University is to be found in the anonymous *Sketches of Cambridge by a Don*, now known to have been written by the late Leslie Stephen.

of the Senior Wrangler and other high wranglers were published, and on these occasions, if no other, Cambridge attracted more attention than its better known rival, Oxford. The Senior Wrangler in particular acquired great reputation in academic circles and some popular fame.*

From 1882 onwards there has been no "Senior Classic," the successful candidates for the Classical Tripos being henceforward divided into three classes, with some subdivisions. Other Triposes were altered in a somewhat similar way about the same time or a few years later, and none of the newer Triposes which were inaugurated to meet the growth of new studies, adopted the order of merit. In the Mathematical Tripos, however, the order of merit survived for nearly 30 years more.

The examination for the Mathematical Tripos is conducted entirely by means of printed papers of questions to be answered by the candidates in a limited time (three hours per paper), without any assistance from books or notes. The standard type of question consists of two parts; first comes a piece of "bookwork," that is a known theorem, the proof of which is required to be given; this is followed by a "rider," or example, which is, at least in theory, a consequence of the "bookwork," but is in practice not infrequently rather loosely connected with it. Before the last change in the regulations there were also set two papers of "problems," exercises of a more difficult kind with no indication given by the "bookwork" as to the method to be used in the solution. Such "riders" and "problems" are very characteristic of Cambridge Mathematics; they are to be found in abundance in any ordinary English text-book and even treatises of an advanced type. It is enough to refer to two books which are to some extent known outside England, Routh's *Treatise on Rigid Dynamics*, and Dr. Forsyth's *Treatise on Differential Equations*; Salmon's well-known geometrical treatises contain a smaller number of such examples. As far as I know examples of the Cambridge type occur sparingly in German text-books, except in very elementary ones, but are becoming more common in certain types of French books; for example, in M. Goursat's excellent *Cours d'Analyse* there are a good many examples somewhat of the Cambridge type.

The subject matter of the questions covers a large range of subjects in Pure Mathematics and in "Applied Mathematics" (or Natural Philosophy), about equal weight being attached to each of the two divisions. The Pure Mathematics for Part I. of the Tripos, according to the regulations which prevailed up to 1909, comprised a large range of subjects, from elementary geometry, algebra and trigonometry up to such subjects as analytical geometry of three dimensions, (including some of the

* There is a well-known story, the authenticity of which I do not guarantee, that a young man who had recently been declared Senior Wrangler, happening to enter a theatre simultaneously with a Royal party, bowed gracefully in acknowledgment of the applause with which he imagined that he was being received. In some recent years the Senior Wrangler's portrait has been published as a picture postcard.

simpler parts of differential geometry), differential equations, the elements of calculus of variations, and of elliptic functions. No theory of functions was explicitly included, though a good deal of what might be claimed as elementary theory of functions was in fact included as algebra and trigonometry (theory of infinite series and products), or integral calculus (Cauchy's integration theorem, &c.). The Applied Mathematics ranged from the elements of mechanics to quite difficult three dimensional dynamics of rigid bodies, hydrodynamics, electro-magnetism, and spherical astronomy.

Some serious difficulties in carrying out the system of examination are obvious.

In the first place it is clearly very difficult to set questions suitable for a large number of candidates of very varying ability and knowledge, including some who may be the leading mathematicians of the coming generation, and others who with little aptitude for or interest in the subject, acquire the minimum of knowledge requisite for obtaining a degree, and have no intention of paying any further attention to the subject in after life. This difficulty was in part met by dividing the examination into two parts (the "first four days" and the "second four days") separated by a short interval of time, and dealing respectively with the easier and the harder subjects. A candidate who did sufficiently well in the first four days obtained his degree, even though he did nothing whatever in the second four days, but the marks obtained in both parts of the examination were added together to determine the final class list and the order of merit. Thus the weaker candidates could and in practice did study only a small part of the whole range; and the marks obtained by the majority of such candidates in the second four days were often ludicrously small. But the fact that the best candidates also took the easier part of the examination involved the setting of riders of a quite difficult character even on the simplest subjects.* Thus a very large part of the examination was wholly unsuitable for a considerable part of the candidates, while on the other hand the best men were compelled, if they worked for a high place in the final list, to carry in their heads proofs of propositions and to retain facility in the rapid working out of examples belonging to the elements of the subject. A man who might in two or three years be making serious original contributions to mathematical science was liable to lose a much valued place in the list if he had forgotten, say, the proof of some curious but trivial property of the triangle†. Moreover, the good mathematician who would naturally at this

* There is some difference of opinion as to the validity of the cause here assigned for the difficulty of the riders and problems, though there is a general agreement as to the fact.

† I know of a case of a senior wrangler who, having allowed an insufficient number of hours for revising his school knowledge of elementary geometry just before the examination, came out from his first paper a good deal perturbed because he had been unable to reproduce the proof of one of Euclid's propositions.

stage of his career have a bent towards certain departments of mathematics was much discouraged from any kind of specialisation. The pure mathematician inclined to pursue the study of higher analysis would be checked by the necessity of being able to answer questions on geometrical optics, while the man with a real interest in electricity would be diverted from any attempt to read about electrons and be compelled to give his attention to the subtleties of uniform convergence.

This last difficulty was intended to be met partly by the institution of Part II. of the Tripos. This examination comprised more difficult subjects than Part I., and a candidate was expected to specialise to a considerable extent. He might, for example, spend his fourth year studying theory of functions and analytical geometry, or electricity and hydrodynamics. The class list was not arranged in order of merit, but in classes with subdivisions. On the other hand, it is evident that advanced knowledge is peculiarly unsuitable for testing by answers to printed questions to be written in a very short time; and candidates necessarily wasted a good deal of effort in committing to memory results and formulæ which a professed mathematician would naturally take from a book when he wanted them. For example, it is clearly unreasonable to expect a man to carry in his head the elaborate formulæ belonging to lunar theory or to Jacobi's theta-functions, and it is difficult to test by what a candidate could write in half-an-hour, whether he understands the elaborate argument of Gordan's proof of the finiteness of the number of concomitants of a binary form. I am, however, inclined to think, differing in this respect from many of my colleagues, that, indefensible as the examination for Part II. was in theory, the time spent in studying for it was, on the whole, well employed, and that the class lists issued represented, with at least rough justice, the merits of the candidates.

It will be noticed that in the examination system thus sketched, no place is assigned to any kind of original research. The examinations tested knowledge and that limited form of originality which consisted in applying knowledge very rapidly to such application of theory as could take the form of examination questions. Direct encouragement to research has been given by the University by means of the two Smith's prizes (to which the Rayleigh prizes have been added recently) which have been given since 1885 for original dissertations. These prizes are of little pecuniary value, but are much esteemed as distinctions. The dissertations are sent in about the middle of the fifth year, so that a candidate who is not compelled to earn his living at once, has been able normally to give six months to his dissertation after finishing with examinations, and may have been able also to give some consideration to this work while mainly engaged in preparing for Part II. of the Tripos. Encouragement to research in a different form has been given

by the Colleges, which, to an increasing extent, though by no means universally, award the much coveted distinctions of Fellowships* for original work in addition to or in lieu of success in examinations.

Another serious defect, which has been very generally attributed to the system of the Mathematical Tripos, is the almost complete divorce between mathematics and experimental physics. Although the mathematician has given about half his time to "Applied Mathematics" he need have, and in fact frequently has had, no knowledge of experimental physics. Normally, he goes to no experimental lectures, he does no work in a laboratory, and the experimental facts which he learns in his mathematical text-books are usually of the simplest character, reduced to an abstract and almost conventional form, suitable for the direct application of mathematical analysis. A high wrangler may be able to solve elaborate problems in spherical trigonometry or optics without having seen a telescope or handled a lens; he may be able to calculate the potential due to the most curious distributions of electricity, without the least idea of the mechanism of an electric bell or tram. Physics learned in this way is naturally most unreal, and the mathematician who wishes afterwards to devote himself to physics, is at first at a great disadvantage, not only by want of familiarity with physical apparatus and physical data, but by a lack of the "physical instinct," which enables the trained physicist to judge what elements are important or what unimportant in any particular investigation. All his previous training leads him to be prepared to ignore in any problem whatever his teacher or his examiner tells him to ignore in order to obtain a particular mathematical result; and the examination system tends to eliminate from questions all restrictions which cannot be subjected to reasonably short and simple analytical treatment.† Thus the mathematician acquires the habit of regarding all bodies as rigid and most of them as frictionless, of ignoring absorption of light in lenses or in the air, of assuming that an electric charge remains permanently on a conductor of any shape which is susceptible of mathematical treatment.

While our mathematicians study physics in this highly abstract way we have in Cambridge two other important groups

* A "Fellowship" is a most difficult institution to explain to anyone who is not familiar with Oxford and Cambridge. I can only state here that a Fellow receives from his college an income not usually differing very much from 200*l.* a year for six years or more, and that most appointments in the colleges, as well as a large number of University appointments, fall to Fellows.

† The typical Mathematical Tripos question on mechanics is supposed to begin: "An elephant, whose mass may be neglected . . ." A more serious illustration is that a mathematician frequently has difficulty in remembering whether the enormous factor $e = 3 \times 10^9$, which occurs on passing from electrostatic to electro-magnetic units, should be put into the numerator or denominator of a formula.

of students studying the same or closely allied subject matter in different ways. We have large and flourishing schools of Experimental Physics and of Mechanical Sciences (Engineering). Experimental Physics is one of the subjects in the Natural Sciences Tripos. A student taking Part I. of the Tripos, which is the ordinary examination giving an Honours degree in the subject, usually takes three or four branches of natural science, of which one may be Physics. With Physics he may take perhaps Chemistry and Botany. If he goes on to take Part II. (normally at the end of his fourth year) he may take one subject, such as Physics. A student of Mechanical Sciences passes normally a preliminary examination in Mathematics and mechanics, usually about the end of the first year, and then devotes himself to work for his Tripos, which is ordinarily taken at the end of the third year and comprises mainly various branches of mechanics and physics, as connected with mechanical engineering. A student who takes Physics either for the Natural Sciences Tripos (especially in Part II.), or for the Mechanical Sciences Tripos is expected to be able to use a fair amount of Mathematics and is clearly at a great advantage in the further pursuit of either subject if he is an accomplished mathematician, but the exigencies of the course which he has to pursue make it difficult for him to devote much time to systematic mathematical work, and he is liable to study the subject superficially and too much from the main point of view of immediate practical utility.*

It might seem natural that a student interested in physical and engineering subjects should study them *pari passu* from the mathematical and experimental side. The tripos regulations have, however, made this difficult, and the nearest approach to a combined course has been pursued by the men who have taken the Mathematical Tripos first, and then one or two years later the Natural Sciences or Mechanical Sciences Tripos. Though this plan has been pursued with excellent results, it is liable to involve an undue prolongation of the University course, while there is at the same time a serious risk of a loss of places or of a class in one or the other tripos through insufficient time for preparation, and the mathematical course not only suffers from the physical unreality already referred to, but involves the study of branches of Mathematics which, though interesting in themselves, are of little use towards the comprehension of the principles or the practical use of the kind of Mathematics which the physicist needs.†

* I should explain here that I have little acquaintance with the course of study for these two triposes, and the impressions that I here give might be challenged by the responsible teachers in these departments.

† For example, my engineering and physical friends speak very slightly of all but the most rudimentary knowledge of the properties of curves and surfaces, to which a good deal of attention is properly paid in the Mathematical Tripos.

In giving this preliminary account of the Mathematical Tripos as it was before the last change, I have incidentally referred to the main objections brought against the old system. Briefly, it was urged that the examination, and particularly the riders and problems set, were much too hard for the weaker men; that the better men were compelled to keep fresh to the end of their career elementary and trivial knowledge, and spend over difficult, though essentially trivial riders and problems, much time that might more profitably have been given to higher theory; that such specialisation as consisted in paying more attention to congenial than to uncongenial parts of the subject was discouraged by the whole system, and particularly by the order of merit; and, lastly, that Mathematics was studied apart from experimental physics and thereby suffered in reality.

One other point connected with the order of merit deserves mention. It was argued that a definite numerical place in the Mathematical Tripos, particularly amongst the high wranglers, was a "hall-mark" placed by the University on its finished product, which was much prized in itself and was of great value in obtaining educational and other appointments. To this it was replied that the stamp thus given corresponded very insufficiently to reality. Apart from the fact, that the kind of ability tested by examination was only of a very special kind, and might be much less important than other kinds, notably that of producing original work, the mere accidents of examination, such as indisposition on particular days, resemblance of questions to those answered before by the student's teacher, speed of handwriting, tastes of individual examiners, &c., were enough to make minute classification largely illusory. If it was conceded that the senior wrangler of any year was probably abler than the 10th wrangler of the same year, there was but little presumption that under slightly different conditions the 10th and the 11th wrangler would not have come out in a different order; and still less could it be asserted that a 10th wrangler of one year was equal to the 10th and superior to the 11th of another year. It was, in fact, not uncommon to have experienced teachers judging the senior wrangler of one year to be inferior in every way to one who was several places lower in another year.

In addition to the criticisms that might be made *a priori*, the case for a reform was supported by the undoubted fact that the Mathematical Tripos was ceasing to attract students in the same numbers as before. A fairly steady and continuous decline in numbers had been going on for years. Between 1882 and 1909, a period during which no changes of first rate importance had been made in the examination, the number of successful candidates in Part I., had fallen from 116 to 74; the corresponding numbers in Part I. of the Natural Sciences Tripos, were 54 and 174, while the class list of the Mechanical Sciences Tripos contained 7 names when it began in 1894 and 31 in 1909. It seemed clear that mathematicians were drifting

away to other subjects; part of the drift was doubtless due to other causes, but it was none the less disturbing to those responsible for the teaching of Mathematics in the University.

An attempt to remodel the tripos and in particular to abolish the order of merit was defeated in 1900. A different scheme was shortly afterwards set on foot and, after elaborate discussion in committee and more publicly, was passed in 1906.

Although the authors and supporters of this scheme were doubtless influenced by various reasons it may be said that the leading ideas of the scheme were to enable those who would not devote their whole University course to Mathematics to specialise earlier than before and to make it easy for others to devote part of their time to Mathematics as a preparation for study of the allied experimental sciences. With these objects in view a new and much simplified Part I. was instituted which could be taken at the end of the 1st or 2nd year and did not carry with it a degree. The subjects comprised bore a general resemblance to those of the "first four days" of the old Part I., but with some important modifications designed to check the over-elaboration of the details of the elementary subjects, replacing these by the simpler parts of more advanced subjects. Thus the schedule of dynamics was extended to include elementary rigid dynamics and analytical geometry was no longer restricted to two dimensions.* At the same time a resolute effort was made materially to alter the character of the questions traditionally set, by regulations laying stress on the importance of simple riders, by the abolition of the problem paper, by reference to numerical work, by the requirement of some knowledge of physical phenomena, and lastly by a regulation that one of the examiners should in general be a teacher of physics or engineering. At the same time the Pure Mathematics was to some extent lightened by a schedule which substantially excluded the theory of convergence, though some power of handling certain important infinite series was still to be expected. Successful candidates were to be placed in three classes with no order of merit and no subdivision of the classes. It was hoped that work for the examination would be a useful course of Mathematics for physicists and engineers, who after taking it could in general give two years to their main subject, while the better class of mathematicians could take it without any serious interruption to their main work, and then abandon the effort to keep their knowledge of elementary subjects in a form fit for examination purposes. Further, it was pointed out by some supporters of the scheme, that weaker mathematicians with no special bent for the physical sciences might with advantage take this simplified course of Mathematics and then spend one or two years more profitably on some quite different subject, such as history.

* The detailed schedules of this examination and of Part II. are given in the Appendix

The new Part II. was to be taken normally at the end of the third year, and to be like most of the other triposes a degree-giving examination. The subjects of examination were divided into two groups, enumerated in Schedule A, and Schedule B,* which may briefly be described as compulsory and optional. Schedule A covers approximately the same ground as the "second four days" of the old Part I., but regulations were again introduced which abolished the "problem" paper, aimed at the simplification of the riders and at increasing the reality and physical significance of the questions on applied Mathematics. Schedule B consisted of a number of loosely defined branches of Higher Mathematics, not differing very materially from the corresponding list in the old Part II., though the authors of the scheme insisted on the importance of simplification in the type of question and framed regulations to ensure this result as far as possible.

Candidates were free to ignore Schedule B altogether or to take one or more of the subjects in it, and were further at liberty to define more precisely the range of their work in these subjects, so as to guide the examiners in the choice of questions to be sent. There was no order of merit, the successful candidates were to be placed in those classes in the first instance entirely on the results of the examination in Schedule A, while proficiency or distinction in subjects in Schedule B was to be indicated by special marks. Thus a man in the second class might be declared to have shown a competent or even distinguished knowledge of subjects of Schedule B and one in the first class might have obtained no such distinction. A limited power was, however, given to the examiners to raise a man on the border line into a higher class on account of his work in the subjects of Schedule B. The object of this rather elaborate method of drawing up the class list was to ensure that the better men, while they acquired a fair all round knowledge of the subjects of Schedule A should nevertheless be encouraged to specialise according to their tastes in some of the subjects of Schedule B, while men of less ability could devote themselves exclusively to the subjects in Schedule A. To give further protection for weaker men, whose interests were felt by many to be seriously jeopardised by the new scheme as a whole, a regulation was added that questions of a specially simple character, marked as such and partly on subjects included in Part I. of the tripos, should be set in the examination on Schedule A, satisfactory answers to which should qualify for a place in the class list.

The first examination for the new Part I. was held in 1908, and for the new Part II. in 1910. Any attempt to estimate the success of the new regulations on the basis of experience and merit must therefore be very tentative. The papers set in the first

* See Appendix.

four years for the new Part I. appear to have met with general approval* ; the riders have in general been simple and those on Applied Mathematics at any rate more physical in character than had been the case in the past. Moreover, the tripos has succeeded in attracting candidates, the average number of successful men in the five examinations being a little more than a hundred.

The new Part II. has so far been less successful ; the papers set on the first occasion were very generally condemned by teachers in the University as much too difficult† ; there was some improvement in the following year, but it still seems very doubtful whether the examiners will be able to free themselves from most of the evils of the old tradition. The average number of successful candidates in the three years 1910-1912 has been about 50. These figures seem to show that about half the students who pass Part I. then proceed to other studies. Although I have no statistical material available my impression is that the large majority take engineering or natural sciences, while a few take other subjects.‡ It seems, therefore, as if substantial progress had been made in bridging the gap between Mathematics and the allied Physical Sciences.

The new scheme seems to me also, on the whole, to have worked well for the better class of mathematicians. A large number have taken subjects of Schedule B, and many of them have done well. It has been a gain to teacher and pupil to save much of the wearisome "revision" towards the end of the academic course of elementary subjects. On the other hand, the absence of any fourth year examination seems likely to curtail the period of mathematical study for some men who might profit by it. British parents are generally reluctant to provide funds for studies leading to no tangible result in the form of an examination record, and it seems not unlikely that all but the very best students, who have hopes of winning a fellowship by research, will either leave the University at the end of their third year, or devote their fourth year to some other subject. Either result seems to me in general a matter of regret.

Lastly, such experience as we have had of the working of the new system seems to have done little to comfort those, who like myself, have from the beginning feared that the weaker men, who under the old regulations would have spent three years at Mathematics and obtained a second or third class,

* The papers set in 1912 have appeared too recently to be subject of criticism.

† I can report this unfavourable judgment the more easily as I was one of the examiners responsible for the papers.

‡ In my own college there have been this year 11 first year men who have been reading Mathematics for Part I. Of these, 2 intend to proceed to Part II., 1 intends to take the Natural Sciences Tripos, 7 the Mechanical Sciences Tripos, and 1 the Theological Tripos ; but these figures are obviously not a safe basis for generalisation.

would be discouraged from taking up the subject seriously from the difficulty of the new Part II., and either take some less difficult subject from the beginning or content themselves with the somewhat slender range of knowledge necessary for Part I. Much, however, still depends on the action of the examiners for Part II. during the next two or three years.

ARTHUR BERRY.

APPENDIX.

SCHEDULE OF SUBJECTS OF PART I. OF THE MATHEMATICAL TRIPOS.

Pure Geometry.—Lines, planes, circles and spheres : including questions on reciprocation, cross-ratios and inversion. The methods of orthogonal projection and of perspective.

Algebra and Trigonometry.—Exercises involving simple algebraic computation, the use of the slide rule and logarithmic tables being allowed. Elementary properties of equations. Use of the binomial, exponential and logarithmic expansions: † applications to trigonometric functions. Use and properties of trigonometric functions. Trigonometry of simple figures.

Analytical Geometry.—Lines, planes, circles and spheres, excluding questions requiring the use of oblique co-ordinates. Elementary properties of conic sections, and of an ellipsoid referred to its principal axes, including purely geometrical methods in suitable cases.

Differential and Integral Calculus.—Differentiation and simple integration, partial differentiation. Simple applications to plane curves. Volumes of solids of revolution. Use of Taylor's and Maclaurin's theorems: Maxima and minima for one independent variable. Curvature: plotting of curves from their equations. Simple linear differential equations.

Dynamics.—Mass, momentum, force, energy, moment of momentum, and their fundamental relations. Equilibrium without and with friction. Easy exercises on conservation of energy, of momentum, and of moment of momentum. Motion under gravity, and under simple central forces. Simple and compound pendulums. Elementary uniplanar applications to equilibrium and stability. Simple graphical statics: funiculars, frameworks. Statics of liquids and gases.

Elementary Electricity.—The elementary parts of electricity: namely, properties of the potential: lines of forces; air-condensers; Ohm's law with simple application galvanometers.

Optics.—Reflection and refraction of light: applications to mirrors, prisms, lenses and simple combinations, excluding spherical and chromatic aberrations.

SCHEDULES OF SUBJECTS FOR PART II. OF THE MATHEMATICAL TRIPOS.

SCHEDULE A.: (COMPULSORY).

Plane and Solid Geometry, including methods of Pure Geometry. Curvature of curves and surfaces. Plane and Spherical Trigonometry.

Algebra, including Theory of Equations.

Differential and Integral Calculus, including the first variation of integrals.

Elementary parts of the Theory of Functions, including the properties of simple functions of the complex variable, circular functions and their inverses: simple applications to doubly periodic functions. Simple properties of Legendre's functions and Bessel's functions. Use of Fourier's series.

Differential Equations.—Integration of ordinary differential equations of the first order, and of ordinary linear differential equations having constant co-efficients. Elementary theory of linear differential equations of the second order, having only regular integrals; solution of simple types of such equations of the second order, having integrals that are not regular. The methods of solving partial differential equations of the first order, and the use of methods of solving simple partial differential equations of the second order.

Dynamics, including Lagrange's equations and simple problems of motion in three dimensions. Elementary analytical statics. Elementary gravitational attractions. Vibrations of a stretched string.

Hydromechanics, including hydrostatics, the general principles of hydrodynamics with simple applications, elementary questions on the vibrations of elastic fluids, plane waves of sound.

Astronomy.—The elementary parts so far as they are necessary for the explanation of simple phenomena.

Electricity.—Fundamental principles of electrostatics, distribution of currents, magnetism, electro-magnetism.

Elementary optics, including the geometrical treatment of interference of waves.

SCHEDULE B.: (VOLUNTARY).

Theory of Numbers. Invariants and Covariants. Synthetic Geometry. Algebraic Geometry. Differential Geometry. Groups. Theory of Functions. Elliptic Functions. Differential Equations. Dynamics. Hydrodynamics. Sound and Vibrations. Statics and Elasticity. Electricity and Magnetism. Geometrical and Physical Optics. Thermodynamics. Spherical Astronomy and Combination of Observations. Celestial Mechanics.

THE UNDERGRADUATE COURSE IN PASS MATHEMATICS, GENERALLY, AND IN RELATION TO ECONOMICS AND STATISTICS.

The objects of this paper are (a) to describe the work done at present in Mathematics by external* candidates for the London Pass Degrees of B.A. and B.Sc., and to consider *de novo* what courses would be most suitable for students with their knowledge and opportunities, and (b) to discuss the mathematical preparation necessary for those who deal in applied statistics, biological or economic, and for those who wish to apply mathematical reasoning to the theory of political economy.

I.—THE UNDERGRADUATE COURSE IN PURE MATHEMATICS FOR EXTERNAL STUDENTS READING FOR THE LONDON B.A. (PASS).

These students (in general) must have passed the London Matriculation Examination. The normal course is then to take the Intermediate Examination after one year's preparation and the Final Examination after two years' further work. Many of the students are taught in University Colleges or other institutions throughout the country, others obtain correspondence lessons or private help. To the University they are merely unidentified units whose knowledge is tested by examination.

At matriculation the candidates must show moderate knowledge of arithmetic; of algebra, including primary operations, solution of quadratic and linear simultaneous equations, and progressions, with very elementary graphic work, but not indices; and of the plane geometry of triangles, quadrilaterals, and circles, but not of similar figures.

The Intermediate Examination extends the geometry to include similar figures and solids, and the algebra to include indices, logarithms, interest and annuities, permutations and combinations, the binomial theorem for a positive integral exponent, and the theory of quadratic equations. The mensuration of triangles, polygons, circles and regular solids, and trigonometry to the end of logarithmic solution of plane triangles are included. The graphic representation of simple algebraic functions is taken as part of the algebra.

In the Final Examination knowledge of the binomial, exponential and logarithmic series in algebra, of the sine and cosine series in trigonometry, and some acquaintance with

* The University of London deals with *Internal* students, who are taught in the Colleges and the Schools of the University, and with *External* students for whose teaching the University takes no responsibility.

simple tests of convergency are required; these form the subject of one paper. A second paper deals with the pure and analytical geometry of the conic sections, including simple projective methods in the one and a discussion of the general equation of the second degree in the other. Trilinear co-ordinates and abridged methods are expressly excluded. There is no calculus.

It should be added that, in all, five subjects must be taken for the Intermediate and four for the Final Examination, and that Mathematics is not an obligatory subject, so that students who have no mathematical capacity can avoid it and still have a wide choice.

It cannot be assumed that the students who work for this degree have more than the minimum knowledge required at the time they matriculate, for in many cases (especially with women students) Mathematics beyond arithmetic is not carried to any high standard during their school career. For such students the Intermediate Course provides a rather heavy year's work, (regard being had to the necessity of studying four subjects in addition to Mathematics), not so much because the actual contents of the syllabus is excessive or the examination papers artificially loaded, but because of the nature of text-books. Many of the text-books in common use deal in great detail with all the applications and complications the writers can devise of the simple principles involved, and teem with unimportant formulæ, with little discrimination of the essential from the bye-products; and the student, unless carefully guided, will spend weeks at the various types of useless conundrums on permutations, or highly technical problems on annuities, or complicated and artificial trigonometrical formulæ, or will load his memory with formulæ relating to truncated cones or areas of quadrilaterals. The following illustrate the examples and formulæ to be found in such books:—

EXAMPLES.

(1) Find the number of permutations that can be made from the letters of the word *examination* taken three at a time.

(2) I have three copies of Homer, two copies of Virgil and Dante and single copies of Shakespeare and Milton. In how many ways can I distribute them among nine friends, if each is to get a book.

(3) If $a, b, c,$ and d are any four consecutive terms in a binomial expansion, prove that $(bc + ad)(b - c) = 2(ac^2 - b^2d)$.

(4) An artisan rents a house from a Building Society for 40*l.* a year on the understanding that he is to become the possessor of the house in 20 years' time. If the Society reckons to pay 5 per cent. on its capital as dividend to its shareholders and the expenses of management amount to an extra $\frac{1}{2}$ per cent., find what the rent of the house should be if it were to remain the property of the Society.

(5) If $a, b, c,$ and d be sides of a quadrilateral taken in order, prove that $d^2 = a^2 + b^2 + c^2 - 2ab \cos \alpha - 2bc \cos \beta - 2ca \cos \gamma$ where α, β and γ denote the angles between the sides a and b, b and $c,$ and c and a respectively.

FORMULÆ.

(6) The volume of the zone of a sphere is

$$\frac{\pi h}{6} \{3(r_1^2 + r_2^2) + h^2\}$$

where r_1 and r_2 are the radii of the plane surfaces and h the distance between them.

(7) If S is the area of a cyclic quadrilateral whose sides are a, b, c, d the radius of the circumscribed circle is

$$\frac{\sqrt{(ab + cd)(ac + bd)(bc + ad)}}{4S}$$

These questions, of course, are of use in giving facility of manipulation and experience, and the formulæ are neat and are entitled to a place in a book of reference. But where time is so limited and there are so many important mathematical ideas which a student can hardly reach, it is a pity to devote much time to permutations (Ex. 1 and 2), which are mainly of use as leading to the Binomial Theorem, so far as elementary work is concerned; or to the terms and coefficients of the Binomial Theorem (Ex. 3), when its use is almost entirely for approximations; or to the technical subject of annuities (Ex. 4) which should only have a place as supplying illustrations of geometric progressions; or to artificial relations of a quadrilateral's sides and angles (Ex. 5) when all that is wanted in practical mensuration is done more easily otherwise, and the whole of spherical trigonometry and of the development of trigonometric series, is still untouched. With pass students the aim should be directness of expression and ready application of important principles, not dexterity in manipulation.

The work required for the Final Examination is very easy and could be mastered by a student with only moderate mathematical ability in one year instead of two, except perhaps in the case of pure geometry, where a long sequence of propositions and new methods must be followed. In the trigonometry no knowledge of complex quantities is compulsory, but it is expedient to have enough for simple developments from De Moivre's theorem; the rest is only a slight extension of intermediate work. The three algebraic infinite series named in the syllabus involve (for examination purposes) but a superficial knowledge of the nature of limits and convergency. The co-ordinate geometry depends on a first reading of simple principles, which need not take more than a few months if the student is not misled by the diffuseness of the text-books.

II.—THE UNDERGRADUATE COURSE IN PURE MATHEMATICS FOR EXTERNAL STUDENTS READING FOR THE LONDON B.Sc. (PASS).

The Intermediate Science syllabus is the same as that of the Intermediate Arts already described. A candidate only

offers four subjects in all; Mathematics is not obligatory and can be avoided by students taking a group of biological subjects and chemistry.

It may generally be assumed that Science students are better prepared in Mathematics prior to matriculation than are Arts students, and if they are taking physics or mechanics as one of their subjects they are helped by immediate practical use of their mathematical formulæ. The syllabus is therefore unduly light for students whose ability is sufficient for the science degree course.

The syllabus for the Final Examination is considerably heavier than in the case of Arts and the questions set are more searching. The pure algebra is nearly the same in the two cases, but the trigonometry involves more definitely the use of complex quantities, and of the formulæ depending on them. The examination includes a paper on geometrical drawing, necessitating considerable knowledge of projective properties and of conic sections. In addition the calculus is included—differential to Taylor's expansion in one variable, integral to the evaluation of areas and simple volumes. The co-ordinate geometry differs little from that of the Arts syllabus.

The course necessitates in the case of ordinary students two years' careful reading after the standard set by the Intermediate Examination is reached; but capable students who have had an adequate school training before matriculation could (if not hampered by the Intermediate Examination) obtain a fair working knowledge of the subjects, except perhaps of the geometry, in a year less. Those who have special difficulty in Mathematics frequently drop the subject after their Intermediate Examination.

III.—THE IDEAL UNDERGRADUATE COURSES IN MATHEMATICS.

In the remarks that follow it is not intended to criticise the procedure and regulations of the University of London in an adverse sense. The University has been limited by the conditions under which it works; the necessity for devising examinations where there is no control over the teaching is a serious handicap; the place of Mathematics must be decided with reference to the demands of other subjects. But, merely taking the courses just described as a text, it is proposed to consider *ab initio* the principles which should govern mathematical study in the abstract, and to suggest the best way in which these principles could be applied in the case of such students as do in fact present themselves for three years' study in letters or in science.

Arts (Letters) Pass Students.—Let us take as typical a student who selects two foreign languages, history, English, and Mathematics for Intermediate subjects, and the same (without English) for his Final Examination. The Mathematics

is alternative to logic, followed by philosophy, in the pure Arts course. (A mixed course also exists in which a natural science can be substituted for Mathematics.) In such a group of subjects there is no application of Mathematics to anything outside itself, unless to mensuration and algebraic extensions of arithmetic. We may assume that these elementary uses have already had their place in pre-matriculation work, and that they need only be kept in intermediate work to make easier the passage from the concrete to the abstract. In the London regulations, dynamics and statics form a distinct subject, and can in practice only be taken at the expense of a literary subject with the effect of making the whole course hybrid. It is an open question whether a wider course, embracing the more elementary parts of mechanics, would be more suitable; the present writer would incline to the negative both because of its inconsistency with the ideas which are indicated below, and because, with students unfamiliar with applied science, the mechanics would tend to degenerate into an imperfectly comprehended application of abstract principles to an unrealised universe.

The objective of the mathematical course should be to complete the education of a student whose other studies are linguistic or historical, and is comparable to that of the alternative study of philosophy. Applications need not be considered as important in themselves; these students will never have to solve a triangle, evaluate an annuity, construct a conic section, or use natural logarithms, in after life. The education should then be frankly abstract and deal in ideas. It is the logical, critical, analytical and generalising faculties that should be developed by the course. For this purpose the principal conceptions evolved by mathematicians—such as functional relationship, continuity of algebraic expressions and their relation to space, limiting values, generalisation from the simple to the complex with the help of conventions, the generalisations obtained by projection and by anharmonics, the nature of incommensurables, the final unity of geometry and analysis—should, so far as time allows—be presented to the student's thoughts. Naturally a good teacher would develop these ideas very gradually, building up the structure with examples and simple theorems; but these preliminaries should be definitely chosen with a view to the understanding of the broad idea. If the ability or time of the student does not permit him to get beyond the details and special cases to the general conception, the whole section of the work should be cut out; for example, it is useless (from this point of view) to treat of the parabola, ellipse and hyperbola unless it is to be shown that they are special cases of a general curve, the first being the limiting form of each of the others; or to take another instance, complex quantities should not be touched unless they are to be applied to the extension of a considerable part of the already known theory of the algebra and trigonometry of real quantities.

As a useful aid to generalisation and as invaluable examples of exactness and conciseness of expression, symbols (of functions, limits, summation, ranges, determinants, and so on) should be freely used.

When it has been decided in what direction and how far it is expedient to go, the means of travelling can be so chosen as to give the best scope for studying the relationship of the various ideas, for applying sheer analysis to obtain results from defined data, and for selection of the most suitable tools for the solution of problems. These are the processes which make the mind agile and resourceful. Further, accuracy should be trained both as to definiteness of ideas and as to numerical and geometrical results. There is, perhaps, no part of Mathematics in which these processes cannot be illustrated. In the existing syllabus of co-ordinate geometry, for example, relationship can be shown by obtaining the equations relating to the eccentric angle by orthogonal projection, analysis in reducing the general equation of the second degree to its special forms, selection of methods by suitable choice of axes, and accuracy by the tracing of curves from their equations.

It may be remarked that knowledge of what are here called "the means" are alone tested in the existing London examinations, which are confined to asking for reproductions of book-proofs and for the carrying out of simple mathematical operations. A B.A. student might pass very creditably without having understood or been aware of the existence of any of the broad conceptions enumerated above.

It is not the purpose of this paper to suggest a syllabus, but it may be well to outline the region which might be covered, during the three years' course from Matriculation to Graduation, by a student of fair ability, who had selected Mathematics as a more congenial study than philosophy. To begin with, the course should be lightened by discarding complicated but elementary equations, annuities (unless as illustrations), logarithmic solution of triangles, mensuration formulæ, and a good part of both the analysis and geometry of the separate conic sections. The algebra can then include that part of the convergency of infinite series (real variable) which follows most readily from the easier limits, and a study of the rational integral function both in relation to the real roots of an equation and to graphic representation. The co-ordinate geometry should in its earlier stages be closely related to the algebra, in the middle stages to geometrical conics, and in its final stages to harmonic properties; the simpler cases of families of curves, using abridged notation, might be included. The geometry should deal carefully but briefly with similar figures (both plane and solid), should deal as one subject with elementary geometry of three dimensions and projection, and should pass rapidly through the few significant properties of the special conics to a

simple projective treatment of the general conic. The trigonometry may be confined to the relations of the circular functions to each other, to their periodicity and functional nature, and to their development in series (without using complex quantities).

The more capable students could add to this one or more of the following: complex quantities, so far as the interpretation $\cos z$, e^z and e^{zi} (z complex), and the expression of $f(z)$ in the form $a + bi$ in simple cases; reciprocation and duality; homogeneous co-ordinates; three dimensional co-ordinates. In each case the subject should proceed by the simplest path to the more accessible of the important ideas.

It will be easily recognised by any teacher that each of the subjects named (except perhaps complex quantities) is suitable for progressive study, since in all cases the general ideas can be first reached by induction through simple illustrations, and that most of the subjects afford numerous facilities for problems of graded difficulty, leading to results that can be tested graphically or numerically.

Answers should in general not be given in text-books in this course; but the examples should be such as to prove themselves or as to give results that can be tested. By a lazy student an example is often done by guesswork and a vague use of formulæ, and if the book answer is obtained no further thought is spent upon it.

Examination papers are mainly devoted at present to the applications of methods, and they test memory, dexterity and accuracy. There should be a further paper in which answers more of the nature of essays are required, and adequate time should be allowed for candidates to develop their ideas. As examples of such questions the following may be suggested:—

(i) With what type of theorem is the geometrical method more appropriate than the analytical?

(ii) Consider the statement that "theorems involving only intersections, ratios, and tangency, and not angles, can be expressed with oblique axes as simply as and more generally than with rectangular axes."

(iii) Give a short account without proofs, of the geometrical properties that follow either from the theorem that a point, and the intersections of a line through the point with a conic and with its polar, make a harmonic range or from the theorem that the ratio of the rectangles $OP \cdot OP'$, $OQ \cdot OQ'$, where OPP' , OQQ' are lines meeting a conic at P , P' , Q , Q' , depends on the directions of the lines only.

(iv) In what respects are the properties of a parabola intermediate between those of an ellipse and of a hyperbola, and in what respects is a parabola the limiting form of an ellipse and of a hyperbola?

(v) Discuss in detail the method by which the exponential theorem is extended from the expansion of e^x to that of a^x .

Easier questions could, of course, be framed; but if a student has devoted one-fifth part of his time for three years to the study of Pure Mathematics of a University standard, and has also had a literary training, these subjects should be within his range.

IV.—SCIENCE PASS STUDENTS.

We may divide these into two classes: those who are studying physical sciences (physics and chemistry) and those whose work is mainly biological (zoology, botany and physiology). Geology belongs to both groups. Astronomy tends to be specialised applied mathematics and physics. Candidates may in London select any four subjects for their Intermediate and three for their Final Examination. Since four laboratory subjects involve too much time, Mathematics is commonly selected for the Intermediate Examination.

Students whose interest is biological are, it is alleged, frequently but poor mathematicians, and they generally give up the subject at the end of a year's work. For these it is well to have a straightforward common-sense course, involving simple and well-defined ideas. This is provided by graphic algebra, elementary trigonometry (not including logarithmic solution of triangles), and mensuration involving geometry, algebra and trigonometry. Considerable importance should be attached to accuracy of drawing, measurement and calculation. It should be borne in mind that their study of Mathematics ends with the year, and little should be begun that cannot in that time be brought to a definite stage. With a view to the developments of Mendelian experiments it might be well to spend some time on permutations, combinations and the binomial theorem; and able students might be introduced to the quite elementary parts of the theory of probability. Great care should be taken that time should not be wasted on uncomprehended methods and proofs, nor on formulæ whose utility is not evident. The case of biometrical studies is considered later (p. 208).

On the other hand students who are to deal with physics and physical chemistry can hardly hope for success unless they have mathematical intelligence and at least moderate ability. For these the effective standard of matriculation should be raised so as to include logarithms, a fair amount of graphic algebra, plane similar figures, and the trigonometry of one acute angle. These subjects are in fact commonly learnt by boys of this class before they leave school. There seems no adequate reason why some knowledge of mechanics should not be demanded at matriculation. Many students who choose physics have, in the writer's experience, a school knowledge of elementary statics (resolution of forces, parallel forces, moments, centre of gravity, friction and simple machines) and could easily have learnt the kinematics of uniformly accelerated motion. We will assume a preliminary knowledge of these.

Science students should obtain in their laboratory subjects a considerable part of the training described above as suitable for Arts students. They follow the progress of discovery, learn the relations of phenomena, and the uses of hypothesis,

induction and deduction. Their mental training will not suffer if the mathematical ideas, whose comprehension is so important for Arts students, are limited to those which are of most immediate application in their science. Within the region chosen, the theorems and the theoretical accuracy of methods should be rigidly demonstrated, to correct the tendency of the empirical mind to be satisfied with looseness of demonstration if the result seems practical. Among other things the precision of mathematical tables should be tested and the idea of relative accuracy of measurements inculcated.

It has very unfortunately become the custom to separate Applied Mathematics from Pure. If statics and dynamics were treated as experimental sciences this would be natural; but even after their segregation they remain abstract, depending throughout on mathematical methods, of which at the same time they form the best illustrations. It actually happens in London that a student who takes both physics and chemistry is obliged to drop his Applied Mathematics after his first year; for he is examined in three subjects only, so that Pure and Applied Mathematics are alternative, the latter involves the former, and the ordinary student cannot or will not find time for work far outside his syllabus. In fact Applied Mathematics is dropped before many of the principles used in ordinary physics are reached.

There seems no reason why Pure and Applied Mathematics should not be united for such students. For those with special mathematical ability they can still make two examinational subjects, viz., "General" Mathematics and Advanced Mathematics; these combined with Physics and with an intermediate knowledge of Chemistry would make as coherent a group as Physics, Pure Mathematics and Applied do in London now. On the other hand, "General" Mathematics (Pure and Applied), Physics and Chemistry would make an equally good group.

Intermediate Course.—Allotting one-third of the student's time to Mathematics, the year's work might cover, in Applied Mathematics, a revision of statics, and the laws of force, momentum, energy, work and direct collision in dynamics, and just enough hydrostatics to supplement the physical teaching of the subject. In Pure Mathematics it is most important to get to the differential calculus as soon as possible, and it is worth sacrificing much for this purpose. It is now generally recognised that the calculus does not need a preparation of advanced mathematics for its earlier and more immediately applicable stages, and that the ideas are sufficiently simple for youthful comprehension. In the first half of the year the algebra can be carried past the binomial theorem (positive integral index) to simple limits illustrated graphically, with but scant attention to permutations and combinations, and none to annuities. The calculus should follow at once, with illustrations from graphic algebra and from kinematics. There will still be time

with the better students to carry on trigonometry to the logarithmic solution of triangles, to do what pure geometry is necessary for the other subjects, and to take the mensuration of the surfaces and solids used by the physicist. So far as possible rigidity of proof should be preserved, but where results are involved, which are elementary from the physicist's and advanced from the mathematician's point of view, incomplete proofs may be temporarily given and their validity established in subsequent years.

Final Work.—A student equipped with this knowledge at his Intermediate Examination is well advanced towards the bare minimum needed in the physical sciences, and he need not give a large proportion of his time in his second and third years to Mathematics. For those taking *Chemistry, Physics and only "General" Mathematics*, the further work should include (i) the calculus carried forward to include the differentiations, integrations, differential equations and results generally needed in physics, with omission of those theorems and applications not so needed; (ii) the dynamics should include motion in a circle and simple harmonic motion at once, and a rapid outline course in rigid dynamics in two dimensions should reach the principle by which moments of forces are related to acceleration about a fixed axis, and careful work should be done on those simple rotation problems which involve easy moments of inertia only. Dynamics of a particle should stop short after a demonstration of Kepler's Laws; (iii) statics, to which little attention has so far been paid, should be revised and extended; (iv) the theory of attraction and potentials should only be taken to the extent required in elementary physics; (v) in algebra, the elementary theory of convergency (with its application to the binomial, exponential, logarithmic, sine and cosine series), and some knowledge of hyperbolic functions will be wanted in connection with the calculus. Unless considerable time is devoted to the elementary theory of complex quantities, it is better to manage without them.* (vi) co-ordinate geometry should be taken mainly from the graphic point of view, and should not be restricted to the conic sections, nor to two dimensions; but the three dimensional work must be very simple.

For these students, whose main interest is not mathematical, it seems advisable to arrange their training in the ideas of Pure Mathematics round the calculus as the central subject, and not to lay stress either on analytical manipulation or pure geometry. The proofs of the theorems they use should be rigid, but these theorems should be as few as possible, and their choice should be utilitarian. This apparently wide course could then be covered. "General" Mathematics is thus intended

* The treatment of complex quantities offered in many text-books is of very doubtful validity.

to mean a practical and elementary training in that part of general mathematical knowledge which presents no great intellectual difficulty and demands little analytical skill.

Perhaps some lectures should be given to them showing the scope of the ideas in Mathematics, both pure and applied, beyond those which are within their reading, and explaining without proof the more practical of the results achieved by mathematicians.

The remaining class of pass students, who would take "*General and Advanced Mathematics and Physics*", would be selected for their mathematical ability, and will not have been kept back (as happens with such students now) to a low stage at their Intermediate Examination. All the work enumerated for "general" mathematics can be done by them from the mathematician's as well as from the scientist's standpoint, and they can obtain the kind of training in ideas, advocated above for Arts students, with the advantage of having the whole field of dynamics to add significance and illustrations to them. For the individual it would be best to have no hard and fast syllabus, but to allow him to go on as far as he could; but as it is necessary to define a course, we should suggest that the London syllabus be lightened by putting little emphasis on the properties of the special conics (whether treated geometrically or analytically) and omitting everything that is not essential in dynamics of a particle. Analysis and statics and dynamics could then be extended to include the simpler properties in three dimensions; it is not necessary to learn all that has been done in a subject before the useful general elementary results can be used. Modern geometry and refinements of expression in co-ordinate geometry and algebra should have their place; but no one method or subject should interfere with the main intention of the course, which is to give students in physics a wide knowledge of mathematical principles and results.

Nothing in these suggestions is intended to apply either to engineering students or to honours students whether in Mathematics or any other subject.

V.—MATHEMATICS FOR THEORETICAL POLITICAL ECONOMY.

In this case we need not think of an undergraduate course, but consider simply (a) what preliminary mathematical training is good for a student of political economy in general, (b) what tools and methods are of most value in the specialised mathematical analysis of economic problems, (c) the minimum knowledge necessary for following simple mathematical exegesis of economic theory.

Political economy depends on the accurate definition of terms, which connote complex and abstract conceptions. Economic theorems are generally quantitative, though the quantities are often not measured. The theorems are often

illustrated by numerical examples (fictitious or otherwise), and the special cases taken may easily involve fortuitous relations not proper to the theorem. For accuracy the theorem must be thought of in general terms, and the general expression of the theorem should be algebraic; in fact, the possibility of the translation of the theory into mathematical language is sometimes a test of its truth. Theorems can often, however, be realised without actual notation, though they are essentially mathematical; and economists very generally state their arguments in words, whether in fact they can or cannot use notation with facility.

The mathematical conceptions most generally used are those of functions of one or more variables with their geometric representation, of limiting values and of maxima and minima. These involve the first and second (rarely the third) derived functions, total and partial, some use of Taylor's theorem, with the preliminary ideas of limits and the mathematical use of the word "infinite." Mathematical economists often illustrate their arguments by geometrical figures of two or three dimensions. Dynamics and statics are only used for illustration of less mathematical ideas by analogy.

As a mere matter of training, for preparing the mind to grasp the theory of political economy and without reference to its mathematical expression, the course sketched above (pp. 199-202) for B.A. pass students would probably be very useful. The searching out of the consequences of exact definition of difficult conceptions, the study of the number and character of conditions necessary to a theorem and the different lights given to a proposition by different methods which that course involves, have their close counterpart in economic theory. But the same results could be obtained, with no more effort, by a choice directed to the principles actually used by mathematical economists; thus, omit the whole geometry (pure and co-ordinate) of the conic section, omit the trigonometry beyond generalised definitions of the circular ratios, and omit harmonic ranges. Taking enough examples and illustrations from all available sources, pass through the elementary theory of limits and convergency to the differential coefficient, and carry on the calculus to partial differentiation, maxima and minima, and Taylor's theorem, including nothing that is not directed to these results except as subsidiary and illustrative. Hardly more than the definition of the process of integration is necessary. On the other hand treat the equations $y = f(x)$ and $f(x, y) = 0$ very much from the point of view of graphic algebra, and, taking as great a variety of functions as the student can easily be acquainted with (e.g., a^x , x^n , $\log x$, $a + bx + cx^2 + dx^3 + \dots$, $\frac{a + bx + cx^2}{d + ex}$, $\sqrt{a^2 - x^2}$, $\sin x$, $\cos x$, $\tan x$), find the directions at various points and maxima

and minima with the help of the calculus. With the examination of the nature and development of simple functions as the main line of work, it is easy to bring illustrations from any branch of Mathematics, and to make a coherent course involving geometry, algebra, trigonometry and analysis in any proportions.

Experience shows that an adequate knowledge on the lines indicated can be obtained by fairly mature students with rather marked but not unusual mathematical capacity, and with preliminary knowledge of only a matriculation standard, in two years, without specialisation or very serious demands on their time. The first year (the heavier of the two) would be spent on the syllabus of the Intermediate Examination in the Faculty of Economics,* the second year on the calculus.

For one who wishes to investigate economic problems mathematically, for himself, little more knowledge, but a good deal more experience, is necessary. This can be obtained in various ways, but a concentration on co-ordinate geometry and the calculus, with special reference to finite differences and to continuity, is perhaps the most suitable.

VI.—MATHEMATICS FOR STATISTICS, THEORETICAL AND APPLIED.

Many branches of Pure Mathematics are needed for the modern treatment of statistics, whether from the point of view of the biometrician or of the economist or sociologist; but we may instance as primarily useful the theory of probability, co-ordinate geometry in its widest aspect, including all graphic

* The syllabus is as follows:—

Algebra.—Theory of indices, nature and use of logarithms. Arithmetic and geometric progression. The binomial, exponential, and logarithmic series. Simple limits and approximations. The use of Horner's method of solving equations in simple cases. Simple applications of the above.

Graphic Algebra and Co-ordinate Geometry.—The straight line. Solution of linear simultaneous equations. The graphs of $a + bx + cx^2 + dx^3$ and of $\frac{A + Bx}{a + bx}$, with special reference to the approximate solution of equations in which y is an explicit function of x . Equation of the circle, equation of the ellipse and hyperbola, referred to axes parallel to their principal axes. (Focal properties are not required.) Simple applications of the above.

Pure Geometry.—Relations and properties of similar plane figures. Orthogonal projection; the properties of the ellipse as the projection of the circle.

Trigonometry.—Definitions and graphs of sine, cosine, and tangent. Use of four figure tables of natural sines, cosines, and tangents. The formulae for the sine and cosine of the sum of two angles. Circular measure. Approximate values of the sine, cosine, and tangent of small angles.

General.—The slope of curves as determined by the limiting value ($h = 0$) of $\{f(x + h) - f(x)\}/h$. Effect of change of scale on the slope and on the area of the curve. Simple cases of derived curves, illustrated by the functions and series named above, and simple cases of maxima and minima.

representation of functions, the calculus with special reference to expansions and to maxima and minima, and the theory of equations.

The Intermediate syllabus (*see* footnote on p. 208) was devised with a view to statistical needs, and (as already indicated) is a year's work for a capable student. In point of fact, students at the School of Economics have frequently more than a matriculation knowledge at entry; if not, they generally need a longer time. The calculus follows readily on to this course and should include the differentiation and integration of standard forms, the various methods of integration by reduction, application to tracing plane curves and to areas and volumes, and expansions. A first reading occupies a year, as the ideas need time and reflection, but little time per week is required. It is well to continue co-ordinate geometry, to include the analysis of the general equations of the second degree, to take simple three dimensional equations and generally to amplify the Intermediate syllabus. At the same time the elementary algebraic theory of chance should be very carefully studied, including inverse probability, and with exercises in permutations and combinations.

Besides these definite subjects, experience in manipulation should be obtained by every means possible. The theory of equations, determinants, and co-ordinate geometry of two and three dimensions (especially transference of axes) are specially valuable.

With this preparation the student may begin seriously the study of statistical theory. As he proceeds he will find methods and notation with which he is unfamiliar and from time to time he may have to read new sections in Pure Mathematics; but the actual application of Mathematics to statistics, whatever school of investigation is followed, is specialised both in notation and methods, and the necessary experience can be better obtained by working through the text-books and treatises relating to mathematical statistics than by further general mathematical training.

Throughout this paper the intention has been to define the aim of mathematical training for various groups of students, and to suggest the course which will lead most directly and furthest towards that aim, avoiding the complexities, digressions and irrelevancies with which mathematical text-books and courses are often crowded.

A. L. BOWLEY.

EXAMINATIONS FOR MATHEMATICAL SCHOLARSHIPS.

I.—INTRODUCTORY.

At first sight it seems one of the oddest of paradoxes that, although poverty was the badge of the whole tribe of the early Universities, Faculties, and Colleges of Europe, yet that very poverty lay at the root of their power. Savigny has suggested that to the purely spiritual character of the existence and power of the Universities must be attributed their independence of the temporal authority. The ancient rhyme—

Les bons enfants orrez crier,
Du pain ! n'es veuil oublier,

is in itself a vivid picture of the material conditions under which learning thrived at the *Collège des Bons Enfants* in the opening years of the thirteenth century. And our own Chaucer, in his deeply bitten etching of the Clerk of Oxenforde, tells us how he

. bisily gan for the soules preye
Of hem that yaf him wher-with to scoleye.

But it was not long before a sound national instinct led patrons of learning to mitigate, as far as in them lay, the bitterness of this secular union of learning and poverty. Thus we find, for instance, that funds were provided at Exeter College, Oxford, for such scholars as were "*pauperiores in facultatibus*"; and that Trinity College, at the same University, was founded "*ad perpetuam pauperum scholarium in academia degentium sustentationem.*" From that time forth the leaven has never ceased to work. There is no University in this country in which there is not an annual commemoration of the pious benefactors who, consciously or otherwise, have made it possible for the poor man who has shown evidence of capacity to receive the advantages of a University education. Universities, Colleges, Public Bodies—all are doing their utmost to see that the best brain-power and intellect of the young generation shall not be wasted for want of opportunity. Even the State itself, so long notorious among civilised nations for its attitude of scarcely qualified indifference to a matter of such vital importance to the community, has at last awakened, as it were, out of sleep. And while we cannot rest satisfied with what has been done, considerable as it may appear to some, it is clear that public opinion is at last being aroused, and that before long we may count upon a full atonement for the neglect of the past.

The Universities and the Secondary Schools.—The influence of the Universities on the schools which is exercised both directly and indirectly through a variety of examinations of different standards culminates in that which is due to the examinations as a result of which scholarships are awarded to students for the purpose of continuing their training at Oxford, Cambridge, and elsewhere. As the subjects selected for these competitions are Mathematics, Classics, Natural Science, Modern Languages, and History, and in any or all of these a student has been, as it were, directly under the eye of the Universities from almost the age of twelve, it will be gathered that in no limited sense can the Universities be said to be responsible for the training received from first to last in the secondary schools.

By the scholarship system the Universities get the pick of the brains of the secondary school. They will not award scholarships to boys who do not promise to prove to be above the average in the final degree examinations. And those boys must come to the University with a certain standard of attainment and with preparation of a specified kind. Thus a conventional system has grown up, and by an almost irresistible tendency the lines laid down for secondary training have been adapted to the needs of the future University student rather than to the requirements of the average boy or girl. The school runs the danger of becoming merely "the ante-room to the University."

British parents, for the most part, perceive dimly whither this is leading. They measure the value of a school by the number of University Scholarships it wins. Each has possibly the scarce concealed idea that his duckling may turn out to be a swan. Nor is he indifferent to the hope that he may be spared some of the expenses attending his boy's University life. And of course in a very large number of instances the emoluments will alleviate a serious domestic strain, and will frequently be the deciding factor in the final resolution to allow the boy or girl to proceed to the University. Nor must we forget the status that attaches to the dignity of being a "Scholar of his College." To parents this is the outward and visible sign of the worth of the successful candidates, and fills both them and their children with high hopes for the future.

The schoolmaster has for some fifty years been accustomed to the control of the Universities over curricula and methods. The *kudos* attached to the winning of a scholarship is considerable. With each fresh success the reputation of the school is increased. Specially qualified masters are engaged for the promising boys, and in a large number of instances it may be said that almost the sole measure of the school's success is the number of these scholarships that it wins. So that neither master nor parent is likely to protest against a system which produces such agreeable results. The walls of the school are hung with "honours boards" on which are inscribed in letters

of gold the names of those who have thus distinguished themselves and "the old school."

These are, however, the names of the very few. What has become of the many who have passed through the whole or part of the routine, and have not achieved this kind of success? Is that system the best in which the treatment of children of all capacities is practically the same as that devised for the securing to the nation its most brilliant and promising cadets? A recent writer has said that the honours boards which are now the pride of a school will in the near future be regarded as "a record of offences demanding condemnation."

It is but fair to add that the evils inherent in such a system have already been realised and that the process of mitigation is in full swing. If the policy pursued by the Board of Education of recent years be permitted its normal development, this reproach of the sacrifice of the interests of the many to those of the few will be removed. For the Board are in the main responsible for the fact that the "success" of a school is more and more being measured in terms of the extent to which the average level has been raised. In the domain of Mathematics there are not wanting signs that the evils have been fully realised. The movement in the direction of the improvement in the teaching of the subject, initiated some years ago by Professor Perry, has already been productive of considerable results. A distinction is now drawn between the type of training which is necessary for the few who are to be professional mathematicians, and that for the many who do not intend to go to the Universities. The content of the mathematical curriculum for the average boy will not be quite the same as that for the average girl, and neither will be on the same lines as that for the candidate for University scholarships. The Committee of the British Association, the Committee of the Mathematical Association, the Committee of the Association of Science Masters in Public Schools, have taken part in a development which will be far-reaching in its influence.

The view so characteristic of the British mind in the past that Mathematics beyond the barest Arithmetic is of no use to anyone, is doomed to extinction. The public has perceived that perhaps to engineers such a subject may be of use, but hitherto they have not realised that it comes into touch at a multitude of points with their daily life. Nor can we much wonder that this has been the case. The subject had never been presented to them as one connected with realities. Thanks to the movement for reform the old distaste is disappearing. Already the young generation is entering into its heritage.

Technical Terms for Monetary Awards.—Before entering into further details it may be as well to define the various terms that are used to describe the monetary awards to successful candidates at examinations intended for those who require financial aid to enable them to pursue their studies at the Universities.

A donation, legacy, or revenue appropriated to endow an institution, and constituting a permanent fund for its support, was called in the olden time a *Foundation*. The term has come in these days to be applied to the institution itself, which is founded or supported by such endowments. An award of money from such a source for the support of a student selected by examination, whether he be already a member of the foundation or not, is called a *Foundation Scholarship*.

The institution may also possess other funds than those mentioned above. If money awards are made for the same purpose from these other funds, they are called Scholarships—*tout court*.

It may be roughly said that *Major Scholarships* and *Minor Scholarships* are in descending order of value and prestige. If the awards are of less money value than these, they are called *Exhibitions*, and below these, again, are awards to which the name of *Sizarships* is attached.

Exhibitions and Sizarships may be awarded at examinations held for the purpose. They may also be given at examinations for Scholarships, being bestowed upon candidates who have not quite reached the scholarship standard, but who have done work worthy of commendation.

The Scotch *Bursary* may be looked upon as broadly equivalent to the smaller Exhibition or Sizarship.

II.—EXAMINATIONS FOR MATHEMATICAL SCHOLARSHIPS.

Throughout Great Britain and Ireland a large amount of monetary assistance is given to promising students who wish to obtain a University education. This assistance is given partly by the Universities themselves—either out of their own funds or from private benefactions—partly by Schools which award school-leaving “exhibitions,” and partly by the Government, the County Councils, and other Public Bodies. It would be difficult to form an estimate of the number of students who obtain help in this way, but one or two examples will give some idea of their extent and distribution.

Scotland.—In the four Universities of Scotland, viz. Edinburgh, Glasgow, Aberdeen, and St. Andrews, more than 800 bursaries, in addition to scholarships, are held by students in the faculty of arts alone. A proportionate number are awarded to students in the other faculties. At least one out of every four university students in Scotland appears to be the holder of a bursary attached specifically to his University. The scholarships are awarded for proficiency in some special subject or combination of subjects. The bursaries are given on the results of a general examination in which Mathematics is obligatory. Students enter the Universities at an earlier age than they do in England, and the standard of attainment at entrance is consequently not so high; but the Scottish boy probably reaches a higher mathematical standard than the

English boy of the same age, and this is mainly due to the influence of the University bursary system. The bursaries are tenable for three or four years, and are of all amounts from 5*l.* to 100*l.*, the average being about 25*l.* This is only a modest sum, but the students can live very inexpensively, and many of them reside with their relatives and friends.

Ireland.—In Ireland the assistance given to University students is largely supplied from Government funds. Some 50 scholarships were attached in this way to each of the three Queen's Colleges in Belfast, Cork, and Galway, which formerly constituted the Queen's University. This University was dissolved in 1882, but was practically re-created by the Irish University Act of 1908 in the form of the Queen's University of Belfast and the National University of Ireland. These two Universities are now able to offer for competition fairly liberal scholarships. The remaining and the principal University in Ireland is the University of Dublin, which is the only one possessing valuable endowments. It has 70 men-scholars on the foundation, 30 sizar, and 100 exhibitioners; and about 10 scholarships, not on the foundation, are allotted to women. Two studentships of 100*l.* each, tenable for five years, are also annually awarded, one in Mathematics and Physics and the other in Classics. "Royal Exhibitions" are attached to certain schools to enable their students to proceed to a University, and a number of scholarships and exhibitions have been founded at the Universities by private beneficence. The dominating factor, however, in the education in Irish schools is the annual examination by the Board of Intermediate Education, on the results of which grants of money are awarded to managers and exhibitions to pupils. The Board has recently added inspection to its functions. The system, to judge by results, has not worked altogether satisfactorily. The mathematical attainments of scholarship candidates have not reached as high a standard as might have been expected. But there is little doubt that the standard will gradually improve as the Universities obtain a more controlling influence. They are already taking steps in this direction by holding conferences with schoolmasters, and by other means. It is hoped that in this way a broader and more rational system of teaching Mathematics will be encouraged.

England: Public Bodies.—As regards the assistance given by Borough and County Councils to qualified candidates, it will be sufficient to take the London County Council as an example. This Council annually awards a large number of senior scholarships, which were in the past year (1912) 100 in number, and will in the future be maintained at that level. The scholarships are open to both sexes, and are only awarded to students who stand in real need of assistance. The candidates are not tested by examination, although the Council reserves the right to impose

such a test if it thinks fit. Candidates are required to give particulars of any public examination they have passed, to state the course of study they intend to pursue, and to submit reports from their teachers and other persons as to their attainments, character, and capability to turn their proposed studies to a profitable and useful account. As a rule, a candidate is not elected unless he has already won a scholarship in open competition. The scholarships are awarded for three or four years, and in special cases are extended to a fifth year. The value varies from 30*l.* to 90*l.* per annum, according to the needs of the candidate, and the total cost is estimated at 20,000*l.* a year.

Smaller Corporations.—There are numerous other bodies that award scholarships, either out of their own funds or as trustees of charitable bequests. Among these a notable example is the Goldsmiths' Company of the City of London, which has made many and various gifts to the cause of higher education. It offers annually 16 or more exhibitions of 50*l.*, tenable for three or four years, to students in their first year of residence at Oxford or Cambridge. They are awarded on the results of an examination in Classics, Mathematics and Divinity.

Smaller Universities.—The smaller Universities in England and the University of Wales do not possess many scholarships in their own gift. We need, therefore, refer only to the Universities of London, Cambridge, and Oxford. The two latter may be taken together, as their systems are alike.

University of London.—The University of London has only a comparatively small number of scholarships. These, no doubt, would be increased if the financial position justified such a step. Formerly the scholarships were open to all comers, subject only to the restriction of an age-limit and of attaining a certain standard. But the London University has of late become not only an examining but a teaching University, and the system on which scholarships are awarded is changed. They are still open to both sexes, but are no longer tenable by students in any other University, or in any college or institution unconnected with the University of London, except under special circumstances and with the express permission of the Senate. The University holds an examination in June, at which five exhibitions of 40*l.* per annum, tenable for two years, are offered to students under 18 years of age, who are not of more than one year's standing from Matriculation. One of these is offered for Mathematics combined with a subsidiary subject. In July an examination is held at which 19 scholarships of 50*l.* a year, tenable for one year, are offered to students of not more than one year's standing from the Intermediate degree examination. Three of these are for Mathematics. The Colleges and Institutions of the University also hold examinations at which scholarships are awarded,

including three or four in which Mathematics is the principal subject. University College offers a post-graduate studentship of 50*l.* for one or two years to a student who has passed through the three years' mathematical course at the College.

Oxford and Cambridge.—The Entrance Scholarship examinations at the Oxford and Cambridge Colleges are conducted on practically identical systems and differ but slightly in detail. At a rough estimate, the Cambridge Colleges award 50 or more scholarships, exhibitions, and sizarships every year for proficiency in Mathematics. Their value varies from 30*l.* to 80*l.* per annum. The Oxford Colleges annually award for Mathematics about 20 scholarships of 80*l.*, and 10 exhibitions of from 30*l.* to 60*l.* Most of these are tenable for two years, but they are renewed for one, two, or three more years, subject to satisfactory progress being made. The age-limit at the time of the examination is 19. The subjects of examination in both Universities are:—Geometry, Algebra, Trigonometry, Differential and Integral Calculus, and Mechanics.

At Cambridge, to which we shall now more particularly confine ourselves, the value of a scholarship may be increased, and new scholarships given, while the student remains in residence. A few of the colleges hold separate examinations, but most of them have combined into two large groups: the Trinity College group, comprising five colleges, and the Pembroke and St. John's group in which are seven colleges. Both groups hold their examinations in December and on the same dates, so that a candidate who fails to do himself justice at the first attempt has no choice left for a second trial at any of these Colleges in that year. Apart from this, which may or may not be considered a drawback, the system has many advantages. The questions are set with the greatest care. Every precaution is taken to avoid inaccuracy and to make the wording of each question clear. Incorrect questions are of rare occurrence.* In recent years I have noticed but one—in one of the two easier papers of 1908 in the Trinity group:—"The four lines which bisect at right angles the sides of a quadrilateral are given; prove that the locus of each vertex of the quadrilateral is a straight line."

Examination Questions.—It may be said that Trinity College has generally taken the lead when any changes have been made in the style of examination. Twenty or thirty years ago the papers contained a more or less equal admixture of "book-

* "There have been cases, though rare, in which a single set of questions contained several errors; . . . Cambridge tradition still hands down warnings . . . pointing to instances where a slight inaccuracy in the enunciation of a problem, or a latent fault in the process of solution, lured a candidate on to such an extravagant waste of time as to lose a senior-wrangler'ship." (Todhunter, *The Conflict of Studies*, p. 97. London: Macmillan & Co. 1873.)

work" and "riders."* Then Trinity made an innovation by setting papers which consisted almost entirely of easy "problems."† At the time the change was probably an improvement. The bookwork questions had tended to become too stereotyped. It was the opinion of some that such questions formed no more than a test of memory, while the capacity to solve problems was the only real test of a candidate's ability. This view has been abandoned. The bookwork questions now set afford plenty of scope for originality of treatment. It should not be forgotten that they also form the best test of the candidate's ability to express himself intelligibly—a matter to which increasing, but not sufficient, importance is being attached. The candidate has also to write an essay, and, in the case of the Trinity group has to take a paper of questions on literary and general topics. Knowledge of mathematical bookwork is not considered in itself an important thing; but rather a knowledge of results, and of general processes and ideas. This receives special recognition in the paper devoted to bookwork only, now set in both the large groups at Cambridge.‡ In this paper the candidate is limited to the selection of three or four questions from perhaps nine or ten; they are for the most part on general theory, and the answers to them are in the nature of essays. This change was first introduced at Trinity in 1905. Still another change, also at Trinity, was made the next year, when two of the four papers were set for "Candidates in Mathematics, in Mechanical Sciences, and in Mathematics applicable to Physics." These papers contain rather easier questions. Some of them have been too long, and have favoured the rapid worker rather than the candidate who sees and thinks. But the great merit of the change is clear. It brings the non-specialist into the fold, and tends to widen the horizon of the specialist.

In the papers of the Trinity group the questions cover a very broad ground. A candidate cannot gather by the study of previous papers what he may expect to be set when his turn

* "Bookwork" is the name given to questions which a candidate may be expected to answer from his knowledge of the text-book. When to a bookwork question an illustrative example is attached, implicitly intended to be solved largely by means of the material afforded in that bookwork, it is technically called a "rider." "Problems" are original exercises not set in connection with bookwork, and may be attacked by the candidate in any way he pleases. It will be seen that the implied restriction in the case of the rider may make it more difficult than if it were set as a problem. "It is not infrequently very difficult to trace the influence of the bookwork on the rider. . . . A distinguished living mathematician is reported to have been the first who reversed the natural process in the formation of the binary compound; instead of selecting a bookwork question and then constructing a simple rider to it, he is said to have invented the rider first and then sought for the piece of bookwork to which it might be least unnaturally attached." (*loc. cit.*, page 204.)

† This is not now correct. The Pembroke group seems to have abandoned the "bookwork" paper and substituted for it a "general" paper.

arrives. He may anticipate having a question on frameworks, and another on asymptotes, or the tracing of a curve; but he will receive credit for any sound work, especially if it is out of the usual routine, and he will be wise to avoid any attempt at "cramming." * In the Pembroke group the questions show much greater uniformity from year to year, and are within distinctly narrower limits. There are four papers: one on Algebra, Trigonometry, and Differential Calculus, on which four, four, and three questions respectively are set; one on Geometry, containing five questions in Pure and six in Analytical Geometry; one on Statics and Dynamics containing ten questions; and the theoretical or bookwork paper of about ten questions, of which not more than four may be attempted. These papers may be regarded as typical; and the purely personal opinions which we venture to express upon them are intended to be of general application.

Algebra.—The number of questions on Algebra seems very inadequate, considering what a large number of branches this subject is supposed to include: such as convergence of series, partial fractions, continued fractions, determinants, probabilities, theory of numbers, and theory of equations. The deficiency is scarcely made up in the special bookwork paper, in which one, or possibly two, questions on Algebra may be included. A better apportionment of the questions in this paper would appear to be to assign half to Algebra, and a quarter each to Trigonometry and Elementary Calculus, including Integration.

Geometry.—In devoting a whole paper to Geometry, the subject receives liberal treatment. The questions on Pure Geometry cover a very narrow range, and there is practically no Modern Geometry. In this respect Oxford has always been more advanced than Cambridge. The paper on Geometry should not only include Modern Geometry, which has the advantage that it is equally amenable to purely geometrical and analytical methods, but it should also contain geometrical applications of the Calculus. It is a curious fact that the English mind, which is supposed to be instinctively geometrical, has a distaste for the study of Pure Geometry; and it is a matter of opinion whether this is the cause or the result of the somewhat antiquated view of the subject taken at Cambridge. To give an instance: little or nothing is heard at Cambridge of the Principle of Duality, and it is difficult to understand why the principle is not given its rightful place, and why it is replaced by its barely recognisable substitute, Reciprocation, most

* "Cramming" is the process by which a subject is taught not to bring out a boy's intelligence, but to enable him to make the best possible show on a certain day. The subject is "got up." The result may be intellectual indigestion. The metaphor is taken from the treatment accorded the unhappy Strassburg goose, from which great results are expected in the way of *pâté de foie gras*.

of the questions on which are merely academic applications devoid of any suggestiveness, such as the following :—

“Reciprocate the following with respect to the point D : the centre of the rectangular hyperbola $ABCD$ is on the auxiliary circle of the conic which touches the sides of the triangle ABC , and of which D is a focus.” (Trinity College, 1903.)

“Reciprocate the following theorem with respect to a focus: if tangents are drawn to a system of confocal conics from a fixed point on the line joining the foci, their points of contact lie on a circle.” (Trinity College, 1906.)

Graphs and Graphical Methods.—Fortunately Cambridge and Oxford have refused to accord to graphs more than can be legitimately claimed for them, notwithstanding the insistence with which enthusiasts have extolled their merits in an educational training. On the whole, more zeal than circumspection has been shown in the introduction of graphs into school work. They are, no doubt, excellent when used in moderation, as assisting a boy's comprehension of principles and relations, but they do little to advance his knowledge, and take up a great deal of time.

It is sufficient in the earliest stage that the boy should understand what a graph is, and that he should be given a few illustrations; it is not necessary that he should work examples. In doing elementary Geometry he should learn what rectangular and polar coordinates are; and he may be given some notion of the variation of a function by having graphs drawn for him. He may draw graphs himself in elementary Trigonometry, and afterwards use them whenever they are helpful, making a natural and not a forced use of them. By the time he has reached this stage he may be able to decide whether he has a scientific or a technical aim in learning Mathematics; and, in the latter case, whether or not it is needful for him to begin specialising at once in the use of graphical methods. He should, however, be made to understand that the earlier he does this the greater will be the sacrifice he will have to make in the loss of insight into his work, and the lack of acquirement of more general and powerful mathematical methods.

Graphical methods of solving questions in Statics are useful for the engineer, but only of slight importance for the mathematician. They do not help towards the understanding of mechanical principles. They are technical and special; and work done by them is slow until the worker has become expert by practice.

Important as it is in some aspects, Graphic Statics is not an essential part of theory; and the student who wishes to make the best use of his time in learning as much Mathematics as possible would be well advised to leave this subject alone.

We do not wish that any obstacles should be put in the way of graphical teaching, provided it is not overdone. Questions which can be solved graphically should certainly be set in examinations, and a graphical solution should rank as of equal merit with any other, provided it attains sufficient accuracy.

Statics and Dynamics.—The paper on Statics and Dynamics has in recent years been a beneficent guide rather than a mere source of perplexity to the enquiring student. The paper might, perhaps, cover rather more ground, such as that recommended by the Mathematical Association (*see* Appendix II.); but the questions form a very good test of the candidate's ability and knowledge of the subject. We have the greater pleasure in expressing this opinion, because many of the questions in the past were useless conundrums, depending for their solution upon tacit assumptions which were demonstrably false.

Examinations and the School.—In the foregoing, some views have already been expressed as to the effect of the scholarship examinations upon school teaching. On the whole we regard them as having a very beneficial effect, although still admitting of several improvements. We believe that the majority of teachers who prepare candidates adopt what appears to them to be the best course of instruction for their pupils, irrespective of the examinations ahead; and we maintain that this is not only the most satisfactory way, but also the surest way to enable a candidate to do his best. The candidate should go through a course of the papers, doing one or possibly two a week, some time before the examination. These papers can be utilised by the teacher for imparting many items of theory and technique which have previously escaped notice, or which have not been sufficiently dwelt upon. If this incurs the charge of "cramming," we can only say that such a course of papers will not avail much if the previous preparation has not been sound. We think that Cambridge has succeeded in a very remarkable degree in making cramming ineffective so far as the winning of mathematical scholarships is concerned.*

Wider Recognition of Geometry.—What seems to us to be the most desirable improvement is a wider recognition of the claims of Pure Geometry, not by devoting a larger number of

* That a contrary view is held by many teachers is illustrated by the following quotations from a correspondent:—"A good deal of cramming and special coaching for the Scholarship Examinations takes place. The boy's success depends less on his own ability than on the experience and method of preparation adopted by his teacher." . . . "The strain resulting from working to reach so advanced a stage brings about such a reaction that it is necessary that a candidate should take a rest after winning a scholarship, sometimes for nine months." . . . "The method of teaching in schools is so different from the lecture system at the Universities that many boys are unable to keep up with the lectures. Each year I have had complaints from old pupils that they have found the strain of the first two terms almost unbearable."

questions to it, but by widening the range. If a student is not well grounded in Geometry at the beginning of his career he is too often apt to neglect it later. The methods of Pure Geometry are not cut and dried like those of elementary analysis. They require the learning of a greater mass of details, and also demand a greater amount of practice. But the gain to a mathematician who is able to make full use of Pure Geometry can scarcely be overestimated. Apart from its utility it has a special value as a training of the intellectual faculties. Many, however, grudge the time necessary for a thorough grounding in the subject, and Cambridge, instead of counteracting this tendency, appears to approve of it. It seems also desirable that what is intended to be included in Algebra should be more clearly defined.

The Aims of Examinations in Mathematics.—More important than the particular subjects to be included is the question of the aims that should be kept in view in a mathematical examination. Those aims should be to encourage soundness of work and breadth of reading, and to discourage waste of time in acquiring acquaintance with the bye-paths of a subject, and an unnecessary degree of skill in manipulation. Such aims have scarcely been acknowledged, and certainly not been realised, until within the last few years. It is to be hoped that they will occupy the chief place and become more and more widespread in the future.

Text-books.—One of the evils that inevitably follow from an examination system is the writing of text-books to fit the examination; and of course these reflect all the faults of the system. In this respect the examinations of the past have much to answer for. Teachers are driven to use these text-books as they have to get their pupils through the examination. In most of our elementary text-books the bookwork is disconnected, principles are relegated to the background, difficulties are shirked, and a good deal of matter is included that is not of the slightest use to any reader. The examples are not less at fault than the bookwork; they are too numerous and too academic. A student should be supplied with a text-book in which he can do all the general questions without spending too much time; and to these should be added a few special sets of questions from which he can select. Excellent text-books have been written having no special examination end in view, but they have not met with the success they deserved.

The Examiners and the Questions.—The quality of an examination, and its influence on the teacher, depend entirely on the questions set. The responsibility of deciding on them should be undertaken only by a large and competent Board, on which teaching experience and expert knowledge are more or less equally represented. The duty of examiners should be merely to suggest questions, and of the Board to accept or reject them; but the examiners should receive for this, the most important

part of their work, higher remuneration than has been customary. The questions should be most carefully scrutinised ; they should not be allowed to become stereotyped ; they should be rejected without scruple if too difficult, or tricky in wording or solution, and if they do not involve some important principle or method.

We do not think it a good thing to set optional questions such as those in the Exhibitions Examination at the University of London, printed in Appendix III. Such questions tend to become too advanced, and to encourage scrappy reading on the part of the candidates. This objection does not apply to a "bookwork essay paper," for a candidate may with advantage specialise in a few subjects which interest him most ; and a paper entirely on bookwork gives him an opportunity of showing such special knowledge as he may possess, if the questions are sufficiently numerous to select from ; and, further, in such a paper the time of the candidate is not wasted in selecting questions or in making attempts which he is obliged to relinquish ; for he can see at a glance whether to "write a sketch of the history of the theory of inversion," or to "discuss the geometrical representation of imaginaries on the Argand diagram." In an ordinary paper the questions should not be so numerous that it is not practicable for a good candidate to attempt them all. He should in fact be prepared and expected to answer all the questions set ; but there might not be much harm in limiting him to six questions out of eight or eight out of ten. In the matter of the difficulty of the questions there may be a certain amount of latitude ; but at the same time the setting of too difficult or intricate questions is the chief thing to guard against. An immature mind cannot grasp a subject in its entirety ; this will come slowly and naturally of itself as new subjects and new ideas are unfolded. The attempt to proceed too far along any one particular line dulls and contracts the intellect to a much greater extent than it contributes to its expansion.

In 1904 the Mathematical Association, which confines its activities chiefly to Elementary Mathematics, issued a Report on Advanced School Mathematics, and in 1907 a Report on Entrance Scholarships at the Universities. These are reproduced in the Appendices, together with specimen papers set at two groups of Cambridge Colleges, two groups of Oxford Colleges, and the University of London. Permission to reproduce these has been courteously granted by the authorities in each case.

F. S. MACAULAY.
W. J. GREENSTREET.

APPENDIX I.

REPORT OF THE COMMITTEE OF THE MATHEMATICAL ASSOCIATION ON ADVANCED SCHOOL MATHEMATICS. 1904.

In considering this subject the Committee have taken account of the different classes of boys who study what may be termed "advanced mathematics" at school. These include candidates for army examinations, science students, engineering students, as well as boys who intend to read for mathematical schools at the universities. Those who intend to take up engineering will often be candidates for mathematical scholarships.

The Committee consider it desirable that, in order to secure the highest possible efficiency in teaching mathematics in schools, the schedules of examinations should be so framed that candidates may, so far as possible, all follow the same preliminary course.

The Committee think that a preliminary advanced course suitable for all classes of students should include :

(1) The following portions of Algebra: partial fractions, elementary manipulation with complex number, and geometrical applications thereof, the theory of equations so far as it treats of the numerical solution of equations, the notation and easy properties of determinants, the simpler tests of convergency, and the binomial, exponential and logarithmic series.

The following parts of Algebra should be excluded from the preliminary course:—The theory of numbers, probability, continued fractions (with the exception, perhaps, of simple continued fractions), and advanced theorems on inequalities, and on indeterminate equations, and on summation of series.

The preliminary course should further include :

(2) An early introduction to the differential and integral calculus and a free use of the same in subsequent work. The use of differentials should be permitted.

(3) Graphical illustrations of De Moivre's theorem. Simple work in trigonometrical series and factors.

(4) A treatment of the elementary parts of conic sections in which either the geometrical or the analytical method is used, that method being used in each particular case which is most suitable for the problem under discussion.

(5) The elementary geometry of the plane, cone, cylinder, sphere and regular solids, including practical solid geometry.

(6) An introduction to the dynamics of rotation (in two dimensions), viz., the motion of a rigid body round a fixed axis with uniformly accelerated angular velocity, together with other simple cases of the motions of rigid bodies.

The Sub-Committee also wish to express their opinion of the importance of :

(7) A more intimate union between the teaching of Mathematics and Science, whereby theoretical and practical work may be brought into relation with one another.

The above course would form an excellent preliminary course for scholarship candidates, who would then proceed to more advanced work in the various subjects so as to arrive at about the standard which is now required.

Regarding examinations for scholarships at the Universities in their twofold capacity, (1) as a means of selection, (2) as providing a guide to work at schools, there is no reason to doubt that the former function is efficiently performed by the papers as now set.

With regard to the second function, that of directing school work, if it is decided that the above preliminary course is on the right lines, it would be desirable for scholarship examiners to recognise it by :

- (a) Setting questions involving applications of the integral calculus ;
- (b) requiring work in theoretical solid geometry as above indicated ;
- (c) abolishing the restriction to " methods of pure geometry," and leaving the method employed to the judgment of the candidate ;
- (d) discouraging the cramming of bookwork in advanced trigonometry by not requiring candidates to reproduce long and difficult proofs (such as that needed to establish the infinite product for the sine).

APPENDIX II.

REPORT OF THE COMMITTEE OF THE MATHEMATICAL ASSOCIATION ON THE COURSE OF MATHEMATICS REQUIRED FOR ENTRANCE SCHOLARSHIPS AT THE UNIVERSITIES. 1907.

1. It is very desirable that the subjects of examination for Mathematical Entrance Scholarships at Cambridge and Oxford should admit of the same course of preparation.

2. In order to allow of a broader syllabus the exercises and riders should be simplified.

3. Questions on elementary bookwork ought not to be entirely omitted. Restricting bookwork questions to the higher bookwork would encourage the neglect of the more elementary parts, which are of at least equal importance.

The Committee considers that the essay type of bookwork questions (as set in the Trinity group of Colleges at Cambridge in December, 1905) is of special value.

4. A knowledge of the general principles and application of graphical work should be required, but only a moderate degree of manipulative dexterity.

5. The range in Pure Geometry should approximate to the Oxford standard: purely geometrical methods should be encouraged by a fair number of questions being set which can best be solved by Pure Geometry. Purely geometrical methods as opposed to analytical should not be insisted upon, but there is no objection to the appropriate method of solution being recommended.

6. In the Calculus, geometrical proofs, which assume that the functions considered are such as can be represented by curves, should be permitted, provided that the necessary conditions for their validity are stated.

7. In Geometry, Trigonometry, Theory of Equations, and Dynamics the use of Infinitesimals and the methods of the Calculus should be freely permitted.

8. The subjects of the examination should be in accordance with the Schedule appended, due regard being paid to the second recommendation of the committee.

SCHEDULE.

PURE GEOMETRY.—Geometry of Straight Lines, Circles and Conics: Inversion, Cross-ratios, Involution, Homographic Ranges, Projection, Reciprocation and Principle of Duality; Elementary Solid Geometry, including Plans and Elevations of Simple Solids.

ANALYTICAL GEOMETRY.—Straight Lines and Curves of the Second Degree; Tangential Coordinates.

Excluding Homogeneous Coordinates; Invariants; and Analytical Solid Geometry.

ALGEBRA, including Elementary Theory of Equations.

Excluding Recurring Continued Fractions; harder tests of Convergence Theory of Numbers; and Probability.

GEOMETRICAL TRIGONOMETRY. *Excluding* Spherical Trigonometry.

ANALYTICAL TRIGONOMETRY.—Properties of Circular, Hyperbolic, Exponential and Logarithmic Functions, with real and complex argument.

Excluding the proofs of the Infinite Products for Sine and Cosine, and of the Series of Partial Fractions for the other Trigonometrical Ratios.

CALCULUS.—Total and Partial Differentiation; Taylor's and Maclaurin's Theorems; Elementary Integral Calculus; Simple Applications to Plane Curves (especially to such as are of intrinsic importance), to Maxima and Minima, to Areas and Volumes, and to Dynamics; Curve Tracing, not as a rule to scale.

Excluding the Theory (but not the use) of Differential Equations.

DYNAMICS.—Elementary Statics, including Simple Graphical Statics; Elementary Kinematics and Kinetics, including motion of a rigid body about a fixed axis, and motion of cylinders and spheres in cases where the centre of gravity describes a straight line.

Excluding Hydrostatics and Hydrodynamics.

APPENDIX III.

ENTRANCE SCHOLARSHIPS EXAMINATION PAPERS.

(a) CAMBRIDGE UNIVERSITY.

(1) EXAMINATION FOR SCHOLARSHIPS, EXHIBITIONS, AND SIZARSHIPS.

Trinity, Clare, Trinity Hall, Peterhouse, and Sidney Sussex.

Wednesday, December 7, 1910. 9—12.

1. The lines AD , BE , CF are drawn perpendicular to the sides BC , CA , AB of the triangle ABC , and EF , FD , DE are drawn to cut BC , CA , AB respectively in X , Y , Z . The tangents at A , B , C to the circle ABC cut BC , CA , AB respectively in P , Q , R and L , M , N are the middle points of AP , BQ , CR respectively. Prove that the six points X , Y , Z , L , M , N are on the radical axis of the circles ABC and DEF .

2. A , B are the points of contact of a common tangent of two given circles, and any line parallel to AB cuts one circle in P and the other in Q . Prove AP and BQ intersect on a fixed circle coaxial with the given circles.

3. Find three points X , Y , Z on the sides BC , CA , AB of the triangle ABC , such that YZ , ZX , XY will pass respectively through three given collinear points L , M , N .

Hence, or otherwise, find the points of contact of three given tangents to a conic having also given the pole of a given straight line.

4. Prove that, if n is a positive integer,

$$1 - \frac{n}{1^2}x + \frac{n(n-1)}{1^2 \cdot 2^2}x^2 - \frac{n(n-1)(n-2)}{1^2 \cdot 2^2 \cdot 3^2}x^3 + \dots$$

$$= e^{-x} \left\{ 1 - \frac{(n+1)}{1^2}x + \frac{(n+1)(n+2)}{1^2 \cdot 2^2}x^2 - \frac{(n+1)(n+2)(n+3)}{1^2 \cdot 2^2 \cdot 3^2}x^3 + \dots \right\}.$$

5. Prove that, if α , β , γ are the three roots of the equation

$$x^3 - 21x + 35 = 0;$$

then will

$$\alpha^2 + 2\alpha - 14 \text{ be equal to } \beta \text{ or to } \gamma.$$

6. Prove that

$$\cos 2\alpha \sin(\beta - \gamma) + \cos 2\beta \sin(\gamma - \alpha) + \cos 2\gamma \sin(\alpha - \beta)$$

$$= 4 \sin \frac{1}{2}(\beta - \gamma) \sin \frac{1}{2}(\gamma - \alpha) \sin \frac{1}{2}(\alpha - \beta) \{ \cos(\beta + \gamma) + \cos(\gamma + \alpha) + \cos(\alpha + \beta) \}.$$

7. The diameter AB of a circle is produced to C so that BC is equal to the radius OB of the circle; CD is drawn perpendicular to OBC and $CD = BC$. A point P is taken on the circle on the same side of AB as the point D , and such that $\angle AOP$ is half a right angle. Prove that, if Q is the point where PD cuts the circle again, $\angle BOQ$ is 1.001 radians, very nearly.

8. Prove that chords of the ellipse $x^2/a^2 + y^2/b^2 - 1 = 0$ which subtend a right angle at a given point P of the ellipse intersect the normal at P in a point P' such that PP' is equal to $2abd/(a^2 + b^2)$, where d is the semi-diameter conjugate to CP .

Prove that as P varies the locus of P' is a similar ellipse, and that the normals to the ellipses at P and P' respectively intersect either ellipse in four concyclic points.

9. Show that the locus of the intersection of normals at the extremities of chords of a parabola which pass through a fixed point is a parabola, and find the direction of its axis.

10. Find the equation of the tangent at the point $(p^2, 1, p)$ on the conic $a\beta - \gamma^2 = 0$; and prove that, if the sides of a triangle touch the conic and two of its vertices are on the lines $\alpha - \lambda_1\beta = 0$ and $\alpha - \lambda_2\beta = 0$ respectively, the locus of the third vertex is the conic

$$(\lambda_1 + \lambda_2)^2 a\beta - 4\lambda_1\lambda_2\gamma^2 = 0.$$

11. Show that if θ and ϕ lie between 0 and π , a^2 is less than 1, and
 $(1 - 2a \cos \theta + a^2)(1 + 2a \cos \phi + a^2) = (1 - a^2)^2$,
 then

$$\frac{d\theta}{d\phi} = \frac{\sin \theta}{\sin \phi} = \frac{1 - 2a \cos \theta + a^2}{1 - a^2}.$$

12. Prove that if the chord of curvature through the origin is $2r^n/a^{n-1}$ ($n \neq 1$), for any point of a curve at distance r from the origin, then the radius of curvature is proportional to

$$r^{n+1} \frac{a^{n-1}}{e^{(n-1)r^{n-1}}}.$$

Wednesday, December 7, 1910. 1—4.

BOOKWORK.

[Candidates are requested to attempt at least ONE question from each section of the paper, and not more than THREE in all.]

1. Write a short account of the method of projection in geometry. Include in this account the properties of a projected figure corresponding to (a) circles, (b) right angles, (c) a pair of equal angles, (d) middle points of lines, and (e) foci of conics, in the original figure, and illustrate your theory by stating in a form true for all conics the property that the angle at the centre of a circle is double that at the circumference.

2. Prove that if m is prime to a the least positive remainders of the series of integers

$$k, k + a, \dots, k + (m - 1)a$$

with respect to m are a permutation of the numbers of the series

$$0, 1, 2, \dots, (m - 1),$$

and that

$$a^{\phi(m)} - 1 \equiv 0 \pmod{m},$$

where $\phi(m)$ is the number of integers less than m and prime to it.

Prove also that $(m - 1)! + 1$ is divisible by m if, and only if, m is a prime.

Show that if m is a prime, and $p < m$,

$$(p - 1)! (m - p)! + (-1)^{p-1} \equiv 0 \pmod{m}.$$

3. State and prove the leading propositions in the theory of determinants and indicate some applications of the theory.

4. Starting from the definition of a differential coefficient, develop methods and results which will enable you to differentiate any function obtained by combining exponential functions, circular functions, powers, and the inverses of these functions.

5. Establish formulæ for the curvature at any point of a plane curve, including the cases when the curve is defined (a) by a Cartesian equation, (b) by equations of the type $x = \phi(t)$, $y = \psi(t)$, (c) by an intrinsic equation, (d) by a p , r equation, and (e) as the envelope of a line whose equation contains one variable parameter. Apply such of these formulæ as are suitable, to the cases of the ellipse and the parabola.

6. Starting with any definition of $\exp x$ or $\log x$, where x is a real variable, develop the principal properties of these functions, including among your results the expansions for $\exp x$ and $\log(1+x)$, the equation

$$\frac{d}{dx}(\log x) = \frac{1}{x},$$

and the relation

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \exp x.$$

Explain also the connexion between $\exp x$ and e^x .

A. Prove the theorems on which graphic methods as applied to statical problems depend.

B. Investigate the various theorems concerning the Conservation of Energy and the Conservation of Momentum under appropriate conditions, for a system of two particles.

Apply your theorems to the solution of the following problem: Two particles of masses m and m' are connected by an elastic string, of natural length l and without mass, and are initially at rest at a distance $\frac{1}{2}l$ from each other at points A and B respectively. Blows P and Q are applied to the particles in directions perpendicular to AB and towards A respectively. Discuss the subsequent motion.

C. Investigate the various problems arising from the collision of smooth elastic spheres. In particular consider the loss of kinetic energy.

A mass m of water issues per unit time from a pipe with uniform velocity u , and strikes a pail which retains it, there being no elasticity. Initially the pail is at rest, and at a subsequent instant is moving in the direction of the stream with velocity V . Prove that

$$\frac{dV}{dt} = \frac{m(u - V)^2}{M u^2},$$

and that the loss of energy up to this instant is

$$\frac{1}{2} M u V,$$

where M is the mass of the pail, and gravity is omitted from consideration.

D. Investigate the theory of the isochronism of oscillations, considering harmonic motion in general, motion on a cycloid, and the small oscillations of a pendulum.

Thursday, December 8, 1910. 9—12.

FOR CANDIDATES IN MATHEMATICS, IN MECHANICAL SCIENCES AND
IN MATHEMATICS APPLICABLE TO PHYSICS.*[Mathematical Tables and Squared Paper are provided, and mathematical instruments may be employed for calculation.]**Candidates for Mathematical Scholarships are to omit questions A—C.*

A. Prove the formula

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

Show that

$$\sin A \sin(B - C) + \sin B \sin(C - A) + \sin C \sin(A - B) = 0.$$

B. Find the length of the perpendicular from the origin to the line

$$3x + 4y - 1 = 0.$$

Find also the equations of the two lines which are parallel to this line and at twice its distance from the origin.

C. Trace the curve $y^2 = 16x$, and find the points where it is cut by the line $y = 2x - 6$. Prove also that the line cuts the curve at angles ϕ and ϕ' given by

$$\tan \phi = \frac{1}{7}, \quad \tan \phi' = \frac{1}{3}.$$

1. The bisector of the angle A of a triangle ABC passes through the centre of the square described externally on BC . Show that A is a right angle, or that $AB = AC$.2. $ABCD$ is a quadrilateral not inscribable in a circle. Show that

$$AC \cdot BD < AB \cdot CD + BC \cdot AD.$$

3. The roots of the equation

$$x^4 - 12x^3 + 51x^2 - 92x + 60 = 0$$

are all integral; find them.

4. Prove that if x is the n th root of a , the error involved in taking x for the $(n + q)$ th root of a is

$$\frac{qx \log_e x}{n + q} \left\{ 1 - \frac{q \log_e x}{2(n + q)} \right\} \text{approximately,}$$

where

$$\frac{q \log_e x}{n + q}$$

is small.

Estimate roughly the extreme magnitude of this error when n and q are not greater than 1000 and 1 respectively, and n is not less than 200.

5. Solve (graphically or otherwise) to two significant figures the equation

$$x^3 - 10x - 11 = 0.$$

6. Find the general solution of the equation

$$\operatorname{cosec} 4a - \operatorname{cosec} 4\theta = \cot 4a - \cot 4\theta.$$

7. Solve completely the triangles in which

$$a = 113, \quad b = 152, \quad A = 27^\circ.$$

8. In an acute-angled triangle ABC , if LMN is the pedal triangle and ρ its in-radius, show that

$$LM \cdot MN \cdot NL = 2\rho\Delta,$$

where Δ is the area of ABC .

9. Interpret the equations obtained by eliminating x and y in succession between the straight line $x + 2y = 3$ and the conic $x^2 - 3y^2 = 22$, and hence (or otherwise) prove that the circle described on this chord as diameter has for its equation

$$x^2 + y^2 + 18x - 12y - 128 = 0.$$

10. Find the relation between the eccentric angles at the extremities of conjugate diameters of an ellipse.

If the ellipse be

$$x^2/a^2 + y^2/b^2 = 1,$$

prove that the tangents at the ends of conjugate diameters intersect on the ellipse

$$x^2/a^2 + y^2/b^2 = 2,$$

and that the corresponding normals meet on the curve

$$2(a^2x^2 + b^2y^2)^2 = (a^2 - b^2)^2 (a^2x^2 - b^2y^2)^2.$$

11. Prove the rule for differentiating the quotient of two functions.

Find the differential coefficients of

$$\frac{x-2}{\sqrt{2x^2-2x+5}}, \quad \sin\left\{\frac{1+x^2}{1+x}\right\}, \quad \tan(x - \sin^{-1}x).$$

12. A bell-tent consists of a conical portion above and a cylindrical portion near the ground. For a given volume and a circular base of given radius, prove that the amount of canvas used is a minimum when the semi-vertical angle of the cone is $\cos^{-1}(\frac{2}{3})$.

13. Integrate the following expressions:—

$$\frac{x}{(x^2 + 3x + 2)}, \quad \tan^2x, \quad x^2 \log x, \quad \frac{1}{\sqrt{2-x} + \sqrt{1+x}}.$$

14. Apply the integral calculus to evaluate

- (i) the area of a parabola cut off by a line perpendicular to the axis;
- (ii) the volume of a sphere.

15. Trace the curve $xy^2 = 4a^2(2a - x)$ and prove

- (i) that the area between the curve and the axis of y is $4\pi a^2$;
- (ii) that the volume generated by the revolution of the curve about this line is $2\pi^2 a^2$.

Thursday, December 8, 1910. 1.30—4.30.

FOR CANDIDATES IN MATHEMATICS, IN MECHANICAL SCIENCES
AND IN MATHEMATICS APPLICABLE TO PHYSICS.

[Mathematical Tables and Squared Paper are provided, and mathematical instruments may be employed for calculation.]

1. A ship leaves a certain port and steams N.W. at 15 knots; ten hours later another ship leaves the same port, and steams W.S.W. at 12 knots. Their wireless instruments are capable of communications up to 500 nautical miles: how long may the ships expect to remain in touch with one another?

[A knot = a nautical mile an hour.]

2. A train passes a station *A* at 40 miles per hour and maintains this speed for 7 miles, and is then uniformly retarded, stopping at *B*, which is 8 miles from *A*. A second train starts from *A* the instant the first train passes and, being uniformly accelerated for part of the journey and uniformly retarded for the rest, stops at *B* at the same time as the first train. What is its greatest speed on the journey?

3. A tramcar starts from rest and its velocities at intervals of 5 secs are given in the following table:—

Time	0	5	10	15	20	25	30
Velocity in miles per hour. } }	0	8·1	11·8	14·6	16·3	17·7	19

Calculate the distance in yards travelled in the above time. Also, if the car weigh 8 tons, estimate the effective pull exerted on the car at the end of 20 seconds.

4. A particle is projected from a given point with a velocity whose vertical component is given. Prove that the initial angular velocity about the focus of the path is greatest when the angle of projection is 45°.

5. A smooth sphere is tied to a fixed point by an inelastic string, and another sphere impinges directly on it in a direction making an acute angle with the string. Show that, if the second sphere is reduced to rest by the impact, the ratio of the total kinetic energy after the impact to that before is equal to the coefficient of restitution.

6. A light spring is such that *m* lbs. weight compresses it *a* feet. It is compressed *c* feet (*c* > *a*). If *m'* lbs. is placed on the top and the spring is released, find the condition that the weight will leave the spring, and its velocity when this happens.

7. Four equal masses are attached at equal distances *A, B, C, D* at points on a light string, and so placed that $\angle ABC = \angle BCD = 120^\circ$, and the various parts of the string are straight; an impulse *I* is given to the mass at *A* in the direction *BA*, show that the impulsive tension in *AB* is $\frac{1}{4} I$.

8. A heavy uniform beam, 8 feet long, weighs 20 lbs., and at one end it carries a load of 20 lbs. and at the other end a load of 10 lbs. The beam thus loaded is to be carried on a certain support. Find by a graphical construction the position of the support.

If the beam is to rest on two supports which are to divide the load in a given ratio, show how to find on your diagram any number of suitable pairs of points, and find the position of the supports when they are at a distance 4 feet apart, the pressures on them being in the ratio of 3 to 2.

9. A tripod consists of three equal rods, each of weight *w*, smoothly jointed at the upper ends. It is placed symmetrically on a rough horizontal table, for which the angle of friction is ϕ , and a weight *W* is put on the top of the tripod. Show that the rods cannot make an angle with the vertical greater than

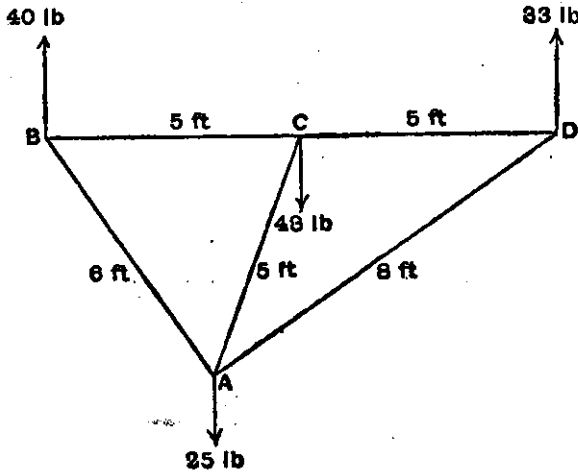
$$\tan^{-1} \left(2 \frac{W + 3w}{2W + 3w} \tan \phi \right).$$

10. Two equal uniform rods are fastened together so as to bisect one another at right angles. They rest in a plane at right angles to a rough wall, the one rod resting on the edge of the top of the wall, and an end of the other against the vertical side. Prove that the limiting inclination θ of the second rod to the vertical is given by the equation

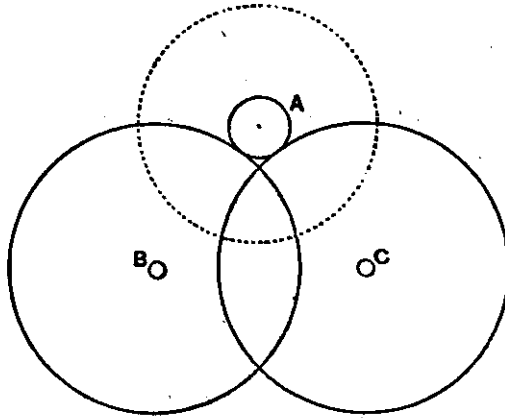
$$\tan \theta \cos^2 \lambda = \cos(\lambda + \theta) \sin(\lambda - \theta),$$

where λ is the angle of friction.

11. The figure $ABCD$ represents a freely-jointed light plane framework, with forces acting in the plane at right angles to BCD . Show that it is in equilibrium, and prove, graphically or otherwise, that the stresses in AB, AC are equal and of opposite sign.



12. Two equal friction-wheels of radii R inches turn on axles B, C of radii r inches; the coefficient of friction between wheel and axle is $\tan \phi$. A wheel of weight W is attached to an axle of radius a inches which rests on the circumferences of the first two wheels as in the figure. Neglecting the weights of the wheels B and C , show that the least couple which will rotate the wheel A is $Wa \frac{\sin 2\theta}{\sqrt{3}}$, where $\sin \theta = \frac{r}{R} \sin \phi$: the centres of the three wheels forming an equilateral triangle.



(ALTERNATIVE QUESTIONS IN PHYSICS.)

13. A diving bell has the form of a paraboloid of revolution cut off by a plane perpendicular to its axis at a distance h from the vertex, where h is the height of the water-barometer. If the bell be lowered so that its vertex is at a depth $5h/2$ below the surface of the water, find how high the water will rise in the bell.

14. A battery is connected to a galvanometer of 40 ohms resistance, and a certain current is observed. The galvanometer is now shunted with a resistance of 10 ohms, and the current in the galvanometer falls to one-half of the former value. Find the internal resistance of the battery.

15. The coefficients of cubical expansion of mercury and brass are $\cdot 00018$ and $\cdot 00006$ per degree Centigrade. A mercury barometer with a brass scale reads correctly at 15° C., standing at 30 inches; what will be the error at 35° C. ?

16. One cubic foot of air at a temperature of 500° C. absolute is expanded isothermally from a pressure of 120 lbs. per sq. in. to twice the initial volume; it is then expanded adiabatically to three times the initial volume. Find the pressure and temperature at the end of each stage and calculate, graphically or otherwise, the work done and the heat units taken in during each stage, taking the mechanical equivalent of one thermal unit as 1,400 ft. lbs., and the equation for adiabatic expansion of air as $pv^{1.4} = \text{const.}$

Friday, December 9, 1910. 9—12.

GENERAL QUESTIONS.

SECTION A.

This section is to be taken by all Candidates.

SECTION B.

Candidates may write on not more than FOUR subjects in this section; and must choose their subjects as far as possible from different groups.

SECTION C.

Candidates may take any or all of the questions in this section, but they must do justice to Sections A. and B. before they pass on to C.

SECTION A.

This section is to be taken by all Candidates.

1. Translate :

Un jeune homme—traçons son portrait d'un seul trait de plume :—figurez-vous don Quichotte à dix-huit ans; don Quichotte décornelé, sans haubert et sans cuissard; don Quichotte revêtu d'un pourpoint de laine dont la couleur bleue s'était transformée en une nuance insaisissable de lie de vin et d'azur céleste. Visage long et brun; la pommette des joues saillante, signe d'astuce; les muscles maxillaires énormément développés, indice infailible auquel on reconnaît le Gascon, même sans béret, et notre jeune homme portait un béret orné d'une espèce de plume; l'œil ouvert et intelligent; le nez crochu, mais finement dessiné; trop grand pour un adolescent, trop petit pour un homme fait, et qu'un œil peu exercé eût pris pour un fils de fermier en voyage, sans la longue épée qui, pendue à un bandrier de pean, battait les mollets de son propriétaire quand il était à pied, et le poil hérissé de sa monture quand il était à cheval.

DUMAS.

SECTION B.

Candidates may write on not more than FOUR subjects in this section; and must choose their subjects as far as possible from different groups.

2. Discuss the chief characteristics of two of the following schools of Art :—Dutch, Venetian, Tuscan, Spanish and English.

3. "The successful novelist cannot be a successful dramatist."
4. "Opera is a bastard art : it attempts to combine music with drama."
5. Compare the merits of Classic and Gothic architecture.
6. "The wealth of nations, as of men, consists in substance, not in ciphers : and the real good of all work, and of all commerce, depends on the final intrinsic worth of the thing you make or get by it."
7. "It is in general a necessary condition of free institutions, that the boundaries of governments should coincide in the main with those of nationalities."
8. How far do you consider impartiality in history possible ?
9. "Perfect human liberty is possible only to a despot, who is absolutely and universally obeyed."

10. What theories have been advanced as to the maintenance of the radiation of energy by the sun ?

11. Did Sherlock Holmes reach his conclusions by the "method of Science" ? If not, in what way did his method differ from that of Science ?

12. Some philosophers believe that matter is an hallucination, others that it is real. Others, again, believe that its qualities are distorted and perverted by the media through which we become aware of it. Discuss these points of view.

13. Discuss the possibility and probability of life on the moon. What directions must evolution have taken to make it possible ?

14. The evils of deforestation.

15. The influence of mechanical flight on the mutual relations and boundaries of nations.

16. How are the prevention, the diagnosis and the cure of diseases aided by research ?

17. Discuss the problem of street traffic, and make suggestions for relieving congestion in the more crowded areas.

SECTION C.

[Candidates may take any or all of the questions in this section, but they must do justice to Sections A. and B. before they pass on to C.]

18. Translate :

Dors dans cet endroit pauvre où les archiducs blonds
Sont vêtus d'un airain que le Temps vert-de-grise.
On dirait qu'un départ dont l'instant s'éternise
Encombre les couloirs de bagages oblongs.
Des touristes anglais traînent là leurs talons,
Puis ils vont voir, plus loin, ton cœur, dans une Église.
Dors, tu fus ce Jeune homme et ce Fils, quoi qu'on dise.
Dors, tu fus ce martyr ; du moins, nous le voulons.
... Un Capucin pressé d'expédier son monde
Frappe avec une clef sur ton cercueil qui gronde,
Dit un nom, une date,—et passe, en abrégeant...
Dors ! mais rêve en dormant que l'on t'a fait revivre,
Et que, laissant ton corps dans son cercueil de euvre,
J'ai pu voler ton cœur dans son urne d'argent.

19. Translate.

Während auf dem Damme von beiden Theilen mit der hartnäckigsten Hitze gefochten wurde, hatte man die Scheld-Brücke von Antwerpen aus mit neuen Maschinen bestürmt, um die Aufmerksamkeit des Herzogs auf dieser Seite zu beschäftigen. Aber der Schall des Geschützes vom Damme her entdeckte demselben bald, was dort vorgehen mochte, und er eilte, sobald er die Brücke befreit sah, in eigner Person den Deich zu entsetzen. Von zweihundert spanischen Pikenierern begleitet flog er an den Ort des Angriffs, und erschien noch gerade zu rechter Zeit auf dem Kampfplatz, um die völlige Niederlage der Seinigen zu verhindern. Eiligst warf er einige Kanonen, die er mit gebracht hatte, in die zwei nächsten Redouten, und liess von da aus nachdrücklich auf die feindlichen Schiffe feuern. Er selbst stellte sich an die Spitze seiner Soldaten, und in der einen Hand den Degen, den Schild in den andern, fuhrte er sie gegen den Feind.

SCHILLER.

20. Compare the use of Blank verse and the Hexameter in English poetry.

21. Assign any or all of the following to their authors on the ground of style:—

- (a) " We drove in a body to Science the other day for an antidote; which was as if tired pedestrians should mount the engine-box of headlong trains; and Science introduced us to our o'er-hoary ancestry—them in the Oriental posture: whereupon we set up a primæval chattering to rival the Amazon forest nigh nightfall, cured, we fancied. And before daybreak our disease was hanging on to us again, with the extension of a tail."
- (b) Homer's Epos has not ceased to be true: yet it is no longer *our* Epos, but shines in the distance, if clearer and clearer, yet also smaller and smaller, like a receding Star. . . . So likewise a day comes when the Runic Thor, with his Eddas, must withdraw into dimness; and many an African Mumbo-Jumbo and Indian Pawaw be utterly abolished.
- (c) The natural advantages which arise from the position of the earth which we inhabit with respect to the other planets, afford much employment to mathematical speculation, by which it has been discovered, that no other confirmation of the system could have given such commodious distributions of light and heat, or imparted fertility and pleasure to so great a part of a revolving sphere.
- (d) While all melts under our feet, we may well grasp at any exquisite passion, or any contribution to knowledge that seems by a lifted horizon to set the spirit free for a moment, or any stirring of the senses, strange dyes, strange colours, and curious odours, or work of the artist's hands, or the face of one's friend.
- (e) Roses whose lips the flame has deadened
 Drink till the lapping leaves are reddened
 And warm wet inner petals weep;
 The flower whereof sick sleep gets leisure,
 Barren of balm and purple pleasure,
 Fumes with no native steam of sleep.
- (f) Were others angry: I excused them too;
 Well might they rage, I gave them but their due.
 A man's true merit 'tis not hard to find;
 But each man's secret standard is his mind,
 That Casting-weight pride adds to emptiness.
 This who can gratify? for who can guess?

Saturday, December 10, 1910. 9—12.

ENGLISH ESSAY.

Write an Essay on *one* of the following subjects:—

1. "The doctrine of self-reliance and self-denial which is the foundation of political economy."
2. Compare the poetry of the eighteenth and nineteenth centuries.
3. Cecil Rhodes.
4. "The aim of education is to form character: its result is to destroy it."
5. Hobbies.
6. "To form habits is to fail in life."

(2) EXAMINATION FOR ENTRANCE SCHOLARSHIPS 1910.

Pembroke, Gonville and Caius, King's, Jesus, Christ's, St. John's, and Emmanuel Colleges.

Wednesday, December 7. 9—12.

ALGEBRA, TRIGONOMETRY AND DIFFERENTIAL CALCULUS.

[*Mathematical Tables are supplied.*]

1. Show that the expression

$$(cx - as)^2 - (ay - bx)(bz - cy)$$

is the product of two linear functions of x, y, z and that, when a, b, c are real, the coefficients in the linear functions are real provided

$$b^2 < 4ac.$$

2. Prove that, if n positive quantities be restricted to have a definite sum ac , their product cannot exceed c^n and the sum of their squares lies between ac^2 and n^2c^2 .

Show also that, if there be four positive quantities and they be restricted to have a sum $4c$ and a sum of squares $8c^2$, no one of them can exceed

$$(\sqrt{3} + 1)c.$$

3. Show that, in the expansion of $(1 + x)^m$ by the Binomial Series in the case when x is a positive fraction and m lies between the positive integers $r - 1$ and r , the series begins with $r + 1$ positive terms, and the numerically greatest term cannot occur later than the middle term or terms of these positive terms.

4. Determine the range of values for which the two infinite series

$$1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \dots,$$
$$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} \dots$$

are respectively convergent.

Prove that in the approximate formula used by Napier for calculating logarithms

$$\log \frac{a}{b} = \frac{a-b}{2} \left(\frac{1}{a} + \frac{1}{b} \right).$$

where $\frac{a-b}{a}$ is small, the error is of the order

$$\frac{1}{6} \left\{ \frac{a-b}{a} \right\}^3.$$

5. Find a general expression for all the angles which have the same tangent as a given angle.

If $\tan 5a$ is given, find the number of different values of $\cos a$.

Show that if $\tan \theta = \tan^3 \phi$ and $\tan 2\phi + 2 \tan \psi = 0$, then $\theta + \phi + \psi = n\pi$.

6. If in a triangle ABC , the angle A , the side BC and the perpendicular from A on BC are given, find formulæ to determine the other angles and sides.

Calculate them when A is 62° , BC 125 feet, and the perpendicular 72 feet.

7. Express the radii of the inscribed and escribed circles of a triangle in terms of the radius of the circumcircle and the angles of the triangle.

Show that two circles can be drawn within the angle BAC of a triangle to touch AB and AC and to cut off a chord of given length from BC , provided the given length does not exceed twice the geometric mean of the radii of the inscribed circle and the escribed circle opposite A .

8. Express in real form the partial fractions of $\frac{1}{x^2+1}$.

Show that the general solution of $a \cos \theta + b \sin \theta = c$, where $c > \sqrt{a^2 + b^2}$ is

$$\theta = 2n\pi + \tan^{-1} \frac{b}{a} + i \cosh^{-1} \frac{c}{\sqrt{a^2 + b^2}}$$

provided the value assigned to $\tan^{-1} \frac{b}{a}$ is also a value of $\cos^{-1} \frac{a}{\sqrt{a^2 + b^2}}$.

9. Discuss by means of the differential coefficient the variation in value of $x \log x$ for positive values of x , and show that $x \log x = c$ has one real root if c is positive and two real roots if c is negative, and numerically less than $1/e$.

10. Show that

$$y = \frac{1}{2} (\sin^{-1} x)^2 \text{ satisfies } (1-x^2) \frac{d^2y}{dx^2} = x \frac{dy}{dx} + 1.$$

and obtain the expansion of the function y in ascending powers of x .

11. Define the curvature of a curve at any point, and from the definition find the curvature of the curve $y = f(x)$ at any point.

Trace the curve $y^2 = x(x-1)^2$ and show that the circle which touches it at the points where the ordinate has its maximum and minimum values is the circle of curvature at both points. Draw the circle.

Thursday, December 8. 9—12.

STATICS AND DYNAMICS.

1. Prove that the resultant of forces represented by $m \cdot OA$ and $n \cdot OB$ is represented by $(m + n) OG$, where G is a point on AB such that $m \cdot AG = n \cdot GB$.

The line of action of a force P cuts the sides of a triangle in given points. Show how to find three forces acting along the sides of the triangle which shall have P for their resultant.

2. Find the resultant of a couple and a force in its plane.

A right circular cylinder, of weight W and radius r , stands on a horizontal plane, and a smooth circular wire of weight W' and radius a ($a > r$) is placed over it so as to encircle it. Prove that, provided $W > W'$, the ring may be supported by a vertical string held in the hand, without touching the plane; and find the pressures on the cylinder. Why is the restriction as to W introduced? Would it be required if the string were attached to the cylinder?

3. Two uniform rods AB and CD , each of weight W and length a , are smoothly jointed together at a point O , where OB and OD are each of length b . The rods rest in a vertical plane with the ends A and C on a smooth table and the ends B and D connected by a light string. Prove that the reaction at the joint is $\frac{aW}{2b} \tan \alpha$, where α is the inclination of either rod to the vertical.

4. Establish the existence of the centre of a system of parallel forces.

A hollow wedge of negligible weight is full of water and has its edges parallel to the edge of a horizontal table on which it rests. The cross-section ABC has its side BC on the table and B projects over the edge of the table. Prove that the length h of BC on the table must exceed $\frac{1}{2}(a + d)$, where $d = b \cos C$, and that, if $h < \frac{1}{2}a$, and if the water leaks slowly through a small hole at B , the wedge will fall over as soon as the proportion of water escaped to the original amount is n^2 , where

$$n^2(a - 2d) - (n + 1)(3h - a - d) = 0.$$

5. Two equal rough circular cylinders of weight W are placed upon a horizontal plane with their axes parallel, and a plank of weight W' is laid across them perpendicular to their axes so as to project equally beyond each. The plank is then tilted about a line parallel to the axes, while the upper cylinder is held fixed. Prove that, if the coefficient of friction μ is the same at all points of contact, equilibrium will be broken when the inclination of the plank to the horizontal is

$$\tan^{-1} \frac{\mu W'}{2W' + W}.$$

6. Show how the velocity of the centre of mass of a number of particles moving in one plane can be found when their velocities and masses are known.

Two particles A and B are moving in one plane with given velocities and are not acted on by any external forces; if they impinge upon one another and after the impact A 's velocity and the direction of B 's velocity are given, show how to determine the magnitude of B 's velocity and the ratio between their masses.

7. Prove that the increase of kinetic energy of a particle is equal to the work done on it, in the case when the particle is moving under the action of a constant force.

Two particles of masses $7m$ and $3m$ are fastened to the ends A, B respectively of a weightless rigid rod, 15 ft. long, which is freely hinged at a

point O , 5 ft. from A : if the rod is just disturbed from its position of unstable equilibrium, prove that the velocity with which A will pass through its position of stable equilibrium is $\frac{1}{16} \sqrt{380g}$.

8. If a train travelling at 30 miles an hour picks up 10,000 lbs. of water in a quarter of a mile, find the back-pressure thereby produced on the train and the extra horse-power required on this account to maintain the speed.

9. Find the loss of energy when two smooth spheres moving in the same straight line come into collision.

A wedge of mass M and angle α is lying with one face in contact with a smooth horizontal table, and when on the other face a particle of mass m is placed, motion ensues: if the wedge when moving with velocity v is suddenly stopped, determine the loss of energy, e being the coefficient of restitution between the particle and wedge.

10. Determine the acceleration of a point describing a circle with uniform speed.

A small ring fits loosely on a rough spoke (length a) of a wheel which can turn about a horizontal axle and the ring is originally at rest in contact with the lowest point of the rim: if the wheel is now made to revolve with uniform angular velocity ω , prove that the angle θ through which the wheel will turn before the ring slides is given by the equation

$$\cos(\theta - \lambda)/\cos \lambda + \omega^2 a/g = 0,$$

where λ is the angle of friction.

Thursday, December 8. 1.30—4.30.

GEOMETRY.

1. In plane geometry, prove that if O is the middle point of AB , the distance of O from any line is either half the sum or half the difference of the distances of A and B from that line. Distinguish the two cases.

The base AB of a triangle is fixed while the vertex C moves, and CP , CQ are the diagonals of squares (external to ABC) described on the sides CA , CB . Prove that the middle point of PQ is a fixed point.

2. A point P describes a circle, and a second point Q moves so that its distance OQ from a fixed point O is always proportional to OP and the angle POQ is constant. Prove that Q also describes a circle.

A point A is taken within a given circle, and through each point P of the circle is drawn a chord PQ inclined at a constant angle α to AP . Show that the line through Q which makes an angle α with QP but is not parallel to AP passes always through a second fixed point within the circle.

3. Prove that if a figure is reflected in lines AB , BC successively the result is the same as rotating the figure about B through twice the angle ABC .

†† Show that if the reflections are accompanied by displacements parallel and equal to AB , BC the rotation is the same as before and takes place about the circumcentre of the triangle ABC .

4. Fixed tangents to a parabola at points P , P' equidistant from the axis are cut by a variable tangent in points Q , Q' . Prove that either the sum or the difference of the lengths PQ , $P'Q'$ is constant.

5. Prove that tangent cones to a sphere with collinear vertices have planes of contact which pass through a line.

Two perpendicular lines meet perpendicularly the same diameter of a sphere at points inverse with regard to the sphere. Prove that any chord of the sphere which meets both lines is harmonically divided by them.

6. Find the equations of the tangent and normal at the point $(am^2, 2am)$ of the parabola $y^2 = 4ax$.

The tangent at any point P of this parabola is met in Q by a line through the vertex A at right angles to AP , and Z is the foot of the perpendicular from A on the tangent at P . Show that there are three positions of the point P on the parabola for which the point Z lies on the straight line.

$$Ax + By + C = 0,$$

and that the corresponding points Q lie on the line

$$(2Aa - C)x + 4Bay + 2Ca = 0.$$

7. Find the equation of the circle, of which the chord of the ellipse

$$ax^2 + by^2 = 1,$$

intercepted on the line $lx + my = 1$ is a diameter.

Prove that this circle will touch the ellipse at a third point if the given chord touches a similar, similarly situated and concentric ellipse.

8. A straight rod APB has its ends sliding on fixed perpendicular lines OA, OB . Show that each point P of the rod describes an ellipse and that the normal at P passes through the vertex I of the rectangle $AOBI$.

Show also that if K is the centre of curvature at P , and KX parallel to OA meets AB in X , then the triangles PXI and PIA are similar.

9. Prove that the conic

$$9x^2 - 24xy + 41y^2 = 15x + 5y$$

has one extremity of its major axis at the origin and one extremity of its minor axis on the axis of x . Find the coordinates of its centre and foci.

10. The perpendiculars from fixed points P_1, P_2, \dots on a variable line have lengths p_1, p_2, \dots , and the line moves so that the sum $\sum k_i p_i^2 = 0$. Show that the envelope of the line is a conic; and that if $\sum k_i = 0$ it is a parabola, and if $\sum k_i k_j P_i P_j^2 = 0$ it is a rectangular hyperbola.

11. Show that, if the polars of P and Q with respect to a conic are parallel respectively to QR and PR , the polar of R must be parallel to PQ ; and that the triangle PQR is self-conjugate with respect to a second conic having the same asymptotes as the former.

Friday, December 9th. 9—12.

MATHEMATICS.—GENERAL PAPER.

1. Give a short account, without proofs, of the method of inversion in plane geometry. Apply the method to prove that a system of coaxial circles, which cut in two real points A and B , are cut orthogonally by a second system of circles which do not intersect one another and one set of which surround the point A , while the other set surround the point B ; and show how to determine in the inverse figure the circle which corresponds to the straight line separating the two original sets of circles from one another.

2. Two projective (homographic) ranges of points on a line are such that no (real) point on the line is self-corresponding. Prove that two positions of a point P may be found in any plane through the line, such that all pairs of corresponding points of the ranges subtend the same angle at P .

3. A polynomial $\phi(x)$ is defined by the identity

$$\phi(x+1) - \phi(x) = (1+x)^r$$

with $\phi(1) = 1$, where r is a positive integer. Prove that $\phi(0) = 0$, $\phi(-1) = 0$, and from $\phi(-x) - \phi(-x-1) = (-1)^r x^r$ deduce that $\phi(x) = (-1)^{r-1} \phi(-x-1)$ for all integral and therefore for all values of x .

Hence show that the sum of the r th powers of the first n natural numbers is an integral function of n divisible by $n(n+1)$ and that it is also divisible by $(2n+1)$ if r is even.

4. Two perpendicular lines in an inclined plane make angles α, β with a horizontal plane; prove that the angle γ between the two planes is given by

$$\sin^2 \gamma = \sin^2 \alpha + \sin^2 \beta,$$

and find the angles between the given pair of lines and a line of greatest slope in the plane.

5. The tangents to an ellipse at A and B , extremities of the major and minor axes, meet in K . Show that the osculating circles at A and B have K as an external limiting point, and subtend supplementary angles at K .

Find any such ellipse in relation to the circles

$$\begin{aligned}(x^2 + y^2) \cos^2 \alpha + 2cx + c^2 &= 0, \\(x^2 + y^2) \sin^2 \alpha + 2cx + c^2 &= 0.\end{aligned}$$

6. A point P moves in a plane in such a way that

$$VP - SP = SK,$$

where S is a fixed point in the plane, V a fixed point not in the plane and K the foot of the perpendicular from V on the plane. Prove that the locus of P is a parabola with focus at S ; and that, if the vertex of this parabola is A , the point V lies on a second parabola with vertex at S and focus at A .

7. Show that the function $\sin x + a \sin 3x$ for values of x from 0 to π has no zeros except the terminal ones if $-\frac{1}{2} < a < 1$. Show also that it has two minima with an intermediate maximum if $a < -\frac{1}{2}$, one maximum if $-\frac{1}{2} < a < \frac{1}{2}$, two maxima and an intermediate minimum if $a > \frac{1}{2}$.

Indicate roughly the types of the graph of the function in the four cases.

8. Five weightless rods, AB, BC, CD, DA and BD , smoothly jointed at their ends, form a framework: AB is vertical with B above A , $BAD = 30^\circ$, $ABD = 60^\circ$, BC is horizontal and $BD = DC$. A weight W is suspended from C and the framework is supported by a horizontal force at B and a thrust at A . Find the stress in each of the rods, distinguishing between thrust and tension.

9. State the general principle of virtual work and prove that when applied to the case of a single rigid body in two dimensions it leads to three independent necessary and sufficient conditions of equilibrium and deduce these conditions in one of their usual forms.

Four equal uniform rods, of length $(a + b)$, are hinged at A, B, C, D , and are suspended by two strings of equal length, so that the diagonal AC is vertical and A is the highest point. The strings are attached to two pegs in the same horizontal line and to two points in AB, AD at distance a from A . Prove that if the rods rest in the form of a square the inclination of the strings to the horizontal is $\tan^{-1}(a/b)$.

10. Prove that if two particles of masses m_1, m_2 are moving in a plane, their kinetic energy is

$$\frac{1}{2} (m_1 + m_2) V^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v^2,$$

where V is the velocity of their centre of mass and v is their relative velocity.

A shell of mass $(m_1 + m_2)$ is fired with a velocity whose horizontal and vertical components are U, V , and at the highest point in its path the shell explodes into two fragments m_1, m_2 . The explosion produces an additional kinetic energy E , and the fragments separate in a horizontal direction: show that they strike the ground at a distance apart which is equal to

$$\frac{V}{g} \left\{ 2E \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \right\}^{\frac{1}{2}}.$$

Friday, December 9. 1.30—4.30.

ESSAYS FOR CANDIDATES IN MATHEMATICS.

Write an essay on *one* of the following subjects:—

1. The place of Mathematics in a general scheme of education.
2. Money.
3. "He that will not apply new Remedies must expect new Evils: for Time is the greatest Innovatour." LORD BACON.
4. The highway and the motor-car.
5. The value of a hobby.

(b) OXFORD UNIVERSITY.

(1) MATHEMATICAL SCHOLARSHIPS.

Balliol, Queen's, Corpus Christi, and St. John's Colleges, 1910.

I.

1. Draw the graph of the function

$$y = \frac{(x-1)(x-3)}{(x+1)(x-5)}$$

Prove that $\frac{1}{2}$ is a *maximum* value of y and that 1 is a *minimum* value and explain the apparent paradox.

2. Find the sum of n terms of the series

$$1 + \frac{x}{a}(1+x) + \frac{x^2}{a^2}(1+x+x^2) + \frac{x^3}{a^3}(1+x+x^2+x^3) + \dots$$

State the conditions under which the series has a sum to infinity, and find this sum when it exists.

3. If a, b, c, \dots, k are positive proper fractions, prove that $(1-a)(1-b)(1-c) \dots (1-k)$ lies between 1 and $1-a-b-\dots-k$.

If x is a positive quantity, n a positive integer, and

$$u_n \equiv 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n}$$

prove that

$$\left(1 + \frac{x}{n}\right)^n \text{ lies between } u_n \text{ and } u_n - \frac{x^2}{2n} u_{n-2}.$$

4. Resolve

$$\frac{1}{(x-a)^2(x-b)}$$

into partial fractions.

If ${}^p C_q$ denote the number of combinations of p things q at a time, prove that

$$\begin{aligned} n \left(\frac{1}{n} - \frac{{}^r C_1}{n+1} + \frac{{}^r C_2}{n+2} - \dots \pm \frac{{}^r C_r}{n+r} \right) \\ = r \left(\frac{1}{r} - \frac{{}^{n-1} C_1}{r+1} + \frac{{}^{n-1} C_2}{r+2} - \dots \pm \frac{{}^{n-1} C_{n-r}}{r+n} \right) = \frac{n}{r} \frac{r}{n+r} \end{aligned}$$

5. Show that if p_n/q_n be the n^{th} convergent to the continued fraction

$$\frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}}$$

then

$$p_n = b_n p_{n-1} + a_n p_{n-2}, \quad q_n = b_n q_{n-1} + a_n q_{n-2}.$$

Show that the value of the continued fraction

$$\frac{1}{2} - \frac{2}{3} - \frac{3}{4} - \dots - \frac{n}{n+1} \text{ is } \frac{1 + |2 + \dots + n|}{1 + 1 + |2 + \dots + n|}.$$

6. Show that a real root of the equation $f'(x) = 0$ lies between each pair of real roots of the equation $f(x) = 0$, whether these pairs are distinct or not.

If the equation $x^n + p_1 x^{n-1} + \dots + p_n = 0$ has three roots each equal to a , prove that a is a root of the equation

$$n^2 x^{n-1} + (n-1)^2 p_1 x^{n-2} + (n-2)^2 p_2 x^{n-3} + \dots + p_{n-1} = 0.$$

7. If

$$\tan \frac{1}{2}(a + \theta) = \frac{a - \sin \phi}{b - \cos \phi}, \quad \tan \frac{1}{2}(a + \phi) = \frac{a - \sin \theta}{b - \cos \theta}$$

and $\theta - \phi = \beta$, show that

$$a^2 + b^2 = 2a \sin a + 2b \cos a + 2 \cos \beta + 1.$$

8. (1) If A, B, C are the angles of a triangle ABC , prove that

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1.$$

(2) If D is a point dividing BC in the ratio $m : n$, and if D' is a point taken on BC such that the angle BAD is equal to $D'AC$ and the angle CAD is equal to the angle $D'AB$, show that

$$(i) \tan \frac{1}{2} DAD' = \frac{cn - bm}{cn + bm} \tan \frac{1}{2} A;$$

$$(ii) (m + n) bc \cdot AD = (b^2 m + c^2 n) AD'.$$

9. Find the sum of n terms of the series

$$\cot \theta \cot 2\theta + \cot 2\theta \cot 3\theta + \cot 3\theta \cot 4\theta + \dots$$

Prove that the value of the fraction

$$\frac{\sin n\theta - \sin(n-1)\theta + \sin(n-2)\theta - \dots + (-1)^{n-1} \sin \theta}{\cos n\theta - \cos(n-1)\theta + \cos(n-2)\theta - \dots + (-1)^{n-1} \cos \theta}$$

is $\tan \frac{1}{2}(n+1)\theta$ or $-\cot \frac{1}{2}(n+1)\theta$ according as n is odd or even.

10. Express $\cos n\theta \sec^n \theta$ in ascending powers of $\tan \theta$, when n is a positive integer.

Prove that $\cos(n + \frac{1}{2})\theta \sec \frac{1}{2}\theta$ is the coefficient of x^n in the expansion in ascending powers of x of

$$\frac{1-x}{1-2x \cos \theta + x^2}$$

II.

1. Find the greatest value of $7x^2 - 12xy - y^2$, where x and y are positive integers satisfying the equation

$$3x - 5y = 7.$$

2. Prove that

$$1 + \frac{a^2}{(b+c)^2 - a^2} + \frac{b^2}{(c+a)^2 - b^2} + \frac{c^2}{(a+b)^2 - c^2} = \frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$

3. If

$$a \cos \beta \cos \gamma + b \sin \beta \sin \gamma + c = 0$$

and

$$a \cos \gamma \cos \alpha + b \sin \gamma \sin \alpha + c = 0$$

and $\alpha - \beta$ is not zero or a multiple of 2π , show that

$$(a^2 \cos^2 \gamma + b^2 \sin^2 \gamma) (a \cos \alpha \cos \beta + b \sin \alpha \sin \beta + c) \\ = (bc + ca - ab) (a \cos^2 \gamma + b \sin^2 \gamma + c).$$

4. O is a point outside a circle, OAB is the diameter through O , OLM is any other chord, and C is the centre of the circle; prove that

$$\tan \frac{1}{2} LCA \cdot \tan \frac{1}{2} MCA = OA / OB.$$

5. Prove that the radical axis of the inscribed circle and one of the escribed circles of a triangle and the radical axis of the two other escribed circles are perpendicular.

6. P is any point on a conic whose foci are S, H . PS and PH cut the conic again at Q and R . Show that the tangents at Q and R intersect on the normal at P .

7. A variable circle is drawn to touch the tangent at the vertex of the parabola $y^2 = 4ax$ and to cut the parabola in four points. Show that the difference of the abscissæ of the centre of the circle and of the centroid of the points is always the same.

8. A variable circle passes through the centre of a fixed rectangular hyperbola and meets it at the four points P, Q, R, S . Show that, if the tangents to the hyperbola at R and S intersect on a fixed straight line, those at P and Q intersect on a fixed circle.

9. If $t > a$, show that

$$te^{-\frac{t}{a}} < ae^{-1}.$$

10. P is the point whose rectangular coordinates are (x, y) and P' is the point whose coordinates are $(a^2/x, b^2/y)$, where a and b are constants. P describes a curve and the tangent to this curve at P meets the axis of x at the point T , and M is the foot of the perpendicular drawn from P to the axis of x . T' is the point where the tangent at P' to the curve described by P' meets the axis of x and M' is the foot of the perpendicular drawn from P' to the axis of x . If O is the origin, show that $MT : MO :: M'T' : M'O$.

11. Three uniform heavy rods AB, BC, CD are smoothly jointed at B and C , the ends A, D moving smoothly on a horizontal rod. If BC is not vertical, show that its inclination to the horizon is

$$\sin^{-1} \{ (CD - AB) / BC \}.$$

12. An elastic ball is projected horizontally from a point on an inclined plane and in a direction perpendicular to the intersection of the inclined plane and the horizon. The latera recta of the paths of the ball after and before it strikes the plane again are in the ratio $(3 + e)^2 : 4$, where e is the coefficient of restitution; show that the inclination of the plane to the horizon is 45° .

III.

1. If a straight line meets the sides BC, CA, AB of a triangle ABC in the points D, E, F respectively, show that $BD \cdot CE \cdot AF = CD \cdot AE \cdot BF$; signs as well as magnitude being taken into consideration.

A second straight line meets BC, CA, AB at D', E', F' respectively; EF' meets BC at D'' , FD' meets CA at E'' , and DE' meets AB at F'' . Show that $D'' E'' F''$ lie on a straight line.

2. Show that the locus of a point in the plane of two coplanar circles at which they subtend the same angle is their circle of similitude.

Find the locus of a point such that the tangent cones drawn from it to three given spheres have the same vertical angle.

3. Show that the inverse of a circle with respect to a point, whether lying in the plane of the circle or not, is a circle.

Show that the inverse of the circle escribed to the side BC of a triangle ABC with respect to a coplanar circle with centre A and radius $\sqrt{AB \cdot AC}$ is the circle which touches AB and AC and the circumscribing circle of the triangle ABC internally; and hence, or otherwise, show that the radius of this circle is $r \sec^2 \frac{1}{2} A$.

4. The circle drawn through the real foci of a central conic and any point P on the conic meets at the points U and V the minor (or conjugate) axis. Show that PU and PV are the tangent and normal to the conic at P .

D is the middle point of the side BC of a triangle ABC ; the internal and external bisectors of the angle BAC meet at E and F respectively the perpendicular to BC drawn through D ; AL is the perpendicular drawn from A to BC . Show that

$$(1) (AB + AC)^2 - BC^2 = 4FD \cdot AL;$$

$$(2) BC^2 - (AB - AC)^2 = 4DE \cdot AL.$$

5. Give any method by which, if we are given a tracing of an arc of a conic, we can determine whether it is an ellipse, parabola, or hyperbola.

If the conic is a parabola, how can its axis be constructed?

6. Show that the equation to the straight line joining the feet of the perpendiculars drawn to the straight lines $ax^2 + 2hxy + by^2 = 0$ from the point (p, q) is

$$\{(b - a)p - 2hq\}x + \{(a - b)q - 2hp\}y - aq^2 + 2hpq - bp^2 = 0.$$

7. Find the coordinates of the points of intersection of the parabola $y^2 = 4ax$ with the straight line

$$2x - y(m_1 + m_2) + 2am_1m_2 = 0.$$

A triangle is inscribed in a parabola so that two of its sides pass, one through each of two fixed points on its axis. Show that the third side touches a parabola having the same axis and vertex as the first parabola.

8. Show that the equation to any tangent to the ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$ may be written in the form

$$x \cos \theta + y \sin \theta = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta};$$

and find the equation to the corresponding normal.

If the rectangle contained by the perpendiculars drawn from the centre to two tangents is equal to ab , show that the perpendiculars drawn from the centre to the corresponding normals are equal.

9. Show that the locus of the middle points of chords of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ which are parallel to $x/l = y/m$ is

$$lX + mY = 0,$$

where $X \equiv ax + hy + g$ and $Y \equiv hx + by + f$.

Defining an axis of a conic as a straight line about which it is symmetrical, show that the above conic has two axes, the equations to which are

$$h(X^2 - Y^2) - (a - b)XY = 0.$$

(The axes of coordinates are supposed to be at right angles.)

10. Find the equation of the tangent at the point $\theta = a$ to the conic $l/r = 1 + e \cos \theta$.

S is a focus of an ellipse, P and Q are any two points on the ellipse, and T is the intersection of the tangents at P and Q . Show that ST^2 is greater than $PS \cdot SQ$.

11. Show that the equation

$$\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$$

is the equation to a conic inscribed in the triangle of reference; and that, if $lx + my + nz = 0$ is any tangent to this conic, $fmn + gnl + hlm = 0$.

Show also that the conic is an ellipse, parabola, or hyperbola according as fgh ($f + g + h$) is positive, zero, or negative.

(The coordinates are areal, that is, $x + y + z = 0$ is the equation to the straight line at infinity.)

IV.

1. If p is a prime number such that $p^2 > r$, show that

$$\frac{p^2 (p^2 - 1) (p^2 - 2) \dots (p^2 - r + 1)}{r}$$

is a multiple of p .

2 Prove that if x, y, z , are any real quantities

$$\frac{x^2 + y^2 + z^2}{y^2 + zx + xy}$$

cannot lie between 1 and -2 .

3. Show that

$$(\sqrt{x+1} + 1)^{2n} + (\sqrt{x+1} - 1)^{2n} = 2\pi \left(x + \operatorname{cosec}^2 \frac{r\pi}{4n} \right);$$

where r has the values 1, 3, 5 ... $2n - 1$.

4. The tangents to a circle at the points P and Q meet at the point T . Show that, if a point lies in the same angle between TP and TQ as the circle, then the square on the perpendicular drawn from the point to PQ is greater than, equal to, or less than the rectangle contained by the perpendiculars drawn from the point to TP and TQ according as the point lies outside, upon, or inside the circle.

5. If a tetrahedron touches a sphere at the centroids of its faces, show that it must be equilateral.

6. PSP', QSQ' are chords of a fixed circle which pass through a fixed point S and include a constant angle; Y, Y' are the feet of the perpendiculars from S on $PQ, P'Q'$. Prove that YY' envelops a fixed circle whose centre is S .

7. A chord of a conic is drawn in a given direction. Show that the locus of the orthocentre of the triangle formed by the chord and the normals at its extremities is a straight line.

8. Show that, by suitably choosing k , the equation

$$k(x^2 + y^2 - 2by) = (by - ax - ab)^2$$

can be made to represent a pair of common tangents of

$$x^2 + y^2 = 2ax \text{ and } x^2 + y^2 = 2by.$$

9. If two conics have double contact, show that the distances of their centres from the chord of contact are proportional to the rectangles under the diameters parallel to the common tangents.

10. If n is an integer, prove that generally

$$\frac{\log(n+1)}{n+1} < \frac{\log n}{n}$$

What are the exceptions?

11. Two rods AB , CD are uniform and in all respects equal. The points A , C are fixed in the same horizontal line, and B , D are connected by a string of length equal to AC . Show that, if the length of each rod be less than AC , there are two positions of equilibrium in which the rods are at right angles.

12. A series of equal particles are attached to a smooth string at equal intervals, each equal to l , and are placed, coiled up in a heap, at the edge of a smooth table. A smooth semicircular tube, open at one end, is fixed with its plane vertical and its bounding diameter horizontal and with its open end close to the heap of particles. If the last of the particles is dropped gently into the open end of the tube, and descends the tube dragging the others after it, show that, at any time before this particle reaches the lowest point of the tube, V_{n+1} , the velocity of the particles in the tube immediately after the $(n+1)^{\text{th}}$ particle has entered the tube, is given by the equation

$$(n+1)^2 V_{n+1}^2 - n^2 V_n^2 = 2agn \sin(nl/a).$$

V.

1. (1) Given that

$$\frac{d}{dx} (\sin x) = \cos x,$$

deduce the value of

$$\frac{d}{dx} (\cos x).$$

(2) If s is a value of y which satisfies the equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - m^2) y = 0,$$

when $m = 0$, prove that $\frac{dz}{dx}$ satisfies the equation when $m = 1$.

2. (1) Differentiate $\log(1-x)$, n times.

(2) If $x = \sin y$, show that when $z = 0$

$$\left(\frac{d}{dx}\right)^{r+1} (y^r) = (r-1)^2 \left(\frac{d}{dx}\right)^{r-1} (y^r), \text{ if } r > 1.$$

3. (1) If $(x+a)^y = (y+b)^x$, express $\frac{dy}{dx}$ in terms of x , y , and a .

(2) If u is a function of the product XYZ , where

$$\frac{X}{z \sin x} = \frac{Y}{y \sin y} = \frac{Z}{s \sin s} = \frac{1}{xyz}$$

show that

$$\frac{x \frac{\partial u}{\partial x}}{x \cot x - 2} = \frac{y \frac{\partial u}{\partial y}}{y \cot y - 2} = \frac{s \frac{\partial u}{\partial z}}{s \cot s - 2}$$

4. (1) If the equation of a curve is $y=f(x)$ and also $u=F(\theta)$, show that

$$\frac{xf'(x) - f(x)}{x + f(x)f'(x)} + \frac{F'(\theta)}{F''(\theta)} = 0.$$

(2) OY is the perpendicular from the origin O on the tangent to a given curve at the point P . Show that the normal to the locus of Y bisects OP .

5. The coordinates of any point on a curve are given by the equations

$$x = f(t), y = F(t);$$

express ρ in terms of t .

In the curve $x^2 + y^2 = a^2$, show that $\rho \propto p$, p being the perpendicular from the origin on the tangent at (x, y) .

6. (1) In the case of a system of forces in one plane, if the sum of the resolved parts of the forces along a given line is zero and the sum of the moments of the forces about a given point is also zero, show that the system of forces, if not in equilibrium, is equivalent to a single force; and find the exact position of the force.

(2) Two forces are represented by CA and CP , C being the centre, A a vertex, and P any point on a given ellipse. Find the positions of P when the resultant is (i) greatest, (ii) least.

7. (1) Find the centre of gravity of the curved surface of a hemisphere.

(2) A bent rod ABC (in which ABC is a right angle) is suspended from A . Given the inclination of AB to the vertical, find the ratio $AB : BC$.

8. What is meant by the "Cone of Friction" ?

A lamina, the curved boundary of which is a parabola with its focus at the centre of gravity of the lamina, rests with its curved boundary in contact with a rough vertical wall and a rough horizontal plane. If the friction is limiting at both points of contact and if the coefficients of limiting friction between the lamina and the wall and the lamina and the ground are each equal to μ , show that $\mu^2 \sin^2 \alpha + \mu \tan \alpha = \cos^2 \alpha$; where α is the inclination of the axis of the parabola to the horizontal.

9. (1) Find the acceleration of the centre of gravity of the moving masses in Atwood's machine.

(2) A particle of mass 1 lb. is projected vertically up with velocity g ($= 32$) ft. per sec. After $1\frac{1}{2}$ seconds another particle of mass 2 lb. is projected upwards from the same point with velocity 64 ft. per sec. If after meeting the particles coalesce, after what time will they return to the starting point ?

10. (1) Show that "Newton's Experimental Law" concerning the relative velocities of two spheres after direct impact may be obtained from the hypothesis that the ratio of the impulses of the forces of restitution and of compression is equal to e , the coefficient of restitution.

(2) Two spheres (in which $e = \frac{1}{2}$) have their centres in the same horizontal at a distance $2a$ apart. They are projected directly towards one another with equal velocities and move under gravity. After collision each describes a parabola of latus-rectum $\frac{3}{4}a$. Find the velocity of projection.

(2) MERTON, EXETER, NEW AND HERTFORD COLLEGES
SCHOLARSHIPS, UNIVERSITY COLLEGE EXHIBITION.

December 1910.

MATHEMATICS I.

1. A variable straight line through a fixed point O meets two fixed straight lines in P, Q ; and R is the middle point of PQ . Prove that the locus of R is a conic passing through O and having its asymptotes parallel to the fixed lines.

2. If s_r denote the sum of the r th powers of the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0,$$

show that, (where $r > n$),

$$s_r + p_1 s_{r-1} + p_2 s_{r-2} + \dots + r p_r = 0.$$

If α, β are roots of this equation find $\sum \alpha^{-2} \beta^{-1}$, the summation being for all pairs of roots.

3. ABC is a fixed triangle; from P , a variable point on BC , a line PL is drawn parallel to AC meeting AB in L , and PM is drawn parallel to AB meeting AC in M . If lines are drawn through M and L in fixed directions, prove that they intersect on a fixed straight line.

4. A point moves in a straight line from rest so that its velocity after a time t is $at - bt^2$ (where a and b are positive). What is the acceleration of the point at time t , and what is its greatest positive distance from its starting point?

5. A uniform cylinder, length one yard, diameter one foot, mass one ton, the planes of whose ends are perpendicular to its axis, rests in limiting equilibrium with a point of its upper end against a smooth vertical wall, a point of its lower end against a rough horizontal plane, and its axis inclined at 45° to the vertical. Find the coefficient of friction.

A gradually increasing horizontal force, acting towards the wall, acts on the cylinder at the point where it touches the plane. What is the magnitude of this force at the instant when the cylinder begins to slip?

6. P and Q are any two points on the curve

$$y = \frac{c}{2} \left(e^{\frac{x}{c}} + e^{-\frac{x}{c}} \right).$$

Show that the area included by the ordinates of P and Q , the axis of x , and the arc PQ is proportional to the length of the arc PQ .

7. Find the equation of the chord joining the points α, β on the conic $l/r = 1 + e \cos \theta$.

Two points P, Q on a conic subtend a constant angle at a focus S . Prove that the locus of the point in which PQ meets the internal bisector of the angle PSQ is a conic with the same focus and directrix.

8. If the centre of the nine point circle of a triangle lies on an external bisector of one of the angles of the triangle, prove that the angle must be 120° .

9. From a point on a horizontal plane a heavy article is projected in a direction inclined at an angle α to the plane. t seconds later a second heavy particle is projected from the point in a direction inclined at an angle β to the plane, so as to strike the first particle at the instant when the latter reaches the plane again. Show that, if the range of either particle on the plane is equal to the distance through which a heavy particle would fall freely in time t , then

$$\sqrt{\tan \alpha} - \sqrt{\tan \beta} = 1.$$

10. Find the sum to infinity of the series whose n th terms are

$$(i) \{n(n+2)\}^{-1};$$

$$(ii) \{2^n n(n+2)\}^{-1}.$$

11. Find the equation of the normal at the point $(am^2, -2am)$ of the parabola $y^2 = 4ax$.

Prove that two normals can be drawn from any point on a parabola which are distinct from the normal at the point, and that they cannot be real unless the abscissa of the point exceeds twice the latus rectum.

12. State the rule for the multiplication of two determinants of the same order.

Find the value of

$$\begin{vmatrix} (a_1 - b_1)^2, & (a_1 - b_2)^2, & (a_1 - b_3)^2 \\ (a_2 - b_1)^2, & (a_2 - b_2)^2, & (a_2 - b_3)^2 \\ (a_3 - b_1)^2, & (a_3 - b_2)^2, & (a_3 - b_3)^2 \end{vmatrix}.$$

MATHEMATICS II.

1. Find the factors of

$$(b + c - 2a)^5 + (c + a - 2b)^5 + (a + b - 2c)^5.$$

2. If α, β, γ are positive integers which may have any value from 0 to n , show that the number of solutions of $\alpha + \beta + \gamma = 2n$ is the same as the number of solutions of $\alpha + \beta + \gamma = n$.

3. A quadrilateral $ABCD$, which varies in magnitude, but is always similar to a given quadrilateral $AB'C'D'$, moves with three of its angles A, B, C on three given lines: find the locus of the fourth vertex.

4. When a motor-car is travelling at v miles per hour the consumption of petrol is $a + bv + cv^2$ gallons per hour, where a, b and c are positive. Show that the least number of gallons of petrol which can be used in covering a distance of s miles is

$$(3 \cdot 2^{-\frac{1}{2}} c^{\frac{1}{2}} a^{\frac{3}{2}} + b)s.$$

5. In calculating the radius of the circumscribed circle of a triangle by measuring the sides of the triangle small errors da, db, dc are made. Prove that the corresponding error in the radius is

$$\frac{1}{2} \cot A \cot B \cot C (\sec A da + \sec B db + \sec C dc).$$

6. Find the asymptotes of the curve $x^2(y - a) = y^2(x - 2a)$, and trace it.

7. A variable circle passes through a fixed point and cuts a fixed circle at a constant angle. Prove that the locus of its centre is a conic.

8. State Pascal's theorem concerning a hexagon inscribed in a conic.

From any point on a diagonal of a parallelogram parallels are drawn to the sides. Prove that a conic can be described to pass through the extremities of the other diagonal and the four points where these parallels intersect the sides.

9. Find the equation of the two straight lines joining the origin to the points of intersection of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

with the line $lx + my = 1$.

C is the centre of an ellipse and CP, CQ are radii vectores making equal angles with a fixed straight line. Prove that PQ touches a fixed rectangular hyperbola.

10. If

$$\phi(x) = \frac{d}{dx} \psi(x) \text{ and } \phi(x), \psi(x)$$

are finite and continuous between the values a and b of x , show that the limiting value of

$$(x_1 - a)\phi(a) + (x_2 - x_1)\phi(x_1) + \dots + (x_{r+1} - x_r)\phi(x_r) + \dots + (b - x_n)\phi(x_n),$$

when each of the quantities $x_1 - a, \dots, x_{r+1} - x_r, \dots, b - x_n$, approaches the limit zero, is $\psi(b) - \psi(a)$.

11. Find the condition that the line $lx + my = 1$ should touch the curve $x^3 = ay^2$.

A variable tangent to the curve $x^3 = ay^2$ meets the axes in L and M , and perpendiculars to the axes at L and M meet in P . Prove that the locus of P is a curve similar to the original curve and $\frac{4}{27}$ the linear dimensions.

12. If

$$P + Qx^{\frac{1}{2}} + Rx^{\frac{3}{2}}$$

(where P, Q, R are rational and integral in a, b, c, x) is a factor which rationalises $a + bx^{\frac{1}{2}} + cx^{\frac{3}{2}}$, prove that the product is divisible by

$$a^2 + (b^2 - 3abc)x + c^2x^2.$$

One root of the equation

$$x^6 - 9x^4 + 18x^4 + 9x^2 + 27x^2 - 54x - 36 = 0$$

is $3 - 3^{\frac{1}{2}} - 3^{\frac{3}{2}}$. Find all its roots.

MATHEMATICS III.

1. Find the number of different values which

$$(a + i\beta - \gamma - i\delta)^4$$

can take when the unequal quantities a, β, γ, δ are permuted in all possible ways. [$i^2 = -1$].

2. PQ is a focal chord of a parabola. The normal at P meets the curve again at R , and the line through Q parallel to the axis meets this normal at S . Prove that, if PQR is a right angle, the triangle PQS is equilateral.

3. Find the equation of the line joining an angular point of the triangle of reference to a point whose trilinear coordinates are given.

P_1, P_2 are two given points [$(x_1, y_1, z_1), (x_2, y_2, z_2)$ in trilinear coordinates]. AP_1, BP_1, CP_1 meet the sides BC, CA, AB in X_1, Y_1, Z_1 respectively, and AP_2, BP_2, CP_2 meet them in X_2, Y_2, Z_2 . Prove that the six points $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ lie on a conic, and find the equation of this conic.

4. If x, y, z are three functions of t connected by the relations

$$xy = x^2; \quad \frac{dx}{dt} \cdot \frac{dy}{dt} = \left(\frac{dz}{dt}\right)^2;$$

show that the ratios $x : y : z$ are constant.

5. At a place where the height of the water-barometer is H a cylindrical diving-bell is fixed under water with its axis vertical, so that the length of the bell not occupied by water is a , and the level of the surface of the water inside the bell is h below the surface of the water in which the bell is immersed. The upper part of the bell contains a body denser than water, whose volume is $\frac{1}{n}$ that of the part of the bell not occupied by water. This body becomes detached and sinks. Find an equation for the distance through which the surface of the water inside the bell will rise, and show that, if n is large, it is

$$\frac{a(H + h)}{n(H + h + a)}, \text{ nearly.}$$

6. On the normal at any point P of a fixed hyperbola a point Q is taken whose distance from P is equal to the length of the semidiameter of the conjugate hyperbola which is parallel to the tangent at P . Show that Q lies on one or other of two fixed hyperbolas, whose eccentricities are connected by the relation

$$e_1^{-2} + e_2^{-2} = 1.$$

7. A train running at 60 miles per hour commences to take up water at the rate of 6 gallons per second, without change of speed. What is the least increase of horse-power necessary to maintain the speed constant?

(Take the mass of a gallon of water as 10 lbs. and one horse-power as 550 foot-pounds per second.)

8. The sides of the triangles ABC and $A'B'C'$ are parallel: if x is the perpendicular distance between the sides BC and $B'C'$, and y and z have similar meanings for the other sides, prove that the difference between the radii of the circumscribing circles is

$$\frac{x \sin A + y \sin B + z \sin C}{2 \sin A \sin B \sin C}$$

9. If x and y are integers which have no common factor, show that

(1) x and $x + y$ have no common factor.

(2) If p is an odd prime number, $x + y$ and $\frac{x^p + y^p}{x + y}$ have no common factor, with the possible exception of p .

10. ABC is a triangle and D, E, F are points on BC, CA and AB respectively such that AD, BE and CF subtend right angles at a point P . Prove, by reciprocation or otherwise, that D, E, F are collinear.

Prove also that, if the line DEF touches the circle inscribed in the triangle ABC , the point P lies on a concentric circle, whose radius is $\sqrt{2}$ times that of the inscribed circle.

11. P is the point of intersection of the curves $y = \sin x$, $y = k \tan x$ ($k < 1$), whose abscissa lies between 0 and $\pi/2$. The tangents to the two curves at P meet the axis of x at Q and R respectively. Show that the circumcircle of the triangle PQR touches the ordinate of P at P .

Show also that, if α and β are the acute angles between the tangents to the two curves at P and at the origin respectively, then $\alpha = 2\beta$.

MATHEMATICS IV.

1. A parabola, whose vertex is A and focus S , has double contact with a circle whose centre is C . P is the point of contact of a tangent from A to the circle. Show that

$$2CS = CA + AP.$$

2. Prove that the coefficient of x^m in the expansion of

$$(1+x)^m(1-x)^{-m-1}$$

in ascending powers of x is the same as the coefficient of x^m in the expansion of

$$(1+x)^m(1-x)^{-m-1}.$$

3. Prove that curves intersect at the same angle as their inverse curves.

From one of the points of intersection of two fixed circles which are touched by a variable circle tangents are drawn to the variable circle: prove that the locus of the middle point of the chord of contact is two fixed circles.

4. Given the length of the minor axis and the trilinear coordinates of one focus of a conic inscribed in the triangle of reference, find the coordinates of the other focus.

If a focus of a conic inscribed in a triangle lies on the circumcircle, prove that the conic is a parabola.

5. One side of a parallelogram is of length a and subtends angles α and β at the two extremities of the opposite side of the parallelogram: prove that the perpendicular distance between these sides is

$$\frac{2a(\cot \alpha + \cot \beta)}{(\cot \alpha - \cot \beta)^2 + 4}$$

6. Enunciate the principle of virtual work.

A weight W is placed upon a smooth horizontal plane and has attached to it two light strings, which pass over two smooth pegs equidistant from the plane and support equal weights (too small to lift W off the plane) hanging freely. Examine whether the equilibrium position of W is stable or unstable.

7. O is any point in the plane of a triangle ABC , and X, Y, Z are the middle points of OA, OB, OC respectively. Prove that the lines drawn through X, Y, Z perpendicular to the sides BC, CA, AB respectively meet in a point P such that the line joining P to the circumcentre of the triangle is bisected by the centroid of the four points A, B, C, O .

8. The first term of a series is a , the second term is b , and every other term is the geometric mean between the two preceding terms. Find the n th term of the series.

9. If

$$y = x \sin^{-1} x \text{ and } p = \frac{dy}{dx} - \frac{y}{x}$$

show that

$$(1) x \frac{dp}{dx} = p + p^2;$$

$$(2) x^2 \frac{d^2y}{dx^2} = 4p^2 + 3p^4.$$

10. Find any value which

$$a \tan^{-1} a + \beta \tan^{-1} \beta + \gamma \tan^{-1} \gamma$$

can have, when a, β and γ are the three cube roots of unity.

11. A, B, C, D are four collinear points. One of the points of intersection of circles through AB and CD is on a fixed straight line. Prove that the locus of the other is a circle.

12. A thin uniform rod has small smooth rings attached to its ends. One of the rings slides on the upper half of a thin wire in the form of a circle in a vertical plane, and the other on a thin wire which coincides with the vertical diameter of the circle. Show that, if l is the length of the rod and a the radius of the circle, the inclination of the rod to the vertical in a position of equilibrium is

$$\sin^{-1} \sqrt{\frac{4l^2 - a^2}{3l^2}}, \text{ provided that } a > l > \frac{a}{2}.$$

(c) UNIVERSITY OF LONDON.

(1) EXHIBITIONS EXAMINATION: 1910.

[*For Internal and External Students.*]

APPLIED MATHEMATICS.

Examiner—J. B. DALE, Esq., M.A.

Wednesday, June 8. Morning, 10 to 1.

[*Candidates may not attempt more than SIX questions in all, of which not more than TWO must be in the same section.*]

SECTION A.

1. Show that three forces represented in all respects by the lines joining a point P to three points A, B, C , have a resultant represented by three times PG , where G is the mean centre of A, B, C .

If the three points lie on a horizontal plane, and P be a rough particle resting on the plane acted on by the above forces, show that it will be in equilibrium anywhere within a certain circle.

2. A framework consists of five bars AB, BC, CD, DA, BD , freely jointed at their extremities. $ABCD$ forms a trapezium, of which the upper side BC and the lower side AD are horizontal. Unequal weights are suspended from the joints B, C , and the framework is supported at A and D . Construct a force polygon showing the stresses in the bars.

3. What is meant by the Dimensions of a Physical Quantity?

By consideration of the dimensions involved, show that if a particle moves towards a fixed point in a straight line with an acceleration μ/x^3 , where x is the distance of the particle from the point, the time occupied by the particle in moving to the point from rest at a distance a varies as a^2 .

4. Describe the Common Hydrometer.

When immersed in a liquid of density 1.02, the hydrometer marks a graduation 36, and when immersed in one of density 1.08 it marks 82. Find the density corresponding to a graduation 53.

SECTION B.

5. Two equal uniform rods, rigidly jointed at an angle β , are suspended from one extremity. Show that the action at the joint consists of a force and a couple whose magnitude is

$$\frac{3Wa \sin \beta}{\sqrt{10 - 6 \cos \beta}}$$

where W is the weight and $2a$ the length of one rod.

6. A triangle is immersed in water with the vertex in the surface and the base horizontal. Show that if the triangle be divided into two parts by a horizontal line drawn at a distance $\frac{1}{2}a$ from the vertex, the moments of the pressures on the two parts of one face of the triangle about the line are equal; and the pressures are equal if the distance is $a/\sqrt{2}$, a being the height of the triangle.

7. Prove that the central acceleration of a particle describing a circle of radius a with velocity v is v^2/a .

A particle is suspended from a fixed point by a string of length a , and is projected horizontally with velocity $\sqrt{7ga/2}$. Show that it will describe an arc of 120° in a vertical circle, and will then leave the circle, falling in a parabolic path which passes through the point of projection.

8. State the laws of statical friction.

A cylinder of radius a is placed with its curved surface on a rough horizontal plane. To the middle point of the uppermost generating line is attached a string whose other extremity is fastened to a point connected with the plane at a height $2a$ above it, the string being at right angles to the axis of the cylinder. The plane being tilted about a line parallel to the axis, show that the cylinder will slip when the inclination of the plane is $\tan^{-1}(2\mu)$, μ being the coefficient of friction.

SECTION C.

9. Explain the Principle of Archimedes.

A cylinder of radius a , length h , and density ρ , floats with axis vertical in a liquid of density σ contained in a deep cylinder of radius b . Find the work done in lifting the cylinder just out of the liquid.

10. A particle of mass m is placed upon a smooth wedge of mass M resting upon a smooth table. If h be the height of the wedge, and a the angle at the base, show that the particle slides from the top to the bottom in the time

$$\sqrt{\frac{2(M + m \sin^2 a)h}{(m + M)g \sin^2 a}}$$

and that the wedge then moves with constant velocity

$$m \cos a \sqrt{\frac{2gh}{(m + M)(M + m \sin^2 a)}}$$

11. Define the instantaneous centre of rotation of a body.

Describe and prove the theory of any linkage by means of which the circular motion of one point may be connected with the rectilinear motion of a second point.

12. A ball of mass m moving on a small table with velocity u , impinges directly upon a vertical plane of mass M held in position by a massless horizontal spring at right angles to the plane of unstretched length a , and such that its length is doubled by a weight equal to that of the plane. The coefficient of resilience being e , show that the spring will be compressed by the amount

$$\frac{mu(1 + e)}{M + m} \sqrt{\frac{a}{g}}$$

PURE MATHEMATICS.

Examiner—J. B. DALE, Esq., M.A.

Wednesday, June 8. Afternoon, 2 to 5.

(Tables of logarithms allowed.)

[Candidates may not attempt more than SIX questions in all, of which not more than TWO must be in the same section.]

SECTION A.

1. Draw the graphs of

$$y = \frac{2(x^2 - 6x)}{x^2 + x - 6}$$

and

$$y^2 = \frac{2(x^2 - 6x)}{x^2 + x - 6}$$

2. Show that any integral algebraic function of x of degree n can be expressed in the form

$$a_0 + a_1x + a_2x(x-1) + \dots + a_{n-1}x(x-1)(x-2)\dots(x-n+1).$$

Express thus $x^2 - 3x^2 + 1$.

3. Given the area S of a triangle, a side a , and the opposite angle A , prove that

$$\frac{\tan \frac{B-C}{2}}{\tan \frac{B+C}{2}} = \pm \frac{\sqrt{a^2 - 4S \tan \frac{A}{2}}}{\sqrt{a^2 + 4S \cot \frac{A}{2}}}$$

and hence show how to solve the triangle completely.

4. By means of a geometrical construction cut off from a given triangle, a triangle whose area is $\frac{1}{n}$ th part of the area of the original triangle, by means of a line drawn parallel to a given line which is not parallel to any of the sides of the triangle.

SECTION B.

5. Solve the equations

$$ay - bx - ay = bc - ca - b^2.$$

$$yz - cy - bz = ca - ab - c^2.$$

$$zx - ax - cz = ab - bc - a^2.$$

6. Find the least value of $x^2 + y^2$ subject to the condition that

$$ax + by = c,$$

where a, b, c are constant.

Hence show that if D is a point in the base of a triangle ABC , such that the sum of the squares of the perpendiculars DE, DF to the sides AB, AC is a minimum, then

$$\frac{DE}{AB} = \frac{DF}{AC} = \frac{2S}{AB^2 + AC^2}$$

when S is the area of the triangle.

7. If n be an integer, positive or negative, prove that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

In a triangle in which $b > c$ show that

$$\frac{\sin B \sin C}{\sin A} = \left(\frac{c}{b}\right) \sin A + \left(\frac{c}{b}\right)^2 \sin 2A + \left(\frac{c}{b}\right)^3 \sin 3A + \dots \text{ad inf.}$$

8. Find the equations of the bisectors of the angles between the lines

$$ax^2 + 2bxy + by^2 = 0.$$

Show that if one of the three lines whose equation is

$$ax^2 + bx^2y + cxy^2 + dy^3 = 0$$

bisects the angle between the other two, the equation of the bisecting line is

$$3(ax + dy) + (by + cx) = 0.$$

SECTION C.

9. A cypher code is to be formed using m letters and n figures, and each word is to contain p letters and q figures, $p+q$ being a given number. Show that under these conditions the code containing the greatest number of words will be formed by taking p to be the greatest integer in

$$\frac{(r+1)(m+1)}{m+n+2}$$

where $r = p + q$.

10. A series of points is taken upon the axis of x so that x_n the abscissa of the n^{th} point is related to the abscissa of the two preceding points x_{n-1} x_{n-2} by the equation

$$x_n = \frac{1}{2} (2x_{n-1} + x_{n-2}).$$

Show that the differences between the abscissæ of pairs of successive points are in geometrical progression, and that as n is indefinitely increased x_n approaches the limiting value

$$\frac{1}{2} (x_1 + 3x_2),$$

when x_1 and x_2 are the abscissæ of the two points first taken.

11. Find the ratios of the volume and area of surface of a regular tetrahedron to those of the inscribed sphere.

12. Prove that the directions of the tangents drawn from the point P whose coordinates are (x', y') to the parabola $y^2 = 4ax$, are determined by the equation

$$x' \tan^2 \theta - y' \tan \theta + a = 0;$$

and the rectangle contained by the tangents is equal to

$$(y'^2 - 4ax') SP/a$$

S being the focus.

(2) SCHOLARSHIPS EXAMINATION: 1910.

[For Internal and External Students.]

APPLIED MATHEMATICS.

Examiners { L. N. G. FILON, Esq., D.Sc., M.A., F.R.S.
H. W. RICHMOND, Esq., M.A.

Tuesday, July 19. Morning, 10 to 1.

1. A flat board of any shape lies on a smooth table, its edge being pressed against two pegs P and Q fixed in the table, and is in equilibrium under the action of a force F along the table. Prove that, if the pegs are smooth, the line of action of F must pass from a certain fixed point to some point of PQ ; while, if the pegs are rough and the coefficients of friction are not too great, the line of action of F must pass from some point of a certain finite straight line to some point of PQ .

2. Two smooth spheres impinge obliquely; lines OP, Op represent their velocities before impact, and lines OQ, Oq represent their velocities after impact. Prove that PQ and pq are parallel to the line of centres of the spheres at the moment of impact, and express the consequences of the laws of impact by geometrical relations among the points P, p, Q, q .

Consider also the case when the velocities are not all coplanar.

3. Prove that in uniplanar motion of a lamina the acceleration at a given instant of any point Q relative to any second point P is proportional to PQ , and inclined at a constant angle to PQ .

4. Six light rods form the edges of a tetrahedron $ABCD$, and two points, P on AC and Q on BD , are connected by a string of tension T . Find the thrusts X_1, X_2, X_3, X_4 produced in the rods AB, BC, CD, DA , and prove that

$$\frac{X_1}{AB} + \frac{X_2}{BC} + \frac{X_3}{CD} + \frac{X_4}{DA} = \frac{T}{PQ}$$

5. A heavy uniform chain of length l has one end fixed at a height h above the ground (which is level), and is partly stretched in a straight line along the ground and partly hanging in a catenary. Prove that, if μ is the coefficient of friction between the chain and the ground the length of the straight portion of the chain cannot exceed

$$l + \mu h - (h^2 + 2\mu hl + \mu^2 h^2)^{\frac{1}{2}}$$

6. A light string ABC , which has two equal masses attached to its end C and its middle point B , is rotating on a smooth horizontal plane about the end A , which is fixed, with AB, BC in a straight line. Suddenly a point D of AB is seized and held fixed. Prove that in the subsequent motion the greatest angle between the directions of the strings DB and BC is

$$2 \sin^{-1} \frac{AD}{2\sqrt{5DB \cdot BC}}$$

7. Matter is removed from a homogeneous attracting sphere till a spherical cavity (not concentric with the sphere) is formed. Find the positions of equilibrium of a particle (i) within the cavity, (ii) outside the sphere; find also the pressure between the particle and the surface of the attracting matter in each case.

8. A homogeneous cube can turn about one edge which is fixed in the surface of water. Find an equation giving the inclination to the horizon of one of the sloping faces of the cube in the position of equilibrium (i) when its specific gravity is less than $\frac{1}{2}$, (ii) when it is greater than $\frac{1}{2}$ but less than one.

9. A lamina lying at rest is set in motion by jerking a string tied to a given point of the lamina. Show that whatever be the direction of the jerk the instantaneous centre lies on a certain fixed line in the lamina. Determine this line when the lamina is a rectangle and the string is tied to a corner.

PURE MATHEMATICS.

Examiners { L. N. G. FILON, Esq., D.Sc., M.A., F.R.S.
H. W. RICHMOND, Esq., M.A.

Tuesday, July 19.—Afternoon, 2 to 5.

1. Prove that if Σu_n be a convergent series of positive terms, and if, after a certain value of n

$$(v_{n+1} / v_n) < (u_{n+1} / u_n), \text{ numerically,}$$

the series Σv_n is convergent.

Deduce that the series Σv_n is convergent if

$$\lim_{n \rightarrow \infty} \left(1 - \frac{v_{n+1}}{v_n} \right)$$

exists and is greater than 1.

Investigate the convergency of the series for which

$$v_n = e^{-2} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right).$$

2. Explain how to sum a series of the type

$$\sum \frac{\phi(n)}{n!} x^n,$$

where $\phi(n)$ is a rational integral function of n

$$\frac{x}{2} + \frac{2x^2}{1.3} + \frac{3x^3}{1.2.4} + \frac{4x^4}{1.2.3.5} + \dots$$

to infinity.

3. By considering the expansion of $(1 + 1)^p$ by the Binomial Theorem (or otherwise) prove that

$$(p - 1)! [2^p - 2]$$

is divisible by p .

Hence prove Fermat's theorem that, if p be a prime number,

$$2^{p-1} - 1$$

is divisible by p .

4. On a series of parallel chords of a circle as diameters circles are described. Find the envelope of such circles.

5. Show that the equation of any conic circumscribing the quadrangle of which $a = 0, a' = 0; \beta = 0, \beta' = 0$ are pairs of opposite sides, can be put into the form

$$aa' + \lambda\beta\beta' = 0.$$

Find the inclinations to the axis of x of the axes of the two parabolas through the four points $(0, 0), (0, 1), (2, 0), (3, 4)$, the coordinate axes being at right angles.

6. Explain the method of polar reciprocation for transforming one curve into another.

If reciprocal polars are taken with regard to a circle of radius unity, centre the origin, show that if $u =$ inverse radius vector of a point P , $\theta =$ vectorial angle of P , $p =$ perpendicular from origin on tangent, $\omega =$ angle p makes with initial line, then u and p , θ and ω , reciprocally transform into one another.

Hence (or otherwise) show that the radius of curvature of the reciprocal curve at the point corresponding to the tangent at P is

$$\frac{d^2u}{d\theta^2} + u.$$

Deduce that the polar reciprocal of a conic with regard to a circle whose centre is a focus is a circle.

7. Evaluate the integrals:

$$(i) \int \frac{dx}{(x-3)(x-1)^2(x+2)^2}$$

$$(ii) \int x^2 \tan^{-1} x \, dx.$$

8. Obtain a reduction formula for the integral

$$\int \frac{dx}{(ax^2 + 2bx + c)^{n+1}}$$

Hence (or otherwise) work out the value of

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)^{3/2}}$$

9. Prove that the tangents drawn from any point P to an ellipse subtend equal or supplementary angles at the foot of the perpendicular from P on the major axis.

10. Prove that, if

$$(1+x^2) \frac{d^2y}{dx^2} = y,$$

and if, when $x = 0$,

$$y = 1, \frac{dy}{dx} = 1,$$

then

$$\begin{aligned} y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} - \frac{5x^5}{5!} + \dots \\ + (-1)^{n-1} \frac{x^{2n}}{(2n)!} 11.29 \dots, (4n^2 - 10n + 5) \\ + (-1)^{n-1} \frac{x^{2n+1}}{(2n+1)!} 5.19 \dots (4n^2 - 6n + 1) + \dots \end{aligned}$$

MATHEMATICS.

Examiners { L. N. G. FILON, Esq., D.Sc., M.A., F.R.S.
H. W. RICHMOND, Esq., M.A.

Wednesday, July 20.—Morning, 10 to 1.

1. The imaginary cube roots of unity being denoted by x and y , prove that, if

$$\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c} + \frac{1}{x+d} = \frac{2}{a},$$

and

$$\frac{1}{y+a} + \frac{1}{y+b} + \frac{1}{y+c} + \frac{1}{y+d} = \frac{2}{y},$$

then will

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} + \frac{1}{1+d} = 2.$$

2. The extremities of any diameter of a given circle are joined to a given point by lines which cut the circle again in P and Q . Prove that PQ passes through the fixed point.

3. A regular plane hexagon whose position in space is unknown is orthogonally projected upon a plane. Given the projections of three consecutive vertices, show how to construct the complete projection of the hexagon upon the plane.

4. Find the maximum value, the minimum value, the greatest value, and the least value of $x - 4 \sin^2 x \cos x$, as x ranges from 0 to $\frac{1}{2}\pi$.

5. Find the coordinates of the centre, foci, and vertices of the hyperbola

$$x^2 - 60xy + 33y^2 + 20x + 12y = 0.$$

6. A particle moves under an acceleration μ/r^2 towards a fixed centre of force. Prove that the semi-latus rectum of the orbit described is

$$h^2/\mu,$$

where h is twice the area swept out by the radius vector to the centre of force in unit time.

If the particle be projected in a direction making an angle α with the above radius vector, with a velocity u times the velocity from infinity, show that the eccentricity of the orbit described is

$$\sqrt{1 - 4u^2 \sin^2 \alpha (1 - u^2)}.$$

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7. A projectile moves under gravity in a medium whose resistance is constant and is equal to n times the weight of the projectile.

Show that if U be the initial horizontal velocity of projection, a the angle of projection, then the velocity at the highest point is

$$U \left(\frac{\cos a}{1 + \sin a} \right)^n.$$

8. A uniform solid, in the shape of the portion of a paraboloid of revolution, of semi-latus rectum $2a$, cut off by a plane perpendicular to the axis of symmetry at a distance h from the vertex, is placed with the vertex downward and in contact with the highest point of a fixed sphere of radius b .

Show that the equilibrium is stable if

$$h < 3ab/(2a + b).$$

9. A smooth horizontal tube OA is free to revolve about a vertical axis through O . Inside the tube is a particle of weight W , attached to O by an elastic string of natural length l , which is just taut without being stretched. A constant angular velocity ω is now impressed upon the tube and maintained. Show that, if a weight nW is required to stretch the string its own length, it will describe harmonic oscillations of period

$$2\pi \sqrt{l/(ng - l\omega^2)}$$

and that the greatest tension in the string will be

$$2ln\omega^2 W/(ng - l\omega^2).$$

10. A uniform heavy bar AB is placed with a point A resting on the ground at a distance d from a wall of height h , and with a point P of AB resting on the top of the wall.

Prove that if the bar is not to slip at P , the angle between AB and the top of the wall must not be less than

$$\cos^{-1}(\mu d/h),$$

μ being the coefficient of friction between the bar and the wall.

EXAMINATIONS FOR MATHEMATICAL SCHOLARSHIPS AT OXFORD.

Examinations for Open Scholarships and Exhibitions in Mathematics are conducted by three groups of Colleges:—(1) Balliol, Queen's, Corpus, St. John's; (2) University, Merton, Exeter, New College, Hertford; (3) Magdalen, Brasenose, Christ Church, Worcester. The first two groups examine simultaneously in December, at the same time as the two largest Cambridge groups. The third group examines at the end of February. Jesus College examines by itself in the Christmas Vacation.

At Hertford College there are one, sometimes two, Scholarships of 100*l.* a year, with no age limit but confined to members of the Church of England (or the Protestant Church of Ireland, or the Episcopal Church of Scotland), tenable for five years if the holder is in residence; at Balliol every four or five years there is a special scholarship, the Skynner, of 90*l.* a year with no age limit, tenable for five years during residence; and at Jesus College there is no age limit. All the others are restricted to candidates under 19; they are of the value of 80*l.* a year, and, whatever are the differences in the formal regulations of the various Colleges, actually they are tenable for four years and may be extended to a fifth to enable the holder to read for some other University examination, usually the Final School of Physics. A Scholarship is rarely renewed for a fifth year for the purposes of research alone, and there is no scheme of Senior Scholarships, at any rate for Mathematics.

Candidates of great and even exceptional ability occasionally present themselves, but it is undeniable that the average candidate is not as good as the average candidate at Cambridge. The genius from the small grammar school, the promising student from a provincial university, the ablest boy at the large public school, all are sent to Cambridge in preference to Oxford as a rule. Some of the candidates sent to Oxford from large public schools are occasionally so bad that one can only suppose that their masters think that a willingness to come to Oxford is a sufficient qualification for a Mathematical Scholarship there. There is no doubt that Oxford does not offer much of a career to a mathematician. Fellowships are very few, no prize fellowship has been offered in Mathematics since 1907, and, in the absence of senior scholarships, a man has to proceed to earn his living immediately after taking his degree, usually by teaching in a school, though a few are so fortunate as to get into the Civil Service by taking up other subjects. As the usual system of appointing mathematical masters appears to be to get a Cambridge man for the highest work, in order to obtain scholarships at Cambridge, and to consider a man's athletics

before his Mathematics for the other posts, the Oxford man does not get many of the prizes of the profession. At the present moment there are two pupils of mine of very great capacity, of one at least of whom I can say that had he been at Cambridge he would almost certainly have obtained a fellowship, who are teaching in schools where in all probability they will never have a pupil to prepare for a mathematical scholarship, any mathematical talent discovered early being diverted into science or into a bank.

Until the last few years the type of examination was much the same everywhere. There were papers on algebra, trigonometry, pure geometry, analytical geometry, and mechanics, or a paper that was half differential calculus and half mechanics. The questions usually consisted of a piece of book-work with a problem attached. There has been a considerable change since the appearance of the recommendations of the Mathematical Association. These recommendations have not been adopted entirely by any means, and it may be remarked that the gentlemen who drew them up seem to have overlooked the fact that the object of Scholarship Examinations is not to award certificates of proficiency to candidates who attain a certain standard of merit, but to discover the candidates of greatest natural ability, and, if possible, to put the candidate of natural mathematical instincts above the candidate who owes more to his teacher than to himself. Practically no schoolboy can have made much progress in the dynamics of rigid bodies, and such portions of the integral calculus as a schoolboy usually learns are concerned with finding simple areas and similar "practical" applications, so that the questions that could be set on these subjects, with any prospect of their being answered, would not separate candidates of average ability. Such questions are excellent as a mathematical test for candidates for science scholarships, but they do not furnish much indication of natural mathematical genius. Effect has been given, however, to several of the ideas embodied in these recommendations. Questions on probability have disappeared, and also questions on the theory of numbers that involve more acquaintance with the subject than a knowledge of the difference between a prime and a composite number. Any questions set on convergency depend for their solution on common sense and not on the application of elaborate tests. Proofs of expansions in infinite series are not asked for. Questions on modern geometry, projection, reciprocation, &c., appear occasionally, but there is no systematic development of this subject as a part of the examination. Groups 2 and 3 have adopted the integral calculus, group 1 has not. No group appears to set questions on elementary rigid dynamics, and questions on dynamics to which the calculus can be applied are very rare. The arrangement of papers has altered. Group 1 sets three papers involving book-work, one on algebra and trigonometry, one on

geometry pure and analytical, one on mechanics and differential calculus, and two problem papers, the second of which is generally much harder than the first. A piece of book-work set in the other papers is frequently merely a hint for the solution of a problem which accompanies it and is not expected to be a test of knowledge. Group 2 sets four papers that are practically problem papers. Book-work rarely appears in them, and only as a hint. Group 3 sets five papers much on the old lines, algebra, trigonometry and pure geometry, analytical geometry, mechanics, differential and integral calculus. The papers at Jesus College vary considerably, and general comments on papers do not necessarily apply to them. There is not much real difference, apart from the reduced quantity of book-work, between the type of question set now and set formerly. The questions on mechanics remain of a distinctly old-fashioned character. No group sets a book-work or essay paper such as is set by some of the Cambridge groups. There are very few questions that will give the candidate or the teacher something to think about outside his ordinary routine. Such questions are certainly suggested but rarely permitted to appear. There is sometimes the fear that, if the papers of a group contain questions that the candidates have not been directly trained to meet, preference will be given to some other group. The addition of some suggestive and educative questions and of some live mechanics would give the papers more value from the point of view of the teacher.

In estimating, by means of the performances of candidates in Scholarship Examinations, the nature and value of the teaching in Mathematics given in schools, it is necessary to take into account the fact already alluded to, that schools send their ablest boys to Cambridge rather than Oxford, and that the average candidate here does not represent the very best product of school teaching. However one can only report on what one sees and knows and hope that, where one's impressions are unfavourable, they are formed from data that do not do complete justice to the teachers' efforts.

The impression acquired is that the teachers do not try to teach their pupils Mathematics but to "prepare them for scholarship examinations." Most of the work suffers from this defect, that the candidates rely almost entirely on formulæ, and know very little of principles. A reason for this may be found in the fact that, in learning book-work, they learn the shortest proofs. These proofs, when they are proofs at all, are almost always purely "*ad hoc*" and lead nowhere. The candidate is also allowed and sometimes encouraged to pass over any theorem that is not likely to occur as an examination question. Consequently candidates have no consistent and progressive knowledge of any subject. A subject means to them so many detached things to be got up. Problems that do not fit in with one or other of these are quite beyond them. Identities and equations

in algebra and trigonometry are often handled with considerable skill and ingenuity, and there is frequently good work in elementary geometry, in fact it is in these departments that the candidates' natural abilities have to be discovered; but the work in the subjects that lead to higher things, analytical geometry, calculus, and mechanics is far from good on the whole. The work in analytical geometry is frequently deplorable. Candidates have no idea whatever of setting about a question that is not purely numerical. They show no intelligence in the choice of axes of co-ordinates, they rarely express the co-ordinates of a point on a conic in terms of one parameter, they never seem to perceive when from considerations of symmetry or dimensions their results are patently wrong. For the most part their work is a wilderness of algebra, and they simply shake things up together in the hope that something will drop out. Results that come right are frequently verifications, not solutions, and the tendency to work from both ends and meet in the middle, a process often leading to dishonesty, is far too prevalent in this subject, and indeed in others. No candidate dreams of attacking a problem by means of tangential co-ordinates, and direct questions on them are generally ignored. Trilinear co-ordinates are always employed in preference to areals, and a wholly unnecessary insistence on the use of actual trilinear co-ordinates usually results at a very early stage in a tangle from which the candidate never emerges. The reason for this state of things is probably to be found in the undue prominence given to squared paper and plotting in the teaching of the subject, and in the fact that almost all the exercises that boys are called upon to do are metrical. This treatment of the subject may be admirable for the engineer or scientist who is learning Mathematics as what Professor Ray Lankaster recently called an "equipment" study, but for the boy who proposes to read for Mathematical Honours at a University, which is naturally expected of a mathematical scholar, it is worse than useless. By perpetual contact with numerical examples he fails to acquire the important idea of dimensions, by metrical examples he acquires the notion that oblique axes are things to be left severely alone, and he gets firmly, almost ineradicably, into his mind the abominable idea that the general equation of a straight line is not $lx + my + n = 0$, but $y = mx + c$. It is due probably to this last circumstance that he finds tangential co-ordinates so completely beyond his comprehension. If candidates were encouraged to do examples leading to descriptive results and to be familiar with general equations, more progress would be made. Such work as the average candidates can do on the differential calculus is restricted practically to the use of formulæ. They can often differentiate accurately enough, and they can verify differential equations satisfied by functions if the result is given (they rarely attempt any direct method) and

hence obtain expansions; they can find the equations of tangents and normals to curves, and occasionally do examples that involve the use of these equations; they can do examples on curvature that can be done by quoting some one of the numerous formulæ for the radius of curvature; but they cannot do an example that involves a thorough comprehension of what a differential coefficient is. Very few candidates can answer such a question as: "Show that if α is an acute angle $(\sin \alpha)/\alpha$ decreases as α increases." Most candidates will be able to give a proof of the fact that the tangent of one of the angles between the tangent and radius vector at a point of a plane curve is $r \frac{d\theta}{dr}$,

but very few of them have the smallest idea of how to decide which of the two angles it is, if they have to deal with a particular curve. Partial differentiation they simply do not understand at all. If they are given u and v as two explicit functions of x and y , they will be able to find the partial differential coefficients of u and v with respect to x and y , but, if they are asked to find from these equations the partial differential coefficients of x and y with respect to u and v , they will almost certainly suppose them to be the others put upside down. Some candidates trace simple curves very successfully, but the attempts of others are poor and ignorant, and show that they have never been taught to approximate. With regard to the related subject of graphs, there is certainly some improvement, but there is room for much more, especially in the application of graphs to questions that are not entirely numerical. Many candidates could find a root of $\tan x = x$ by graphs, but would be unable to do the question "Show that, if k is real, the equation $\tan x = kx$ has an infinite number of real roots." A surprisingly large number of candidates cannot draw the graphs of the hyperbolic functions. In questions on maxima and minima candidates are seen at their worst in the blind use of formulæ. The idea of a function increasing or decreasing seems to escape them altogether. Their method is simply "Put $f'(x) = 0$. If this gives a value of x , then this makes $f(x)$ a maximum or minimum according to the question." No test is used to discriminate; it is "obviously" a maximum or minimum as may be required. The simplest question which involves the consideration of increase or decrease or a greatest or least value that is not a critical value, is entirely outside their experience. Since the greatest and least values required in practical and physical applications are as often as not non-critical values, it seems unfortunate that this blind faith in the critical value, a faith that can never be shaken, should be so universal. It must, however, be acknowledged that it is a matter on which they will get very little help from books, even the best.

With regard to mechanics, there has certainly been a very great improvement of recent years, and the teaching in some schools is excellent. But many candidates do poorly because the subject means nothing to them whatever. They acquire no knowledge of principles because they never see how these principles are applied to actual phenomena. There is no necessity for school-boys to be limited to graphic work in statics and to problems in dynamics that involve huge arithmetical details, but they might surely know something beyond the triangle of forces and uniformly accelerated motion in a straight line. Many candidates have not the faintest idea of applying the principles of the conservation of linear and angular momentum and of the conservation of energy to any questions. Very few candidates have any notion of friction that is not limiting, or of determining whether a body tilts or slides. To the vast majority of scholarship candidates $F = \mu R$ always. When asked to show, in the course of a problem, that the coefficient of limiting friction must not be less than some given amount, they always begin with the limiting case, and, if they find the coefficient in that case equal to the given amount, say at once that the coefficient must not be less than this. If the question were deliberately misprinted, they would say just as glibly that the coefficient must not be greater. Proceeding from a limit instead of to a limit is a characteristic of much work that one sees in these examinations.

In the actual solution of problems in mechanics there is room for much improvement. In trying to solve a problem on the equilibrium of a body under the action of given forces, candidates rarely take moments. They prefer to attempt it geometrically, making various lines in the figure meet at a point, thereby often involving themselves in awkward trigonometry. Very few candidates ever employ analytical geometry either in statics or dynamics. This is probably due to the almost total absence of its use in the text-books that they use. Questions on virtual work are rarely done, for two reasons; one, candidates work without the notation or the methods of the calculus, they get their small displacements by geometry and therefore often inaccurately, and they get confused between finite and infinitesimal quantities; two, they rarely seem to understand that, if they lengthen a heavy rod in a framework, they cause the stresses at the joints to do work. If they have four rods hanging in the form of a square and kept in that form by a string, the only displacement that they make, to find the tension in the string, is to expand the square into a larger square. There do not appear to be many schools where the drawing of stress diagrams is taught. Few boys seem to be thoroughly aware that the equation $P = Mf$ is not true if f is the acceleration of M relative to other bodies in the question. As to the use of the calculus for dynamics, it is a sad fact that

in the examination of the Balliol Group last December no candidate could write down the differential equation giving the motion of a particle moving in a straight line under a force varying as the distance in a medium in which the resistance varies as the velocity.

Some general defects in mathematical teaching may be noticed. One does not expect from a school-boy the absolute rigour that would be expected from a candidate in the advanced part of the Mathematical Tripos, though one does not like to see " $\dots = \infty - \infty = 0$, Q.E.D.," even in a school-boy. But though one has to be content with a little in that direction, yet there are other directions in which one may reasonably expect very much more than one ever gets. The important thing in Mathematics is "the truth, the whole truth, and nothing but the truth." Yet logical order and clearness of statement are often lacking; candidates confuse a question with its converse or take it for granted that if a proposition is true so is its converse; if a result contains factors other than those required by the question, they discard them without explanation; they rarely explain why they take one solution of an equation rather than another. Teachers seem to be unaware of the importance of eliminating these faults at the earliest possible stage.

Reference has already been made to the almost total ignorance of such important considerations as dimensions and symmetry. That there is no exaggeration in this may be seen by the following examples. In January last the following question was set at Jesus College:—

Without attempting to discover what is correct, give obvious reasons why each of the following statements is wrong.

(a) The locus of the intersection of perpendicular normals to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is

$$(a^2 + b^2)(x^2 + y^2)(a^2y^2 + b^2x^2) = (a^2 - b^2)^2(a^2y^2 - b^2x^2)^2.$$

(b) The real foci of the conic

$$9x^2 + 2xy + 9y^2 - 30x - 30y + 25 = 0$$
 are (1, 2) and (-2, 1).

To save trouble, it may as well be stated that in (a) $(a^2y^2 + b^2x^2)$ should be $(a^2y^2 - b^2x^2)^2$, and in (b) the foci are (1, 2) and (2, 1).

The first part appeared as a misprint in the papers of the Balliol Group in the previous December. One reason for setting it here was to see whether any candidates at Jesus College who had competed in the Balliol Group (there were in fact several) had noticed the error or had had their attention drawn to it by their teachers.

There were about two dozen candidates; they were not a brilliant lot by any means, but there were very few who had no claim whatever to be competing for a mathematical scholarship. With regard to (a) only one candidate, a man above the average age and an undergraduate of a provincial University,

noticed that the dimensions were wrong. With regard to (b) only one candidate (not the same one) stated that, as the equation was symmetrical in x and y , if (a, b) was one real focus, the other must be (b, a) . Some candidates calculated the co-ordinates of the centre and showed that it could not be midway between $(1, 2)$ and $(-2, 1)$; this method shows some intelligence, but misses the point of the question by a very long way. Three candidates thought an obvious reason to be found in writing down the general equations giving the foci of a conic, viz.,

$$\frac{(ax + hy + g)^2 - (hx + by + f)^2}{a - b} = \&c. \dots$$

and stating that the co-ordinates given did not satisfy them; an unfortunate line to take, for they all three selected $(1, 2)$, which does happen to be one of the foci.

These facts speak for themselves. Perhaps the schoolmasters would account for them by saying that the candidates had never seen a question like this before. That is precisely what one is complaining about.

The general sacrifice of essentials to details, the concentration on short and "tippy" proofs of theorems and "dodges" for certain types of problem, the failure to acquire the broad principles of any subject or to consider its treatment from the point of view of its developments beyond the stage of the scholarship examination, the justification of the means by the end, the lack of intelligent anticipation, all these defects seem never absent from the work of the average boy, and it is exceedingly difficult to eradicate his faults when he comes to the University. The belief that the "proof" which occupies three lines is better than the proof which occupies a page and the desire to have a "rule" for doing a problem rather than to consider on what principles it depends die very hard indeed. That there should be such widespread ignorance of so important a principle as that of dimensions is simply a scandal. This principle should be insisted upon in all subjects, pupils should be made to verify their results by means of it, and, whenever an error of dimensions is detected in a pupil's work, he should be made to trace his steps backward to the point where it has started.

There is no desire to belittle the work of schoolmasters, a very hard-worked and often underpaid body of men, but the system on which Mathematics are generally taught in English secondary schools is not likely to promote any great interest in or appreciation of them. The proportion of teachers with first-class degrees to boys is probably not one to a hundred, and in the lower and middle parts of schools, a capacity to work sums out correctly on the board appears to be the only qualification required for teaching Mathematics. The master's object seems to be to find a "rule" and to fix it firmly in the boy's heads. I have heard a master tell his boys: "Now when you are going to do questions of this sort, write these formulæ at

the head of your paper." It is not a method of teaching likely to promote any interest in the subject or any idea that Mathematics have any relation to actual life. A boy taught to work in formulæ alone is not necessarily stupid because he finds from his formulæ, without any surprise, men walking at the rate of 100 miles an hour or oranges costing $3/9\frac{1}{2}$ each. The whole subject of Mathematics, as far as it has ever been presented to him, is not less absurd and profitless.

Examinations are often held to blame for all defects in teaching. The teacher has to prepare for a certain examination, and therefore cannot attempt to teach the subject properly. But, if the pupil is properly taught, why should he find any difficulty in applying his knowledge to examples? Truly he will not find a sound knowledge of analytical geometry of much avail for a question, professedly on this subject, which depends on a tricky elimination or ingenious manipulation of determinants, nor will a sound knowledge of the principles of mechanics help him to do a problem, professedly on mechanics, which is really an intricate exercise in trigonometry. But questions of this type are becoming very rare and may safely be neglected. Insistence on the use of methods of general application, such as must be employed later, on logical order of statement, on the fact that "necessary" and "sufficient" are not the same, and on the knowledge of some common-sense methods of checking results, is sorely needed, and will be found of more value for examination purposes than trying to give the boy an illustration of every type of problem that has ever been set and forbidding him to learn anything that may not occur in an examination paper.

The question of text-books is doubtless a great difficulty for many schoolmasters. There are very many poor text-books written in English. They are written solely with a view to examinations. The authors aim at brevity in book-work and the omission of everything that is not likely to be asked for in an examination. They deliberately avoid any suggestion of anything beyond the syllabus of certain examinations. The examples they work out in the text are selected for their difficulty, not to illustrate any principle. On the other hand, it must be admitted that the authors of some books, excellent from the point of view of clear exposition of principles and as introductions to what is beyond, do not realize that the school-boy likes to be amused as well as instructed, and that he needs a little novelty and occasion to display his ingenuity if he is to develop an interest in a subject. A boy, who at the end of some months course of the calculus has done nothing beyond finding the area of an ellipse, the equation of the tangent to $x^2 + y^2 = a^2$, the expansion of $\sin x$, and perhaps the radius of curvature of the conic $3x^2 - xy + y^2 = 5$ at the point (1, 2), is rather bored. If he is occasionally allowed to prove something about the curvature of the pedal of $r^n = a^n \cos n\theta$, he certainly does not acquire

any very important knowledge, but he sees that he can do something with the calculus that he could not do without it, and will be attracted by the hope of doing more. There is a distinct need for books that shall be rigorous in all their discussions and shall give some hint of developments of the subject beyond the school stage without taking the school-boy out of his depth, and yet shall be neither too practical nor too severely formal, but shall give him a few pretty things to do to amuse him and to provide some scope for his ingenuity. The schoolmaster cannot very well be blamed if he is reluctant to use a book when it contains no examples of the kind that are to be found in any set of scholarship papers and its author prides himself on this omission.

Examinations are improving. They are by no means ideal, but there is a growing tendency to require of a boy the things he ought to know, not the things that he may possibly know if he has had the "tip." Let us hope that the schoolmaster will do his best to realize what these things are and to see that his pupils know them.

A. E. JOLLIFFE.

EXAMINATIONS FOR MATHEMATICAL SCHOLARSHIPS AT CAMBRIDGE.

A good deal of the substance of what I should naturally say, on the subject of Scholarship Examinations and Scholarship Candidates, has been said already in Mr. Jolliffe's paper.* The author and the Secretary to the British Sub-Commission have very kindly allowed me to see this paper before writing my own, and I shall assume that the reader is also acquainted with it. I shall thus be able to save a good deal of rather tiresome repetition.

I find myself, for the most part, in tolerably close agreement with Mr. Jolliffe. When we differ, it is as a rule simply because I take, on the whole, a rather more cheerful view of the situation than he does. This is only natural. Mr. Jolliffe himself points out, and it would be affectation for a Cambridge man to deny, that the general level of the candidates in these examinations is higher at Cambridge than at Oxford. This is no reproach to Oxford. It is due to a variety of causes, one of which is, no doubt, the specially privileged position that Mathematics occupied for many years among subjects of study at Cambridge. Nor would any Cambridge man pretend that Cambridge possesses a school of Mathematics great in the sense in which the schools of Paris or Göttingen are great. But, so far as these examinations are concerned, there is no doubt that the general level of our candidates is higher, and the average level of those near the top much higher, than at Oxford.

This fact alone would go far to explain any difference there may be between the estimates of the quality of mathematical teaching in public schools formed by Mr. Jolliffe and myself. But there is another fact which must be borne in mind. Not only do the public schools send us their best boys, but also a large and increasing proportion of the ablest of our candidates do not come from public schools at all. Probably the bulk of our best mathematicians come now from small schools or provincial universities. A good many of those who come from other universities, of course, are a little above the average in age. But the difference is slight; and there is no disguising the fact that they are, as a rule, much better trained than the boys from the schools. They have been taught by mathematicians of a higher calibre than the average schoolmaster, and in a more modern spirit; and they are not overwhelmed, as even the ablest schoolboys often are, by the contrast between school and university Mathematics. I feel, therefore, that the standard of ability displayed in our scholarship examinations, often undeniably very high, may tempt one to estimate the standard of teaching in the schools too hastily and more leniently than it deserves.

* See p. 263 above.

For the purposes of Scholarship Examinations, there are two groups of Colleges in Cambridge. One may call them, shortly, the Trinity group and the Pembroke group. Naturally I have experience only of the form^{er}, and it would be futile for me to attempt to compare their methods in detail, especially as the grouping is on the point of change. So far as I know, there is only one important difference between the two examinations in Mathematics. We have, for some years, included in the examination an "essay" paper, in which candidates are expected to write at length on two or three questions only, and to show the power of summarising and explaining fairly extensive regions of mathematical theory. The other group of Colleges did the same at one time, but have now, I believe, abandoned the paper—why I do not know. Our own experience has been that the paper is invaluable. It is a test peculiarly well adapted for an examination of this kind, in which selection of the best candidates is the only object. Such a paper, if it is to be effective, must be hard; not, of course, tricky, and difficult in the sense in which an old fashioned problem paper was difficult, but advanced and searching. Easy book work which can be reproduced by rote, whatever its merits may be in examinations of a different kind, is quite out of place in a scholarship examination. Our practice is to set a straightforward but difficult paper, and to mark it with extreme severity. We find then that it separates out the best candidates more effectively than any problem paper.

There are three papers in addition to the "essay" paper. The elementary paper on Pure Mathematics is, so far as mathematical candidates properly so called are concerned, nearly useless. It is a paper really intended for candidates in Science and Engineering, and there is very little point in its being taken by mathematical candidates at all. It should be observed that a fair amount of Integral Calculus is included in this elementary paper. It might be inferred from Mr. Jolliffe's remarks that he regards the subject as unsuited for a scholarship examination, on the ground that such applications of it as are within a school-boy's powers are too straightforward. This view, whether it be Mr. Jolliffe's or not, is one with which I disagree decidedly. One of the chief reasons why the Calculus is, on the whole, so badly taught in schools, is that the old Oxford and Cambridge tradition gave a quite absurd priority to the Differential Calculus, and overweighted its geometrical side. Hence the wilderness of puzzling trivialities of which so many books are full. I need hardly say that I agree heartily with Mr. Jolliffe as to the importance of giving the school-boy "a little novelty and occasion to display his ingenuity"; and I admit that it is here that some otherwise excellent text-books fail. But it is not only trivialities that are fascinating and that afford scope for ingenuity. I cannot imagine a subject whose elements afford a better field for the simultaneous

exercise of critical power and technical ingenuity than do those of the Integral Calculus.

Of our two remaining papers, one is a "problem paper" in Pure Mathematics, more or less of the traditional type. The last is a paper on Applied Mathematics. This, like the elementary paper on Pure Mathematics (but not the problem paper), is taken by many candidates whose claims rest for the most part on other subjects. But it is a good deal more difficult, and is a really serious test of the mathematical candidates. As I have never had a hand in setting it, I may fairly say that I think it is generally an excellent paper, striking a reasonable mean between the old and new ideals of what a paper in Applied Mathematics ought to be.

I do not, of course, profess to believe that our examination is in any respect an ideal one. But I do think that it is, on the whole, reasonably well adapted for the purpose it professes to fulfil; and on the whole the results appear to justify it. It must, of course, be remembered that we never elect to a full Senior Scholarship at entrance*; we have always further opportunities of reconsidering our views. There are, and no doubt there always will be, boys whose mathematical abilities stop short about where school Mathematics ends. Such boys are, I think, more likely to be exposed by our present examination than by an examination which contains little but a selection of old-fashioned problems. But it is inevitable, under any system, that cases should occur in which it soon becomes clear that we have allowed ourselves to be imposed upon. Such an Entrance Scholar remains an Entrance Scholar throughout his university career; to obtain a Senior Scholarship in a resident examination on University subjects, conducted by lecturers already well acquainted with the work of the candidates, is humanly impossible to any one but a real mathematician. It should be observed that these "disappointments" are invariably public school boys.

I should now, I suppose, say something more in detail as to the impressions of school teaching left in my mind by these examinations. Here I agree, on the whole, with Mr. Jolliffe: I recognise very well the kinds of confusion and limitation to which he refers. He has drawn a very accurate picture of the average scholarship candidate; I would only add *unsuccessful* candidate. I cannot imagine myself as consenting to the election of a scholar of whom all that Mr. Jolliffe says is true.

Mr. Jolliffe is no doubt right in regarding the general level of teaching in the schools as deplorably low. There are, of course, exceptions. There are schoolmasters for whose knowledge of Mathematics I have a great respect, and I have seen books written by schoolmasters which could only have been written by genuine mathematicians. But of the general level, whether of the books or of the teaching,

* I believe the practice is different at Oxford.

all that one can say is that it is a little better than it used to be. Schoolmasters are no doubt, as Mr. Jolliffe says, "a very hard worked and often underpaid body of men"; and certainly, so far as my experience goes, they are keen and conscientious to the last degree, interested in and patient with their pupils, and sometimes very enterprising in their methods. In short, they have all the pedagogic virtues. But there is one thing too many of them have not, and that is a decent knowledge of Mathematics. They have, as a rule, learnt a good deal of Mathematics at the universities, but apparently it is forgotten as soon as they find themselves back again in the schools. If only schoolmasters would devote to the serious study of Mathematics, to the continuance of their reading, and even, in exceptional cases, to a little independent research, a tithe of the time that they now devote to the organisation of examinations, the framing of schedules, the discussion of details of pedagogic method, and the writing of inferior text-books, I am sure that the results would soon become visible in our examinations.*

Mr. Jolliffe has given a considerable number of very happy illustrations of the deficiencies of the candidates whose work he examines, and it would be a waste of time for me to add a large number of my own. But I may perhaps be allowed to add a few remarks about the teaching of special subjects.

Pure Geometry.—The teaching of the more advanced parts of the subject seems to me to have improved considerably. Many candidates show a very intelligent appreciation of some of the general ideas of projective geometry. Their work, however, is rather stereotyped; I am inclined to think that the teaching of the subject is almost too well organised, and that an undue proportion of time is spent upon it.

The teaching of *elementary geometry* is, I should say, abominable. Here I have to judge rather from the text-books that I see from time to time than from work in examinations. The dethronement of Euclid has been the signal for the appearance of a flood of elementary books whose authors do not seem to have the most rudimentary knowledge of the principles of the subject. There is, of course, a great deal of geometry which is capable of proof, but proof quite unsuited for beginners—and in an elementary course a great deal may quite rightly be treated as obvious which is in reality far from obvious. This is no excuse for professing to prove results by methods, such as the method of superposition, which rest merely on elementary logical confusions.

Algebra.—As regards the teaching of elementary algebra, I need only refer to Mr. Barnard's paper† in this series. "Higher

* The schoolmaster who wishes to keep up his mathematical knowledge must be prepared to read foreign books, in French at any rate. It is ridiculously easy to read Mathematics in any language, and the number of really good English books is very small.

† See *Special Reports on Educational Subjects*, Volume 26, p. 312.

Algebra" has always been a scandal; I shall have some further remarks to make on this subject under the heading of "analysis."

These remarks apply also to *Trigonometry*.

Analytical Geometry.—I have nothing to add to Mr. Jolliffe's remarks on this subject.

Analysis.—Under this heading I include Differential and Integral Calculus, Higher Algebra, Theory of Series, and Analytical Trigonometry. The teaching in all these subjects shows signs of slow but real improvement. It would be unfair to blame schoolmasters too severely for the inadequacy of their treatment of the fundamental ideas of analysis, for it is only in quite recent years that there has been any serious attempt to treat them properly even at Cambridge, and there has not yet been time for a supply of mathematicians with a proper training in analysis to reach the schools. Many schoolmasters realise that there has got to be a change; but it is naturally difficult for a man, encouraged to think loosely as a schoolboy and an undergraduate, to acquire the habit of accurate thinking at the age of 40 or 50. It is only an exceptional mathematician who can do that.

I think that a good many schoolmasters seriously misunderstand the attitude of those of us who are trying to teach analysis in a way that would be regarded as serious in France or Germany. We have not the least desire to drive every mediocre schoolboy crazy with δ and ϵ . We can quite well understand that there is a stage in everybody's education when perfect rigour is a quite unpracticable ideal, and that there are many boys who never can or ought to attempt to reach it. But there are two demands that we do make. The first is simply destructive. We ask that schoolboys' minds shall no longer be stuffed full of the pitiable nonsense that we were taught ourselves at school. The sort of rubbish that used to fill the pages of treatises on "Higher Algebra" and "Higher Trigonometry" had absolutely nothing to recommend it. It had no sort of æsthetic merit, and it was very much harder than any accurate treatment of the subjects need have been. It ought not to be necessary for a university teacher to spend hours of his time trying to make his pupils forget what they have learnt laboriously at school. We ask, therefore, that, when it is clearly impossible that reasonably accurate proofs should be given, the pretence of proof should be abandoned. In this, I am sure, we have with us the majority of the best schoolmasters; the difficulty is simply that there are so many who have not the slightest notion as to what is a proof and what is not.

Our second demand concerns the best boys only. With regard to them we do ask that they shall have decent reasoning put before them from the beginning of their studies in these subjects. But, even in the case of these boys, we are not asking that they should be taught anything intrinsically more difficult

than a great deal that they are taught already. It is a profoundly ignorant superstition that slipshod reasoning is always easy, and accurate reasoning always impossibly difficult. Let me give a concrete example of what I mean.

Schoolmasters are never tired of disputing as to what is the best way to prove that

$$\frac{d}{dx} x^m = mx^{m-1}.$$

I confine myself to the case in which m is rational. This is quite enough for a schoolboy, unless (as may well occur) he raises the question of irrational values himself. In any case, whatever method of proof is adopted, the irrational case introduces further difficulties.

One may proceed as follows. Differentiate x^m when m is a positive integer (using either the Binomial Theorem or the formula for the differentiation of a product). Then, if $y = x^{p/q}$ we have

$$y^q = x^p, \quad qy^{q-1} \frac{dy}{dx} = px^{p-1}, \text{ etc.}$$

But that is too difficult for the schoolmaster. Let us see how he proposes to simplify it.

We have, he says

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^m - x^m}{h} \quad (m = p/q) \\ &= \lim_{h \rightarrow 0} \left(mx^{m-1} + \frac{m(m-1)}{1 \cdot 2} x^{m-2}h + \dots \right) \\ &= mx^{m-1} \end{aligned}$$

Now, in the first place, we are assuming the Binomial Theorem for a non-integral exponent. This is a theorem infinitely more difficult than that which we are trying to prove. Not only this, but the easiest and most natural way of proving the Binomial Theorem (when m is not a positive integer) is by means of Taylor's Theorem, and the application of Taylor's Theorem to the Binomial Series of course involves the very result we are now using it to prove. But, even when the Binomial Theorem has been assumed, we are still confronted by a more serious difficulty. We have to "put $h = 0$ " in the infinite series; and to do this we must use either a direct " ϵ " argument, or some general theorem concerning the uniform convergence of power-series. And yet it is we who are trying to make analysis too hard for schoolboys!

If you give a small boy the armament of a Dreadnought as a toy, and send him out into the fields to shoot sparrows with it, it is certain that a serious disaster will follow. But this is just what the mathematical schoolmaster insists on doing. The result is that the school-boy's sense

of logic and proportion is blighted from the beginning, and that when he has an opportunity of learning something better, the opportunity often proves to have come too late. I have at this moment able pupils who, I can see, will *never* really understand anything in analysis, simply because they will never recover from the poison administered to them early in life. That a better method of teaching is practicable is shown, not only by the example of France and Germany, but also (if I may judge by the results that I see here) by that of our own provincial Universities.

I have, I feel, expressed myself very strongly. My apology must be merely that I feel very strongly. For my own part, I escaped from the English tradition just in time. I was lucky enough, for two terms before my Tripos, to find a coach with a really wide knowledge of Mathematics, who first made me feel that there was a whole world of fascinating theory of which I was completely ignorant. I shall never forget the experience of reading Jordan's *Cours d'Analyse* and other foreign books, and realising for the first time what mathematical reasoning really was. Undergraduates nowadays have better opportunities. The tyranny of the old Tripos has been abolished, and the way is clear to the acquisition of a real knowledge of Mathematics. It is for the schoolmasters to see that their pupils are qualified to take advantage of it.

G. H. HARDY.

APPENDIX.

I add, as illustrations of my remarks on the teaching of analysis, a few quotations from two recent books on the calculus. I need hardly disclaim any particular grudge against these two particular books; no doubt they are no worse than many others of their kind. I refer to the books as A and B. The passage which follows is extracted from A.

"To prove that $\frac{d(x^n)}{dx} = nx^{n-1}$, for all values of n .

Let $y = x^n$, and Δx , Δy be corresponding increments of x and y , so that $y + \Delta y = (x + \Delta x)^n$.

By subtraction

$$\begin{aligned} \Delta y &= (x + \Delta x)^n - x^n \\ &= x^n \left[\left(1 + \frac{\Delta x}{x} \right)^n - 1 \right] \\ &= x^n \left[\frac{n \Delta x}{x} + \frac{n(n-1)}{2} \left(\frac{\Delta x}{x} \right)^2 + \text{terms involving higher powers of } \Delta x \right]. \\ \therefore \frac{\Delta y}{\Delta x} &= x^n \left[\frac{n}{x} + \frac{n(n-1)}{2} \frac{\Delta x}{x^2} + \text{terms involving powers of } \Delta x \right]. \end{aligned}$$

Hence proceeding to the limit $\frac{dy}{dx} = \frac{nx^n}{x} = nx^{n-1} \dots$

On the next page all this is repeated in an example (find $\frac{dy}{dx}$ when $y = (x-3)^{-6}$). Oddly enough, the lines of a satisfactory proof are indicated in three examples a little further on; here the terms " m/n ", m and n being positive integers" and "any positive quantity" (the author's italics) seem to be treated as synonymous; but this is more pardonable.

In the treatment of "areas" simple matters are obscured with even greater ingenuity and success. If we assume the existence theorem for areas (as surely the author proposes to do) the proof that the area function is the integral of the ordinate presents no difficulty whatever, as indeed the author seems to feel when in another place he considers a special case. The elaborate argument used in the general sense is not merely nonsense but absolutely useless nonsense. But it would be waste of time to criticise this book further; I pass to another.

This book "B" is no doubt largely a *jeu d'esprit*, and should not be judged too seriously. But the "ease" of the treatment is announced with so imposing a flourish of trumpets that it is worth enquiring how far this promise is fulfilled. I pick a few examples more or less at random.

Differentiation of x^n .—There is nothing to choose between A and B.

Differentiation of a quotient.—The author solemnly divides $u + du$ by $v - dv$ in seven lines of long division.

The exponential function.—All the difficulties of the old "Higher Algebra" are crammed into three or four pages. The only point of originality in the treatment is the author's habit of calling e "epsilon."

Partial differentiation.—Taking $y = uv$ the author obtains

$$dy_v = vdu, \quad dy_u = udv$$

where (e.g.) dy_v means an increment of y when v is treated as a constant. Also

$$\frac{dy}{du} = v, \quad \frac{dy}{dv} = u$$

"If we put in these values for v and u respectively, we shall have

$$\left. \begin{aligned} dy_v &= \frac{dy}{du} du \\ dy_u &= \frac{dy}{dv} dv \end{aligned} \right\} \text{which are } \textit{partial differentials}.$$

"But, if you think of it, you will observe that the total variation of y depends on *both* these things at the same time. That is to say, if both are varying, the real dy ought to be written

$$dy = \frac{dy}{du} du + \frac{dy}{dv} dv,$$

and this is called a *total differential*."

The area of a rectangle depends, "if you think of it," on its base a and its altitude b . Its area is therefore $a + b$. And what an opportunity for rubbing in a good point has been lost! For the fact that "small increments" may be "compounded linearly" is one of the most striking and fruitful ideas in elementary mathematics.

"The professional mathematicians," the author remarks, "will rise up as one man and damn it as being a thoroughly bad book . . . It commits several most grievous and deplorable errors." This is true enough;

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but these errors are not, as the author seem to imagine, that it shows "how ridiculously easy most of the operations of the calculus really are," and that "what one fool can do, other fools can do also," nor is it that "there is an utter failure . . . to demonstrate with rigid completeness . . ." The fault of the book is its lack of originality, its thoughtless adoption of bad traditional methods, and the perverse (though traditional) ingenuity with which it makes difficult and obscure what ought to be easy and illuminating.

THE RELATIONS OF MATHEMATICS AND PHYSICS.

During the last century the mutual relations of the sciences of Mathematics and Physics were characterised by a very close harmony. In many senses it might be said that the 19th century has been the age of Mathematical Physics. Such names as Fresnel, Fourier, Helmholtz, Ampère, Stokes, Clerk Maxwell, and Lord Kelvin are but a few which rise to one's lips if we try to recall the pleiad of workers who have brought in rich harvests in this field. The creation of the sciences of Thermodynamics and of Electromagnetism, and the unification of Optics with the latter have been among the chief triumphs of this alliance of the experimental and the analytical methods. At one time these successes even led some thinkers to imagine, somewhat prematurely perhaps, that all the problems of the universe were capable of being put into equations and solved by the methods of Dynamics.

The spirit which animated these great men of science is best illustrated by the following passage from Fourier* :—

“The deeper study of nature is the most fertile source of mathematical discovery.

“Not only does such study, by giving to research a specific objective, exclude indefinite questions and insoluble problems ; but it also provides a certain means of building up Analysis itself and of discovering those elements of that science which it is most important for us to know and which should ever be carefully preserved.

“These fundamental elements are those which are constantly recurring in all natural phenomena.”

Signs are not wanting that this deep and fruitful harmony is at the present day being gradually weakened. The traditional sympathy, of course, remains, and no one doubts that Physics still requires mathematical methods, in order to reduce its theories to order. The trouble is that nowadays mathematical methods, which in the past were stimulated by the need of the physicist, no longer appear to respond to that need. The phrase of Fourier, quoted above, no longer represents the feeling of many of the younger mathematicians. The tendency is growing to separate Mathematics as far as possible from its physical substratum, to refine away all appeals to

* *Théorie analytique de la Chaleur. Oeuvres, Ed. Darboux, Vol. I., p. xxii.*

intuition and experience, and to expend the energy of the researcher upon ideas and concepts increasingly abstract. In itself this need not necessarily be a bad thing: in particular the recent revision of the foundations, when at last it percolates into elementary teaching, will eventually be found to remove many stumbling blocks and absurdities. We should also bear in mind that new branches of Pure Mathematics, such as the theory of sets of points or of groups of transformations, even when purely abstract in their inception, may at any moment turn out to be valuable auxiliaries of Physical Science, as indeed has already happened in the case of the theory of the complex variable. Sometimes, as with the theory of integral equations, the new branch has had its ultimate origin in work connected with physical problems, and its development, even if carried out on purely abstract lines, will probably in the end react favourably on the solution of these problems.

Unfortunately while all these new branches of Pure Mathematics flourish, the interest of researchers in Mathematical Physics appears to have largely flagged. No doubt much work is still published which comes under the latter heading, but this work is mostly of the nature of the solution of particular cases, the principal difficulty being the heaviness or the awkwardness of the algebra, and, although often both interesting and valuable in its results, it is rarely stimulating or instructive and frequently leaves one with a strong impression of the hopelessness of any attempt to disentangle things still further on similar lines. Very few papers on Mathematical Physics nowadays suggest new or powerful methods. Certainly our generation has seen nothing comparable with the theorems of Fourier, Green, or Stokes.

On the other hand, if we turn our eye upon Physics, we find that owing to the growth of new experimental branches such as radioactivity and meteorology, and owing to the great development of what may be termed Technical Physics, a large number of new facts, new methods, and new technical terms have to be taken into account. In fact, Physics has now become so highly specialised that a considerable training is necessary before an ordinary mathematician can attack intelligently a physical problem. A hundred years ago any mathematical specialist could acquire, with a minimum of trouble, the training necessary to turn himself into a fairly creditable mathematical physicist. With the increasing specialisation of modern times, Mathematics and Physics in their advanced domains are getting to speak different languages, and the brains which can be masters of both are becoming fewer.

Now there can be little doubt, I think, that the secret of the great victories of Physics in the past century lay in the combination of the powers of analysis and experiment in the mind of the same individual. Should these powers, owing to the increased complexity of Science, become separated, the edge of

the instrument of research will have been singularly blunted. Let us then try to analyse more carefully the causes at work to-day which tend to isolate the mathematical from the experimental mind.

Consider first a little more closely a cause already mentioned, namely, the growing tendency of Mathematics to become metaphysical.

Modern Mathematics is characterised by a complete revision of all those results which depend upon infinite processes. Under this heading may be classed Dedekind's theory of the irrational numbers, the theory of sets (and closely connected with it the foundations of the differential and integral calculus), the theory of infinite series and products, of continued fractions and infinite determinants, the modern theory of divergent series, and Cantor's theory of transfinite numbers.

Concurrently with this work on the infinite and in close touch with it has been the rediscussion of the axioms of Geometry, which started with the discovery of Non-Euclidean Geometry by Lobatschewsky and Bolyai.

Now this perpetual digging at the foundations, this anxiety to justify the known processes, has had a two-fold result.

It has revealed that normal, every-day Mathematics are really, from the purely logical point of view, very restricted in application, and that outside them there exists a whole universe of mathematical laws and forms, quite as interesting to the mathematical mind as those we deal with in practice. This new universe of Mathematics has largely dwarfed the interest that researchers felt in the old problems.

The second consequence has been that, owing to this hypercritical condition of the modern mathematician, the proof of the possibility of a result has gradually assumed an importance equal to, if not greater than, the working out of the result itself. Thus Mathematics is at present overrun with what are known as "existence theorems," showing that a solution of a problem *can* be obtained; but with the actual formal solution very much less progress has been made than might in many cases have been anticipated.

It is clear that to the physicist such a mental attitude is of no value, since in his case he is certain beforehand that a solution is possible, this being already provided by Nature, and his only difficulty is to express this solution in analytical form.

The next contributory cause is to some extent the reaction of the first. This tendency of Mathematics to fly off into metaphysics, apparently on the slightest provocation, was hardly appreciated by the physicists, most of whom had received their mathematical education at a time when intuitional proofs were accepted as orthodox. They not unnaturally suspected scholasticism, and a growing distrust, if not of

Mathematics, at least of mathematicians, began to set in. The attempt at making mathematical teaching rigorous has frightened many young physicists into something like hostility. The result has been that a number lack the training necessary to develop mathematical methods for themselves or to use existing ones with safety. Also the fact that these men have been repelled at an early stage has prevented much that is valuable in advanced Mathematics from being applied to Physics. Thus the method of conformal representation, which gave remarkable results in the hands of Sir J. J. Thomson and Professor Lamb, is practically unknown to the average researcher in a Physics laboratory.

I have already alluded to the increasing volume of physical material, physical technical terms, and physical theories, which make it increasingly difficult for the ordinary mathematician to apply his mathematical knowledge to physical problems. In this connection it may be remarked that most solutions in Physics are only approximate, and the "physical instinct" required to select the right approximation demands a practical knowledge of the particular problem which is not always accessible to the man who has the ability to discuss it theoretically.

Parallel with this is another tendency, which has been a good deal fostered by that modern agnostic philosophy of Science which looks upon theories and equations as mere transitory clothing for the facts. This is the tendency to accumulate facts as such without interpretation keeping pace with them. Examples of this are meteorological records and, to a lesser extent, the mass of spectroscopic measurements, both still waiting for an adequate theory. In such a case it is doubtful whether premature accuracy or multiplication of observations does not discourage, instead of stimulating, progress. It has been aptly remarked that, had Tycho Brahe possessed a telescope endowed with modern refinements, Kepler would in all probability—being confused by small inequalities—have failed to deduce the simple elliptic law from his measurements; and Newton's law of gravitation, which was largely based on Kepler's result, might never have seen the light.

A third cause that I think operates with the two last named in discouraging mathematicians from attempting to deal with physical problems nowadays is the growth of Electrical, as against what I may call Mechanical, Dynamics. In the old days, the laws to which it was endeavoured to reduce physical questions were few and well understood: it was comparatively easy for the mathematician who undertook such work to check it by reference to first principles. Nowadays, Newton's laws appear as secondary effects, and the foundations are much more hypothetical and incomplete; in many cases even essential data seem to be wanting. Thus the proper relations between æther and matter are still wrapped in mystery, and, accordingly, interface conditions never seem to be written down with any

degree of certainty. Again, the distribution of electricity on a moving conductor ought to be determinate when the shape and motion of the conductor are given. Yet in the mathematical treatment of such a problem the actual distribution appears usually to be arbitrarily assumed. The effect of motion on material bodies remains unexplained, and to account for the negative results of Michelson and Morley and their successors a number of hypotheses have been put forward, all of which strike one as strained and unnatural; some of these even revise the notion of time and make it a function of position.

All these things do not tend to create confidence, and this may explain why some who have been trained in the old ideas should be repelled by the difficulties of the new and turn their mathematical energy to more familiar problems.

The difficulty felt by these men brings us to what is probably a very important cause of the present lack of co-ordination between mathematical and physical workers, and that is the gradual decline and collapse of Applied Mathematics as a degree subject in most Universities. Fifty years ago Applied Mathematics was a highly flourishing and stimulating subject, and was taken up by large numbers of candidates. During the last twenty years, as the records of the University of London show, the subject has been going steadily down in importance and popularity. As an Arts subject it has practically disappeared. As a Science subject it still exists, but it shows no vitality, and is taken up by mathematical candidates as a stop-gap, to save working up a more unfamiliar subject like Chemistry or Botany.

This gradual decay of Applied Mathematics is probably due partly to the fact that it laid too much stress on the "hard rider," or trick problem, apart from the application of broad principles: much of what passes as Applied Mathematics is mere algebraic contortions, with little or no bearing on the practical problems of Physics.

For example, what real physical interest attaches to the evolutions of a bar moving with its extremities on two "perfectly smooth" rods? Or to the antics of a "perfectly rough" fly crawling on a disc lying on a "perfectly smooth" table? Nor can one see the usefulness of pages of analysis in order to find the initial radius of curvature of the path of a point connected in some quaint way with a smoothly-jointed framework (quite unlike anything ever constructed), one of whose points is seized and made to move with a given velocity.

What fundamental physical ideas and methods are really brought out by the calculation of the combined electrical resistance of wires in the shape of a regular icosahedron?

Again, what are we to say of a Geometrical Optics which attends to curiosities such as the curves seen on a bicycle wheel in the sunlight, but neglects the proper theory of image formation in modern optical instruments?

Many of the results are even an outrage on common-sense. Much time is spent in hydrodynamics on irrotational motions which make liquid move past an angle with infinite velocity and a solid body in a uniform stream experience no force—the last a statement which a glance at any anemometer contradicts at once, and which, if true, would theoretically preclude all possibility of aviation.

Much ingenuity is expended on complicated examples depending upon a theory of resilience and impact now entirely obsolete: nor is the theory of friction in very much better case.

Whole chapters are devoted to parabolic motion of a projectile, which anyone who has ever fired a rifle at various ranges knows to be not even an approximation to the truth.

No doubt it may be said that in these latter cases the artificiality complained of is due to the great difficulty of the treatment of the problem as it actually occurs in nature, and to the consequent necessity for “idealisation.”

One must, however, bear in mind that the objection is, not to the introduction of the simplified treatment, but to its application to problems where it is obviously inadequate and to the disproportionate amount of time spent on such applications, to the great discouragement of those students who pursue Mathematics for the sake of Physics, and who find that the mathematical and experimental parts of their work are hopelessly out of touch.

It has been argued that the better type of recent text-book is becoming more or less free from these defects. Even if this be admitted, and it appears still doubtful, the improvement, such as it is, has hardly percolated down to those elementary text-books on which the education of the average student is founded; and a glance at examination papers on Applied Mathematics in most British Universities will, I think, generally confirm this impression.

Partly also the decay of Applied Mathematics is due to the unprogressive nature of the teaching and examining in this subject. Thus, while the whole theory of Mechanics is being revolutionised by electrical ideas, electricity is still excluded from many syllabuses in Applied Mathematics. Potential is still often treated exclusively from the gravitational point of view, the electrical facts, which give to the theory of potential its real importance, being left out. Again we have plenty of unreal and entirely useless examples on Boyle's law, but about the elements of the kinetic theory of gases or of thermodynamics a discreet silence is maintained, or else we are informed that to speak of these things would be to trespass on the domain of Physics, as though Applied Mathematics meant anything else than the mathematical treatment of physical problems. This predominance of the mechanical standpoint seems to have diverted whatever living force was left in the subject towards Engineering and away from Physics.

Be this as it may, so long as Applied Mathematics remains an archaic and a neglected subject it will probably be hopeless to expect again such a flowering of mathematical genius applied to physical problems as we have seen in the past. We may get, and indeed we have got, experimenters of the first grade: we may get pure mathematicians of no mean order, but the combination of the two, instead of being comparatively frequent, will be a freak—a rare accident.

Finally, we have to consider those causes which are bound up with the growth of Physical Science. It has been noted that as science progresses there occur epochs of alternate simplicity and complexity. Thus, at first, rough data are gathered unsorted, and all is chaos: then the laws of first approximation emerge and the chaos resolves itself into apparent order. As the means of observation improve, the original data are modified and the original laws are found to be inadequate. Once more we are faced by chaos, total or partial. Then modified laws are found to meet the case, and the process is repeated.

It would seem that in Physics at the present time we are coming to an epoch where new facts and closer observation render the old laws inadequate. New problems arise and new mathematical methods are needed to cope with them. For many of these methods we are still waiting. To take two examples alone: linear differential equations, which were for many years the backbone of Mathematical Physics, are becoming an exhausted field. Residual phenomena like permanent set, &c., are clamouring for attention. Experimentally they are receiving it, but very little progress has been made with the discovery and solution of the non-linear equations involved.

The same may be said of the atomic structure of matter. In the classical Mathematical Physics this atomic structure was a purely theoretical thing; matter, being studied in the mass, was treated mathematically as continuous, and we had the paradox of an infinitesimal element, $dx dy dz$, infinitely great with regard to the ultimate particle of matter. In this way all the well-known space-differential equations of elasticity, of hydrodynamics, of electromagnetism, were obtained. In the present day, however, ionisation and ultra-microscopy are beginning to isolate the individual corpuscles experimentally, and it is obvious that our Mathematics must follow this evolution if they are to be helpful. It is true the æther still remains as a continuous medium, but some of our younger physicists are in favour of abolishing it altogether.

All this implies a complete change of front, for which mathematicians trained in the old methods are ill-prepared. Hence the need for reorganisation and development, for an interval of preparation, during which little effective co-operation is to be looked for from Mathematics. We must in fact learn

once more to think atomically, even as the mathematicians of the 18th and early 19th centuries, whose thoughts ran on actions at a distance, had to adapt themselves to continuous media and contact stresses. In this connection few things are more instructive than the way in which early writers deduced the equations of motion of, say, an elastic medium, from the mutual attractions of molecules. We are now, in effect, going through a sort of converse process.

Having now reviewed in some detail the causes of the phenomenon we set out to investigate, we may inquire in what way and to what extent they may be counteracted.

It is clear at once that causes such as: (1) the need of new mathematical methods to cope with the new facts of Physics, (2) the uncertainty and novelty of the new electrical theories, (3) the interest created by the opening of new fields in Pure Mathematics, belong to the domain of the inevitable, and that it is neither possible nor desirable to interfere with them. They are a necessary phase in the history of science, and have to be accepted as such.

On the other hand: (1) the mutual misunderstanding due to over-specialisation, (2) the accumulation of uninterpreted material in Physics and of abstract concepts in Mathematics, (3) the neglect of Applied Mathematics—these are tendencies which are, in varying degrees, capable of regulation. Much might be done to remove the first two by educating teachers, examiners, and researchers in both subjects to see the dangers of the present courses; but above all things, it seems to me that the remedy is to be sought in a sound, broad, up-to-date curriculum of Applied Mathematics, backed up by a series of progressive books on modern lines, a curriculum which shall concern itself with really practical problems and not with intellectual gymnastics. At the same time we must never forget that what the mathematician requires to know of Physics in order to help the physicist to do his work is to understand the laws established by the physicist, his nomenclature, and the significance of his experimental results. It would be a mistake to transform this course of Applied Mathematics into one of Experimental Physics by dragging in a laboratory experience which the mathematician does not need and which is here quite a secondary consideration. We hear not infrequently teachers of Physics or Engineering complaining of the mathematical courses being too abstract. It might be pointed out that a good mathematical physicist's time might be equally wasted by too severely practical a course. Nevertheless, enough should be given to impress him with the significance of the problems that are set before him.

Such a course of Applied Mathematics, which should initiate the learner to all the more fundamental physical methods and

results of a mathematical nature, including the more recent developments, but without going into any overwhelming analytical difficulties, would surely provide an excellent training for the young men who propose to pursue either Mathematics or Physics. It would give the mathematician and the physicist a common meeting ground, and enable either of them, when the occasion arose, to throw useful light on the problems of the other.

L. N. G. FILON.

THE TRAINING OF TEACHERS OF MATHEMATICS.

Two distinct subjects fall within the scope of this memorandum: (1) the existing arrangements for producing teachers of Mathematics; and (2) the principles that should determine the character of their training. The treatment of the first consists merely in a recital of facts; that of the second involves expressions of opinion which must be understood to commit no one but the writer. It will be convenient to begin with some consideration of the latter.

I. THE COURSE OF TRAINING.

(a).—*Professional Studies.*

It is now beginning to be recognised generally that a knowledge of Mathematics is not by itself a sufficient equipment for the business of teaching the subject in a school. A candidate seeking his first post has an appreciably better chance of success, even in a "public" or a secondary school, if his qualifications include, in addition to a good degree, evidence that he has studied teaching methods under proper guidance. The change of opinion upon this subject now in progress implies the recognition that "training" is concerned with something more than the tricks of management needed to keep a class of 80 children in reasonable order on a hot afternoon, and that "method" is not merely a system of devices for insinuating unwelcome facts into dull minds. It implies, at lowest, the recognition that efficient teaching involves sound judgment as to the things which should be taught, and the time and manner of teaching them; and that to leave the young teacher to acquire this judgment by "trial and error" is a plan at once wasteful and dangerous. When this growing opinion has developed into clearness it will probably assume something like the following form.

The aim of the teacher of Mathematics is most profitably regarded as being not so much to communicate certain "truths" as to cultivate a typical form of intellectual activity. Mathematical facts are, of course, the medium in which the teacher works, but (to employ Herbert Spencer's classical antithesis) his ulterior purpose should always be the production of faculty. That is, his success is to be measured by what his pupils can *do* rather than by the reasonings to which he has persuaded them to give their assent. These (even if mastered for reproduction in an examination) may often lie outside the scope of the pupil's genuine spontaneity—may be facts which have escaped conversion into faculty. The mathematical activity, in common

with other features of the pupil's mental being, has a developmental history. It springs from definite embryonic origins and has definite principles of growth. While maintaining its historical identity it may yet exhibit changes so striking as to deserve the name of metamorphoses. Many of the developmental features here in question are common to all forms of intellectual activity; others are peculiar to Mathematics or assume in this region a special character.

The first task of the young teacher during his novitiate should be to assimilate this view of Mathematics and of mathematical instruction. Frequently he is startled if not shocked by it, especially if he comes fresh from advanced studies at the University. He has there learnt to think of Mathematics as containing things new and old, easy and difficult, but all products of a definite kind of "reasoning," with unchanging canons and universal ideals of "rigour." He finds it difficult to exchange this static conception for the dynamic conception of Mathematics as a developing activity. For example, it is not easy to learn that a "proof" may be, for the school teacher, neither good nor bad abstractly; that its value is to be determined largely by reference to the stage of growth of the pupil's mind. Nevertheless the young teacher must be led to think of proof as a critical *function* which plays a rôle of varying character and importance at different stages of mathematical progress.

This argument points to a definite conclusion. The course of training should include general genetic psychology, and, in addition, a detailed study of the mathematical development of the normal boy or girl. In other words, the student in training must think his way afresh through the mathematical curriculum from the genetic standpoint; logic, psychology, and the history of the science being his guides. The problem thus indicated is not to be solved merely by reading and attendance at lectures. Constant practice and observation in a school are at least as necessary. The reason is not merely the obvious one that the student is seeking to acquire an art, not to learn a subject. To try to master the science of teaching, apart from a school, is like trying to learn chemistry without laboratory work. The analogy may be pursued to a further conclusion. The school must be so intimately associated with the training college that its mathematical curriculum and methods of teaching are simply the ideas of the lecturer in being; so that, in fact, the "theory" of the lecturer is an exposition and analysis of the practice of the school. This requirement may be met in two ways. The school may be a demonstration school in which the teaching is inspired and controlled by the training college. Alternatively, the chief mathematical master in the school may assume charge of the instruction of the students in the theory and practice of his department. Upon either plan the conditions of efficiency are the same. The lecturer must be a

teacher who has not allowed his philosophical studies to detach him from the practical situation, or to rob him of his craftsmanship. The schoolmaster must be not merely a teacher, but must have cultivated those "unusually obstinate attempts to think clearly" about his craft which lead to philosophic insight into its problems.

The foregoing considerations have had the specialist teacher in a secondary school chiefly in view, but the principles indicated are of universal application. Teachers preparing for work in an infants', an elementary, or a preparatory school, may do without a mathematical degree, but cannot, without detriment to their future pupils, neglect the critical study of mathematical development. The range to be covered by their theoretical and practical work will, of course, be less than that which the specialist should compass, and their studies must, as a rule, be of a more elementary character than his. Nevertheless the spirit and method of the training must be the same for teachers of all grades. Moreover, it is very important that even the specialist should spend much of his time with classes of younger children. In these classes teaching problems present themselves in their clearest form; the need of the guidance of theory is most obvious, and its soundness can be most directly tested.

(b).—*The Mathematical Knowledge of the Teacher.*

There is an important practical difference between a course of training which includes actual instruction of the student in Mathematics and one which presupposes the necessary knowledge and is concerned only with methods of teaching. The problem before the training college is in the second case obviously simpler than in the former. Nevertheless, a review of the whole preparation of the teacher must in each case take into account the "academic" as well as the "professional" part of his training.

Two questions arise for consideration: (1) Is it, in general, better that the academic and professional work should be taken concurrently or successively? and (2) What are the best courses of actual mathematical study for the intending teacher? To the former question a fairly confident reply can be given. There can be little doubt that the better plan is the one in which the professional course follows the academic. This plan should, if possible, be adopted even where the training college is responsible for both courses of instruction. As was pointed out above, the student as student and the student as teacher have to consider Mathematics from very different points of view. The main interest of the work lies in the one case in synthesis, in the other in analysis. These two interests in the same subject-matter can hardly both flourish at the same time; one of them—generally the professional—is likely to be robbed of its due. The weight of this argument is felt by many who are

concerned with the training of non-graduate teachers, and the opinion is growing that training colleges should become, if not purely professional schools, at any rate, institutions in which the continuation of the student's school education plays a less conspicuous part than at present. Even if it is impossible to dispense altogether with actual mathematical instruction it should eventually be possible to confine it, as Mr. Raymont and others have suggested, to the reconsideration of Elementary Mathematics from the pedagogical standpoint.

The question as to the most appropriate mathematical course is much more difficult. The problem includes several cases: the infants' school teacher, the elementary school teacher (with whom may be bracketed the teacher in the preparatory school and the "non-specialist" teacher of the lower forms of the secondary school), the technical school teacher, and the "specialist" teacher in the secondary school.

The case of teachers preparing for the infants' school is the easiest. If the standard of mathematical proficiency represented by a school leaving certificate could be exacted before admission to college there would be no need to give these students any further "academic" instruction in the subject. The regulation of the Board of Education which has hitherto required all women students to pass an examination in Mathematics has, perhaps, been inevitable in view of the low standard of attainment prevalent among entrants to the training college. On the other hand, a relaxation of this regulation in the case of teachers in infants' schools might, even in the past, have been a wiser policy. For a compulsory course of instruction in Mathematics a further study of the psychology of the subject in the earliest stages might, with great advantage, have been substituted.

On the other hand the question of the best preparation in Mathematics for teachers in elementary schools offers many difficulties. As is shown below, two alternative courses are open to the student—a course for non-university students leading up to the Board of Education's examination, or an approved substitute, and a course for undergraduates leading up to a degree in Arts or Science. In the former examination Mathematics is at present compulsory for all students, although men are required to take a wider course than women. On the other hand, there is no compulsion to take Mathematics as part of the degree course. Thus, in the case of a university student the Government's certificate guarantees mathematical knowledge only up to the standard demanded in the entrance examination to the university. In both cases the minimum mathematical course may be supplemented at the choice of the student. The non-university student may present himself for an "optional" examination; the university student may offer Mathematics as part of his course for the degree. It may be assumed that students who take the more advanced courses seek to qualify themselves for the special opportunities

which are open to the teacher in the "central" or higher elementary schools, and are beginning, with increasing frequency, to be offered even in the ordinary elementary school. It must be added that all students working under the Board's regulations are at present expected to study, theoretically and practically, the methods of teaching Mathematics in schools.

With regard to these arrangements the following important questions arise: (1) Do the compulsory examinations of the Board and the entrance examination of the University (which may be regarded as also the leaving examination of the secondary school) determine suitable courses for the teacher who is to take Mathematics as part of the routine of general class-instruction? (2) Do the optional examinations of the Board and the Mathematics for the University degree tend to produce teachers properly equipped for giving special instruction in Mathematics? The difficulty in answering these questions arises chiefly from the fact that expert opinion upon the aims, curriculum and methods of elementary instruction in Mathematics is at present in a very unsettled state. Changes of a drastic character in the requirements of public examinations are being widely advocated but do not command universal approval. At such a time criticism, whether negative or constructive, must be largely tentative and largely individual. In the case of the compulsory examination of the Board of Education there is the special difficulty that the regulations are now undergoing revision, so that detailed criticism would be unprofitable. In view of this revision the Training College Association have made recommendations to the Board which show the general trend of opinion among those immediately concerned with the training of teachers. These recommendations include the following proposals: (i) that Mathematics shall no longer be a compulsory subject; (ii) that the present "compulsory" and "optional" examinations shall be replaced by a "pass" and an "advanced" examination of which the latter may be taken either in lieu of or in addition to the former; (iii) that the existing difference between the work required of men and women shall be removed; (iv) that students who do not offer Mathematics shall not be required to show a knowledge of the methods of teaching the subject.

The first and last of these recommendations are no doubt intended to meet the needs of two classes of students: those who wish to teach in infants' schools and those (especially women) who have no taste for Mathematics and would be better employed in increasing their efficiency in other departments of the curriculum. The recommendations seem also to imply the correlative assumption that "specialist teaching" in the elementary school is not undesirable and is likely to increase.

The Association have also drawn up alternative syllabuses which have been submitted to the Board as "samples of the kind of syllabus which the Association think desirable" for the

examination of non-university students. They express the opinion that arithmetic "should be treated solely or predominantly from the point of view of method," and it is therefore not included in the syllabuses of Mathematics. One of the alternative sample syllabuses in this subject differentiates between teachers of infants and other teachers and expects on the part of the former a more detailed knowledge of such topics as the psychology of number concepts.

The syllabuses for the "pass" and "advanced" examinations in Mathematics include one which was written originally for publication in this report and is given here as an appendix (page 303). It will be observed that the "pass" course includes the elementary uses of the trigonometrical ratios and the fundamental ideas of the calculus. The "advanced" course includes a common section and four other sections from which a choice may be made. The common section develops further the subject-matter of geometry, algebra (including the calculus) and trigonometry. It also introduces the student to the ideal of Mathematics as a rigid, logical system. The special sections are intended to indicate directions in which students with mathematical tastes, but of differing temperament and outlook, may profitably continue their studies. Groups I. and IV. differ only slightly, if at all, from familiar schemes; in Groups II. and III. an attempt is made to direct the student's studies towards topics which are of undoubted importance and are very attractive to certain types of mind.

It may be suggested at this point that by the addition of further alternative sections some such syllabus as this might meet the needs of teachers in technical schools, and of the schools with a technical bias which are now being developed within the elementary system.

Criticism of the mathematical part of the school examinations which are accepted as entrance examinations by the Universities must also be limited by the consideration that their scope and character is likely to be modified before long. The Mathematical Association have recently issued reports which have already affected materially the syllabus of at least one important examination. It is a fairly safe prophecy that the movement thus initiated will spread, and that its effect will be to widen the range of mathematical topics studied in schools and to make the treatment less "academic." Eventually a knowledge of simple trigonometry and of the fundamental methods of the calculus will, perhaps, be included in all school-leaving examinations. In other words these examinations will define a field of instruction roughly equivalent to that of the "pass" division of the appended syllabus. They will then be acceptable as indicating a satisfactory minimum preparation for the "non-specialist" teacher in the elementary school and the junior classes of other schools. At present they are from this point of view inadequate in several respects. They fail to introduce the student to

mathematical ideas and processes which would often illuminate and enrich his work as a teacher, and they favour methods of treatment which destroy the natural vivacity and fertility of the fundamental mathematical notions.

These criticisms recur when one considers the pass degree course as part of the training of the specialist teacher in the elementary school or the non-specialist teacher of Mathematics in the secondary school. It may be doubted whether it is really suitable for the purpose. As a rule it is too narrow in scope and too "disciplinary" in intention. It aims too much at forming the student's mind and too little at informing his vision of the world around him. There can be no question of the value to the future elementary schoolmaster of contact with university life, teaching, and ideals. It is, however, arguable that the present degree course might be replaced in his case by one more directly adapted to his needs. As far as Mathematics is concerned, such a course, while conserving all the elements of high value in the ordinary curriculum, might well be broader in its range, and richer in concrete topics, such as those suggested in Groups II. and III. of the appended syllabus.

Of the honours degree as a preparation for the specialist teacher in the secondary school little or nothing need be said. The student who has sufficient ability to do well in such a course can without difficulty supplement his academic studies in the directions most profitable to him as a teacher. The important thing in his case is that he should learn to understand and to deal sympathetically with the earlier stages of intellectual development.

II.—THE EXISTING PROVISION FOR TRAINING.

Students are trained for the profession of teaching under various conditions and in various types of institution. Moreover, the courses of training are determined or affected by a number of professional examinations. For the purpose of this report it will suffice to distinguish between courses supervised by the Board of Education and subsidised by Government grants and those which are independent of official support. The latter may be referred to very briefly. They include courses of training designed to meet the requirements of certain external examining bodies. The examinations held by the National Froebel Union and the Cambridge University Teachers' Training Syndicate may be regarded as typical of this class. The former are intended for "teachers in kindergartens and the lower forms of schools," and include two grades. The elementary certificate is sometimes sought as an additional qualification by certificated teachers in the infants' departments of elementary schools. The higher certificate is sought also by women who wish to take up posts in the preparatory departments of secondary schools. Evidence of an "adequate general

education" is required as a condition of admission to both examinations. The higher examination contains a compulsory paper intended to "test the candidate's knowledge of (a) the facts and principles of such elementary mathematics as may appropriately be taught, either incidentally or formally, to children under twelve; and (b) the best methods of teaching the subject." In 1911, out of 1,769 candidates, 155 gained the elementary certificate, and 767 one or other of the two parts of the higher certificate. Almost all the candidates were women. The regulations for the examinations advise (but do not require) a period of training of one year for the lower examination, and of two years and a term for the higher. The examinations are exerting a wide influence upon the teaching of Mathematics in kindergartens, the preparatory departments of schools where women are employed, and in the lower forms of secondary schools for girls.

The examination of the Cambridge Teachers' Training Syndicate is also open to all persons who can present evidence of an adequate general education, although the majority of the candidates now come from training colleges inspected and approved by the Syndicate. The certificate is sought largely by non-graduate teachers in secondary schools, but is also taken as a post-graduate diploma by students working under the regulations of the Board of Education and by others. An important feature of the examination is the paper on Method. It is possible for a candidate to confine himself to questions on the teaching of Mathematics, and to earn an endorsement of his diploma guaranteeing his qualifications as a specialist in the teaching of the subject.* Mention should also be made at this point of the post-graduate courses of training in the Universities (Oxford, London, &c.) which come into the present category because they do not necessarily comply with the conditions of recognition by the Board of Education.

Courses of training "recognised" and subsidised by the Board of Education, fall into two classes: those which comply respectively with the Board's regulations for the training of teachers for elementary and for secondary schools. Courses belonging to the former class fall again under two main headings: the two-years' non-university course and the university course which has hitherto extended over three years, but may now, under regulations issued in 1911, extend over four.

The Board's Regulations define the range and general nature of the training given both in the "University Day" and the "Two-Year" training colleges. The students sign on admission an engagement to serve in "approved" schools for a term of years after leaving college†, receive aid from Government grants towards the cost of their training, and are certificated by

* Similar arrangements hold good in the case of the Oxford Teachers' Diploma.

† Seven years in the case of men. 5 years in the case of women.

the Board when they have completed their course satisfactorily. The University Day Training College must be a department of, or be connected with a University, in which the students must qualify for a degree in Arts or Science, in addition to working through the Board's programme of professional study. Thus, while the training college is responsible entirely for the professional course, it stands merely *in loco parentis* as regards the student's work for his degree. On the other hand, the Board of Education withholds its certificate from the university student who does not, at the end of his course, obtain the degree in addition to satisfying its requirements on the professional side*. The "Two-Year" training colleges differ from the former, not only in giving a shorter course, but in having responsibility alike for the academic and the professional studies of their students. The college leaving examination covers both these departments and is generally conducted by the Board of Education itself, though alternative examinations may be taken. In 1911 there were 20 university day training colleges, containing 1,739 men and 1,529 women students, and 65 non-university colleges (14 day and 51 residential), containing 2,496 men and 6,336 women students.

Candidates for the two-year course are required to have passed either the Board of Education's own preliminary examination or one of a number of alternative examinations. In the Board of Education's preliminary examination, arithmetic is a compulsory subject. If the candidate presents an alternative examination as his qualification for admission, he must in every case have passed in arithmetic, but need not have been examined in any other branch of Mathematics. In the entrance examination no distinction is made between men and women, nor between women who intend to teach girls and those who intend to teach infants.

The Board's Regulations determine the course by prescribing a syllabus for the examination at the end of the student's training. It has already been explained that the scheme for this is under revision. At present the examination in Mathematics must be taken by all students in training, but women are not subject to the same requirements as men. There is a common paper for men and women which contains questions on arithmetic and algebra as far as simple equations of one unknown. Women are not required to face any other examination in Mathematics, but may take a second paper if they wish to obtain a mark of distinction. The second paper, which is compulsory for men, carries algebra as far as the binomial theorem for positive integral exponents and the use of logarithmic tables. It also includes work in practical and theoretical geometry covering the substance of Euclid, Books I.,

* This regulation is relaxed under the new scheme for the four-year course.

IV., and VI. In addition, the candidate is expected to be acquainted with the more usual solid forms. Besides the pass paper, an additional "optional" paper on Mathematics is set. The syllabus includes the co-ordinate geometry of lines and circles, a fuller study of logarithms and of the binomial theorem for positive integers, the solution of triangles by plane trigonometry and a course of mechanics which covers such topics as virtual work, the motion of pendulums and projectiles and circular motion. In most of the non-university colleges the range and standard of instruction in Mathematics is determined by this syllabus. Of the students examined in 1910, 2·9 per cent. passed in the optional paper, 2·1 per cent. being men and 0·8 per cent. women; 5·4 per cent. of the women obtained the mark of distinction for which the second paper in algebra and geometry is required. It should be added that the questions in the examination do not aim merely at determining the candidate's knowledge of the subject matter. They frequently have reference to the methods of presenting it to children. Questions of this kind may be regarded as supplementary to the methodological questions which are included in the general paper on the Principles of Teaching taken by all the candidates.

Students taking the two-year course are required to spend a total of six weeks at least in teaching and in observation of teaching in approved public elementary schools. This part of their work must be supervised by members of the college staff. While the Board's Regulations prescribe that each student shall receive instruction in the methods of teaching Mathematics, they do not require that every student shall teach the subject in his school practice. It is probable, however, that practically all students obtain a certain amount of practice in teaching arithmetic, at any rate, under expert observation and criticism. In former years it was customary to make a rather sharp separation between the academic and the professional instruction of the students. The special tutor in Mathematics confined himself to teaching the students the subject-matter of his syllabus, but took no part in the direction and supervision of their practical teaching. For this purpose and for instruction in method, the students looked to the master of method or "normal master." It is becoming the custom to break down this distinction. The person who teaches the students in the college lecture room now more often follows them into the practising schools and directs and criticises their performance as teachers.

In the case of students taking the university course, no study of Mathematics after admission is compulsory. That is to say, a student who takes a degree course which does not contain Mathematics as a subject need not acquire any knowledge of Mathematics beyond the standard represented by the matriculation examination of the University. Since all matriculation examinations require algebra and geometry in addition

to arithmetic, the effect is, roughly, that for women who take the university course the minimum of Mathematics is considerably greater, and for men rather less, than the minimum looked for in the case of women or men who take the two-year course. As a matter of fact, a very large proportion of the university students take Mathematics in the intermediate stage, and a considerable proportion in the final stage of the degree course.

Under the conditions of the three-year course the university student has generally taken his academic and professional courses concurrently. For example, in the London Day Training College, a school in pedagogy of the University of London, two-thirds of the yearly entrants have taken the courses concurrently. The remaining third, having passed an intermediate examination before admission, have been able to complete the degree course in two years, and, therefore, to give their undivided attention during the last year to their professional work. The expressed intention of the new regulations for a four year course is to make the latter state of affairs the universal rule. The change will be obviously to the advantage both of the degree work and of the professional training. On the administrative side the force of the new regulations is to bring the training department and the rest of the university into closer relations with one another. Under the old scheme the university student was examined in professional subjects by the Board. Under the new scheme the University will itself undertake the examination of the students at the end of their career, the Board's officials acting merely as assessors.

The Board's Regulations require that all students, whether they take Mathematics in the degree course or not, shall be instructed in the art of teaching the subject in the elementary school. The minimum amount of school practice is eight weeks. This professional instruction must necessarily be given by a member of the staff of the training department, though, of course, in special cases this member may also be a teacher of Mathematics in the University. The Board of Education lays down no instructions to govern this pedagogical work except the general one, that the needs of the elementary school are to be borne in mind. The various colleges and departments differ considerably in their practice, but the arrangements in the London Day Training College may perhaps be regarded as typical. The students of this College receive a course of lectures extending over at least two terms and dealing with the general principles of mathematical teaching with special reference to the more elementary parts of the subject. The bulk of these lectures are attended by all students. The last few are intended for students who aim at becoming special teachers of Mathematics in the higher types of elementary school.

In addition to the lectures, members of the staff specially concerned with Mathematics give a series of demonstration

lessons in the elementary schools attached to the College. All students who are paying special attention to the teaching of Mathematics have the opportunity of following these demonstration courses. Finally, students are placed in rotation in regular charge, for at least a term, of classes in one or other of the demonstration schools. Here they teach, under supervision, from a syllabus in Mathematics, drawn up by the College, and prepare their lessons daily under the guidance of the lecturers.

The Board of Education also makes grants to institutions which train teachers for service in secondary schools. The chief conditions upon which students are "recognised" for the purposes of these grants are: (i) they must be graduates of a University or have equivalent qualifications; (ii) the course must extend over at least a year; (iii) the student must spend at least 60 school days "in contact with class work, under proper supervision," and two-thirds of the practice must (in general) be taken in a secondary school; (iv) the student must include in his course a special study of the teaching of one definite branch of the curriculum of a secondary school—which branch may, of course, be Mathematics. The Regulations do not prescribe any course or tests of efficiency, but the courses usually followed are those laid down by the various Universities as their requirements for their examinations for a teacher's diploma.

The arrangements made to meet the needs of students who select Mathematics as their special professional subject are so various that no general description of them is possible. They range between the alternatives indicated on page 292. The plan described in the case of the elementary students of the London Day Training College is probably followed in principle very widely.

It should be added that the student who has been trained in the elementary department of a college is permitted by the terms of his undertaking to serve in any secondary school which is in receipt of public funds. As a consequence of this freedom many secondary schools are at present drawing their specialist teachers of Mathematics and other subjects from the University Day Training Colleges.

The organised courses of training described above are, in many places, supplemented by courses in the teaching of Mathematics, provided either by the local Universities, with or without the financial aid of the local Education Authority, or by the Education Authority itself. Thus, in London, in the session 1910—11, courses of lectures upon the following subjects were organised by the London County Council and delivered by teachers of the University in its various schools:—

"The Teaching of Mathematics."

"The Rational Teaching of Practical Geometry."

"The Teaching of Number to Young Children."

“Practical work in the Arithmetic Lesson” (for head teachers of elementary schools).

“Practical Work in the Mathematics Lesson” (for masters in boys’ secondary schools).

“The Teaching of Algebra in Girls’ Secondary Schools.”

T. P. NUNN.

APPENDIX.

PROPOSED SYLLABUS OF MATHEMATICS FOR A TWO-YEARS’ COURSE.*

Practical measurements and the study of models should accompany the work at all stages.

I.—First Part of Syllabus. For all Students who take the Subject.

(A) GEOMETRY.

NOTE.—In the proofs of theorems and deductions from them the use of hypothetical constructions is permitted. A figure accompanying a demonstration should be drawn neatly, but need not be drawn accurately unless an accurate figure is expressly required. Nevertheless, candidates must be prepared to execute skilfully any figure and any construction assumed in the proof of a theorem or deduction, and to give a demonstration of its validity. Any theorem may be assumed in any such demonstration.

The theorems of Section 1 may be assumed in the demonstration of any theorem in the subsequent sections. Any demonstration of a proposition will be accepted which appears to the examiners to form part of a logical order of treatment of the subject, but candidates are recommended to base their demonstrations, as a rule, directly upon the theorems of Section 1.

In dealing with proportion it may be assumed that all magnitudes of the same kind can be treated as commensurable.

Theorems on the following subjects and the more important deductions from them :—

1. The conditions of congruence of plane triangles. The conditions of perpendicularity and of parallelism between two straight lines, a straight line and a plane, and two planes. The relations between the internal and external angles of plane rectilinear figures. The sum of the external and of the internal angles of such figures. The definition and conditions of similarity of plane triangles.

2. Application of the properties of similar triangles to simple problems in measurement and surveying, including numerical problems.

The conditions of similarity of plane polygons. Graphic and numerical problems based upon the linear relations between such figures,

The constant ratio between the sides about corresponding angles in similar right-angled triangles. Use of tables of tangents, sines and cosines in determining the sides or angles of a right-angled triangle from given data. Application to simple problems in mensuration and surveying, including the determination of the angle between two planes.

Numerical problems in the solution of triangles other than right-angled triangles. (No formulae will be required. The problems will be restricted to those which may be solved by regarding the given triangle as the sum or difference of two right-angled triangles.)

* See p. 296.

3. The concurrence of the bisectors of the sides and angles (internal and external) of a triangle, of the perpendicular bisectors of the sides, and of the perpendiculars from the vertices upon the sides. The more important circles connected with the triangle. Ratio of sections of the side of a triangle by the bisector of the opposite (internal or external) angle.

4. The circle as a symmetrical figure. The fundamental chord, tangent and secant properties of the circle. The relations between an angle at the circumference and the angle at the centre standing on the same arc. The properties of polygons described within and about a circle. The circle as a "limit" between the series of internal and external polygons. Circles as similar figures. Possibility of calculating the ratio of circumference to diameter by consideration of the perimeters of the internal and external polygons. (The actual calculation will not be required.)

5. The areas of rectangles, trapeziums, parallelograms, triangles. Numerical problems on the areas of these figures, and of figures resolvable into them.

The relation between the areas of similar rectangles, similar parallelograms, similar triangles, and of figures resolvable into these.

The area of the circle.

The theorem of Pythagoras.

Simple numerical and graphic problems concerning the edges, surface areas, and cross-sectional areas of right and oblique prismatic and pyramidal solids.

The properties of rectangular figures which can be expressed algebraically. The following identities:—

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$a^2 - b^2 = (a + b)(a - b)$$

6. The volume of a right solid of uniform cross-sectional area. The equality of volume between such a solid and the corresponding oblique solid of the same vertical height standing upon the same base.

The relation between the volumes of similar solids.

(B) ALGEBRA.

1. The expression of arithmetical generalisations in formulæ. The interpretation of formulæ. Substitution in formulæ. The "graph" as a subsidiary means of analysis and of generalised statement. Simple applications of "graphs" to the treatment of statistics.

2. Factorisation and the reduction of fractions as processes subsidiary to the use of formulæ. Meaning and use of the processes of multiplication and division of algebraic expressions.

The rule for the extraction of a square root in arithmetic. Simple methods of approximating to the values of squares and cubes, square roots and cube roots.

"Changing the subject" of a formula. Simple equations of one or more unknowns.

3. The uses of "directed" (*i.e.*, positive and negative) numbers. Algebraic addition and subtraction. The multiplication and division of directed numbers. Applications in formulæ, including formulæ for the terms and sum of arithmetical progressions.

Extension of the rules of multiplication and division to algebraic sums and differences.

Division as the process of "expanding" a fraction: uses in approximate computation. The expansion of the fraction $a(1+r^n)/(1+r)$: geometrical progressions (including the case when r is less than 1 and n is large without limit).

4. Direct proportion: the straight line through the origin as its graphic symbol. The general "linear relation" $y = ax + b$ in its various cases. The linear relation as a commonly occurring law of increase and decrease: applications to simple problems of motion, simple interest, &c.

Inverse proportion and its graphic symbol. The general "hyperbolic relation" $y = a/(x + b) + c$. Simple applications.

The relation $y = kx^2$ and its graphic symbol. The general "parabolic relation" $y = a(x + b)^2 + c$. Maximum and minimum values of parabolic expressions. Solution and formation of quadratic equations. Simultaneous equations involving one linear and one quadratic equation.

The "exponential relation" $y = ar^x$ when x is a positive integer. The exponential curve as the graphic symbol of the relation when this restriction is removed. The use of negative and fractional indices to describe ordinates of the curve. The exponential relation as a commonly occurring law of increase and decrease. Its applications to the study of compound interest, mortality returns, insurance, &c.

The logarithm as the inverse of the exponential.

The use of tables of logarithms.

5. Proof that for all values of n (the number of terms in the numerator and the denominator being the same) :—

$$\frac{0 + 1 + 2 + 3 + \dots + n}{n + n + n + \dots + n} = \frac{1}{2}$$

Application of this theorem to problems, including the determination of the lengths of lines, of the areas of plane figures and of the volumes of the wedge, the paraboloid of revolution and other solid figures.

6. Proof that by taking n large enough the following equality becomes true to any specified degree of accuracy :—

$$\frac{0^3 + 1^3 + 2^3 + \dots + n^3}{n^3 + n^3 + n^3 + \dots + n^3} = \frac{1}{4}$$

Application of this theorem to problems, including the determination of the lengths of lines, of the areas of plane figures and of the volumes of right and oblique pyramids, of right and oblique cones, and of spheres.

7. The notions of (a) the average (b) the limiting rate of change of a continuously varying quantity. Graphic determination of the limiting rate. Proofs, that if the magnitude of a quantity is given by the law (i) kx or (ii) kx^2 or (iii) kx^3 , the limiting rate of change is measured by (i) k or (ii) $2kx$ or (iii) $3kx^2$ respectively. Application of these results in simple problems.

II.—Continuation for Advanced Students.

(A) GEOMETRY.

1. Geometry as a logical system: the functions of axioms and definitions. Axioms of parallels. The meaning of Euclid's constructional propositions. Incommensurable quantities; Euclid's definition of proportion.

2. Systematic treatment of the subject matter of Euclid, Book XI., upon a basis of axioms and definitions.

3. Fundamental properties of vectors. Vector addition and subtraction. Simple applications including the calculation of vectors by trigonometry.

(B) ALGEBRA.

1. Algebra as a logical system: the axioms of Algebra., The laws of signs. Irrational numbers and their connection with Geometry. Simplification of surds in easy cases. Easy equations involving surds. The laws of indices. Interpretation of negative and fractional indices based on the law of indices. Straightforward examples on indices. Formal proofs of the properties of logarithms.

2. The binomial theorem for a positive integral exponent. Proof by mathematical induction. Proof by induction that the theorem holds good when the index is a negative integer. Verification (by means of the ordinates of the curve $y = (1 + r)^x$ or in other ways) that it may also be extended to fractional indices.

3. The expansion of e^x . Natural logarithms; their connection with logarithms to other bases. (A detailed knowledge of the calculation of logarithms is not required).

4. The limiting value of the ratio of a change in each of the following functions to the change in the variable which causes it: e^x , $\sin x$, $\cos x$, e^x , $\log x$; x being the independent variable. The same when x is itself one of these functions of an independent variable x . Application of the results to problems including the determination of speeds, accelerations, lengths, areas and volumes in simple cases. (The ordinary notation of the differential and integral calculus will not be insisted upon. Any mode of stating the arguments will be accepted if it is clear and accurate.)

(C) TRIGONOMETRY.

The trigonometrical ratios of angles less than 180° . Application to the solution of right-angled triangles. Solution of other triangles by reduction to the sum or difference of two right-angled triangles. Relations between the ratios of one angle. Simple trigonometrical identities. Simple trigonometrical equations.

2. The circular measure of an angle. Angles of indefinite magnitude. Graphs of the trigonometrical functions when the angle increases indefinitely. Pendulum motion. Simple harmonic waves. The angles corresponding to a given value of a trigonometrical function.

3. The sum and difference formulæ for sines and cosines. Application to simple identities and problems.

4. Formulæ for the cosine and sine of an angle of a triangle in terms of the sides. The in-radius, ex-radius, and circum-radius of a triangle.

5. Study of the forms produced by the superposition of simple harmonic waves in the same straight line and by simple harmonic vibrations at right angles to one another.

(D) ONE OR MORE OF THE FOLLOWING GROUPS OF SUBJECTS:—

(*N.B.*—*Students should aim at thoroughness rather than width of knowledge.*)

Group (I.)—Pure Mathematics.

1. The simpler modern geometry of the triangle. Tests of collinearity and concurrence. Simple treatment of harmonic ranges. Poles and polars of a circle.

2. Fundamental chord, tangent and normal properties of the parabola, ellipse, and hyperbola, established by geometrical or algebraical methods (including the use of differentiation).

3. The graphic representation of simple algebraic functions. Determination of maximum and minimum values in easy cases. Determination of tangents and normals at given points of curves.

4. Complex numbers: their geometrical significance. De Moivre's Theorem. Simple applications. Imaginary roots of equations.

5. Convergence of series: simple tests. Determination of the coefficients of an expansion by differentiation. Application to the expansions of $(1+x)^n$, a^x , $\log(1+x)$, $\sin x$, $\cos x$.

Group (II.)—The Mathematics of Geography and Astronomy.

1. The elementary geometry of the sphere. Relations between a spherical triangle and its polar triangle. Properties and relations of lines of latitude and longitude. Fundamentals of the theory of plane sailing.

2. The gnomonic polar projection of a sphere and its properties. Graphic solution of simple problems in great circle sailing. Solution of such problems by the use of trigonometrical formulæ. (Only the simplest and most commonly used of these formulæ will be required.)

3. The simple theory of map projections—including the cylindrical equal area projection, the sinusoidal projection and Mercator projection. The use (but not the calculation) of a table of meridional parts.

4. The celestial sphere and its circles. Application of geometry and trigonometrical formulæ to simple problems in astronomy, especially those of practical importance.

Group (III.)—The Mathematics of Finance, and Statistical Investigation.

1. The theory of compound interest and discount: nominal and effective rates of interest (including continuous interest). Applications to the Registrar-General's returns.

2. The theory of Annuities-certain; the loans of Public Authorities, Building Societies, &c. Government Bonds. Leases. Reversions.

3. Various methods of graphic representation of statistics. Meaning and use of the terms median, mode, quartile, &c. Variability. Standard deviation.

4. Probability. Calculations of probability in simple cases from statistical data. Probable error. Permutations and combinations. Normal statistical distribution.

5. Fundamentals of the theory of Life Insurance, &c. Contingent Annuities.

6. The study of the correlation between two variables. Simplest methods of measuring correlation. Easy applications.

Group (IV.)—Mechanics.

1. The analysis by statical methods of the use and behaviour of common structures and machines. Friction. Hooke's Law, Young's Modulus.

The vector law of forces; moments; work; virtual work; centres of gravity.

2. Analysis of rectilinear and curvilinear motion under hypothetical conditions. The motion of projectiles and pendulums under gravity.

3. The mutual action of colliding bodies: mass; momentum; impulse; force. Newton's Laws: problems involving them in direct application.

4. The dynamics of circular and of harmonic motion.

5. Dynamics of rotation about an axis. Moments of inertia.

RESEARCH AND ADVANCED STUDY AS A TRAINING FOR MATHEMATICAL TEACHERS.

To foreigners who have attended the Mathematical Congresses of 1904 and 1908 it must have come as a surprise that so few English mathematicians took part in these gatherings, and the surprise will have been greater in view of the excellency of the papers contributed by Englishmen, both to these proceedings and generally to English and foreign mathematical transactions and journals. It would be a natural inference that England is capable of producing mathematicians whose work is not inferior in quality to the corresponding output of any other nation, but that interest in these investigations is confined to a small circle of Englishmen, and that outside this circle there exists a widespread want of appreciation of the value of original investigations in Mathematics often amounting practically to a failure even to recognise the subject as a science.

A further example of this spirit of apathy and indifference is the hostility of certain classes of "practical men" to mathematicians; in proof of the existence of this feeling it would be easy to produce circumstantial evidence. Where I have referred to the matter in print and advocated the claims of the English "unpractical man," I have had letters from many people quite upholding my views, but in some cases expressing the opinion that it was hopeless to carry on the campaign against what I have called "England's Neglect of Mathematics."

On the other hand, the recent foundation of a Mathematical Institution at Göttingen with a large endowment contributed partly by the State and partly by commercial firms, affords evidence that in Germany the claims of the mathematician are widely appreciated, and while it is undesirable to attach too much importance to statements made in private conversation, I have been told by Germans that any notes or articles that I have written advocating the claims of mathematicians would be regarded in their country as superfluous and unnecessary, and as only stating what everybody knew.

There appears to be a widespread belief, even among many mathematical teachers in this country, that Mathematics is of no use except for the mental training and discipline which its study affords. Claims of this kind are, however, apt to be met by counter-claims as to the educational value of the study of even such subjects as Latin and Greek declensions and irregular verbs, which present no very obvious applications to the practical requirements of modern civilisation.

The modern movement for reforming the teaching of elementary mathematics and bringing it into closer touch with instruction in physics will probably do much to dispel this failure to appreciate the usefulness of Mathematics, and the published reports of the Mathematical Association show that this aspect of "England's Neglect of Mathematics" is enjoying the attention of a competent and active band of workers.

A matter of no less importance, and one which properly forms the subject of the present report, is the wide gap which in our country separates the educational from the research aspect of Mathematics. One of the most noticeable features of recent educational movements has been the recognition of the value of "research," not so much for the sake of the scientific results to which it leads, but rather for the educational training it confers on those engaged in its pursuit. There is a general consensus of opinion that to produce the best teachers their course of study ought to include a certain period, say a year, devoted to post-graduate study and research. We see thousands of pounds being spent annually in "research studentships," and in subjects like physics and chemistry there is usually no lack of candidates able and willing to fill the duties attached to such appointments, often even before they have completed their undergraduate course.

It is of course impossible to fix the exact proportion of these post-graduate awards, which might with advantage be allocated to students of any particular branch of science. It may, however, I suppose, be assumed that mathematical science would be over-represented if as many research studentships were given for Mathematics as for the two subjects physics and chemistry taken together, while, on the other hand, the proportion of mathematical awards would be conspicuously low if the total number of students receiving them were less than *half* the corresponding number of students working in *one* of these two subjects.

When we come to examine actual facts, we find that research studentships of the ordinary modern type are rarely, if ever, awarded for Mathematics (no instance had come under my notice at the time of writing), that colleges still exist where research is encouraged in many subjects, but mathematical teaching is entirely confined to an examination syllabus, and that where mathematical research is carried on very few people are aware of its existence, and cases occur in which educational authorities fail even to make proper enquiries into the matter.

It need hardly be pointed out that these remarks do not apply in the same way to the two older universities. At Cambridge many students are stimulated to engage in what is virtually a post-graduate research course by the Smith's Prizes, while the system of College Fellowships is undoubtedly mainly

responsible for the output of original papers and treatises which have hitherto maintained the high prestige of British mathematicians among the mathematicians of the world.

The success of the College Fellowship is probably due in large measure to the following conditions:—(1) Fellows are often elected at a time when their knowledge has had time to mature, and they have recovered from the strain due to examination stress; (2) no conditions are attached to them, consequently a mathematician may embark on an investigation the results of which are uncertain, and may possibly end in failure, but in the course of six years he has even then every probability of arriving at some valuable contribution to science; (3) a Fellow is not disqualified from combining teaching work with his Fellowship, a combination not only needed to prepare him for the time when his term of office expires, but which also affords some protection against the excessive mental strain caused by over-specialisation.

But the tendency of modern legislation has been in the direction of disendowing, or at least reducing the endowments of the College Fellowship, and if it be true, as now seems to me probable, that the modern Research Studentship has failed to promote research in this important branch of modern science, then the outlook is a serious one.

The disendowment of the Oxford and Cambridge Fellowships, or even the reduction of the proportion of these Fellowships awarded to mathematicians, will certainly reduce the quantity of our output of original work. This output is probably less than that of Germany, while a study of the Bulletin of the American Mathematical Society will show how America is pressing forward in the development of university research, particularly in such studies as the Theory of Groups, the neglect of which in England was recently referred to by Burnside⁷

In any case, however, the College Fellowship rarely reaches the large body of students who graduate in Mathematics and then go into the teaching profession without undergoing the research training that their colleagues in physics have enjoyed.

Admitting this failure, what is its cause? Is it (1) because research in Mathematics is not needed, or (2) because there is no demand for mathematical teachers with research degrees; or (3) because there is an essential difference between mathematical research and research in other subjects; or (4) because our graduation courses in Mathematics fail to qualify our students for research; or (5) because the conditions of award of these scholarships are at fault?

In regard to (1) it may be at once stated that mathematical research was never more needed than it is at the present day. Not only have new lines of development been opened for the

* London Mathematical Society: Proceedings, 1909, vii. (London: Francis Hodgson.)

pure mathematician in such subjects as Theory of Groups, and still more recently Integral Equations, but many new branches of study have been brought into the domain of the mathematician, whose connection with exact science would have been ridiculed thirty years ago. We may instance in particular the mathematics of statistics and its application to biology, life insurance, politics and eugenics.

In regard to (2), the small demand which exists for teachers, it is true that vacancies in our universities and university colleges occur but very rarely, and a man with a brilliant mathematical degree may wait five years or more before even an assistant lectureship at £20l. a year becomes vacant. But the number of existing appointments of this class is nearly the same for Mathematics as for other branches of science, such as physics and chemistry, in which the number of students qualifying in research is considerable, so that the difference cannot be accounted for in this way. It is true that of recent years a fair number of new appointments have been created in experimental science, and these have absorbed some of the best "research students"; but, on the other hand, it is certain that the large bulk of the remainder have to teach in schools, and when a state of equilibrium has been reached between supply and demand, it is improbable that the prospects of a specialist in experimental science will be materially better than those of a student who has devoted his attention to pure science.

It follows, then, that we must look to our schools to provide employment for many of our best graduates not only in Mathematics, but also equally in other branches of science. Thus we are led to inquire whether the value of research as a training for teaching of a somewhat elementary character is sufficiently appreciated by those who have control of scholastic appointments, and in particular whether Mathematics is in this respect worse off than other subjects.

Unfortunately it would appear that many of our educational authorities actually discourage research, at least in Mathematics, and even in elections to university chairs it may happen that a really eminent mathematician is passed over in favour of a candidate possessing qualifications of a very ordinary character.

On the other hand, cases frequently occur where the mathematical teaching of our schools is placed in the hands of graduates who have only taken Mathematics as a subsidiary subject in their university course, and who have devoted their best years of study mainly to other subjects. It is true that the immediate results of such an appointment are not always unsatisfactory. A teacher whose knowledge is limited no doubt finds it easy to keep his pupils down to the level of knowledge required for their next examination, and the pupils very often score high marks at the commencement of their

college career, but when it is attempted to continue their course of study they seem to come to a dead standstill and to be unable to proceed further. They have, in fact, been confined so completely to going over and over the same limited range of subject-matter that *they have completely lost all power of assimilating new ideas.*

Now, the object of every education should be to turn out not only brilliant scholars but efficient citizens. In the battle of life problems are constantly presenting themselves which every man has to solve to the best of his ability, not only without the aid of text-books or teachers, but often in the face of opposition. In other words, he is called on to perform *original researches* under conditions far more unfavourable than those which he has encountered in his school or college career. It is essential that he should be trained by a teacher who has himself been engaged in research or in exploring some difficult region of advanced study, either without the help of a guide or under conditions which leave a great deal to individual effort.

I think it may be laid down as a general rule that when a teacher has had his training confined to a certain syllabus his pupils are liable to come to a standstill as soon as they go beyond that syllabus. Thus, if boys are taught by a master who has only passed a matriculation examination they will have difficulty in subsequently going beyond the matriculation syllabus; similarly a teacher who has qualified in Part I. of a Tripos will find most of his pupils shirk off when it comes to studying for Part II.

If any further claim is needed for research and advanced study as a training for the teacher, we have only to study the remarkable movement which has taken place in our country during the last decade for the simplification and improvement of the teaching of elementary mathematics, and we find that the most ardent advocates of common-sense and practical methods of teaching have been Smith's Prizemen and others actively engaged in mathematical research.

On the other hand, a man who has given four of his best years to qualifying for both parts of the Tripos, and has subsequently performed research in the form of a Smith's Prize essay, often finds it rather hard to have to be contented with a poorly-paid appointment, involving long hours of elementary teaching with dull or lazy pupils. And there does appear to be some difference in this respect between the mathematician and the research student in experimental science. The latter has certainly had a good bit more of routine and plodding work in attending to minute details in the arrangement of apparatus, waiting about while experiments are in progress and overcoming failures, and he may perhaps be more patient with his pupils in consequence.

This brings us to the third question, namely, whether a difference exists between research in Mathematics and research in experimental science, and closely connected with this is the fourth question as to whether our graduation courses fail to qualify candidates for research in Mathematics, while the same shortcoming does not exist to the same extent in other subjects.

Now experience shows that students who have obtained a First Class in Part II. of the Mathematical Tripos under the old regulations have often produced excellent researches in competition for the Smith's Prizes, and that they frequently continue to make valuable contributions to mathematical science even when the inducement no longer exists. In this case the necessary preliminary training has been afforded by the course of specialised study required for Part II. which brought the student sufficiently near to the forefront of our existing knowledge in at least *some one branch* of Mathematics to enable him to engage in some original investigation, at least when a subject had been suggested to him by his coach or lecturer. In this case "research" involves the creation of new ideas, the plunging into an unknown region, the following up of a logical sequence of ideas culminating in conclusions previously unknown, the development of fresh fields of study. Such research is appropriately described as "original work." It involves continuous expenditure of thought on the part of the investigator, and there is scarcely a minute when the work can proceed automatically without absorbing his concentrated attention. But, as we have seen, it is generally necessary that the student should have spent one year in specialised study before he can reach the ground he proposes to explore.

I very much doubt whether Part II. under the new regulations will produce such satisfactory results in preparing candidates for the Smith's Prizes. The necessity of giving a large amount of attention in the final year to work of a general character for purely examination purposes will be likely to debar candidates from acquiring that thorough knowledge of a special branch of study that is necessary to enable a candidate to engage in original work.

It is certain that at least in many other branches of science students do not require such a severe preliminary course of study to qualify them for research. For example, in the University of Wales the official published syllabus defines the honours course in most branches of science as nearly identical with that required for a pass degree. In Mathematics, on the other hand, not only are separate pass and honours degree courses in Pure and Applied Mathematics respectively described in the calendar, but the syllabus for honours in either branch consists entirely of a list of new subjects. Yet if a student who has taken all these courses wishes to embark on research, say in Pure Mathematics, he frequently finds that his only

chance of discovering something new would be by giving up about another year to the further study of some special branch of Mathematics, on the chance that at the end he would find some line of investigation which had previously not been worked out. It is small wonder that under these conditions many candidates who have really some aptitude for Mathematics turn to experimental science, in which, according to printed documents, they are frequently put on to research even before completing their examinations.

The advantage offered to the research student by such subjects as physics, chemistry, botany or zoology is that a large proportion of his work consists in performing experiments or making observations. A student who does not possess the creative power necessary to advance any branch of mathematical physics can always be put on to determine the values of some *constants* or *coefficients* and he thus gains a valuable educational training in the use of apparatus. There would appear to be almost unlimited possibilities for this kind of research. In choosing new material for experimenting we have a considerable number of independent variables at our disposal. Such *constants* or *coefficients* may depend on the substance experimented on and on its physical state; moreover, when their values have been determined by one method, a useful exercise is obtained by determining them by another method that had previously not been used, or even by repeating the observations of a previous experimenter.

It is clear that "research" of this kind differs in many important respects from the original work undertaken by the Smith's Prizeman. It may be tedious and wearisome, it may involve the overcoming of serious difficulties when any particular apparatus fails to act, but it does not involve that *continuous expenditure of concentrated thought* required for mathematical original work.

It would hardly be correct to say that there are two kinds of research. It is more correct to say that there are two methods of conducting research, namely, the deductive and the experimental methods, and that these occur in varying proportions according to the nature of the problem under consideration. Mathematical research as a rule depends almost exclusively on the first method, while in subjects like physics it appears possible to find material for study which combines the two methods in almost any desired proportion, or even involves either the first or the second method alone.

We may thus conclude in answer to (3) that a considerable difference exists between original work in Mathematics and research in other branches of science, and in answer to (4) that our graduation courses in Mathematics often fail to provide the necessary preliminary training for research to an extent which does not find an equivalent in other subjects.

Passing on to the fifth question, whether the conditions of award of research scholarships are unfavourable to the mathematician, let us take the case of two candidates, one of whom proposes to study the Theory of Groups or Integral Equations with a view to further investigations, while the second proposes to determine certain thermal conductivities by a new method. There can be no doubt as to which man is most wanted in England at the present time; we ought to do everything in our power to induce more of our mathematical graduates to take up Theory of Groups and Integral Equations, subjects almost unstudied in our country; moreover we are at least better off in having a National Physical Laboratory where the latter investigation could be carried out if other means failed. Now the physicist, whose research has been suggested by his professor, submits a definite statement of what he is going to do; while the mathematician, owing to the very difficulty of his subject, is unable to state at the outset to what conclusions his researches will tend. Consequently the candidate who has undertaken the simpler and more definite task is frequently in a position of advantage.

A further disability of the mathematician is that these research studentships generally require the investigation to be carried out in some definite laboratory, while the mathematician has to do most of his work by himself. It is true that his work ought to be performed under the guidance of a competent authority. This would give rise to no difficulty in a country like Germany, where he would find in some university a professor whose recognised duty was to further research in his special subject of study. In England many of our mathematical professors are too much occupied with undergraduate teaching to be able to attend to the interests of such students, while several of our best mathematicians have had to give up teaching work in order to find time for research, and they could not possibly be expected to undertake the supervision of the work of post-graduate students unless adequate remuneration were offered to them.

For this reason I consider that the conditions of award of research scholarships handicap the mathematician, and not improbably they equally handicap the student who wishes to do research in such a subject as *mathematical* physics or philosophy. They tend to give an advantage to the student who proposes an easy research on well-known ground as against the candidate who wishes to strike out on a more original and difficult exploration.

Indeed the proposition may be advanced at least as a subject for debate: "That the modern system of research scholarships is killing the study of Mathematics in our country by draining our mathematical schools of their best scholars." This is perhaps such an extreme view to take that

I am not at present prepared *definitely* to support it, but there seems certainly something to be said for it.

I think an enquiry might with advantage be instituted regarding the conditions of award of research scholarships with the object of ascertaining whether these conditions tend to the under-representation of research in pure science. In the meanwhile a useful purpose will be served by enquiring (a) whether it is possible to find material for research in Mathematics which shall be as accessible to students of moderate ability as experimental work is in other branches of science; (b) whether our graduation courses may be remodelled so as better to qualify mathematical students for original work.

Now it is probable that if the desirability of including research in the training of our mathematical teachers were generally admitted, a good many subjects of investigation might be unearthed whose difficulty was not so great as to render them inaccessible to an ordinary student. But until a number of people interested in different branches of Mathematics make a serious attempt in this direction the possibilities must remain limited.

So far as Applied Mathematics is concerned, I can suggest an almost unlimited supply of material for research of all grades of difficulty which is at present awaiting development, in the applications of dynamical principles to the motions of aeroplanes. For the plodding student with little creative ability there is just now plenty of work to be done in numerical stability-calculations of a perfectly straightforward character; while for any student capable of concentrated original thought the scope of the investigations is proportionally extended. The many-sided character of the problems requiring solution gives to this subject of study a value as an educational training which is probably not equalled in any other field of research available at the present time.

The recently developed theory of Integral Equations seems to open up considerable potentialities for an English student of moderate ability, especially if his training has been partly mathematical and partly physical. To obtain access to this subject it is essential that he can read German. The student whose knowledge of languages is limited to compulsory Latin and compulsory Greek is thus excluded.

On the other hand there are undoubtedly many branches of Mathematics, especially Pure Mathematics, in which it is far more difficult to suggest subjects for research than to perform the researches themselves. Further, we have to face the difficulty that it is impossible for a student who knows nothing about groups or the calculus of variations to embark straight away on research in one of these subjects. It thus follows, I think, that unless the facilities are increased in our universities for *advanced study and specialisation* as distinct from original investigation of new theorems, many of the most important

branches of modern mathematical science will be altogether unrepresented, and many graduates of our universities will be as far behind the times as a graduate in physics would be who had never heard of the telephone, electrical oscillations, wireless telegraphy, Röntgen rays, radium or electrons.

To the average student who wishes to qualify for teaching Mathematics, the class of investigations to which the foregoing remarks apply will probably present far greater difficulties than would be encountered by a science student of equal ability who embarked on the determination or redetermination of a *constant or coefficient*. Thus arises the necessity for suggesting individual work of an independent character on less original lines. Here again suggestions to be effective should be derived from a large number of independent sources; meanwhile the following may be quoted as instances of what is desirable. Some of the suggestions are due to the Committee.

(a) "Collaboration in the working of incidental details," as suggested by Sir George Greenhill in his address to the London Mathematical Society for 1892,* when he remarked that Professor Tait made a feeling complaint somewhere that in looking back at the time spent on working out a new development he felt that a great part of it had been occupied on analysis that might well have been turned over to a well-trained student.

This kind of collaboration is good for the professor, good for the student, and good for the research. It formed the basis of the original Bryan-Williams paper on longitudinal stability of aeroplanes, an investigation which was cut short by Mr. Williams' subsequent appointment to a studentship in physics. The main difficulty is that the method assumes the co-existence of a professor interested in a piece of original work and a student able and willing to collaborate. Given the former, it is not easy to secure the latter. In "Stability in Aviation," the collaboration was afforded not by a student, but by Mr. Harper, Assistant Lecturer. The method is commonly used in researches in experimental and observational science, where even undergraduates are pressed into the service. One great difficulty in Mathematics is that when students have completed their examination course, they leave College, and indeed honours graduates of our modern universities sometimes go up for a Cambridge Tripos, instead of attempting to do research.

Another difficulty is that while most average students can work out text-book examples by the dozen when the answer is given, they utterly fail to answer the simplest question where the conditions of a problem are given and they are asked

* London Mathematical Society's Proceedings, 1892. (London: Francis Hodgson.)

to find out what happens. The obvious remedy is to send them back again and again till they succeed. This is evidently impossible in a graduation course in which an extensive syllabus has to be covered in a limited time. Surely the prevalence of this type of mind is the strongest argument for including in our mathematical curricula work involving research or individual initiative and not mere slavish adherence to text-books. How can a man be a good teacher who cannot see the way out of his pupils' difficulties at a glance?

The general conclusion is that the collaboration system is of very great value, but owing to the existence of external influences over which the mathematician has little control it can rarely be adopted at the present time. If this view is correct, the removal of these obstacles is a matter of urgency.

(b) "The giving of an account of a mathematical topic in a form both accessible and suited to those whose main interest is in different but allied subjects; *e.g.*, a sketch, to be made as easy as possible, of the mathematical theory of correlation."

Or again "Integral Equations," a subject which has crept into our midst almost unnoticed.

There is an almost unlimited scope for individual effort in this direction. The publication of such papers would do much to awaken the popular interest in Mathematics that is so badly wanted in this country. A thesis on correlation would fill a much felt want.

Now here, again, a great many difficulties arise at the outset if this work is to be efficiently undertaken by the class of post-graduate students contemplated in this report, and not to place a further burden on the small band of eminent English mathematicians now working by themselves. And, again, the existence of these difficulties constitutes the strongest argument for their removal. The difficulties may be classed as follows:—

(i) Is it certain that the preparation of such a thesis would be regarded as "research" qualifying for a post-graduate scholarship. "Research" in modern times has become a vague and elastic term, which is sometimes used to cover up rubbish that would be better consigned to the dust-heap. But it is certain that the initial stages of a course of this kind would be more correctly described as "advanced study," and a board of electors to a scholarship might argue that the student was doing nothing new, although they would admit that a redetermination of a constant or coefficient by a well known method amounted to the performance of a new experiment. It is clear, however, that the production of a thesis of the kind suggested involves an expenditure of original effort and initiative quite out of proportion to that required for the repetition of an experiment, and in view of difficulties such as that suggested

above, it becomes essential that advanced post-graduate study should be recognised as a qualification for student-ships which are at present awarded for what is now described under the ambiguous name of "research." It is, indeed, possible to exaggerate the claims of research and to neglect much valuable work that has been done already in the rush to find and perform something new.

(ii) To undertake this work efficiently the post-graduate student must be able to read French and German. Compulsory Latin and Greek will not help him at this stage. The remedy is obvious.

(iii) To write a simple and popular account of a highly intricate and technical investigation is an exceedingly difficult task. When the author of a paper has proved a result with the use of elaborate formulæ it is difficult to see how any less qualified author can be expected to explain his conclusions from first principles in terms that would be intelligible to ordinary readers. There is thus a considerable probability that before a thesis of this kind is suitable for publication, it will have become almost entirely the work of the students' professor, and will have cost the professor more hours of his time than if he had written it himself. It is not easy to suggest a way out of this difficulty. The only one I can suggest is that the professor's teaching duties should be made so light that he can spare the time to do the work. If the education of the student is the primary object sought this plan will attain the desired end and the training the student will acquire will be of incalculable value to him.

An illustration of the danger of putting this kind of work into inexperienced hands is afforded by some of the modern cheap and nasty calculuses, in which the simplest differentiation formulæ are proved with a wasteful luxuriance of infinite series and a reckless disregard of small quantities.

(iv) As a matter of fact, however, many students who graduate in our universities are very deficient in their powers of lucid exposition and in English composition; sometimes they cannot even write legibly or spell correctly. This is a defect of our system of graduation courses and examinations for which the only remedy is the reform, not the abolition, of our examination systems. I have known cases where examiners have assigned marks for work that was partly illegible or for answers that were absurd. In such cases the examiner first seeks for some passages in the answer that indicate some knowledge of the syllabus. He then mentally re-writes the answer in the light of his own knowledge, marks his own version, and enters the marks on the candidate's mark sheet. For defective handwriting

he may deduct 10 per cent. from the mark. This is a false position. If the examination is a test of mathematical knowledge it is not correct to assert that a man who cannot spell knows 10 per cent. less than one who can, or that he ought to pass if he answers 10 per cent. more of the questions. The proper course is to qualify him in Mathematics, but disqualify him for a degree until he has passed a test in writing and spelling. The root of the mischief, however, is often that the syllabus or the examination paper, or both, are too long, and that, in the scramble to run up marks, English composition and powers of lucid expression go to the wall. Here, again, the remedy is to be sought in a reform of the graduation courses and their examinations. The man who cannot express his ideas intelligibly is not fit to teach others.

* I think many examining bodies might borrow a lesson from the Civil Service Commission and include a test in "précis writing" in their examinations. I learn that an essay paper has been introduced with success in the Cambridge examination, but this is not quite the same thing.

(c) Historical investigations stand on a different footing, and thus present considerable scope for a training in individual work. They appeal to a somewhat different and perhaps more average type of student, namely, the conscientious plodder, whose creative power is not very great, and whose training under existing conditions consists mainly in the memorising of bookwork. The kind of task required is that such a student should take some subject, such as the calculus, analytical geometry or analytical mechanics, and trace the various developments back to the origin, arrange them in chronological order and examine the state of knowledge of older cognate subjects at a time when the first ideas in the selected subject began to dawn. When this method is applied to such a subject as the genesis of logarithms a very different vista is unfolded from that contained in text-books.

It should be clearly understood that in the present article, no attempt has been made to present an adequate statement of the claims of "History of Mathematics" as an educational study. This is a subject which might with advantage occupy a whole article by itself.

(d) There are some cases where results that are obtained in text-books by one method can be established by an independent method. The use of quaternions in physical problems has afforded material for independent work of this kind; integral equations look as if they might do the same. To prove the theorems in Frost's chapter on curvature and torsion of curves by means of velocities and accelerations is a small example I have used in class teaching, though unfortunately it is difficult within existing time limitations to get a pupil to do the whole

work himself. In most cases, the difficulty is that such applications are mostly confined to short isolated problems and do not involve the performance of a substantial connected task.

(e) Where any investigation involves the evaluation of arithmetical details from known data with the use of formulæ a valuable training is afforded by performing the computations. It is a fault both of our text-books and our examination papers that they contain far too few numerical calculations and far too many algebraic symbols; consequently students who can express integrals in the form of "cosh-minus-ones" never evaluate areas in square inches. Some work of the kind suggested thus becomes very desirable.

Physical problems undoubtedly supply the most accessible material for work of this kind. There would be abundant further material in connection with statistical problems, but unfortunately mathematics of statistics is not taught in this country as it ought to be.

It is hoped that further suggestions may be forthcoming indicating other possibilities of supplying the requisite educational training.

Now it will be seen that many of the difficulties referred to above depend more or less on the character of the undergraduate course. Further, the salaries of our mathematical teachers are such that they could not be expected to take a year's specialisation in addition to their existing courses of study. Thus the question turns largely on the possibilities the undergraduate course offers for substituting specialised study for part of the work that is now involved in it.

Of course schemes of study vary in different universities, and it is impossible to lay down any general rules; but they mostly possess one feature in common, in a greater or less degree; Sir W. S. Gilbert, might say they contain "something of everything and nothing of anything." A large amount of time is spent in mastering the contents of text-books, with the result that little opportunity is afforded for the display of individual thought where this exists (frequently it is discouraged), and at the end of this ordeal the newly-graduated student finds himself nowhere near any of the main centres of modern mathematical thought. When I had graduated in Part I., I was horrified at the syllabus for Part II. A good preliminary course of general study is undoubtedly indispensable. But it is equally important that the rank and file of our graduates who go out to teach Mathematics at the end of their graduation course should be equipped with something more than the knowledge of half a century or a century ago, and thus arises the necessity of finding room for a certain amount of specialisation in our undergraduate courses of study.

Now there are many people who condemn specialisation and advocate an all-round course of study. Unfortunately,

however, the mathematical curricula of our universities have a tendency to degenerate into something far worse than specialisation, something which in no sense represents a good all-round training, namely, a scramble to learn up bookwork by heart in order to be able to scribble it down more or less unintelligibly and illegibly in an examination room. There is no more pitiable sight than that of a number of students spending a fine afternoon in June poring over text-books and stunting their mental faculties by excessive memorising just before an examination. And experience tends to show that the more advanced the examination the greater is the premium put on the mere repetition of bookwork, commonly described as "cramming." Of course the examination papers as a rule contain riders, but a limit is soon reached at which three hours is far too little to enable an average candidate, with no books to refer to, to do any good at riders; or, again, very often the bookwork takes so long to write out that he is far too exhausted at the end to answer an easy rider which can be done in half a minute with elementary common sense.

I take it that every student who wishes to succeed either in Mathematics or physics ought to be equipped with a working knowledge of the calculus, some differential equations, analytical geometry of two and three dimensions, the elements of rigid dynamics and hydrostatics, and if possible, particularly in the case of students of physics, some of the analysis connected with solutions of Laplace's equation. It should surely be possible to frame examinations on such a syllabus which would be an effective test of whether a student possessed the qualifications requisite for a successful teacher, such as power of lucid exposition, resourcefulness, intuition, common sense. Such an examination would further indicate whether the candidate was likely to succeed in the study of higher mathematics. Having satisfied ourselves on these points there is nothing to be gained by repeating the tests on a more advanced syllabus, and the passing of such an examination would thus become a qualifying test for admission to a course of specialised study in Mathematics. It is in general desirable that one further qualification should be possessed, namely, some experience and proficiency in the manipulation of physical apparatus.

Now at the present day a lad who intends to qualify for the profession of teacher of Mathematics will certainly have either begun the calculus before entering college or will be ready to learn it quickly after entering. Such a candidate may reasonably complete the qualifying examination above referred to at the end of his second university year. If he does not stop up a fourth year this leaves one year for specialising in some branch of higher mathematics. As different candidates would select different subjects we should thus supply our schools with a body of teachers possessing collectively an up-to-date knowledge of modern Mathematics, whose university training had

qualified them for the study and appreciation of research and for adding their own contribution to our knowledge when their time permitted them.

It will be seen that the above proposal is that the general course of study occupying the first period should be made as short as possible, and should be completed by a searching examination which would test other qualities beyond the memorisation of bookwork; and that in the second period the pupil should make a straight course to the forefront of existing knowledge in some branch of mathematical science by thoroughly studying all the subjects leading up to the desired end.

A general course has been suggested above which might meet the requirements of a student either of Mathematics or of physics. In the actual working out a good many modifications would be required, and it is further desirable that general courses of study should be framed with regard to the requirements of candidates proposing to specialise in other subjects besides mathematics, physics and engineering. The student who intends to become a politician requires thorough training in the mathematics of statistics (and the passing of an examination on this subject should be compulsory for every paid member of Parliament); the student of biology or medicine requires the mathematics involved in the construction and use of the microscope; for Mendelism statistical mathematics is again required. The tendency to overload the general course even in meeting the requirements of the non-mathematician is thus obvious.

On the other hand there are a good many subjects in most existing curricula which might easily be transferred to courses of specialised study. To judge from the recent epidemic of text-books, projective geometry seems to be coming into fashion in schools as well as in colleges. It forms the proper starting-point for a specialised course in higher plane curves and as such may well be transferred to the second period of the graduation course for students taking that subject, but I cannot see that the power of stating and proving Brianchon's theorem helps a man specialising in Integral Equations, or that it is likely to benefit the *average* physicist, engineer, bacteriologist or member of Parliament who requires a general training in Mathematics.

The cutting down of a general course to make room for advanced study is difficult in view of the rival claims asserted by those interested in different subjects. On the other hand the recognition of a period of specialised study should do much to prevent overloading, as it is no longer a question between teaching a subject to everybody and not teaching it at all.

In most branches of education specialisation is carried to far greater excess than is contemplated by such proposals; in many cases it is introduced in the most elementary teaching. The history taught in schools is not history in general, but is

confined to certain periods on which an examination is held. In classics our school children specialise at the very outset on set-books. Once I passed an examination where the Old Testament was represented by the Second Book of Kings. Now these are all branches of study of which a man can acquire a fairly complete general survey in his lifetime. In a subject of which no man can cover more than a small portion we try to impose a common training on all, and so fail to bring them into proper touch with the science they are supposed to be studying.

A further requirement is the provision in our modern universities and university colleges of courses of lectures on some branch of modern mathematics given by a specialist whose time is not too fully occupied with teaching work of a more general character. There are at the present time certain appointments of this kind in existence (for example, in the University of Liverpool Dr. W. H. Young, F.R.S., has given courses of lectures of exactly the kind that is wanted), but far more remains to be done in this direction. Incidentally the establishment of such appointments may do something to mitigate the effects on English Mathematics of a reduction of the College Fellowship system.

We have hitherto considered only the need of advanced study and research in the college training of the mathematical teacher. But there still remains the case of a graduate who has been teaching for several years and who wishes to be brought up to date with the modern developments of his science. This case may be met by following the example of the American Mathematical Society, and holding *colloquia* in which specialists are invited to expound the most recent developments of their subject.

The views put forward in this paper may perhaps be summed up as follows :—

1. It is desirable that research should form part of the training of teachers in Mathematics as well as in other branches of science, and that educational authorities should be made to realise this when they fill appointments.

2. There is a danger that research scholarships may deter students from embarking on researches in such subjects as Mathematics where a considerable course of advanced study is a necessary precedent. To obviate this danger it is important that the relative value of post-graduate study as distinct from, but leading to actual research should be realised.

3. An attempt should be made to obtain lists of suggestions of subjects of research which are easily accessible.

4. For the average student who wishes to obtain educational advantages corresponding to a research training in experimental or observational science, work has got to be found which does

not necessarily involve the creation of new and undiscovered theorems. Suggested directions are: collaboration in the original work of a specialist, preparation of general surveys of existing fields of study, history of mathematics and of its particular branches, proof of known theorems by new methods, numerical work in problems of which practical applications are required.

5. The need of including the power of reading French and German in the training of a mathematical candidate follows inevitably from some of the arguments.

6. Graduation courses in Mathematics should include some period devoted exclusively to the study of modern developments of that science in addition to a general course of preliminary study.

7. The examination on the general course should be framed in such a way as to test the candidates' general capacity and intelligence and to discourage "bookwork scribbling."

8. Courses of lectures by eminent specialists should be provided for in connection with the period of special study.

9. The American system of *colloquia* might very well be introduced into our country.

The object of the "reform" movement by which Euclid was abolished from our schools was to lessen the amount of book-work-memorising, and to enable boys to reach such enlightening subjects as the study of the calculus earlier. This step, once taken, can only be regarded as a first step in a scheme of reform of a much more far-reaching character. It certainly removes a certain educational element from the early stages of a pupil's training, and the loss of this element is deplored by advocates of the old system. The proposals here made would have the effect of restoring this element at a time when the pupils have reached mature years and are more fit and qualified to appreciate and profit by a course of rigorous and logical study carried to its ultimate conclusions.

This paper has been written under the disadvantage that its author is not a pure mathematician, and is therefore incompetent to form any adequate appreciation of the subjects of advanced study and research whose claims should here be advocated most strongly. The only excuse that he can claim is, that it is sometimes easier to advocate other people's claims than one's own, and that English pure mathematicians are often in too weak a position to make their wants known. Under these circumstances the views put forward in this paper are intended to be regarded as suggestive rather than final. At the same time it is hoped that any who do not agree with any suggestion here made will propose something better as a substitute. The subject is one which may well engage the most serious consideration of thinking men.

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My own view is that the study of Mathematics at present occupies an entirely false position in England. In recent years the value of a mathematical training for students of physics and engineering has been brought prominently before the public, but few people realise the importance of the work which is done in *Mathematics* itself as apart from its applications. I believe that nothing would do more to remedy this state of things than a scheme such as is here proposed, which would spread our courses of study over a wider area and enable each of our future mathematical teachers to command some one portion of that great area.

G. H. BRYAN.

THE RELATION OF MATHEMATICS TO ENGINEERING AT CAMBRIDGE.

The scientific analysis of the phenomena presented to the engineer in the practice of his art has been conveniently named "Engineering Science," and it is this which is meant by "Engineering" when that word is used to denote a subject of study or research in a University. Engineering Science occupies a distinct position among such subjects. In so far as it seeks to determine physical laws and properties of materials in the laboratory, it is identical with Experimental Physics. But it has another side, which gives it its distinctive character. It is the province of Engineering Science to investigate the operation of physical laws under the complex conditions of practice—conditions which often cannot be even approximately reproduced in the laboratory. In this respect it has some affinity with the biological sciences, having much the same relation to the art of construction as Physiology bears to the art of medicine. And, just as Physiology must be based primarily on observation of the animal in its natural environment (which is the real subject of enquiry), so the foundation of Engineering Science must be experience of the behaviour of machines and structures under the ordinary conditions of use. Much may be learnt from properly devised experiments in a laboratory, but the application of their results is limited by the fact that the artificial control of conditions which is of the essence of experiment, is of itself a disturbing factor, analogous to that general effect on the subject of a vivisector's operation which always lessens and may destroy the value of his conclusions when applied to the normal animal. But though Engineering Science must be mainly founded on practical experience, the relative simplicity of the conditions and of the fundamental laws make it easier than in Biology to apply rigorous deductive processes to the premisses discovered in the laboratory and so to predict or explain the phenomena actually under investigation. These deductive processes usually take the mathematical form.

That experience is all-important, and deductions from laboratory experiments of relatively little use in Engineering is a common-place among practical men, and it is perfectly true in the sense in which they use the words. By "Engineering" they mean the practice of the art, and in nine cases out of ten the practical engineer relies almost wholly on experience and only makes use of very simple physical ideas and deductive processes. In the design of a machine or of a structure he will be content with rough calculations of the stresses to be expected in the different parts, knowing well that the assumptions on which such calculations are based are subject to greater error

than is involved in the calculations themselves, and that economy of material is often of less importance than ease of manufacture. For such purposes a very small mathematical equipment is adequate. Indeed too much mathematics is apt to be a snare, since the elegance of the calculations may tend to obscure the real issues, as in the case of the young engineer who designed a bracket to carry the weight of a lamp on a locomotive, forgetting the man who would inevitably use it as a step.

In Engineering Science, however, mathematical analysis plays a more important part. Take, for instance, the theory of lubrication. The practical engineer will work out the pressure to be expected on a bearing, but he will not proceed to calculate from first principles what area the surfaces must have in order to sustain such pressure with oil of a given viscosity. For the determination of this point he will consult his experience or that of others as to the bearing pressures which have been found possible in practice under similar conditions. The whole process requires little more than the use of a slide rule and a knowledge of some elementary mechanical principles. But to the student of Engineering Science the problem presented is to explain why these bearing pressures are possible and how they depend on the speed, the nature of the lubricant and the like. It is impossible fully to understand, much less to carry out, such researches as those of Osborne Reynolds on this subject without advanced mathematical knowledge. The same instance illustrates very well the difference between Engineering Science and Pure Physics. The physicist is mainly interested in the properties of the oil, in the variation of its viscosity with temperature, and in the question of whether it slips at the boundary. He is free to choose the conditions of the experiments which he makes for the determination of these things in such a way that the measurements and the deductions from them shall be as simple as possible. So he makes the fluid flow at uniform velocity along a pipe of uniform section, and for the reduction of his results he requires only the simplest elements of the calculus. But the engineering student has to take the conditions as he finds them and has to adapt both his methods of measurement and his mathematics to the conditions instead of *vice versa*. In his problem the flow goes on between cylindrical surfaces whose distance apart is not uniform, the pressure and the temperature changes distort the surfaces and alter the viscosity of the oil, and neither the distance between the surfaces nor the amount of their distortion can be measured. The most powerful mathematics cannot take account of all these factors at once, so he is compelled to proceed by approximation. At every step he has to estimate on insufficient data the probable importance of the things which he is neglecting for the purpose of simplifying the analysis, and his success depends quite as much on the acuteness of his judgment in this regard as on his powers of deduction.

The theory of lubrication belongs to the higher flights of Engineering Science, but the combination which was needed for its development of rigorous mathematical reasoning with judgment in the choice of the data on which to base that reasoning is characteristic of the whole subject. It is because it unites in one study a training of the logical and of the critical faculties that Engineering Science is justified in claiming a place of its own as an instrument of liberal education not inferior to the older studies of Mathematics and Natural Science, but having like each of them its special qualities and advantages. It is this, too, which has made possible in the University of Cambridge the natural growth of an Engineering School as important in its way as any founded in the centres of industry, but different from them and possessing a character of its own. For if it has not, like them, the material for its investigations at its very doors, it has the advantage of being in a centre of mathematical learning which is still second to none.

In 1903, when the author became responsible for the conduct of the Engineering Department at Cambridge, it was already fully established on lines which had been laid down more than ten years before by his predecessor. The policy which Professor Ewing initiated and carried out was consistent with the idea that Engineering Science should be so taught as to be a means of general education and not merely a part of the preparation for a particular profession. Under his direction the study was pursued in such a manner that it gave a valuable mental training to anyone who followed it, even though he should subsequently become a lawyer or a parson and not an engineer. On no other terms could the School have flourished in the University and attracted its fair proportion of under-graduate ability. Its success is shown by the careers of those who have graduated in the Mechanical Sciences Tripos; a majority have, of course, become engineers, but many have entered other professions, or have gone into commercial life, with results that show their education to have been on right lines.

But though it had attained a great reputation and had some 200 students, the Engineering School had not in 1903 that close connection with the general intellectual life of the University and Colleges which must exist before a new study can be said in any true sense to be established at Cambridge. With few exceptions its students got the whole of their instruction within the walls of the Engineering Laboratory. They attended no lectures outside the Department, and the system of College tuition which is by reason of the close personal contact between teacher and pupil so important a feature in the teaching of mathematics and of classics at Cambridge was not extended to them. In particular they had not then the full advantage of their position as students of Engineering Science in a centre of mathematical learning and research.

This was perhaps inevitable. It was essential to the proper development of the School that its curriculum should at first be practically in the hands of one man, who could see that the various parts were properly co-ordinated and due weight given to each. Engineering Science is largely a compound of Mathematics and Experimental Physics, its students must learn something of both, and the facts and theories which they learn are the same as those taught to the ordinary students of those subjects. But the Engineering student regards them in a particular way and his attitude determines not so much what he shall learn as the way in which it shall be taught, and the relative importance of different parts of the subject. For instance, in Elementary Electricity (at any rate in the author's view) he should learn about current first, and proceed later (if at all) to statical charges and the forces between electrified bodies, instead of taking these things in the reverse order as is, perhaps, more usual in Physical Laboratories. His Mathematics is a means for getting numerical results rather than an end in itself, and while this requires that he shall have a sound knowledge of general mathematical methods and of their relations to one another, it makes quite superfluous that facility in the rapid and neat solution of problems which used to be so important for success in the Mathematical Tripos. The pure mathematician can teach engineering students their Mathematics as well as anyone else; but he must first acquire, either by actual practical experience or by contact with those who have that experience, an appreciation of the engineer's point of view. It is not, therefore, surprising that some time elapsed before the Mathematical School at Cambridge was able to be of much assistance to the engineering students.

The establishment of closer relations between the two studies has, however, made great progress during the last eight years, and at the present time the students of Engineering get their foundation of Mathematics and of Elementary Mechanics from College teachers, many of whom have graduated in the Mathematical Tripos. The teaching of these subjects to freshmen has almost disappeared from the Department. At the same time, and perhaps partly as a consequence of this change, the Colleges have come to take a more direct and personal interest in their engineering students. At several of the larger Colleges there is a "Supervisor" or "Director of Studies" in Engineering who is at once a fellow of his College and a demonstrator in the laboratory. College Scholarships are awarded to engineering students on the result of examinations in the subjects of the Mechanical Sciences Tripos. The significance of these things can only be fully appreciated by one who is familiar with the life of Cambridge; they imply that Engineering Science has really taken root in the place.

Among the influences which have helped to bring about this result must be counted certain changes in the Mechanical

Sciences Tripos which came into force in the year 1906. In the syllabus for this examination framed in 1892, there were six subjects each having a separate paper. One of these was Mathematics, which was thus put in the same position as Theory of Structures, Heat-engines, and the other subjects which dealt with actual engineering phenomena.* It was felt that this did not correspond to the real relation of Mathematics to Engineering Science. It was accordingly decided to remove Mathematics as a separate subject from the final examination, and to test the mathematical acquirements of the candidates by their power to solve problems in the other subjects. In this way the amount of mathematical learning which was necessary and sufficient for engineering students was adequately defined—it was to be such that they could fully understand the developments of Engineering Science specified in the Tripos Schedules.

In the Tripos as now arranged a certain minimum standard is exacted in the four subjects—Applied Mechanics, Theory of Structures, Heat and Heat-engines, and Electricity. All the candidates are also required to pass in Drawing, a subject which is defined in the Schedule as “the application of graphical methods to the solution of problems in the subjects” just specified. The Drawing paper to some extent takes the place of the Mathematics paper in the old Tripos; it covers in fact a very important part of the Mathematical training of the Engineering student, but a part which he cannot get from the Mathematical school of the University. The other subjects are represented by easy papers which everybody is required to take. The way in which the ambit of these papers is defined may be illustrated by the following example:—

“Theory of Structures and Strength of Materials.—The elementary theory of elasticity is illustrated in such problems as bending of straight beams, torsion of shafts, struts, and stress in thin cylinder under pressure. Composition and properties of materials of construction. Testing of materials.”

The “elementary theory of elasticity,” limited in this way by reference to some of its practical applications, requires for its development a certain knowledge of mathematical methods, which can be tolerably well defined. A proper discussion of beams and of torsion in shafts is impossible unless the pupils are so far permeated with the motions of the differential and integral calculus that they can appreciate the meaning of continuous variation and its mathematical expression; can write down the expression for a moment of inertia without taking too much thought, and can calculate its value in simple cases. They must also be able to solve some elementary differential equations, and it is desirable that they should understand the methods of solid geometry so far at any rate as the straight line

* The subjects of written papers in the old Tripos were Mathematics, Mechanics, Strength of Materials and Theory of Structures, Principles of Mechanism, Heat and Heat Engines, and Electricity and Magnetism.

and plane. But it is not necessary (so far as this subject goes) that they should know the general theory of differential equations or the analytical geometry of quadric surfaces. A similar specification can be made of the mathematical knowledge required for each of the subjects.

It might be possible in theory to make the instruction in Mathematics follow the examination, and teach that subject as required in the course of its applications. Much can undoubtedly be done in this way, especially in regard to the higher developments. In practice, however, it is found that this method can only be applied to a limited extent. If a lecturer dealing (say) with alternating currents were to make a digression to teach the properties of the circular functions which he uses, his class would lose sight of the main issues. On the other hand, mathematical power cannot be acquired without some general knowledge of methods and of their relations to one another. For this purpose it is essential that the engineering student should receive some teaching in abstract Mathematics, freely illustrated by simple practical and numerical applications of the kind which he will afterwards have to make, but done in such a way as to bring out clearly the general nature and limitations of the processes used.

The mathematical foundation of this kind which has been found necessary and desirable at Cambridge is a good deal more than is acquired at school by the average boy. Broadly speaking it is sufficient, together with the elements of theoretical mechanics, of drawing, and of experimental physics (so far as he requires them) to occupy fully his first year at the University. His second and third years are devoted to the study of Engineering Science proper, the lectures being given on the assumption that he has a sound knowledge of Mathematics within certain limits. In order to define these limits and to ascertain that the student has the requisite knowledge (which is, of course, necessary in order that he may follow the lectures, &c. with intelligence) an examination was set up at the time of the changes in the Tripos which have been mentioned. This consists of two papers, in Elementary Mathematics and Mechanics respectively, and every student must pass it before entering on the second year work for the Tripos. It is simply a pass examination—to quote the Regulations, the examiners are required to publish a list “of those of the candidates who have satisfied them that their knowledge of Mathematics and Mechanics justifies them in becoming candidates for Honours in the Mechanical Sciences Tripos.”*

In addition to its primary function of weeding out those who could not follow with profit the Honours course in Engineering and so making the classes more homogeneous and simplifying the teaching, the “Qualifying Examination”

* The Schedules of the Qualifying Examination and of the Tripos, together with a set of papers in the former, are given in the Appendix.

has served an important purpose in defining the nature and amount of mathematical teaching desired for engineering students. It was indeed quite as much with a view to this as with the object of ploughing inefficient students, that it was set up. Some lead of the kind was necessary to enable College teachers of Mathematics to help engineering students. The syllabus of the Qualifying Examination was drafted, and the papers were at first prepared, with this point especially in view. From the questions set in the examination the mathematical lecturers can see exactly what is required and are able to adapt their teaching accordingly, and of the two examiners one is usually chosen from among them.

The Qualifying Examination represents, as already indicated, a necessary minimum of mathematical knowledge, sufficient with a little further teaching of abstract mathematics which is given by the Engineering Department in the second year, to enable the student to follow the courses for the easy papers in the Tripos. It will be at once apparent from the Schedule; however, that it is far from covering all that is needful for those abler students who wish to follow Engineering Science into its higher developments, nor does it represent at all adequately what they can get from the Cambridge Mathematical School. Such men usually come to Cambridge already prepared up to the standard of the Qualifying Examination and their numbers are steadily increasing, partly no doubt in consequence of the marked effect which the institution of that examination has had on school teaching. They usually spend their first year in working for the First Part of the Mathematical Tripos (by passing which they gain exemption from the Qualifying Examination) and come to the Engineering School in their second year. In 1911, nine of the 45 candidates for the Mechanical Sciences Tripos were trained in this way and the number is increasing.

The close co-operation between the two schools which this represents has been rendered possible by the reform of the Mathematical Tripos which was carried through in 1907. The movement which resulted in this reform was already well-advanced when the author of this paper came to Cambridge, and a committee (of which he became a member) was then drafting a scheme. At that time it was a common practice for a man to spend his first three years in reading for the Mathematical Tripos, and then to work for a fourth year for some other Tripos, such as the Natural Sciences or Engineering. There were on the average two or three candidates per annum for the Mechanical Sciences Tripos who had previously obtained Honours in the Mathematical Tripos. The results in this instance were not entirely satisfactory because the time spent on abstractions was too long, and that devoted to realities too short. The student who had for three years been reasoning rigorously from fully defined data, found it difficult to acquire in one the power of using his

own judgment—even in the simplest cases—as to what is important in an actual engineering problem and what may be neglected for the purpose of calculation. To the mathematician of only a year's standing it is much easier to take up the engineer's point of view and he has twice as long to do it in. The decision, therefore, to reduce the scope of the First Part of the Tripos and to make it a First Year examination was welcomed by those responsible for the Engineering School, as well as by the representatives of other studies who saw the advantages to their students of a short preliminary training in Mathematics.

The character of the new examination was, of course, determined mainly by the mathematicians, but every attention was paid to the needs of the students of Physics and Engineering who were expected to take it. Thus in the Schedule we find that the use of oblique axes in Analytical Geometry is excluded, and so also (by implication) is the rigorous proof of the binomial and similar expansions. On the other hand, simple linear differential equations and the elements of solid geometry are included. The style of the papers also appears to the author to be calculated to encourage a sound knowledge of a variety of mathematical methods, of their relations to one another, and of their practical use for numerical computation rather than that sort of quickness in the neat solution of mathematical conundrums which was such an important qualification for success in the old Tripos. If any criticism may be permitted it would be that the scope of the examination is not quite wide enough. It does not cover much more than the subjects a first-class man—a mathematical scholar, for instance—will have read at school. In the author's view it would be an improvement, at any rate for those who will ultimately take Engineering, to make the Schedule include rather more subjects, especially in Mechanics in regard to which the standard of the examination seems to be unduly low. The average student would not, of course, touch the higher subjects, but the best students, those who form the bulk of the First Class, do now read in their first year subjects outside the schedule of the Tripos, and for the increasing proportion of such men who are taking Engineering it would be advantageous to have these subjects recognised in the examination. The alternative for them, instead of reading Higher Mathematics, is to begin work at once in the Engineering Laboratory, and to devote part of their first year as well as their second and third years to the study of Engineering Science. To a small extent this is now done. This year some 20 freshmen who will take the first part of the Mathematical Tripos in 1912, and the Engineering Tripos two years later, have been spending four hours per week in practical work and drawing at the Engineering Laboratory, a special class having been arranged for their benefit. This class is very popular and it is undoubtedly useful thus to introduce the men to real problems while they are learning Mathematics. The

author would like to see many mathematicians join such a class and believes it would be useful even to those who do not proceed to graduate in Engineering. He is, however, by no means anxious that freshmen who have read most of the work for the first part of the Mathematical Tripos should devote all their spare time to Engineering. He thinks it essential for the fullest development of the Engineering School that such men should be directly influenced by the higher mathematical teaching of the University; and for that reason he would like to see them encouraged by a widening of the Schedule of the first part of the Tripos, to read in their first year some subjects outside its present range.

B. HOPKINSON.

APPENDIX.

SCHEDULES OF SUBJECTS OF EXAMINATION.

MECHANICAL SCIENCES TRIPOS.

Qualifying Examination.

Mathematics.

The paper shall consist of simple questions, mainly numerical, directed to test the candidate's working knowledge of the following mathematical subjects, and his power of applying them to practical problems:—

Use of slide rule and graphs: elementary algebra, including the use of the binomial theorem, the exponential theorem, and logarithms. Elementary trigonometry, including the solution of plane triangles. Representation of directed quantities by vectors. The rudiments of plane analytical geometry, including the simpler properties of conic sections, referred to rectangular axes and polar co-ordinates, but excluding the general equation of the second degree. Use of the elements of the differential and integral calculus, elementary applications to plane curves, maxima and minima of simple functions of one variable, easy areas and volumes, application of Taylor's and Maclaurin's theorems to simple cases of approximations.

Mechanics.

The paper shall consist of simple questions, mainly numerical, illustrating the following:—

Units; composition and resolution of forces; conditions of equilibrium; force diagrams; friction; fluid pressure, centre of stress, floating bodies (excluding metacentres). Force, work and energy; graphic methods of representing motion, including varying acceleration; uniform and uniformly accelerated motion; uniform circular motion; simple harmonic motion; simple pendulum; conservation of momentum; efficiency of machines; moment of inertia; rotation and oscillation of solid bodies about a fixed axis.

Tripes Examination.*Applied Mechanics.*

The principles of statics illustrated by their practical applications, including the calculation of forces in the members of a frame where the strains are not involved. Graphical constructions. The equilibrium of floating bodies, and centre of pressure. The elements of hydraulics, including the flow of liquids through pipes, discharge from orifices, and the elementary theory of pumps and turbines. Dynamics up to and including the rotation and oscillation of solid bodies about a fixed axis. The vibrations of systems having one or two degrees of freedom. The kinematics of machines including plane motions of machine parts, mode of connection, modification, and transmission of motion in machines, and efficiency of machines.

In the paper in Group B questions may also be set on the following :— General kinematics of machines, including valve-gears. The applications of rigid dynamics to practical problems, including crank effort, the stability of governors, and balancing of machines. Waves on the surface of liquids, ship-resistance.

Theory of Structures and Strength of Materials.

The elementary theory of elasticity as illustrated in such problems as bending of straight beams, torsion of shafts, struts, and stress in thin cylinders under pressure. Composition and properties of materials of construction. Testing of materials.

In the paper in Group B questions may also be set on the following :— The equilibrium of thick cylinders and spheres, arched ribs and chains, masonry structures, springs, the bending of curved beams, and structural problems involving the consideration of internal strains. Elementary applications of the theory of vibrations, and of the propagation of waves, in elastic solids. Stresses set up by inequality of temperature.

Heat and Heat Engines.

Elementary theory of heat and the laws of thermodynamics as illustrated in steam, gas and oil-engines and in refrigerators. The composition and combustion of fuel. Properties of vapours and gases. Standard types of engines, condensers and boilers.

In the paper in Group B questions may also be set on the conduction of heat, and on the more advanced applications of thermodynamics, including the flow of gases and vapours, the theory of steam turbines, and the equilibrium of phases.

Electricity and Magnetism.

The principles of electricity and magnetism as illustrated in the theory of the dynamo, transformer, electrical measuring instruments and transmission of power. The laws of electrolysis as applied to the theory of secondary batteries and ordinary Voltaic cells.

In the paper in Group B questions may also be set on the applications of polyphase alternating currents, the applications of electric waves and oscillations, and the theory of telegraphy and telephony.

Drawing.

Plane and solid Geometry. The application of graphical methods to the solution of problems in the subjects specified in this schedule for the other papers of group A.

Engineering Chemistry.

The principles and methods of chemistry as illustrated in the following :—The metallurgy and properties of the commoner metals of industrial

importance. The chief alloys of these metals and the conditions under which they are formed. Furnaces, fuels, and furnace gases. Chemistry of accumulators. The estimation of the commoner metals in their ores, oxides and alloys. Analysis of steel and cast iron, red and white lead and the impurities in water. Limestone, lime, mortars and cements.

GENERAL NOTE:

The questions are to be directed to testing the candidate's knowledge of the physical principles involved and his power of applying them to practical engineering problems. In the papers in Group A questions may be set involving a knowledge of such elements of Chemistry as are applicable to the subjects of the papers.

QUALIFYING EXAMINATION FOR THE MECHANICAL SCIENCES TRIPOS.

WEDNESDAY, *May 29, 1907.* 9—12.

MATHEMATICS.

1. A slide-rule scale, reading from 1 to 10, is 10 inches long. Calculate the length of the interval between the divisions 2 and 2·02.

2. Use the expansion

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \dots + x^n,$$

where n is a positive integer, to prove that the value of $(0\cdot99)^{20}$ is 0·82, correct to two places of decimals.

3. Explain briefly (i) the reasons for the use of 10 as a base of logarithms, (ii) the reasons for the use of e as a base. Prove a rule for deriving the logarithms of numbers to the base 100 from their logarithms to the base 10.

4. Express $3 \sin x - 4 \cos x$ in the form $a \sin(x + \beta)$; and sketch its graph, indicating the scales employed, and finding the co-ordinates of all points in which this curve cuts the axes.

5. Prove that the area of a triangle is $\frac{1}{2} ab \sin C$ where a , b are the lengths of two of the sides and C is the angle between them.

The angle C and the sides a , b of a triangle are measured and found to be 90° , 2,000 ft. and 3,000 ft. respectively. Find the greatest possible error in the estimate of area based on these figures, having given that the measurements of the sides may each be in error by $\frac{1}{10}$ foot.

It is found that the actual error in the estimate of area is about twice the greatest amount derivable from these errors in the measurements of the sides; what is the least error in the measurement of the angle that will account for this?

6. Find the equations to the two straight lines which are parallel to the line $4x + 3y + 1 = 0$ and at a distance 2 from it; also the areas of the triangles which they make respectively with the co-ordinate axes.

7. Find the equations to the four circles of radius unity, each of which touches both co-ordinate axes, indicating on a sketch which circle is represented by each equation.

8. How may the maximum or minimum values of a given function be found?

Prove that $\log_e x + \frac{1}{x}$ is not less than 1 for any positive value of x .

9. Sketch the curve $y^2x = (x + 1)^2$. Find the value of x for which the tangent is parallel to the axis of x ; and show that, for values of x greater than this, the angle which the tangent makes with the axis of x does not exceed the value $\tan^{-1} \frac{1}{3\sqrt{3}}$, which occurs when $x = 3$.

10. The distance from the focus of a parabola to its vertex is 1 inch. Find the area enclosed between the curve and a line passing through the vertex and inclined at 45° to the axis.

11. Evaluate $\int_0^1 \frac{x^2}{4 - x^2} dx$ and $\int_0^a \sin^2 x dx$. Find the average value of the latter integral for all values of a from 0 to π .

12. A certain function of x is equal to ax^2 for values of x less than 1, and to $-ax^2 + bx - 1$ for values of x greater than 1. Find the values of the constants a and b in order that there may be no discontinuity or abrupt change of slope in the graph of the function at the point $x = 1$.

With these values of a and b find the values of x for which the function is zero.

WEDNESDAY, May 29, 1907. 1½—4½.

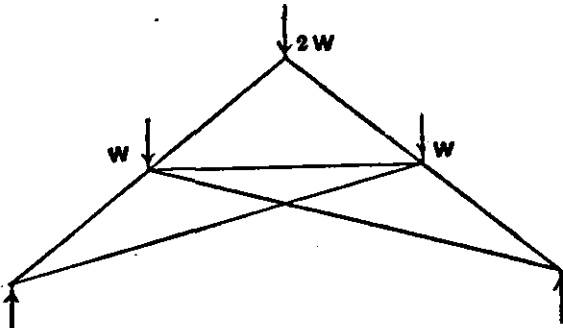
MECHANICS.

[N.B.—A horse-power is 33000 foot-pounds per minute; and the value of g may be taken to be 32 in foot-second units.]

1. A weight rests on a horizontal table, and is pulled by a string which is inclined at 45° to the vertical. If the coefficient of friction be $\frac{1}{2}$ prove that the weight will just slip when the tension in the string slightly exceeds 0.28 times the weight.

2. AC, BC are two uniform beams of equal length and of weights respectively W and W' , inclined to the horizontal at an angle α , and pin-jointed at C , the ends A and B being hinged to two fixed points at the same level. Find the pressure between the two beams at C , shewing that its horizontal and vertical components are respectively $\frac{1}{2}(W + W') \cot \alpha$ and $\frac{1}{2}(W - W')$.

3. Draw a reciprocal force diagram for this frame, loaded and supported in the manner indicated.



4. A truck weighing 1,000 pounds is hauled up a slope rising 1 foot in 20 feet measured along the slope. The truck starts from rest. The acceleration is uniform and the velocity after 10 seconds is 15 feet per second. Prove that the pull on the truck is about 97 pounds weight, and draw the graph shewing the rate of working in horse-power in terms of the time.

5. A body (mass one pound) executes simple harmonic oscillations in a straight line between the points *A* and *B*. The period of a complete oscillation is one second, and the distance *AB* through which the body swings is one foot. Draw the velocity-time curve, the time being reckoned from the instant at which the body is at *A*, and the scales being as follows: for time 1 second = 4 inches, and for velocity 1 foot per second = $\frac{1}{4}$ inch. Also draw the graph shewing the relation of the force acting on the body to its position measured from *A*, the scales being as follows: for force 1 lb. = 2 inches, and for distance 1 foot = 2 inches.

6. A block of wood weighing 10 pounds rests on a horizontal table on which it can move freely in any direction. A bullet weighing $\frac{1}{35}$ pound is fired horizontally into the block with a velocity of 1,500 feet per second. The bullet passes through the block, but is deflected in its course, its direction of motion after emerging being horizontal but inclined at an angle of 30° to the line of firing. The velocity of the bullet after emerging is 1,200 feet per second. Use the principle of the Conservation of Linear Momentum to find the magnitude and direction of the velocity of the block just after the bullet has emerged.

7. The wheels of a bicycle are 30 inches in diameter, the gear-ratio between the crank axle and wheel axle is $2\frac{1}{2}$, and the length of the crank is 8 inches. Find the velocity of the end of the crank and the magnitude and direction of its acceleration, when at its highest point, the bicycle travelling at the rate of 30 feet per second.

8. A weight of 100 pounds hangs freely from the end of a rope. The weight is hauled up from rest by means of a windlass. The pull in the rope starts at 150 pounds and then diminishes uniformly at the rate of 1 pound for every foot of rope wound in. Find the velocity of the weight after 50 feet of rope have been wound. The weight of the rope may be neglected.

9. Obtain from first principles a formula for the kinetic energy of a rigid body moving about a fixed axis with given angular velocity.

Hence calculate the energy in foot-pounds of a disc of 3 ft. diameter and $\frac{1}{4}$ inch thick, of material weighing 500 lbs. to the cubic foot, rotating about an axis through its centre at right angles to its plane and making 2,000 revolutions per minute.

10. A uniform straight rod 6 feet long swings in a vertical plane about one end; if V is the velocity of the free end of the rod in its lowest position, find the least value of V consistent with the rod making a complete revolution.

Compare this with the case of a weight hung by a cord of the same length as the rod, and making a complete revolution in a circle.

11. A cylinder 6 inches long, having a cross-section of 1 square inch floats totally submerged with its axis vertical in a vessel containing water and oil on the top of the water. If the cylinder weighs 0.2 pounds find how much of its length is below the surface of separation of the oil and the water. The specific gravity of the oil is 0.7, and the weight of a cubic inch of water is .036 pounds.

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3. Brush Work in an Elementary School (with illustrations).
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4. The A B C of Drawing: An Inquiry into the Principles underlying Elementary Instruction in Drawing (with illustrations).
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7. The Secondary Day School attached to the Battersea (London) Polytechnic: An Experiment in the Co-education of Boys and Girls.
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ON

EDUCATIONAL SUBJECTS.

VOLUME 28.

SCHOOL AND EMPLOYMENT IN THE
UNITED STATES.



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PREFATORY NOTE TO VOLUME 28 OF SPECIAL REPORTS
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The following report on the Relation between Schools and Industrial Employment in the United States has been prepared by Miss Winefrid Jevons after a visit to America which she undertook on her own initiative and without any definitely formulated plan of writing a report. On her return she submitted the results of her observations to the Director of Special Inquiries and Reports, and, on his recommendation, the Board of Education undertook the publication of her paper. The report deals with questions which are now under general discussion in this country, and the Board considered that the presentment which it contains of the various ways in which the problems of general and specialised industrial instruction are being attacked in the States would stimulate further thought and assist in the formulation of proposals suited to this country.

It will be agreed that in many respects the conditions in America have a greater resemblance to our own than those of neighbouring continental countries. As with us, apprenticeship has declined and its regulation has long been a matter of dispute between employers and federations of labour. Many firms have ceased to employ any apprentices, and very few trust solely to this means for recruiting their staff of workpeople. There is, moreover, an admitted lack of skilled workmen, and it is neither possible nor desirable to make good this deficiency by the continued importation of trained workers from abroad. The difficulty of securing efficient training in the shop has directed attention to the provision of more formal and systematic instruction elsewhere. Naturally the appeal is to the schools. The demand, originating with the employers but generally supported by the Trade Unions and Associations of Labour, includes the provision of Trade Schools, of general industrial training, and even of pre-vocational instruction. This large requisition has not been accepted with unqualified approval by the Education Authorities, and Dr. Maxwell, the City Superintendent of Schools of New York, in an article on "Arrogance in Educational Theorists" in the "Educational Review" for February 1914 makes a strong protest against the "arrogance" of the manufacturers who desire that the public school should take over the whole burden of industrial training. He insists most earnestly that no narrow trade schooling should ever be allowed to encroach upon that general training in literature, arts, science and religion which is the child's rightful inheritance. But Dr. Maxwell is not opposed to all forms of industrial training. He suggests various lines of advance. He is sympathetic towards the Fitchburg plan of co-operation between the high school and the factory, under which alternate weeks are spent in school and workshop. This scheme has aroused an unusual amount of interest in America and is fully described in Miss Jevons' report. Approval

is also given to attempts to modify the curriculum of the Elementary School by the introduction of a larger measure of practical work. In this connection special mention should be made of the work of the school authorities at Gary, Indiana. The developments which have there taken place seem to the Board so fresh and so full of promise that they have supplemented Miss Jevons' account by reports written by Dr. Crowley, one of the Board's Senior Medical Officers, who visited Gary after attending the Fourth International Congress on School Hygiene (Buffalo, 1913), and by Miss Hilda Wilson, who spent some time in America in the same year.

A still more difficult problem relates to those children who leave the Elementary School without completing the course and, in subsequent years, depend for their existence on unskilled desultory labour. Here the subject matter of instruction is not so immediately defined by the worker's occupation as it is in industrial education proper, and this, coupled with the difficulty of obtaining the right type of teacher, makes the solution of this problem perhaps the hardest task with which the educational administrator is at present faced. Even in America, only the barest beginnings have been made, but it is not without interest to compare the methods there employed with those practised in other countries.

There is yet another aspect of the question which has to be kept in view. It is not sufficient to understand the child and his capacity, it is necessary to know "the job" as well. Various investigations have been set on foot with a view to determining the specific qualities which are likely to ensure success in a particular calling. In this connection a passage may be quoted from a report made to the Foreign Office by Mr. Wilfred Powell, H.M. Consul-General in Philadelphia:—

"It is suggested that in respect to Vocational Training a school can do at least two definite things: first, the school can adapt itself in its curriculum and daily discipline to the industrial possibilities of the child, in other words, the school can incorporate in its daily instruction larger elements of vocational discipline; and second, it can place this direction of vocational education in connection with the child leaving school so as to advise him concerning the best possible wage-earning career which is likely to prove to be his vocation in life.

"It might here be stated that it has only been within recent months that a careful study of the type of mind needed for the motorman's job has been made. A year ago in one metropolitan centre there were more than 50,000 street tram accidents and the Tram Company of that City paid more than 13 per cent. of its gross income for accident indemnity. After trying by every possible mechanical means to avoid accidents and after spending large sums of money on safety devices with but trifling results, the Authorities of that Tram Company at last

thought of turning their attention to the motormen who had charge of their cars. The services of a professor from a well-known college were secured to study the circumstances, and after various experiments he was able to apply a test by which he could pick out the type of man who would be likely to make a good motorman, that is to say, 'make his time,' and yet to safeguard the lives in his keeping. The application of these tests by the professor showed that more than 25 per cent. of the men then employed were wholly incapable of using the type of mentality necessary."

Other experiments of a similar nature have been made and it is hoped thus to accumulate experience and by well-founded educational guidance to prevent the waste which the wrong choice of a career involves.

It is, however, clearly recognised that the success of the whole movement is dependent upon two things: on the satisfactory supply of specially qualified teachers, and on the provision of larger funds. To meet the former need, special courses for the training of industrial teachers have already been established at such institutions as Teachers College, Columbia University, and Pratt Institute, New York. The provision of the necessary funds has introduced considerable modification into State systems of education. It is generally agreed that the provision of industrial training cannot be regarded as a purely local matter. Accordingly in those States which have been most active in the last six years—Massachusetts, New York, New Jersey, Wisconsin and Maine—State funds and State officials have been freely provided to assist in the development of local effort. But the imperative national call for improvement has led people to demand a still further extension of the sources of financial assistance by the provision of federal aid. In July 1912 a Bill was introduced into the Senate, which, if carried, would ultimately have provided nearly 3,000,000*l.* a year, of which more than 300,000*l.* was to be devoted to the training of teachers.

The Board desire, on Miss Jevons' behalf and for themselves, to acknowledge the help rendered her in her inquiry by Mr. P. P. Claxton, Commissioner of Education for the United States, and by the City Superintendents of Schools for Boston, Chicago and New York. Miss Jevons also desires to express her gratitude to many others, including State and City School Superintendents, teachers, and social workers, who gave their time and assistance unstintedly in their endeavour to be of service in the cause of those children who leave school early for industrial employment.

It will be understood that the Board do not necessarily endorse any opinions which may be expressed in these papers. For these the authors are alone responsible.

Office of Special Inquiries and Reports.

April 1914.

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THE RELATION OF SCHOOLS TO EMPLOYMENT IN THE UNITED STATES.

CHAPTER I.—INTRODUCTION.

Any study of education in the United States is complicated by the fact that each State is autonomous, legislating on educational matters as best suits itself. Each State, moreover, has its own child labour laws. A comparatively uniform system of general or cultural education has been built up, though even in this, tremendous differences in the amount of expenditure and school attendance can be found between the progressive states of the north and the backward rural states of the south. But American education is now in a period of transition, different conditions in the country are introducing new educational aims, and as this is a time for experimenting, one meets many varieties of school and continuation class in a journey only so far west as Wisconsin and Chicago.

The United States has provided free tuition for its children from the elementary school up to, and in many States through, the State University; it has aimed at providing a liberal education for all who should wish to profit by it; its scheme was built up on the assumption that each child should have the opportunity of fitting itself to become and should become master and mistress of its own domain. The movement of population till 1880 was rapidly westward to the unopened territories lying there, and the immigrants, Irish, English, Scandinavian and German, mostly settled on the land. Since then the movement westward has been less marked and attention is being turned to the industrial centres of the east; greater opportunities of advancement occur there rather than in the western states, and the type of immigration has altered. Southern Italians, Slavs (Poles, Slovaks, Lithuanians, Bohemians, &c.) Jews from Russia and Austria, and Greeks are pouring into the country, the total numbers of immigrants reaching a million a year. Most of these do not go to the land but to the coal mines, the steel mills and the textile works of the east and middle west. It is an immigration of people from a rural life to town conditions, of a people who have not had educational opportunities themselves and who do not realise the value of the education offered to their children as does the American born. They are a people, too, on whom the conditions of town life often tell heavily and whose children therefore may be compelled to take up wage-earning at the earliest legal opportunity. A class is growing up, rather has already grown up, who will always be employed, and Americans are enquiring how far their educational system is meeting the needs of these people.

Elementary education is compulsory in most of the northern states between the ages of six or seven to fourteen, or to sixteen under certain circumstances. It is not possible to consider here the southern states where conditions are still very different from the north, and where an improvement in educational laws and arrangements is greatly needed. Below the age of six, such instruction as the parents desire must be obtained in kindergartens supported by philanthropic institutions and only occasionally by the public school authority. The elementary schools are called grammar or grade schools, the standards of our elementary schools being represented by the eight grades through which the average child is expected to pass at the rate of one a year. Hence he should pass out or "graduate" from the grade school at fourteen years of age or thereabouts. If he is backward in learning he may be compelled to remain in day school to the age of sixteen. The standard required varies very much: in some States he may leave school at fourteen if he has passed the third grade, in others he must have completed the fifth. When he is sixteen he is free to work as he wishes, except that in certain States he must prove he is not illiterate, in which case he must attend an evening school.

If the child* wishes to continue his education during the day after the completion of the elementary work, he passes from the grade school to the high school in which he can take four years of work—he may enter for the academic, commercial, or the technical (manual arts) course, which for girls is usually represented by the domestic arts. Thence, in many States, he may go on to the free State University, taking either the general or the technical degree—all this without other charge than (in some States) the purchase of school books and of such materials as are necessary for some of the practical arts training.

If he wishes to work between the ages of fourteen and sixteen, and some States have a law that between these ages a child must be in school or at work, he must, as a rule, obtain a "work certificate," without which no employer may legally employ a child under sixteen. This work certificate may be obtained in various ways in different States, application being made to a judge, or to the board of education, or the board of health. Only eight States have adequate laws for birth registration and some of these are of recent date. Hence proof of age must be furnished by the production of a baptismal certificate if a birth certificate is not available, or, failing that, by an affidavit made by the parent. In addition a record of health filled in by an appointed doctor, and a schooling certificate given by the head teacher of the last school attended must usually be shown. A certificate from an employer undertaking to give the child employment must also be obtained in five States.

This method at first sight strikes an English visitor, accustomed to a birth certificate as the only necessary proof of

* The term "child" has been used throughout this report, as in America, to include all boys and girls up to sixteen years of age.

the right to work as very clumsy, but a nearer acquaintance with it shows the immense possibilities provided by the scheme for the close supervision of the young workers between fourteen and sixteen years of age, possibilities which are slowly being taken advantage of in a few localities and which are dealt with later in this report.

There is a constant dropping out from the grade schools, of those who being fourteen are anxious to work as soon as possible, and from the high schools of those who for various reasons, cannot or do not wish to continue learning. Americans have been asking why it is that the children do not remain longer at school. Of those leaving the grade schools to work, about 60 per cent. have not completed the school curriculum. Numbers leave the high schools during or after their first year there.

Attention was first directed to the reasons why children leave school by the inquiry made into the relation of children to the industries for the Massachusetts Commission on Industrial Education in 1905.* Cases of children at work in Massachusetts were investigated, and it was shown that 76 per cent. of these might have continued in school if the advantage had been realised. In an inquiry made under the Commissioner of Labor for the United States Senate† in 1909 the cases of 620 children from certain selected schools who had left to go to work were investigated. The children's earnings were considered necessary for the home if the weekly per capita income after payment of rent and sickness and death expenses reached only 1½ dollars a week—if it lay between that sum and 2 dollars they were considered necessary if the standard of management of the home was not the most economical. Taking the four towns in Northern States from which returns were made, viz., Pawtucket and Woonsocket in Rhode Island, and Plymouth and Hazleton in Pennsylvania, one finds the following reasons given for leaving school :—

	No.	Per cent.
Earnings necessary to family	103	} 27·3
Help needed	3	
Self support necessary	9	
Child's help desired though not necessary	125	29·7
Child's dissatisfaction with school	118	} 38·7
Child's preference for work	45	
Other causes	18	4·3
Total	421	100

* Report of the Massachusetts Commission on Industrial and Technical Education, April, 1906.

† Report on condition of Woman and Child Wage Earners in the United States, Vol. VII., Conditions under which children leave school to go to work. Washington, Government Printing Office, 1910.

In this return therefore in only 27·3 per cent. of the cases was the child obliged to leave school. In 38·7 per cent. of the cases he (or she) left of his own choice.

In a careful investigation made into the cases of 302 children leaving school in two typical districts in New York City by the Vocational Guidance Survey under the Public Education Association of New York City* of which a preliminary report only has so far been published, it was shown that only 20 per cent. of the children left school through economic pressure, their families having a weekly per capita income of \$1.50 or less, only 33 per cent. having \$2 or less. More than two-thirds of the children and more than three-fourths of the parents had no conviction that it was worth while to spend more time in school.

An inquiry made of 1,000 boys attending evening schools in New York City† showed a somewhat different result: 64·8 per cent. said they had left to support the family or owing to the death of a parent, 27·6 per cent. because they wished to; but this inquiry can scarcely be regarded as typical, since as the compulsory evening attendance law is not strictly enforced, the boys dealt with would be mostly those wishing to continue learning and therefore attending voluntarily.

An investigation was made by a sub-committee of the City Club of Chicago in 1910 and 1911 which covered this question. It is stated that of 205 boys who had left school and were visited through permission of their employers, 90 per cent. said they had not left school because of the necessity of going to work, but that they were tired of school—75 per cent. said they would have stayed in school if they could have been getting trade training.‡

It should be remembered that the reasons for which any one child leaves school are complex—often several together form an invincible argument when one alone would not have been sufficient to cause the step, but nevertheless it may be safely assumed from the preceding typical investigations that of the children leaving school to work at or about fourteen years of age in the northern states only about 30 per cent. do so through financial necessity. The rest leave because they or their parents are dissatisfied with school.

Following this therefore comes the question, what do these children do on leaving school, what work do they take up, and what prospects lie ultimately before them?

* Bulletin No. 9. Public Association of New York City, reprinted from the 14th Annual Report of the City Superintendent of Schools, New York City, 1912.

† Fourteenth Annual Report of the City Superintendent of Schools. Report on Evening Schools for the year ending July 31, 1912. (City of New York.)

‡ A report on Vocational Training in Chicago and in other cities by a sub-committee of the Committee on Public Education, 1910-11, of the City Club of Chicago, p. 37. (City Club of Chicago, 1912.)

The report of the Massachusetts Commission on Industrial Education shows that 68 per cent. of those found at work under sixteen are in unskilled industries, juvenile occupations and textile mills, 22 per cent. are in the low grade skilled industries and only 9 per cent. in the high grade skilled industries.*

In the report to the United States Senate already quoted from, the 222 boys investigated in the northern towns entered the following grades of work:—

	Nos.	Per Cent.
Wage for adults: \$15 a week and upwards - - - - -	11	5
Wage \$10 and under \$15 - - - - -	15	6·7
Wage under \$10† - - - - -	196	88·3
	<hr/> 222	<hr/> 100

The 302 children investigated in New York City had held by the time they were revisited 406 jobs, in 314 of which there was absolutely no training.

"In 41 there was some chance to 'pick up' if the rush were not too great, in 30 some boys had a chance to work on one process, in 21 there was some supervision, but in the majority of these, the children were either working in a small shop or with relatives. An attempt was made to discover whether there was any educational value in the jobs even where there was no specific training. In 49 cases the work was enervating to the extent of being positively bad in its effect on the boy, in 321 cases it was enervating because it was entirely empty of meaning. In only 36 cases did the work take hold of the children and stimulate them. These children worked in 215 manufacturing establishments, 76 per cent. of them belonging to the 20 largest industries in Manhattan."

The inquiry made of the occupations of 1,000 boys attending the New York City evening schools showed 76·2 per cent. employed as errand or office boys, 15·3 per cent. at unskilled factory labour, 6·2 per cent. unemployed, and only 2·3 per cent. working at a skilled trade; 17·7 per cent. of these boys were over sixteen years of age.

* "Unskilled industries are classified as those in which the work is the repetition of a single or simple operation, easily learned in a short time, and in which the knowledge of one part is not essential to that of another. Consequently the grade of ability and responsibility required is low, and the wages correspondingly low. . . . Among the low-grade skilled industries, the mill industries are similar in character and influence to those of the unskilled industries; but the difference exists in the cleaner and more desirable conditions of work, in the greater care required in similar processes, and in the existence of a larger number of skilled occupations in the industry into which the ambitious employee may force himself In the high-grade skilled industry a high grade of skill is required in any one occupation and must be acquired by some years of training or experience. Or an acquaintance by the employee with one occupation in the industry is often essential to the knowledge of another." Report of the Massachusetts Commission on Industrial and Technical Education, pp. 32-34.

† These wage rates indicate roughly the groups of skilled, low grade skilled, and unskilled occupations, though exact classification is impossible.

In Chicago it was found that 29 per cent. of all the children of the city between the ages of fourteen and sixteen had left school. Half of these were idle when the school census in 1910 was taken. Of the rest 29 per cent. were in factories, 34 per cent. in stores and offices, and 36 per cent. in miscellaneous occupations. In Cincinnati, of all children leaving school before they were sixteen to work, 73 per cent. were only fourteen to fifteen years of age, and almost all went to uneducational employment.

Inquiries made of the social workers, education officials, and Trade Union men whom the writer met during her visit, showed that in the middle west (Wisconsin, Illinois, Indiana, and Ohio) no boy can enter a trade until he is sixteen years of age, while few girls entering as young as fourteen can hope to have any opportunity of advancement. This appears partly due to the labour laws of those States which prohibit children under sixteen from working on moving machinery and restrict their working hours to eight a day and so have made it unprofitable for employers to take them for skilled work; and partly to the attitude of the Trade Unions who throughout maintain that sixteen is a young enough age to enter a trade. The following reasons were given the writer by the men to account for this:—"The more education a boy has before entering a trade the better"; "a boy does not know his own mind before sixteen"; "he wouldn't settle down to a trade before sixteen, eighteen would be better, but we can't make it too high."

In the East, where some of the Unions are not so strong, nor the labour laws so prohibitive, a boy of fourteen has rather more opportunity of entering skilled work, but very few do even here—so few that they are almost a negligible quantity, for employers care less and less to employ children under sixteen years of age for any but the most unskilled work. The relatively high wage expected for children probably helps, too, to make it unprofitable for them to put boys and girls of fourteen at work where teaching is required. Hence the work taken up by the children who leave the grade schools at fourteen is all uneducational, unprofitable work—messenger, errand boy, monotonous factory employment and the like. The children, too, pass readily from job to job, sometimes with a period of unemployment in between. Figures given the writer for the State of Connecticut showed that of 15,000 children to whom work certificates had been granted in the past 18 months, quite 4,000 had held three places during that time; one child had held 12 jobs.

The writer cannot say what proportion of these children ultimately enter skilled trades, nor can she hear of any inquiry made into the careers of adult workers^{*}; when the question was

* Since writing the above, Bulletin No. 17 of the United States Bureau of Education for 1913 has been published in which it is stated (p. 30) that an inquiry made by the Women's Educational and Industrial Union of Boston into the careers of 309 women in skilled trades in Boston and Worcester, Mass., showed that only four began at unskilled work.

put to Trade Unionists and others no one could give an answer. But be that as it may, the fact remains that these two years, fourteen to sixteen, are profitless to the children who work and the good gained by them in school is thrown away. What then can be done to keep these children longer in school? Or if they cannot, or will not, remain in day school, what else can be done to give them some further education and to leave them less helpless to meet the demands and pressure of the wage-earning life which they so thoughtlessly enter at fourteen? Americans are asking these questions and are answering them in various ways.

First the legal age for leaving school can be raised to fifteen or to sixteen. Some reluctance to do this exists since it is felt that as yet the schools are hardly meeting the needs of the industrial class of children, and that more experience is necessary before all should be compelled to remain longer in the day school. In many States, however, Bills to raise the school-leaving age are introduced before the legislatures each year, and in Massachusetts and Illinois there seems a real chance that in two or three years this may be done. Meanwhile where the school-leaving age cannot be raised, the alternative is to make the day schools so attractive to the children that they shall wish to remain after fourteen years of age, and to give such education that the parents shall feel that time spent at school is not lost. The attitude of many is at present summed up in the words of the hotel porter at Chicago who asked the writer what a certain conference was about. "On what to teach children in the schools," was the reply. "Teach them something useful, that's what I say," was his answer; and "something useful" means to the immigrant parent some occupation which the child can follow later or which he or she can take up at home, something related to real life as they know it. Besides, the boy or girl of fourteen is longing to be doing something useful and real, and in various places the school syllabus has already been altered to satisfy this desire. Some form of manual work has long been introduced into most elementary schools, and about an hour and a half a week has been given in the lower grades to paper and cardboard box-making, raffia work, elementary weaving, &c., and in the upper grades to sewing for the girls and woodwork or Sloyd for the boys; but this, good though it is as far as it goes, is not enough, and therefore "pre-vocational" courses, as they are termed, are being tried in certain schools. In these the children devote half the school day to active manual work, house work, cooking and sewing for the girls, preliminary trade training of some kind for the boys. These courses have already been very successful in keeping on in the elementary school those children who would otherwise have left at fourteen without completing the grade work, besides affording them opportunities of trying their powers at various trades.

Those children who have completed the grade school work at fourteen, but to whom the high school does not appeal, and those

who have not graduated but want industrial training may be attracted by regular trade schools. The movement for the extension and development of day trade schools under public authorities dates from the beginning of this century, and is, therefore, still only in its infancy. Many of the high schools have for some years had practical courses in which forge, foundry, engineering and wood work may be taken as part of the general school curriculum, but this manual work has been introduced mainly for educational purposes. For girls this training has been given by courses in domestic arts. Undoubtedly some children who wish to follow manual trades enter these technical high schools first, but the courses have been planned for college preparatory work, and cannot, therefore, be regarded as undertaking to give instruction in a trade. How far it is the duty of public authorities to give definite vocational instruction was a subject much under discussion in the winter 1912 to 1913. In the Eastern States trade schools of two types, the whole-time day trade school and the co-operative half-time school, have been started under State systems of industrial education, and both have undoubtedly proved very successful in drawing children to them who would otherwise have entered some uneducational employment. Enquiries have been made and legislation introduced to increase such schools. The history of the movement and the relation of these schools to conditions of industry and organised labour is discussed later.

The children who have left school to work can be offered opportunity for further education and training at continuation classes. A system of evening classes for elementary and high school work has been in existence some years and is taken advantage of by those, mainly adults, who wish to study further, and sometimes to prepare for college. Evening trade classes to supplement the work of those following trades, have been till lately, little developed by the school authorities, such classes, where they exist, being still often mainly maintained by the Young Men's Christian Association and privately endowed technical institutions. But the United States appears to be in advance of England in the development of part-time classes attended during the day by employees through the co-operation of employers or by the enlightened action of the Trade Unions. Such classes, however, are mainly for those in skilled industries, and do little to touch the children between fourteen and sixteen years of age in unskilled work. In three States, therefore, compulsory continuation classes have been instituted. In the State of New York these were started some years ago in the evening for boys who have not completed the grade work—in Ohio and Wisconsin they have been introduced during the day. It is noteworthy that now in all discussions on the matter no suggestion is ever made that compulsory classes for children under sixteen could be introduced in the evening.

The establishment of the Juvenile Advisory Committees in England has been watched with interest. Little has so far

been done in the States in placing children in work on leaving school—whatever has been done has been undertaken by philanthropic agencies. Opinion is divided as to how far this is the function of the education authority—fear is expressed that any such work might lead to children leaving school earlier than necessary and to their exploitation, since it is only unskilled jobs which are open to them at fourteen years of age. Attention has, therefore, been given to “vocational guidance” rather than to placing, *i.e.*, to the study of the child during his time at school, to a discussion of the methods to be used for noting his characteristics and to interesting and instructing the teachers in the matter. By the gradual collection of information the child is then advised as to the most suitable occupation for him to follow, and urged, if possible, to fit himself for that by further study. This method was first introduced in Boston, and is developed further there than elsewhere.

It will be seen that the methods followed to meet the problem of the fourteen to sixteen year old children are various—one is tried in one State, one in another, and some have not yet passed beyond the period of experiment. Conditions of industry, population and temperament vary between England and the United States, but, nevertheless, a study of the methods followed to tackle a problem in many ways so very similar to ours is suggestive; because of the intense readiness of the education authorities, of employers and the people to adapt old methods to new conditions, to utilise voluntary efforts for official purposes, and to try experiments, it is both interesting and stimulating.

CHAPTER II.—HISTORY OF THE MOVEMENT FOR INDUSTRIAL EDUCATION.

History of the Movement.—The introduction of day trade and other industrial schools into the system of education in the United States has naturally a bearing, not only on the age at which children leave school, but on the demand for and supply of skilled labour. A country with a splendid supply of natural resources, it has been prodigal of them. Its exports have been largely raw or partly raw materials, and it has relied on importation in spite of high tariffs for many of its highly finished goods. Together with this, as has been said, it has neglected the immense national resources lying in its children and the children of the immigrants daily reaching its shores. The development of the country has gone on with a marvellous rapidity and energy, but a time of change has gradually come. It is realised that a more intensive system of production must be followed, sufficient skilled labour must be available to develop the resources of the country within the country, and this skilled labour must be produced from the children already there.

The industries of the United States are more subdivided than those of England, machinery is used to a greater extent, its speed even is greater, and there is less opportunity for the highly trained craft work which we still need. Nevertheless the very amount of machine-made work and the speed at which the machines are run call for the existence of highly trained men to act as foremen and supervisors, while much of the work demands a steadiness and resourcefulness which can only be obtained through education.

These very same conditions make the training of boys in the works more difficult. Immigration of adults does not supply the need sufficiently, since the majority of immigrants are unskilled labourers, while the curiously short-sighted policy of the Trade Unions in restricting the opportunities of apprenticeship to very narrow limits where they are strong enough, reduces still further the opportunities of the American youth for securing training in a trade. That there is a demand on his part and on the part of foreign immigrants for trade training, and insufficient opportunities for acquiring it is shown by the fact that schools which purport to teach a trade are run as commercial enterprises. They charge high fees and give short courses of trade instruction only. These schools are to be found especially in such trades as watch making, jewellery and engraving work, photography, hairdressing, the boot and shoe industry, etc. Other short course schools have been founded by private benevolence. It has also been possible, though it is less so now than formerly, for a youth engaged in unskilled work to attend evening classes in a trade and so pick up

sufficient skill to obtain a position in the trade itself, from whence he may work himself upward. There is a far less hard and fast line between the skilled and the unskilled worker in America than in this country, though anyone entering a trade in this way can hardly hope to rise to any good position.

The early movement for the provision of day industrial training was entirely philanthropic and started among the negroes before the whites. In 1868 at the close of the Civil War, General Armstrong started the Hampton Normal and Agricultural Institute, a residential college for giving training in agriculture and trades to coloured men and domestic work to coloured women, together with some general education. Students were also trained to become teachers of these subjects in the country schools. This Institute grew rapidly and is still doing most useful work. In 1909-1910 241 students were taking the trade classes. In 1880 the Tuskegee Institute, where trades also are taught, was established by an Act of the Alabama Legislature and developed by Mr. Booker Washington, a former pupil at Hampton. The coloured population of the United States is placed in a worse position than the white. It is practically impossible for a coloured boy to become apprenticed to a trade except with his own race. Hence these institutions aim at turning out thoroughly trained workmen.

The establishment of day trade and evening technical schools for white boys dates from the eighties, with few exceptions, and is also due to philanthropic enterprise. For instance, in 1881 the New York Trade School, an independent endowed institution, was founded for giving training in the building trades, both in the day and evening, to beginners and those already in the trades. In 1883 the Hebrew Technical Institute, New York, was started for the industrial training of Jewish boys; in 1887 Pratt Institute, Brooklyn, New York, a polytechnic with day and evening technical courses was founded by the benevolence of a philanthropic manufacturer, and in 1888 Mr. Williamson founded the Williamson Free School of Mechanical Trades in Pennsylvania. This last provides free board and lodging and training in engineering or a branch of the building trades, together with some general education, to 230 boys between sixteen and eighteen years of age for three years. It is regarded as one of the most successful day trade schools and has been closely studied by those wishing to start trade schools elsewhere. The magnificence of its endowment, however, makes it an exceptional institution. Students from those private trade schools which offer thorough training appear to have no difficulty in obtaining situations, indeed at the Williamson School the writer was told that applications from employers are received in excess of the supply.

Manual training was introduced into the high schools about 30 years ago, and shortly afterwards into the elementary schools,

but, as has been said, it was introduced for its educational value as part of a regular high school course, and has therefore done little to help the boy destined to work with his hands to train for a trade.

Several of the large corporations, feeling the want of highly trained mechanics, have instituted apprenticeship courses together with some theoretical training in their own schools. Such is the system of the New York Central Railroad where apprentices in the shops attend school every day from 7 to 9 a.m., or of the General Electric Company, West Lynn, Mass., where apprentices spend a few hours a week in school between the hours of 8.30 a.m. and 6 p.m.

In 1892 and 1902 the United States Bureau of Labor published long reports on the results of its investigations into manual, industrial and technical training in the United States and abroad, but the agitation for the improvement of facilities for industrial education by the public authorities dates from 1906. In that year the Commission on Industrial and Technical Education appointed by the Legislature of Massachusetts made its report, in which was shown the number of children drifting to unskilled work on leaving school, and the failure of the principal skilled industries of Massachusetts to provide adequate training for children under sixteen years of age. The Commission recommended the modification of some of the work of the elementary and high schools so that it might have a more practical (either industrial or agricultural) value, and the introduction of a Bill in the Legislature providing for the establishment of a permanent Commission on Industrial Education which should encourage the establishment of industrial schools, *i.e.*, day trade schools and evening trade classes, to be independent of the existing school system. It recommended that the State should reimburse the local authorities for the expense of the maintenance of such schools up to one half the cost. These latter recommendations became law in the same year, a State Commission being created to supervise the formation of these "independent" industrial schools. The powers of this Commission were later transferred to the State Board of Education.

In 1906 the National Society for the Promotion of Industrial Education was formed :—

"To bring to public attention the importance of industrial education as a factor in the industrial development of the United States; to provide opportunities for the study and discussion of the various phases of the problem; to make available the results of experience in the field of industrial education both in this country and abroad; and to promote the establishment of institutions for industrial training."^{*}

This society by means of conferences, publications and appeals to the Legislatures has carried on a vigorous propaganda, and has undoubtedly done much to increase public interest

* Constitution of the National Society for the Promotion of Industrial Education.

in the need for the better industrial training of workers. Hence, within the last few years, with the typical energy of Americans, volumes of literature, enquiries and reports on industrial education have been issued by public and private bodies and the subject has been discussed from every point of view. Attention has been much directed abroad, and the methods followed by Germany have been especially studied and quoted, and have had a marked influence on the views held and suggestions made in America.

Eight States. Connecticut, Indiana, Maine, Maryland, Massachusetts, Michigan, New Jersey, and Wisconsin, have appointed and received reports from special commissions for the study of industrial education. In 1910, partly at the request of the American Federation of Labor, the United States Commissioner of Labor issued an exhaustive report* dealing with the existing public and private opportunities for industrial education in the United States, while the recent reports of the Commissioner of Education have also dealt with the matter.

The National Association of Manufacturers of the United States of America reported on the subject in 1910.† They state that:—

“The interests of manufacturing industry require a new education for boys who are to work with tools and machines. That this industrial education must consist of skill and schooling and that these two parts are of equal importance—that they must be organically combined—and that each will coordinate and supplement the other.”

They therefore approve the institution of “real trade schools” in which half-time should be devoted to “schooling adapted to the life of the pupil” and half-time be spent in well equipped productive shops where pupils are taught the best methods of rapid high grade production by skilled working mechanics. They insist that the training must be extremely practical, and that the schools need not produce anything but useful high grade products. The fact that instruction has so far been given by a course of “exercises” has held manual training back. They lay stress on the importance of the development of a “shop spirit” in the school. The length of time to be passed in these schools should be four years from the age of fourteen, after which the boy should take a place in the works on his own merits.

The American Federation of Labor, the association of Trade Unions, appointed a Committee in 1909 to consider the question of industrial education. This Committee issued their final report in 1912.‡ They there denounce the private and

* 25th Annual Report of the Commissioner of Labor, 1910. Industrial Education. Washington: Government Printing Office.

† Proceedings of 15th Annual Convention of National Association of Manufacturers of the United States of America in 1910.

‡ This was regarded as so important that it was reprinted as an official paper, viz.: Report of Committee on Industrial Education of the American Federation of Labor. Reprinted as Senate Document, No. 936. 62nd Congress, 2nd Session. Washington: Government Printing Office, 1912.

commercial short course trade schools which, they say, place half trained wage earners on the labour market who lower wage standards and often act as strike breakers in the event of any industrial dispute, but they approve of the development of the public system of giving trade training. The following extracts show their point of view:—

“We believe that technical and industrial education of the workers in trades and industry, being a public necessity, should not be a private but a public function, conducted by the public, the expense involved at public cost and as part of the public school system. In order to keep such schools in close touch with the trades and industries, there should be local advisory boards, including representatives of the industries, the employers and organized labor. . . . (p 11).

“We recommend the establishment of schools in connection with the public school systems, at which pupils between the ages of 14 and 16 may be taught the principles of trades, not necessarily in separate buildings but in schools adapted to this particular education by competent trade-trained teachers. . . .

“If in the course of time schools under public administration with a broad and liberal course of instruction (with an advisory committee composed of employers as well as trade unionists) shall demonstrate practical efficiency in training workers for the highly skilled trades, we favour the recognition of that portion of time spent in the schools, which, after an examination by the union at interest of the practical and theoretical ability of the apprentice, can be considered comparable to actual training in particular trades as a substitute for a period of the apprentice's time spent entirely in the industry” (p. 18).

The Committee claims that the system of education in the United States has not been truly democratic, since it is founded on the preparation for college or commercial life, and therefore demands the introduction of industrial training within the public school system in justice to those who by lack of money, desire or capacity are unable to profit by the high school courses.

The effect of, and need for, the introduction of industrial training are so well described that it is worth quoting fully from the report:—

“The committee believes that there are pressing educational needs which can at least partly be solved by the introduction of industrial training. At present a very large proportion of the children leave school between the ages of 14 and 15. They change from one occupation to another, having no particular qualification for any vocation, and gain little in efficiency. Industrial education between the ages of 14 and 16 ought to awake a new school interest and help to retain them longer in school; moreover, if industrial training took the children between the ages of 14 and 16, when they are of little value in a business way and at a time when such education as they have received is of advantage to them so far as it goes, but hardly fits them for actual working places, then it would give them the proper training to prepare them to enter some branch of trade or vocational work. At the time our present public-school system came into operation it met the needs of the people; the industries were carried on in the home, and the children were taught the manual arts there; the boy was taught his trade by his father, and the girl and her mother carried on in the home much of the work now performed in the factory. Economic conditions have changed and the schools must change with them. The ranks of skilled labor are being depleted and the work of the trades is being done by unskilled men or semi-skilled machine specialists.

“The trade unions have been waiting in vain for 25 years for the manual-training schools to furnish recruits to the “depleted ranks” of skilled labor.

It is time now to take steps to bring back the standard of efficiency. We want a system which will develop the labor power of our people, so that every worker may become interested in his work and approach the limits of human efficiency. Our public-school system of today teaches too much and educates not enough, and fails entirely to prepare its pupils for productive labor. It must be changed, and quickly, and the change must be radical. We can not add a few experiments in trade training in our larger cities or introduce intense manual training in manual-training school departments to supplement a Latin and Greek curriculum. Our boys and girls must leave school thoroughly prepared by industrial training to do well some kind of productive work. A healthy community is impossible without the union of the school-house, the home, and the workshop. Modern life has not yet accommodated itself to the great revolution of our industrial system. Nothing but a thorough industrial education and understanding of economical interest of society can lead to the necessary union between labor and capital, and give peace and prosperity to the present disturbed and suffering industrial world. . . .

"The State has provided schools to teach trades to the mentally, morally, and physically deficient; our corrective institutions, orphan asylums, and blind schools are equipped to teach useful occupations. By what right can we refuse the same chance for the normal boy or girl? Would it not be more sage to engraft industry upon our public school system, and rather prevent pauperism, crime, and premature orphanage than make them the bridge to industry? We think so, and we do not beg it as a favor, but we demand it as a right. The 90 per cent. who are going into manual occupations have the same right to the best preparation for their life's work that the State can give them as has the 10 per cent. who go into the professions" (pp. 18-19.)

Both these organisations of masters and men have declared in favour of the development of part-time and evening trade classes, the latter of which it must be remembered are of a more backward development than in England. The National Association of Manufacturers at their 15th Annual Convention in 1910 said:—

"What shall we do to vitalize and enrich the training of apprentices in shops of all kinds as they now exist throughout the country? Besides the half-time regular course for pupils, already considered above, what supplementary training shall the trade school provide for those apprentices and workmen who are already in the shops and factories?"

"(a) Evening Schools may be established both for schooling in common branches and for special skill and shop practice in various branches of mechanical trades.

"(b) Half-day schooling each week may be provided for apprentices and men from the various shops where the proprietor is willing to allow wages to continue while the apprentice devotes the half-day per week in school."

At their convention in 1912 the National Association of Manufacturers went yet further, and recommended compulsory continuation classes until the 17th or 18th year one half-day a week without loss of wages, such classes to be both liberally cultural and related as directly as possible to the occupations of the students. (See pp. 179-80.)

The American Federation of Labor recommend in their 1912 report:—

"Supplemental technical education for those already in the trades. The demand for such instruction is measured by the necessity for training in particular trades and industries, and the chief aim of such instruction should be to present those principles of arts and sciences which bear upon the trades either directly or indirectly. Such schools are commonly known as continuation schools, whether their sessions are held in the day, evening, or on the part-time plan."

The suggestions made by these associations on the half-time cooperative schools are dealt with later.

Present Position of Industrial Education.—What so far is the result of these enquiries and reports? Of 46 states, 29 have enacted laws relating to industrial training. Many of these refer to the manual training of the high schools or to supplementary trade training only. Seven states have established State systems of industrial education. New Jersey can claim the first enactment, the most advanced legislation has been passed by Massachusetts, Connecticut, New York and Wisconsin. Pennsylvania and Indiana passed Acts to promote industrial education in 1913.* Bills dealing with the subject are laid before other legislatures and efforts are now being made to obtain federal aid. It is pointed out that the training of the worker and the increase of his industrial efficiency brings benefits not necessarily to the special locality but to the whole country, and since there is nothing to restrict a worker trained in one state from following his calling in some other far distant one, it is only fair that aid should be given to localities from federal funds. So long ago as 1862 grants of land were made by Congress to each state to be used for colleges of agriculture and mechanic arts, and appropriations of money were made in 1890 and 1907 which largely helped in the development of these colleges. More than 100 million dollars also has been distributed among the states and applied by each to a permanent school fund, which has been distributed among local communities for the purpose of alleviating and adjusting their burdens. This money has been spent mainly on the salaries of teachers of every kind of school. During the last two sessions of Congress, bills have been introduced proposing an extension of this federal help to schools for directly training industrial wage earners. If such federal help is forthcoming it will undoubtedly prove a stimulus to a rapid development of the means for providing industrial training.

The present position in those states having State systems of industrial education is as follows. It will be seen that with the exception of Wisconsin, all are in the industrial districts of the east.

a. *Connecticut.* †—The State Board of Education was authorised and directed by an Act of the legislature to establish two public day and evening schools, "for instruction in the arts and practices of trades." The State Board has complete power over the schools and may spend \$50,000 annually for the purpose of erecting buildings and maintaining the schools. The town in

* For details of these Acts, see Appendix D., page 183.

† *Connecticut* is the third smallest State in the Union and the 31st by population, but only three States have a greater density of population; 89.7 per cent. of its inhabitants are in cities of over 2,500 inhabitants. The industries employing the greatest numbers of work people are the manufacturing of foundry products and machine shop, brass and bronze products and cotton goods. Bridgeport, with 102,054 inhabitants, is its largest city. It ranks 12th among the states in the value of its products.

which a trade school is established under the terms of the Act, may contribute any sum for the enlargement or improvement of the school. The two trade schools have been established under this Act, viz., at Bridgeport and New Britain.

In this case State initiative came from the State authorities. In the other States aid is given to such municipalities as have taken the initiative in establishing day and evening schools.

*b. Massachusetts.**—Under the State law, industrial schools, day, part-time, or evening, may be established by cities and towns. These schools must be "independent" of the common school system to secure the subsidy of the State which equals one half the cost of maintenance. The State board of education has general supervision over these schools; it may assist in their establishment and maintenance in co-operation with the municipality concerned.

Under this law in the school year 1911-12 there were in Massachusetts 9 full-time trade schools for boys and 5 for girls, 3 of the latter being departments of boys' schools. There was also one co-operative half-time school, that at Beverly. In the autumn of 1912 a full-time day trade school was started at Boston and a co-operative half-time school at Quincy.

There were in addition part-time classes in 4 cities and evening classes in 14 cities receiving state aid.

c. New Jersey† in 1903 authorised the expenditure up to \$7000 of State funds in any one year to aid municipalities in their establishment of any school or schools for industrial training and for their support, provided that the municipality concerned provides an equal sum for maintenance and not less than \$3000 for the establishment of the school.

These schools must have the approval of the State Board of Education, and be under a board of trustees.

Not much progress, however, has been made in providing schools, there being only 2 evening schools, 1 day school for coloured boys and 1 elementary industrial school.

d. New York State‡ provided by an Act on industrial education for the establishment and maintenance by the local Board of Education (or where there is no board, by the officer

* *Massachusetts* is the 6th State in the Union as regards size of population which in 1910 was 3,366,416, but it is 5th smallest in area. It is 2nd as regards density of population. It takes 4th place with respect to the value of its products; the industry employing the greatest number of people is the manufacture of cotton goods. Boston, its capital and largest city, has 670,585 inhabitants.

† *New Jersey* is the 11th State in the Union as regards size of population (in 1910 2,537,167), though it is 45th in size. It is 6th in the value of its products.

‡ *New York State* ranks first in the Union in number of population and 29th in size. The population of the State was 9,113,614 in 1910, 4,766,883 of these being in New York City. It is first in the value of its products, its main industry being the manufacture of men's and women's clothing, this occupying by far the greatest number of wage earners both in the State and the city. Printing and publishing and foundry and machine shop work are the next largest industries.

having supervision of the public school system in any city or town) of:—

- (1) general industrial schools open to pupils who are fourteen years of age or who have completed the elementary school course.
- (2) trade schools for pupils sixteen years of age or over who have completed the general industrial school course or who have met the requirements of the local school board.
- (3) schools of agriculture, mechanic arts and home-making for pupils who are fourteen years of age or who have completed the elementary school course or met such other requirements as the local authorities may have prescribed.

The Commissioner of Education for New York State has power to aid such schools from State funds, through the payment of teachers' salaries.

Under this law 10 day industrial schools have been established for boys and 5 for girls—also a day school at Iroquois for Indian boys and girls: evening classes have been held in 5 schools.

e. Wisconsin^{*}, under an Act of 1907, gave the cities power to levy a tax not exceeding half of one mill for the maintenance of trade schools. Under this act the day trade school for boys at Milwaukee, originally started by manufacturers, was taken over by the city and a day trade school for girls was started there.

In 1911 Wisconsin passed a number of comprehensive acts dealing with industrial education and child labour, compulsory continuation classes, &c. A State board of industrial education was created, together with local boards of industrial education, whose duty it is "to foster and establish and maintain industrial, commercial, continuation and evening schools." The boards may also take over existing schools. By the end of 1912 the boards in many cities had taken up their work with enthusiasm, but were beginning rather with compulsory part-time continuation schools and evening classes than with day trade schools. The legislature will grant aid from State funds to half the cost of maintenance of the schools. An appropriation was made to aid 30 cities, no more applications being expected in the first year, but 39 applied, having already started classes.

The Stout Institute, a privately founded institute at Menomonie, for the teaching of industrial arts and the training of teachers for such arts, has been taken over by the State under a special clause of the new acts.

* *Wisconsin* is the 13th State in the Union in population (2,333,860), and 25th in size. It is largely an agricultural State; the chief products are lumber and timber. It takes 8th place in the value of its products. Only one city has over 50,000 inhabitants, viz. Milwaukee with 373,857. The main industries of this city are foundry and machine shop work, the manufacture of leather and tanned goods and printing and publishing.

There are certain questions in connection with the establishment of a system of industrial education which still remain to be solved, viz. :—

1. Should industrial schools be under the existing education authorities or should specially constituted boards be appointed to found and maintain the schools, boards on which industry should have direct representation through employers and men ?
2. If the boys are to have practical shop training and to get it by making articles for real use, what is to become of the articles made ?
3. How are teachers to be obtained ? Should they be trained teachers who have had if possible some shop experience, or men from the shops who may not know how to teach ?

Authorities for Industrial Education.—With regard to the first point, it is interesting to note that in Massachusetts by the recommendation of the Commission of Inquiry, a special Commission on Industrial Education was appointed in 1906 for three years whose duties were to advise and aid in the introduction of industrial education in the independent schools, these schools having to be independent of the ordinary public school authorities. Grants in aid to these schools were made from State funds on the recommendation of the Commission whose existence was later lengthened to 5 years. In 1909 the duties of this Commission were transferred to the State Board of Education which since that date has had control of both general and industrial systems of education, and a lessening of the interest in the latter does not appear to have resulted. An advisory committee is appointed locally to aid the school board.

In Connecticut, New York and New Jersey control has been given to the ordinary Board of Education. In New York a connection with industry has been established by a provision that the local Board of Education shall appoint an advisory board for each school of five members representing the local trades, industries and occupations.

Wisconsin in its new legislation has created a separate official body to deal with industrial education. The day trade schools of Milwaukee, started under the Act of 1907, still remain under the local school board, but a State Board of Industrial Education has been appointed which shall control all State aid given to the new schools. The State Superintendent of Education, who is ex-officio a member of this board, appoints an assistant who acts as assistant for industrial education. Local boards of industrial education are appointed on which the local superintendent of schools sits ex-officio, the other members being two employers and two employees. This local board determines the amount of money necessary to be raised locally for these schools, and this is raised by a special tax for industrial education. One half the

amount expended is repayable from State funds on the school being approved by the State Superintendent of Education and the State Board of Industrial Education.

Wisconsin claims that by this system the interest of employers and working men has been aroused in a way that would never have been possible had they been simply asked to sit on advisory boards and not to control expenditure. It claims also to have provided for due connection between the ordinary school authorities and the industrial boards by the presence ex-officio of the school superintendents on these boards. At the time of her visit to three cities in Wisconsin in January 1913, the writer gathered that both authorities there were working very harmoniously together, there being no rivalry or canvassing for children between the two.

A suggestion made by the Commercial Club of Chicago that separate authorities should be created in Illinois to deal with industrial education led, however, to great discussions on the subject, fear being expressed that the common school system would suffer by the creation of a separate industrial authority. It is indeed difficult to see why sufficient relation to industry cannot be obtained by the formation of advisory boards for each school, together with the representation of employers and employees on the Board of Education, by co-option if necessary.

Disposal of Produce of Schools.—The question of what to do with the articles made by the boys in day trade schools, though regarded as unsolved, yet appears far easier of solution in the United States than in England, owing first to the Trade Unions being in most places not so strong as with us or so determined to oppose all commercial enterprise, and secondly to the readiness with which the needs of the schools are used to supply work to the pupils. For instance at the Boston Day Industrial School for Boys an addition to the school was needed. This was built in wood by the boy carpenters and wired for electric light by the electricians. At the New York Vocational School, the boys in the wood working department were making boards and tables for the playgrounds department of the Board of Education, and the plumbers undertook any repairs needed in the building, Printing was in many cases being done for the local Board of Education or the school. The Boys' Trade School at Milwaukee was moving at the time the writer visited the city. The boys had been given a good deal of work in the new building and re-erecting the machinery would give the engineering pupils profitable and useful employment for a long time to come. Furniture necessary for any of the schools is handed over to the trade school boys to make. The Trades School for boys at Worcester, Mass., has made parts of machinery for firms in exchange for complete machines needed for the school. Some machines have been sold after completion, others built for use in the school, and jack screws, machinists' clamps,

&c., are made for practice and sold; second-hand machinery is bought and rebuilt. The two trade schools in Connecticut have each built a house for practice. Asked if this had not been opposed by the Trade Unions it was said that at first it had, but that opposition had died down when it was found that commercial prices were being charged by the schools for the work. Certain private trade schools also dispose of their work by sale at trade prices without incurring labour opposition. In certain cases it seemed that the completion of a large job for the school authorities might give the pupils more work of one kind than was necessary, but on the other hand the putting through of such a job gives real trade experience and some practice in speed. A foreman is sometimes chosen from the boys, to whom the responsibility of seeing the job through is given, and he divides it up among the others on the shop plan under supervision of the teacher.

If a big increase in the number of day trade schools is to come, the question of what to do with the produce must necessarily arise, for all the teachers agreed on the value gained by giving the boy a piece of real work rather than an exercise which would be melted down and used again. The National Association of Manufacturers in their report say the trade schools "need not produce anything but useful, high grade products" and believe:—

"Where such a trade school can be established, with modern buildings and equipment and a moderate working capital, well managed, it will not only be an efficient educational institution, covering the high school period, but it will be productive and largely self-supporting."

From this one assumes that they intend the schools to enter into competition in the ordinary market.

The American Federation of Labor in their 1912 report, quoting from the Report of the Commissioner of Labor, refer to the question as a most serious and troublesome one. The entry of the schools into competition with regular firms would be objected to by manufacturers owing to its being a form of subsidised competition, they say, though agreeing that the competition of boys in the schools with working men is no greater than that of apprentices in the shops, and therefore not to be objected to by organised labour.

A practical suggestion is as follows:—

"An agreement between school authorities and contractors who are erecting public buildings, whereby pupils of schools given instruction in building trades, shall be permitted during a part of the time to make practical application of their training, on buildings in course of erection, the pupils to receive credit for such work as part of their course, has been suggested as a feasible and unobjectionable plan. Similarly the work under the various city departments, has been suggested to provide practice for pupils in many other trades."

The co-operative plan in which technical experience is gained in ordinary workshops has the great advantage that it does not cause this problem of disposal of produce. The girls' trade schools work for customers and for stock, which is

sold periodically, and no trouble is experienced in giving the pupils sufficient practice.

Training of Teachers.—How to obtain sufficient teachers is another difficulty. In the manual training high schools the teachers are usually men or women who have qualified for teaching manual work by a course in one of the mechanic arts colleges. They therefore treat the subject from the educational point of view and can bring no shop experience or shop methods to their help. The development of day trade schools and part-time classes is causing a large demand for teachers. Both the National Association of Manufacturers and the American Federation of Labor demand that the training should be given by practical men or women who have had some years' experience in shop work. It is also urged that the heads of the schools should likewise be persons with real trade experience. The combination of sufficient general knowledge with trade experience is difficult to find, but in all the newly started trade schools for boys efforts are being made to secure such men as teachers. They, however, may often lack teaching ability. Short courses in pedagogy have been arranged at Columbia University, New York, and elsewhere which may be taken by such men for a few weeks during the summer or during the evenings while still at work.

The State Board of Education in Massachusetts recommends that evening courses should be offered which would enable workmen and workwomen to determine their suitability for teaching and would give them some slight instruction in the theory of the subject. That then those who wish to take up the work should be appointed as assistants in day trade schools and undertake teaching under direction, while devoting a part of their time to further study of methods and aims. A salary must be paid during that time which should make it possible for these people to leave their trades, and it is suggested that this should be provided by the State.

A course for the training of teachers for girls' schools has been arranged at Simmons College, Boston, in co-operation with the Women's Educational and Industrial Union.* It would seem necessary to offer high salaries to trade teachers in order that the right men and women may be induced to take up such courses and fit themselves for the work.

* For the programme of this course, see Appendix B. to Chapter VII., pp. 98-100.

CHAPTER III.—DAY TRADE SCHOOLS FOR BOYS.

THE VARIOUS TYPES OF SCHOOLS.

Day trade schools may be of two types. They may be "pre-apprenticeship" schools in which the training given is of a preparatory nature, the boys needing further training and experience in a workshop before ranking as journeymen. Such is the type of school desired by the American Federation of Labor. The newer schools, such as the New York Vocational School for Boys are of this nature. These schools, which are also called general or intermediate industrial schools, or trade preparatory schools, offer training in practical industrial processes, including such drawing, science and mathematics as will prepare boys for entrance into the trades or industries as efficient beginners.

"Trade Schools" proper (in the American nomenclature) are schools:—

"That afford specialized practical training in manual trades, with the object of preparing for immediate practical work at the trade as a wage earner. Such schools aim to take the place of apprenticeship in whole or in part."^{*}

The privately founded schools are many of them of this type; so are the girls' trade schools at New York, Boston, and Milwaukee, while the boys' school at Milwaukee approaches it.

"Vocational Schools" are defined in the report of the Commissioner of Labor as follows:—

"Vocational schools in one sense include commercial, technical, agricultural, and professional as well as industrial schools. In the use of the term as applied to a phase of industrial education, however, a vocational school is one which (a) prepares for a manual occupation which is not classed as a trade, or (b) gives training in some line of manual work for the purpose of directing the pupil toward a trade or other manual occupation rather than as a part of direct and intensive preparation for immediate entry into such trade or occupation."[†]

The "pre-vocational schools" or elementary industrial schools give such elementary manual training along with the grade work of the elementary school as shall enable the pupils to determine in what direction their aptitudes lie and shall promote their general educational development.

So many differences however exist in the schools that it is difficult to classify them. The following are two well known schools giving preparatory trade training:—

THE VOCATIONAL SCHOOL FOR BOYS, NEW YORK CITY.

This school was opened in 1909. During the year 1911-12, 812 boys were in attendance. The course is planned for two

^{*} Report of the Commissioner of Education for the year ended June 30, 1911, page 301. (Washington: Government Printing Office, 1912.)

[†] 25th Annual Report of the Commissioner of Labor, 1910. Industrial Education, pp. 14-15. (Washington. Government Printing Office.)

years, but special arrangements may be made to enable a boy to take a one-year course. Instruction is offered in the following trades :—

1. Carpentery and Joinery.—Boys in the two years spend time both in the hand workshop and the machine shop, which is fitted with machine saws, planes &c.
2. Cabinet making.—Boys learn hand and machine work.
3. Pattern making combined with wood turning.—Demonstrations are given the boys in a small foundry in the use of their patterns and the principles of construction.
4. Engineer's machinist.—The school possesses a large equipment of up-to-date motor-driven lathes, shaping, drilling, and milling machines, &c.
5. Plumbing.
6. Smithing.—Here all is hand work, there is no power hammer.
7. Electrical wiring and installation.—This is the most popular course of all, far the greatest number of boys taking this subject.
8. Printing (composing) with which a certain amount of book binding is taken. The boys cut and sew the books with small machines, and bind by hand.

Each boy in any trade takes a certain amount of work in an allied trade; thus a smith takes a little work in the machine shop, and a plumber learns the use by practical work of the carpenter's tools with which he may be called on to work.

The exact apportionment of time varies for different groups of boys, but is roughly the following :—

From 20 to 23 hours a week are given to the trade work and 7 or 8 hours a week to drawing. Each boy takes freehand drawing for about 2 hours a week, mostly drawing industrial objects, and gives the remaining time to mechanical drawing. This is varied of course for the printers. Articles to be produced in the shop are drawn in the draughting room.

The remaining part of the school week, 7 to 8 hours, is given to theoretical work, viz., trade mathematics (*i.e.* arithmetic, use of symbols—elementary algebra—and plane geometry and trigonometry as used in the trades); English (*i.e.* business letters, reading with oral expression, drawing of contracts, writing specifications &c.); industrial history and civics; industrial and commercial geography; applied physics; industrial chemistry; simple book-keeping; elements of commercial law.

In addition to these definite trade courses, instruction is offered in preparation for the work of—

- a. Mechanical Draughtsman. In this course about 16 hours a week are given to mechanical drawing, 8 hours to theoretical work, 5 hours to freehand drawing, and the remainder of the time to mechanical work in the shops.

- b. Architectural Draughtsman. The division of time for these is similar, but the boys take for their practical work only the trades pertaining to house building.
- c. Commercial Designer. In this case most of the time is given to freehand drawing and design.

Every effort is made to run the school on practical trade lines. All the teachers both of theory and practice have followed a trade at some time or other, and a workshop tone is given to the whole school. Work is undertaken for the Board of Education and for use in the school building. The plumbers for instance undertake necessary repairs, and at the time of the writer's visit the wood working department was executing an order for tables and drawing boards for the playgrounds department. Exercises for the sake of practice are done as little as possible; the Principal, Dr. Pickett, believes that the boys not only work with an incentive but learn far more when they are working on a commercial product. His aim is to supply employers with boys who will feel at home at once when they enter a workshop and will settle down to their new occupation in a workmanlike manner. "Your boys take right hold as if they had been doing the work all their lives. They are not at all like school-boys" is regarded as high praise from an employer. Nevertheless the arrangement of the work appears to offer the boys taking trade work an opportunity for securing a wide preliminary training as a basis for further training as an apprentice or learner in some firm.

The trade union leaders approve of the school and have given assistance by addressing the boys on special occasions.

The school does not undertake to place the pupils who complete the course, but many employers are now beginning to apply for boys. It has hardly been in existence a sufficient length of time to allow of a judgment of results, but in the report issued in 1912, instances are given of boys who after a year or two of employment are earning very good wages in their respective trades.

The writer's only criticism of the school would be that the courses for draughtsmen are offered to too young a set of boys and that hardly enough time is devoted to theoretical subjects to allow of these boys receiving a training which would warrant their engaging in these occupations.

The following is the *per capita* cost of the school* for the year ending July 31st 1912.

" (a) on the expenditures from the general school fund for salaries of teachers.		
" On average monthly register	- -	\$64.16
" On average daily attendance	- -	\$70.56
" (b) on the special fund for supplies, materials, janitors' salaries and fuel.		
" On average monthly register	- -	\$5.15
" On average daily attendance	- -	\$5.66

* From the 14th Annual Report of the City Superintendent of Schools, 1911-12, p. 229. (City of New York.)

"The cost of supplies delivered to the school during the school year was \$3,423.07. The value of finished product returned to the Department was \$3,064.40. Hence the net cost of supplies was \$358.67 or 77 cents per capita based on the register and 85 cents per capita based on attendance.

"In the \$3,064.40 is included a credit of \$971.00 for repairing textbooks. At an expense of only 17 cents per book, the school turned back to the Department books valued at about \$4,000. A textbook bound in this way lasts much longer than the book as originally bound; thus a great saving was effected."

The expenditure on salaries for this school and the Manhattan Trade School for Girls are grouped together in the report.

Principals	\$7,249.35.
Teachers	\$65,038.40.

CONNECTICUT TRADE EDUCATION SHOP, NEW BRITAIN.*

The aim of the State of Connecticut in maintaining a shop for apprentices in trade education, is to stimulate a desire in the minds of boys and girls to become trained for some life pursuit by taking advantage of the opportunity the State offers in the system of training the mind, the hand and the eye in certain of the various trades, also to secure for graduates, as far as possible, upon satisfactory completion of the prescribed term of apprenticeship, employment at wages commensurate with the ability displayed.

The following are the trades taught to boys:—Machinist; toolmaking; patternmaking (wood and metal); carpentry; cabinetmaking; drafting; printing and book-binding; plumbing.

The regular day course for boys covers a period of two years, regulated as follows: 49 hours per week, 52 weeks each year, less national holidays.

Part-time courses for apprentices employed in the factories are arranged to suit conditions of present employment.

Regular day courses are free to all in the State of Connecticut.

The following is the course of study for 1912-13:—

Machine Department.

With detail of machine and bench work and tool making. Lathes, shapers, planers, milling machines, drill presses and grinders. On general work of centering, turning, thread cutting, mandrel chuck and taper work, gearing, taps, dies and reamers. The building of machine tools complete and assembling.

Estimating for material required, with time estimate in journeyman's hours.

The latter divided by the hours required by the apprentice to complete the various operations give the percentage of efficiency of the apprentice in rate of journeyman's hours. Time cards are made by apprentices for each shop order and job, and on these are recorded his efficiency rating for quality and speed of work.

* This school was not visited. It has a girls' department in which dressmaking and millinery are taught.

Carpentry and Pattern Departments.

Carpentry to include practical work of house framing, bridge work, window and door framing, outside and inside finishings, stair building, mill construction work and specifications.

Cabinet work. Furniture making, cabinets, files, veneering and finishing.

Patternmaking. Wood and metal patterns, core boxes, shrinkage for various metals and foundry practice.

Estimates.

Shop talks and black-board demonstrations form part of the academic work for each trade. In these the value of close personal contact of student and instructor is emphasized.

In mathematics, shop problems are employed wherever possible. General discussions of ways and means of doing work for the purpose of developing the reasoning activity of the boy's mind are included. Lectures are given on loss and waste. Methods as observed in other shops for doing similar line of work, and the reason as given by foreman or superintendent in charge form part of the instruction.

Plumbing.

In this department the detail consists of work on all metals used in plumbing.

Tinning of metals, soldering new and old metals, joint wiping.

Handling of lead pipe, as well as brass and mickle pipe and fittings, pipe suspension and supports.

Cutting cast iron soil pipe, fitting and caulking, house sewage, waste and stock vents, traps and ventilation, hot and cold water supply, bathroom and toilet installation, house heating, steam and hot water, vacuum installation, testing out of work.

Estimating.

Drafting Department.

Instruction in detail of work for all departments. Use of instruments, lines, (boundary, projection, dimension) simple geometrical figures, one two and three view drawings, plane and elevation of machine parts from free-hand sketching of power machines. Problems in curved surfaces and intersections, isometric projections and perspectives. Tracing, blue-printing and Van Dyke work. Assembly drawings, from detail sketches. Lay-out of spur and bevel gears, worm and worm wheels with calculations for same, plate and cylindrical cams. Straight and quarter turn belt drive and other mechanical movements. Materials as listed under three heads: Suitability, Cost, Availability.

Design of jigs, fixtures, blanking dies, cutter heads, other small tools, power machines, (grinders, lathes, drill presses, vises, moulding machines, gas engines) which are part of the commercial product of our shops.

Under the other departments the instruction in drawing follows the same general scheme adapting it to that particular trade of carpentry, plumbing and patternmaking.

General Regulations.

The applicant must have reached his 14th birthday and be able to read and write English, and to work correctly simple problems in arithmetic.

The trial period will consist of 30 to 90 days. Progress in the trade assigned will be carefully observed and if satisfactory the applicant will be regularly assigned, but if unsatisfactory, advice will be given to take up some other trade or if unfitted for trade work and its general requirements, will be discontinued.

Gymnasium and outdoor sports are a factor of both the boys' and the girls' departments, the apprentices being in control of the arrangements under the supervision of the heads of departments.

Factory Visiting Days are arranged for from time to time to enable all apprentices to observe shop conditions as related to their particular trade.

The objects of the school are explained to parents of intending pupils in the following paragraphs, which find a place in the school prospectus.

"The output of labor of your son or daughter should at all times have some commercial value by which to gauge the efficiency of their training, also by which to gauge the efficiency of the instructor, and to determine whether the methods employed in training are correct. When boys or girls enter a factory, dressmaking or millinery shop, they find there are other minds to please, that it is not alone sufficient to satisfy the instructor, but it is necessary to satisfy the employer and also the customer.

"The wishes of the customer as to correct design and workmanship have a vital bearing as to success or failure of the one doing the work, and it reflects strongly on the earning capacity of the individual tradesman or tradeswoman.

"It is the plan of the trade shop to so instruct all apprentices, that their work each day may be sure of its commercial value and when the article is completed it fills a customer's order.

"If it pleases the customer, it pleases us, and the boy or girl knows they have been taught correctly; that their efficiency is correctly shown and not simply made upon a guess from the instructor.

"The instructor knows he must please his customer both in quality of work and time of delivery of the article, consequently it is his duty to see that these two qualities are brought out and trained in the apprentices.

"The result is lasting. The boy and girl enter their trades as journeymen able to satisfy and please, sure of themselves with their neighbor, and stronger in their earning capacity.'

CHAPTER IV.—THE MANUAL TRAINING AND TECHNICAL HIGH SCHOOLS.

TECHNICAL INSTRUCTION IN THE HIGH SCHOOLS.

Manual training as an educational subject was introduced into secondary schools in the United States just over 30 years ago. The movement for the establishment of manual training schools originated with the opening of the St. Louis Manual Training School in 1880. This was followed by the foundation of other schools of a similar type and by the development of manual training courses in the already established academic high schools. The manual training high school of to-day is a revelation to an English visitor:—fitted with foundry and forge, carpentry, pattern making and wood-turning, mechanical and electrical engineering workshops, with up-to-date motor-driven machinery, it resembles at first sight a trade school, but its aims and its teaching may be in reality very different.

According to the definition of the National Society for the Promotion of Industrial Education, manual training was introduced with four distinct objects in view:—

- “1. To educate the whole boy and to develop the entire area of his brain; 2. To lay a broad and appropriate foundation for higher education; 3. To enable a boy to discover his innate mental and physical aptitudes; 4. To furnish a broad basis for an industrial career should one's aptitude lie in the direction of the mechanical arts.”*

The definition of the National Council of Education of the National Education Association runs as follows:—

“The manual training high school or the manual training school is a school of secondary grade in which a greater or less amount of handwork is included in the curriculum and in which the greater part of the academic instruction is similar to that found in other high schools and college preparatory schools, neither the manual nor the academic instruction being specially planned to be of direct vocational service.”†

Manual training is defined by Professor Woodward, one of its chief supporters, as “the systematic study of the theory and use of common tools, the nature of common materials, elementary and typical processes of construction, and the execution and reading of working drawings.”‡ Its introduction as an educational subject was opposed by many, but it was soon found to have “intellectual and moral value far exceeding all expectations.” In his article Professor Woodward quotes the opinion of Mr. James P. Munroe, who believes that

* Quoted in the Report of the Commissioner of Education for the year ended June 30th, 1911, Vol. 1, p. 301. (Washington: Government Printing Office, 1912.)

† Report of the Committee on the Place of Industries in Public Education, Sub-committee on Industrial and Technical Education in the Secondary School. (Journal of Proceedings and Addresses of the Forty-eighth Annual Meeting of the National Education Association of the United States, held at Boston, July, 1910, p. 732. Winona, Minn., 1910.)

‡ Manual, Industrial and Technical Education in the United States. (Report of the Commissioner of Education for the year 1903, Vol. 1, p. 1,019. Washington: Government Printing Office, 1905.)

"when rightly conceived and carried on, it promotes co-ordination, it develops creativeness, it broadens culture, it strengthens character," and of Professor William James, who states that "laboratory work and shop work. . . . confer precision, they give honesty, they beget a habit of self-reliance, they occupy the pupil in a way most congruous with the spontaneous interests of his age." Manual training has therefore been regarded very favourably in the United States. Nearly all public high schools have manual work, though it may be only wood-work, as an optional subject even in an academic course, and a private school, such as the School of Education, in connection with the University of Chicago, attended by the children of the wealthier classes of the city, maintains elaborate engineering and wood working shops for all branches of the trades.

The demand for a vocational bias to education has had its effect on the manual training high school. The teaching has in places been given a more direct bearing upon industry or the technical high school has sprung up in place of it or alongside. Particularly has this been the case where the idea of the distinct trade school has not yet been accepted. In his report for 1910, the Commissioner of Education gives 28 public high schools in which instruction with a vocational aim is offered.

The technical high school, as distinct from the manual training high school, is defined by the National Council of Education of the National Education Association as follows:—

"The secondary technical school, or the technical high school, is a school of secondary grade having the distinct purpose of preparing its pupils for industrial leadership—that is, for positions in industrial life requiring skill and technical knowledge and of greater importance and responsibility than those of the skilled mechanics. In such a school the instruction deals not only with the important manual operations, but also with those principles of science and mathematics and their direct applications to industrial work that will help the student for successfully mastering the more fundamental processes and problems of those groups of industries which the school is designed to reach."*

Since some hold that the aim of the technical high school should be preparation for technical colleges and engineering schools and others that its main purpose should be to prepare pupils directly for wage-earning occupations, abandoning or making subsidiary the course for a technical college, any exact definition or classification seems impossible.

The difference of the trade schools has been defined in the Sixth Annual Report of the New York State Education Department (pages 535-6) as follows:—

"The Trade Schools [or Vocational Schools] will differ from the manual training courses in the secondary schools in the following particulars:—

"1. Pupils enter the trade school with the definite purpose of preparing for industrial careers.

* Report of the Committee on the Place of Industries in Public Education. Sub-committee on Industrial and Technical Education in the Secondary School. (Journal of Proceedings and Addresses of the Forty-eighth Annual Meeting of the National Education Association of the United States, held at Boston, July 1910, p. 732. Winona, Minn.)

"2. The trade school absolutely abandons all college preparatory work.

"3. There is almost no instruction in pure mathematics or pure science, but instead a fair amount of time is given to such applied mathematics and applied science as is closely related to the trade selected by the pupil, in fact all the instruction whether in class room, shop, or laboratory is designed so as to be directly usable.

"4. The trade schools will necessarily take on varying forms in different localities.

"5. They will not be parallel to our existing high schools in that they will not necessarily draw pupils who have completed the 8 grades of the elementary schools."*

Hence the boy who intends to enter industry as a mechanic, working first at any rate with his hands goes to the trade school where one exists—he may not have passed further in the elementary school than the 6th grade. The boy who has graduated from the elementary school and who has in view college life or a high position in industry naturally chooses the technical or the manual training high school. It is inevitable, however, that the boy who is interested in mechanical work and who may not have to earn his livelihood immediately on graduation from the elementary school, should be drawn to the technical or failing that, the manual training high school, if there is no trade school that he can attend, and the practical work given him at these schools must be of great advantage to him, though he should at 15 or 16 seek work as an apprentice at some trade. The failure on the part of American schools up to the present to make note of the occupations taken up by those who drop out in the middle of the course makes it impossible to say to what extent these schools are used by such boys. The boy who has completed the four years course of a technical high school and does not go to college, may enter a surveyor's or architect's office, become a mechanical draughtsman, a chemist (if scientific training is given in his school), a salesman in some industrial or scientific warehouse, or take a Civil Service Examination for technical work, etc. Whatever it may be, the knowledge he has obtained of tools, materials and machinery will be of great assistance to him.

These schools, it is proved, draw a class of boy who would never have entered one of the academic high schools with cultural courses only, and they succeed in retaining the boys longer. At Cleveland the total attendance increased as follows, when the Technical High School was opened in 1909 and the High School of Commerce in 1910:—

—	1906.	1907.	1908.	1909.	1910.	1911.
Academic High Schools - -	4,983	5,059	4,989	4,787	4,436	5,293
Vocational - - - -				729	1,560	2,005
Total - - - -	4,983	5,059	4,989	5,516	5,996	7,298

* Quoted also in the Annual Report of the Commissioner of Education for 1910, Vol. I, p. 238. Washington: Government Printing Office, 1910.

The increase of children of school age in the city was about 2,000 a year. In the last year there was thus a large increase in the numbers attending the academic schools along with an increase at the vocational high schools.

The principal of the Woodward High School, Cincinnati, said that the number of pupils had trebled since the introduction of the practical courses 5 years ago. At Chicago far more boys drop out from the academic than from the technical high schools during the 1st and 2nd years, and when all were given the choice of attending an academic or technical high school, 40 per cent. attended the only two technical schools out of 18 high schools in the city.

The needs of the boy who has to enter employment early are now being met at Chicago by the introduction of two year courses in the technical high schools. Chicago has no trade schools proper. On the other hand, in the Stuyvesant High School, New York City, the industrial courses have been given up since the foundation of the Vocational School for Boys, the only course in the school now being that preparatory for colleges, higher technical schools and professional schools.

STUYVESANT HIGH SCHOOL, NEW YORK CITY.

The writer had the opportunity of going all over this school. The building is a fine one with large laboratories and workshops for each branch of the various trades taught there. The following is the course of study as given the writer in February 1913.

Course of Study.

(Preparatory for Colleges, Higher Technical Schools, and Professional Schools).

First Year.

Required.
 English.
 Latin or German.
 Elementary Algebra.
 Mechanical Drawing (1st term only).
 Freehand Drawing (2nd term only).
 Joinery and Cabinet Making.
 Shop Physics (1st term only).
 Timber Science (2nd term only).
 Music (1st term only).
 Physiology and Hygiene (2nd term only).
 Oral English.
 Physical Training.

Elective

Second Year.

English.
 Latin or German.
 Plane Geometry.
 Mechanical Drawing (1st term only).
 Freehand Drawing (2nd term only).
 Woodturning and Instrument Making (1st term only).
 Pattern Making and Foundry Work (2nd term only).
 Physical Training.

One of the following:
 Chemistry.
 French.
 German.
 Latin.

Third Year.

English.	One of the following :
Latin or German or French.	Analytical Chemistry.
Physica.	Modern History.
Advanced Geometry and Algebra.	French.
Mechanical Drawing.	German.
Forging (1st term) and Art Metal	Latin.
Work (2nd term).	
Physical Training.	

Fourth Year.

English.	Latin, French or German.
English and American History and	Industrial Chemistry.
Civics.	Advanced Physica.
Physical Training.	Applied Mechanics and Electricity.
	Advanced Mathematics.
	Machine Shop Practice.
	Mechanical Drawing (Architectural
	or Machine Design).
	Advanced Shopwork or Advanced
	Drawing.

The school hours are 9 to 12.30 and 1 to 2.30, each boy having from 1½ to 2 hours' home work. There are over 2,200 boys in the school. The manual work consists in the first year of wood work with hand tools largely. Small articles such as tee squares are first made, the boys passing on to making chairs and tables before the end of the year. In the second year wood turning is taken during the first term, such articles as egg cups, bowls, vases, and chair legs, &c. being made. During the second half of this year, the boys take pattern making, articles for use in the other shops being made. A little foundry work is done with iron or white metal in order that the boys may follow the use of their own patterns. In the forge work taken in the third year tools are made for use in the machine shop for a beginning, art metal smithing being taken up later by those who have time and taste for it. In the second term of this year beaten art metal work is taken, the boys making simple trays, boxes, &c., from designs given to them. In the fourth year the boys specialise in one of the following: chemistry, which is being given more and more of an industrial trend, advanced physica, applied mechanics and electricity (a basis for electrical engineering), advanced mathematics, which includes the practice and principles of surveying: or they give extra time to the milling and turning machines of the engineering shops. A boy may also specialise in mechanical drawing, or in advanced work in one of the other shops, forging, smithing or pattern making.

During the four years, six periods a week (of 45 to 50 minutes each) are devoted to the manual work; and four periods to mechanical and freehand drawing during the first two years, and two periods to mechanical drawing only during the last two years. In freehand drawing the boys take a little design or object drawing and then practise sketching of machinery and industrial rather than artistic objects. In mechanical drawing they first learn the use of blue prints and

then as far as possible make working drawings of the articles to be constructed in the shops. Mathematics is taught largely in its relation to technical and practical problems.

CHICAGO TECHNICAL HIGH SCHOOLS.

Chicago has two technical high schools for boys, the Lane and the Crane. The building of others is recommended. Each of these schools has its four year course of manual training or technical work, and special two year vocational courses.

The programmes of the special technical courses are given in the Appendix to this chapter (pp. 40-47 below).

The following figures taken from the Annual Report of the Chicago Board of Education show the enrolment in all the High Schools and in the Crane and Lane Technical High Schools separately for the year ending June 30th, 1912.

—	Enrolment.	Average Membership.	Average Attendance.	Number of Promotions.	Per Cent. of Attendance.	Per Cent. of Promotions.
Crane Technical High School	874	822.7	792.8	675	96.4	82.0
Lane Technical High School	1,631	1,466.4	1,412.6	1,184	96.3	80.7
Totals for all (21) High Schools boys and girls	19,671	18,252.6	17,379.6	13,332	95.2	73.0

Enrolment by Grades.

—	9th Grade.		10th Grade.		11th Grade.		12th Grade.		
	1st Semester.	2nd Semester.	1st Semester.	2nd Semester.	1st Semester.	2nd Semester.	1st Semester.	2nd Semester.	
Crane	Beginners	131	2	250	2	179	—	107	—
	Advanced	99	—	70	1	60	—	—	2
Totals		232		323		179		109	
Lane	Beginners	357	16	322	1	266	2	196	—
	Advanced	192	4	149	—	60	3	49	—
Totals		569		472		331		245	
All High Schools, Boys only		4,184	97 + 1,525 from elemen- tary schools	2,403	31	1,389	18	975	4
Totals		5,806		2,434		1,407		979	

APPENDIX TO CHAPTER IV.

CHICAGO TECHNICAL HIGH SCHOOLS.—PROGRAMMES OF SPECIAL TECHNICAL AND VOCATIONAL COURSES.

Four Year Courses in Manual Training or Technical Work.
September, 1912.

MANUAL TRAINING COURSE.

The purpose of this course is to prepare students for the technical industries, and for normal schools, technical schools, and colleges.

BUILDERS' COURSE.

The purpose of this course is to prepare students for the building industries. It is believed that graduates from this course, after a reasonable apprenticeship, will be competent to take positions as foremen, superintendents, or general contractors.

ARCHITECTURAL COURSE.

This course is planned for those who desire to become architects, or to do work in drafting rooms, and for those desiring a general course in mechanical drawing.

MANUAL TRAINING COURSE.

<i>First Year.</i>				<i>Second Year.</i>				
	Weeks.	Periods.		Weeks.	Periods.			
First Semester—				First Semester—				
English - - - - -	20	4	English - - - - -	20	4			
Woodworking - - - - -	20	10	Foundry, Forge and Pat-	20	10			
Mechanical Drawing - - - - -	20	4	tern-making - - - - -	20	4			
Freehand Drawing - - - - -	20	1	Mechanical Drawing - - - - -	20	4			
Algebra - - - - -	20	4	Plane Geometry - - - - -	20	4			
Physiology - - - - -	20	5	Physical Education - - - - -	20	2			
Physical Education - - - - -	20	2						
		—					24	
		30	Second Semester—					
Second Semester—				English - - - - -				
English - - - - -	20	4	English - - - - -	20	4			
Woodworking - - - - -	20	10	Foundry, Forge and Pat-	20	10			
Mechanical Drawing - - - - -	20	4	tern-making - - - - -	20	4			
Freehand Drawing - - - - -	20	1	Mechanical Drawing - - - - -	20	4			
Algebra - - - - -	20	4	Plane Geometry - - - - -	20	4			
Physiography (with special reference to woods and ores) - - - - -	20	5	Physical Education - - - - -	20	2			
Physical Education - - - - -	20	2						
		—					24	
		30	Choose one of the fol- lowing:—					
			Foreign Language - - - - -	40	5			
			Biology - - - - -	40	5			
			Elementary Physics - - - - -	40	6			
			Chemistry - - - - -	40	6			
			If chemistry is not taken now it must be taken the fourth year.					
Third Year.				Fourth Year.				
First Semester—				First Semester—				
Machine-shop Practice - - - - -	20	8	United States History - - - - -	20	4			
English - - - - -	20	4	Machine or Architectural					
Freehand Drawing - - - - -	20	1	Drawing - - - - -	20	3			
Mathematics - - - - -	20	4	English - - - - -	20	4			
Physics - - - - -	20	6	Physical Education - - - - -	20	2			
Physical Education - - - - -	20	2						
		—					13	
		25						

	Weeks.	Periods.		Weeks.	Periods.
<i>Third Year—cont.</i>			<i>Fourth Year—cont.</i>		
Second Semester—			Second Semester—		
Machine-shop Practice	20	8	Civics	20	4
Machine or Architectural Drawing	20	4	Machine or Architectural Drawing	20	3
Freehand Drawing	20	1	Trigonometry	20	4
Physics	20	6	Physical Education	20	2
Mathematics	20	4			—
Physical Education	20	2			13
		—			
		25	<i>Electives.</i>		
Choose one of the following:—			Chemistry	40	6
History	40	4	Language	40	5
Language	40	5	English	20	4
			Electrical or Gas Engine Construction	40	6
			Electrical or Gas Engine Construction	20	4
			Freehand Drawing	40	6
			Advanced Physics	20	3
			Advanced Chemistry	20	6

One semester of English must be chosen during this year by those who have not taken a foreign language.

BUILDERS' COURSE.

<i>First Year.</i>				<i>Second Year.</i>	
	Weeks.	Periods.		Weeks.	Periods.
First Semester—			First Semester—		
Business English	20	4	Business English	20	4
Mensurational Geometry	20	4	Arithmetic and Book-keeping	20	4
Physiology	20	5	Chemistry	20	6
Architectural Drawing	20	4	Architectural Drawing	20	4
Carpentry	20	10	Bricklaying, Masonry, &c.	20	10
Physical Education	20	2	Physical Education	20	2
		—			—
		29			30
Second Semester—			Second Semester—		
Business English	20	4	Business English	20	4
Mensurational Geometry	20	4	Arithmetic and Book-keeping	20	4
Freehand Drawing	20	4	Chemistry	20	6
Mechanical Drawing	20	4	Architectural Drawing	20	4
Carpentry	20	10	Bricklaying, Masonry, &c.	20	10
Physical Education	20	2	Physical Education	20	2
		—			—
		28			30
Third Year.			Fourth Year.		
First Semester—			First Semester—		
English	20	4	Sanitation	20	4
Mathematics, including Trigonometry and Surveying	20	4	Building Specifications and Estimating	20	4
Physics	20	6	Industrial History	20	4
Architectural Drawing	20	4	Electrical Wiring	20	10
Metal Work	20	10	Freehand Drawing	20	4
Physical Education	20	2	Physical Education	20	2
		—			—
		30			28

	Weeks.	Periods.		Weeks.	Periods.
<i>Third Year—cont.</i>			<i>Fourth Year—cont.</i>		
Second Semester—			Second Semester—		
English	20	4	Strength of Materials	20	4
Mathematics, including Trigonometry and Surveying	20	4	Building Contracts and Ordinances	20	4
Physics	20	6	Industrial History	20	4
Architectural Drawing	20	4	Electrical Wiring	20	10
Metal Work	20	10	Freehand Drawing	20	4
Physical Education	20	2	Physical Education	20	2
		—			—
		30			28

ARCHITECTURAL COURSE.

<i>First Year.</i>			<i>Second Year.</i>		
	Weeks.	Periods.		Weeks.	Periods.
<i>First Semester—</i>			<i>First Semester—</i>		
English	20	4	English	20	4
Drawing—			Foreign Language	20	5
Mechanical and Free-			Architectural or Machine		
hand	20	8	and Freehand Drawing	20	10
Physiology	20	5	Plane Geometry	20	4
Mathematics	20	4	Physical Education	20	2
Shop	20	4			—
Physical Education	20	2			25
		—	<i>Second Semester—</i>		
		27	English	20	4
<i>Second Semester—</i>			Foreign Language	20	5
English	20	4	Architectural or Machine		
Drawing—			and Freehand Drawing	20	10
Mechanical and Free-			Plane Geometry	20	4
hand	20	8	Physical Education	20	2
Mathematics	20	4			—
Shop	20	8			25
Physical Education	20	2			
		—			
		26	<i>Fourth Year.</i>		
<i>Third Year</i>			<i>First Semester—</i>		
<i>First Semester—</i>			English	20	4
English	20	4	Architectural Design	20	10
Foreign Language	20	5	United States History and		
Mathematics	20	4	Civics	20	4
Architectural Design	20	10	Physical Education	20	2
History of Art	20	2			—
Physical Education	20	2			20
		—	<i>Second Semester—</i>		
		27	English	20	4
<i>Second Semester—</i>			Architectural Design	20	10
English	20	4	United States History and		
Foreign Language	20	5	Civics	20	4
Mathematics	20	4	Physical Education	20	2
Architectural Design	20	10			—
History of Art	20	2			20
Physical Education	20	2			
		—			
		27			

Two Year Courses in Vocational Subjects. September, 1912.

Two-year courses in vocational subjects are offered in the high schools, for those who do not expect or are not able to take a full four years' course in the school. Each two-year course has a major subject, which receives a special large proportion of time and credit. A student pursuing such a course will be required to follow it as outlined, in order to be well fitted for the occupation into which the major subject leads. Studies taken successfully in the two-year courses will receive credit towards graduation from the four-year course. On the successful completion of any of the two-year vocational courses, a certificate (not a diploma) will be awarded.

COURSE IN MECHANICAL DRAWING.

The aim of this course is to give a thorough foundation in the principles of instrumental drawing. It is believed that students completing this course satisfactorily will be well fitted to fill positions as draftsmen in machine shops or architects' offices.

COURSE IN DESIGNING.

This course aims to give the students the elementary principles of design as they are illustrated in its application to commercial art and craft work. It should fit students to become designers of garments, household decorations, ornaments, advertising signs, etc.

COURSE IN ADVANCED CARPENTRY.

The purpose of this course is to offer students who are not able to spend the time required to take a four year course in manual training, an elementary knowledge of wood-working, with special application to carpentry and cabinet making.

COURSE IN PATTERN MAKING.

The purpose of this course is to give students the principles of pattern making. It is believed that students who complete this course satisfactorily may, after a brief apprenticeship, become expert makers of wood patterns for metal castings.

COURSE IN MACHINE SHOP WORK.

The purpose of this course is to give students a knowledge of the elementary principles of machine shop practice.

COURSE IN ELECTRICITY.

This course aims to acquaint students with the elementary principles of electricity and their application, such as electric wiring and testing, the manufacture of electrical appliances, also finding the cost and the efficiency of the same.

COURSE IN PRINTING.

The purpose of this course is to give students an elementary training in practical printing.

GENERAL REMARKS.

A full year's work in United States History is allowed in any course instead of United States History and Civics, if, in the judgment of the principal, such arrangement is preferable; but in the event of this arrangement Civics is recommended as an elective subject for another semester.

Students planning to enter Teachers College with the intention of preparing to become teachers of Manual Training, Household Arts, or Drawing, are required to take the full high school course in Manual Training, Household Arts, or Drawing, respectively; and other students are advised to take the Teachers College Preparatory Course.

TWO YEAR COURSE IN MECHANICAL DRAWING.

<i>First Year.</i>			<i>Second Year.</i>		
	Weeks.	Periods.		Weeks.	Periods.
First Semester—			First Semester—		
Business English -	20	4	English -	20	4
Shop Mathematics -	20	4	(or other modern language)	20	5
Mechanical Drawing -	20	8	Geometry -	20	4
Physiology -	20	5	Building or Machine Con-		
Freehand Drawing -	20	4	struction -	20	4
Physical Education -	20	2	Architectural or Machine		
			Drawing -	20	10
		27	Freehand Drawing -	20	4
			Physical Education -	20	2
Second Semester—			Second Semester—		
Business English -	20	4	English -	20	4
Shop Mathematics -	20	4	(or other modern language)	20	5
Mechanical Drawing -	20	8	Geometry -	20	4
Woodworking -	20	10	Building or Machine Con-		
Physical Education -	20	2	struction -	20	4
			Architectural or Machine		
		28	Drawing -	20	10
			Freehand Drawing -	20	4
			Physical Education -	20	2
					28
					or 29

TWO YEAR COURSE IN DESIGN.

<i>First Year.</i>			<i>Second Year.</i>		
	Weeks.	Periods.		Weeks.	Periods.
First Semester—			First Semester—		
Business English -	20	4	English -	20	4
Shop Mathematics, Busi-			(or other modern language)	20	5
ness Arithmetic, or			Geometry -	20	4
Algebra -	20	4	Applied Design -	20	8
Physiology -	20	5	Freehand Drawing -	20	4
Mechanical Drawing -	20	4	History with special refer-		
Freehand Drawing and			ence to History of Art,		
Design -	20	8	and Civics -	20	4
Physical Education -	20	2	Physical Education -	20	2
					26
		27			or 27
Second Semester—			Second Semester—		
Business English -	20	4	English -	20	4
Shop Mathematics, Busi-			(or other modern language)	20	5
ness Arithmetic, or			Geometry -	20	4
Algebra -	20	4	Applied Design -	20	8
Science -	20	5	Freehand Drawing -	20	4
Mechanical Drawing -	20	4	History with special refer-		
Freehand Drawing and			ence to History of Art,		
Design -	20	8	and Civics -	20	4
Physical Education -	20	2	Physical Education -	20	2
					26
		27			or 27

TWO YEAR COURSE IN ELECTRICITY.

<i>First Year.</i>		<i>Second Year.</i>	
	Weeks.	Periods.	
First Semester—		First Semester—	
Business English	20	4	English
Shop Mathematics	20	4	or other modern lan- guage
Woodworking	20	10	Geometry, or History, with special reference to in- dustrial and economic conditions, and Civics
Elementary Physics	20	5	Applied Electricity
Mechanical Drawing	20	4	Mechanical Drawing
Physical Education	20	2	Freehand Drawing
		—	Physical Education
		29	
Second Semester—		Second Semester—	
Business English	20	4	English
Shop Mathematics	20	4	(or other modern lan- guage)
Elementary Electricity	20	10	Geometry, or History, with special reference to in- dustrial and economic conditions, and Civics
Mechanical Drawing	20	4	Applied Electricity
Physiology	20	5	Mechanical Drawing
Physical Education	20	2	Freehand Drawing
		—	Physical Education
		29	
			26
			or 27

TWO YEAR COURSE IN PRINTING.

<i>First Year.</i>		<i>Second Year.</i>	
	Weeks.	Periods.	
First Semester—		First Semester—	
Business English	20	4	Business English
Business Arithmetic	20	4	Mechanical Drawing
Mechanical Drawing	20	4	Printing
Physiology	20	5	History or Science
Printing	20	10	Physical Education
Freehand Drawing	20	1	
Physical Education	20	2	
		—	
		30	
			29
			or 30
Second Semester—		Second Semester—	
Business English	20	4	Business English
Business Arithmetic	20	4	Freehand Drawing
Mechanical Drawing	20	4	Printing
Proof-Reading and Punc- tuation	20	5	History or Science
Printing	20	10	Physical Education
Freehand Drawing	20	1	
Physical Education	20	2	
		—	
		30	
			29
			or 30

CHAPTER V.—CO-OPERATIVE OR HALF-TIME SCHOOLS.

The co-operative schools, which are in effect half-time schools, are unique, the writer believes, to the United States. They are termed "co-operative" because the system is based on the co-operation of manufacturers with the public school authorities, and they are now beginning to play a recognised part in the industrial training of boys.

The system was first introduced by Dean Herman Schneider into the engineering department of the University of Cincinnati in 1906. The theory of engineering is taught in the University, and practice is obtained in the manufacturing plants of the city, the students working alternate weeks at the University and in the manufacturers' shops. The classes are divided into two sections, one being at work while the other is at the University. The students are paid for their work in the shops. The practical work is co-ordinated with the work of the University by a special set of teachers, called co-ordinators, who spend the mornings at the University and the afternoons at the shops, so that the following week they may give instruction to the students they have seen at work, which bears directly on what they were doing.

FITCHBURG HIGH SCHOOL, CO-OPERATIVE INDUSTRIAL COURSE.*

Several manufacturers from the small industrial city of Fitchburg, Massachusetts (with 37,826 inhabitants), were present at a meeting in New York when Dean Schneider explained his system. They were so impressed with it that they themselves suggested the application of the system to the High School at Fitchburg. Hence in 1908 an industrial course was opened in the High School, and this has proved so successful that the system is now often spoken of as "the Fitchburg plan."

Fourteen firms offered or agreed to take apprentices under the half-time arrangement, and a course was drawn up at conferences between the manufacturers and the school authorities. This course is open to any boys who enter the High School, *i.e.*, before taking it they must have first completed all the grades of the elementary school.

The first year is spent entirely in the school at a special preparatory course. During this preliminary year the boys are taken to visit the various manufacturers' shops in the city, and at the close, in July, they select the trade they wish to follow. They are then apprenticed to a firm, in pairs wherever possible (though employers do not always require this) under a special indenture (*see pp. 55-57*) for three years, the first two months being regarded as a probationary period spent entirely at work, during which the indenture can be broken by either party.

* A first sketch of this section, dealing with the Fitchburg plan, appeared in the "Times" Educational Supplement for July, 1913.

During the three years if, at the end of the probationary two months all is satisfactory, the boys alternate week by week between school and work, one boy of the pair being at school when in session while the other is at work. On Saturdays both boys are at work, one picking up what the other is leaving and during the school vacations both boys work, excepting for two weeks' complete vacation during the year.

The teaching in the school is given by special teachers who, with the exception of the English teacher, devote their whole time to the work. They are under a director of the industrial department who spends two afternoons a week visiting the various firms and planning teaching which, so far as possible, shall correlate with the work the boys are doing. All their school work is related to their industrial work as far as possible. A summary of the work taken from a pamphlet, dated August 1913, by Mr. W. B. Hunter, director of the industrial department, is given in Appendix C to this chapter, pp. 58-9. The department is divided into four classes for: (1) machinists and other metal workers, (2) cotton manufacture, (3) printing, and (4) office work.

The annual report of the Fitchburg High School for the year 1912 gives the following list of trades followed:—

“There are 76 students attending school and working in shops alternate weeks. Forty-eight are taking machine work, five saw-making, five office work, four drafting, four textile work, two printing, two iron molding, one pattern-making, one druggist, one tinsmith, one chauffeur, and one grain dealer. These students are earning this year about \$15,000.”

During their time at the shops the boys are paid for the first year 10 cents (5*d.*) an hour, for the second year 11 cents, and for the third year 12½ cents an hour.

The branches of work through which the student is to pass are entered in the indenture and the writer gathers that no difficulty has as yet been experienced in regard to the boys' work, and the cooperation with manufacturers. It is a pity that the indenture is somewhat one-sided, the employer having the power to suspend regular work entirely and substitute other work, and to dismiss or suspend the apprentice for a time without previous notice for work or behaviour unsatisfactory either to him or the school, while should the apprentice violate any of the terms of the agreement a fine of \$100 must be paid. The indenture is signed by the employer, the apprentice, and a bondsman who may be a parent, friend, or any responsible person interested in the apprentice, who will stand behind the boy and see that he fulfils the requirements of his contract. An indenture might have been used which would have placed the cancelling of the agreement in the hands of an outside authority, perhaps one connected with the school, who could have power to adjudicate in case of any dispute and to cancel the indenture in case either party failed to fulfil his share of the contract.

Thirty-one boys have so far completed the course and appear to be doing well. Of these, seven have passed on to universities

or technical colleges to take a technical degree, four are planning to do so, one has left his trade altogether, and the remainder are working at their trades, twelve as machinists, three as draughtsmen, three as saw makers, one as salesman, and one as teller in an office. Five years ago 20 entered the school, in 1912 the entering class numbered 60.

BEVERLY INDEPENDENT INDUSTRIAL SCHOOL.

A school on the half-time system was started at Beverly, Massachusetts, in August 1909. A local committee appointed by the Massachusetts Commission on Industrial Education made an inquiry into the industries of the neighbourhood and consulted with manufacturers as to the establishment of industrial schools. Beverly (with 18,650 inhabitants) is in a somewhat peculiar position, its main industries being, besides agriculture, the manufacture of boots and shoes and of boot and shoe machinery. The latter work lies entirely in the hands of the United Shoe Machinery Company. Co-operation was secured with this firm and the industrial school was established under an independent board of trustees consisting of the mayor, a representative of the works and five members of the school committee.

The United Shoe Machinery Company set aside a small portion of their works for the use of the school and provided an assortment of over 30 machines for the use of the boys, grinders, drills, millers, lathes, planes, &c. The firm pays the salary of one technical assistant who spends all his time in the works and the salaries of the shop co-ordinators for the weeks they spend in the works. The public school authorities bear the remaining expense of maintenance, one half of which is recoverable from the State under the law of 1909.

The school has its own quarters on the upper floor of the McKay Street elementary school, the science work being done in the laboratories of the high school. It is under a director who devotes most of his time to supervision, doing little teaching, and who has an assistant in the school. The boys are divided into two sets, A and B, of which one set is in school when the other is at work. A machinist instructor or shop co-ordinator teaches each set of boys both in school and at work.

These boys are not indentured to the firm in any way. The work given them to do is that required at the time by the firm, and each boy is paid half piece rates for all work which reaches the necessary standard. Care is constantly exercised to secure that each boy has sufficient variety of work and is not tempted by the prospect of higher earnings to pass too long a time on one job. To this end a card^e is kept on which the work done by each boy each week is entered, together with total earnings and amount rejected as bad work.

* See Appendix D (2) to this chapter, page 64.

In the school, as is stated in the summary of the school work given in Appendix D (1) to this chapter on page 60, instruction is as far as possible individual. Boys are admitted on completion of the 6th grade of the elementary school and at the age of 14 or over. They are consequently of various ages and preliminary schooling, some having spent a year or two in high school and others having come straight from the elementary school. As far as possible topics are taken up as the boy has met them in his preceding week in the shop, either in handling a blue print, operating a machine, using tools, or filling out time cards. The school has nearly 1000 copies of blue prints in sets which are in actual use in the machine shop. Mathematics are taught from these as much as possible. Small groups are formed to discuss common difficulties. For two hours two or three times a week a group of boys adjourns with the director to a room set apart in the factory where they study one machine completely, making notes and writing an account of it afterwards.

The hours at school are 8 to 12 and 2 to 4, at work 7-12 and 1-5. No school work at all is expected of the boys during the weeks they are at work. The Saturday in the school week is a holiday.

No definite length is fixed for the school course. The boys are promoted to full work in the factory at full earnings but under school supervision, at any time that they may be considered fit. Eight boys were thus promoted after spending 2 years and 10 weeks in the school, and have proved entirely satisfactory at their work so that foremen in other departments are asking for school boys. When first entering the factory boys are considered on probation and would be returned to the school if they failed in either work or behaviour. During this period of probation they are to be given work which will continue their training and they remain under the supervision of the director as regards this and their behaviour.

The boys are not bound to enter the works of the United Shoe Machinery Company, though all completing the course have so far naturally chosen to do so, as these works offer the best prospects for engineers in the town. Twenty-four boys have dropped out without completing the course. These have gone to other firms, some as engineer's machinists and some in other occupations.

QUINCY INDEPENDENT INDUSTRIAL SCHOOL.

A school of a somewhat different type again was started at Quincy, Massachusetts, in September, 1912. The Fore River Shipbuilding Co. have large works at Quincy, a small town of 32,642 inhabitants, almost adjoining Boston. This firm had already established an apprenticeship system by which the boys were bound for four years' service and received four hours' class instruction a week given in the morning in the firm's own class room. On the establishment of the Industrial School the firm

sent 34 boys at once and intends, the writer was told, to send its remaining apprentices, about 80 in number, by degrees, as the school can accommodate them. Three other firms had sent seven other boys.

Any boys who have reached the 5th grade of the elementary school may enter, but some had already spent three years at a high school. The length of the course had not been fully determined but would be arranged to suit each boy. This school aims at training mechanics only, and the teaching therefore is not as advanced as that given in a high school.

"The school assumes full responsibility for the boy's training in the shop; sees that he is given opportunity to go through the various branches of his trade under just and skilled direction, and that there is no danger of the boy's being exploited for the benefit of the industry. . . . By agreement between the shops and the school, the shop work for the boy is divided into time proportioned to the importance and the extent of difficulty met with in becoming expert in the various mechanical processes."^{*}

The director visits the works on two afternoons a week and on Saturdays when both boys of the pair are at work. He inspects the conditions of work and secures data on which to base the school work of the following week. A record of work done by each boy in the shop is made each week and tabulated on a form filed for reference, so that it is quickly seen whether the agreement that the boys should be passed through the various departments is kept. A report of the attendance and progress of the boys in school is sent to the firms concerned each week.

An extract from the first report on the school by the director giving some account of the teaching and methods followed is given in Appendix E (1). to this section, pp. 64-5.

This school, like that at Beverly, is so organized as to receive a refund of half the expense of working from the State.

The system of co-operative schools is spreading through the States. Since it was first started at Fitchburg it has been followed, in each case under slightly varying conditions, at the Technical High School, Providence, Rhode Island, at Freeport High School, Freeport, Illinois, York High School, Pennsylvania, and the Woodward High School, Cincinnati, Ohio, at Newton and Worcester, Mass., and other places besides Beverly and Quincy. A course has been arranged at the Lewis Institute, Chicago, a privately endowed technical institute, and some of the boys from the Trade School, Bridgeport, Connecticut, are trying it in their second year at the school.† At the time of the writer's visit to Fitchburg a plan had been made and the

* Annual Report of the School Department of the City of Quincy, Mass., for the year ending December 31st, 1912, p. 14.

† A clause in the Act establishing trade schools gives power to do this. "Said board may enter into arrangements with manufacturing and mechanical establishments in which pupils of such trade schools may have opportunity to obtain half-time practice."

co-operation of employers secured to extend the system to the pupils in the commercial section during their last year at school in order that some real office experience might be obtained. At Cincinnati the system was extended to some girls, who were just entering millinery work rooms for alternate weeks at the time the writer visited the city. The Committee on Industrial Education for the State of Maine, reporting in 1910, recommends the establishment of the half-time system for manufacturing towns.

The plan has some very obvious advantages. For the boys, it is possible for many to stay at school who would otherwise have to leave to work full time, for the wages earned alternate weeks are sufficient to enable the parents to keep them. The schools suit those boys who at 14 are anxious to be doing some real work and who cannot see the value of a general education having no practical application. It is naturally an advantage to them that the work they do should be under supervision, that the why and wherefore of the processes should be explained, and that someone with experience should be at hand to solve difficulties, to encourage, and to stir and direct ambition. How much the boy (or girl) of 14 or 15 values the presence of someone interested in his progress in the works to whom he can report is known to all those in England who have taken part in apprenticeship work or who have followed the after careers of their scholars.

To the education authorities, the scheme is a benefit by its cheapness. There is no expense in providing expensive workshops, machinery, and material for exercises, since all practical work can be done outside, and only half the number of teachers are required that would be wanted for a full time school. It is probable that the actual training is not so methodical and possibly not so complete as can be given in a full time day trade school where all practical and theoretical work can be done on a fixed plan, but on the other hand the boys are getting experience under real conditions—they must learn to work with the degree of efficiency, speed, and freedom from waste which is necessary for shop conditions and may therefore become the better mechanics in the end. Moreover the boys' experience is broadened by the discussion of the difficulties and experiences of those in other shops. To secure a systematic training it is necessary that the thorough co-operation and interest of the firms should be secured, not only of the employers, but of the foremen and journeymen among whom the boys work. That the schools are approved by all who have tried them is evidenced by the fact that the number of manufacturers taking boys under the plan is constantly increasing and that they bear evidence to the greater reliability and steadiness of the school boys to those employed full time in the works, while the objections raised originally by foremen have been withdrawn. Of 62 employers approached in Massachusetts,

40 declared themselves in favour of the half-time plan.* It is necessary to have small classes and extremely capable teachers, men of great adaptability who can give individual instruction and who at the same time can work tactfully and sympathetically with those in control of the boys at work. The schools visited by the writer appear to have been particularly fortunate in the teachers obtained and it is no doubt due to this that the schools have grown. To ensure the fullest efficiency of the combined theoretical and practical instruction the State Board of Massachusetts is of the opinion that the "State supervisory control" over students in cooperative shops is extremely desirable, if not necessary.

The National Association of Manufacturers in their report issued in 1910, in which approval of the institution of day trade schools is expressed, refer to the half-time plan in dealing with the question of those already in the shops as follows:—

"Part-time schools may be provided where a double set of apprentices are employed, making a half-time system; or a one-third-time system when three sets of apprentices are employed, so that each apprentice can have one-half or one-third of his time devoted to schooling.

"In all these extension schools, whether one-half per week or part-time plan with two or more sets of apprentices, the wages earned would admit of supplying industrial education to many who could not afford to take the regular trade school course without earning wages meanwhile. All who are able to take the full course would, however, command higher wages at the end of the course."

This constitutes an approval of the cooperative system.†

The American Federation of Labor in their reports disapprove of the plan on the grounds that selection of the boys is made by the manufacturers and the course is therefore open only to those of whom they approve, and that a spirit hostile to organised labour may be developed among them and their teachers should any dispute or strike break out. It is suggested too that the manufacturers may attempt to control the schools for their benefit only.

In practice, however, there has been no difficulty in these matters. No suggestion has been made that the manufacturers have attempted to control the schools, and the provision of a board of trustees on whom the representatives of the firms are in a small minority, as at Beverly, should be a guarantee against this.

Dr. Herman Schneider, the originator of the system, believes it can be extended indefinitely. At Fitchburg, however, where other occupations besides that of engineer's machinist are included, it is already found difficult to correlate the school work to the outside work without a large increase of the school staff. The schools in their co-operative system of training would seem to be most successful, therefore, when they are confined to one or two industries only; if not, they must be

* See Appendix A. to this chapter, page 55.

† See also pp. 179-185.

sufficiently large to allow of the subdivision of pupils by occupation. But supposing that is not possible, or supposing that the boys or girls are engaged outside in non-educational employment, is not the system nevertheless well worth trying for those children who are obliged to leave school at 14 years of age to earn? It is then only the half-time system, but one under which the child is at school for a complete week and can give his undivided energy and attention to school work during that week, while he is under the discipline and care of the school to a greater age than would have been possible otherwise. The plan as a part of the school system of the United States should perhaps still be regarded as experimental, but the writer would be glad to see it largely extended there and tried in England, with the conditions of age, shop supervision and co-operation under which it is worked in America.

APPENDICES TO CHAPTER V.

A.—ATTITUDE OF MASSACHUSETTS EMPLOYERS TOWARDS PART-TIME SCHOOLING.

Seventy-two employers were approached by the Board of Education, Massachusetts, on their attitude towards part-time schooling. The great majority favoured the half-time plan.

" An exception to this is found in the case of department stores where the only plan submitted was for a period of eight hours per week.

" The following table shows the result.

Industry.	Time for School Work Proposed.	Number in Favour.	Number Opposed.	Total.	Percentage in Favour.
Department stores -	Eight hours per week.	10	—	10	100
Printing - -	Half-time	9	1	10	90
Machine work -	Half-time	16	3	19	84
Shoe manufacturing -	Half-time	10	3	13	77
Bookbinding - -	Half-time	2	4	6	33½
Textile work - -	Half-time	3	11	14	27

" The above table seems to show that where skill and training are required the manufacturers are, on the whole, in favour of co-operating with the school. The plans submitted suggested the alternating weekly scheme."

B.—FITCHBURG HIGH SCHOOL.—FORM OF INDENTURE USED FOR APPRENTICES AS ENGINEER'S MACHINISTS.

RULES AND CONDITIONS

Under which special apprentices taking the four year co-operative industrial course at the High School of Fitchburg are received for instruction at the works of _____

* Commonwealth of Massachusetts Board of Education. A Special Report on the needs and possibilities of Part-time Education, p. 100. Boston, 1913

1st. The applicant for apprenticeship under this agreement must have satisfactorily met requirements for entrance to this course at the High School.

2nd. The apprentice is to work for us continuously, well and faithfully, under such rules and regulations as may prevail, at the works of the above company, for the term of approximately 4,950 hours, commencing with the acceptance of this agreement, in such capacity and on such work as specified below.

Lathe work.

Planer work.

Drilling.

Bench and floor work.

And such other machine work, according to the capability of the apprentice, as pertains to our branch of manufacturing.

This arrangement of work to be binding unless changed by mutual agreement of all parties to this contract.

3rd. The apprentice shall report to his employer for work every alternate week when the High School is in session, and on all working days when the High School is not in session, except during vacation periods provided below, and he shall be paid only for actual time at such work.

4th. The apprentice is to have a vacation, without pay, of two weeks each year, during school vacation.

5th. The employer reserves the right to suspend regular work wholly, or in part, at any time it may be deemed necessary, and agrees to provide under ordinary conditions other work at the regular rate of pay, for the apprentice during such period.

6th. Should the conduct or work of the apprentice not be satisfactory to employer or to the High School authority, he may at any time be dismissed or suspended for a time by the employer without previous notice. The first two months of the apprentice's shop work are considered a trial time.

7th. Lost time shall be made up before the expiration of each year, at the rate of wages paid during said year, and no year of service shall commence till after all lost time by the apprentice in the preceding year shall have been fully made up.

8th. The apprentice must purchase from time to time such tools as may be required for doing rapid and accurate work.

9th. The said term of approximately 4,950 hours (three-year shop term), shall be divided into three periods as stated below, and the compensation shall be as follows, payable on regular pay days to each apprentice.

For the first period of approximately 1,650 hours, 10 cents. per hour.

For the second period of approximately 1,650 hours, 11 cents. per hour.

For the third period of approximately 1,650 hours, 12½ cents. per hour.

10th. The above wage scale shall begin the first day of July preceding the apprentice's entrance upon the first year of shop work of the High School Industrial Course.

These papers, subject to the two months' trial noted in paragraph 6, shall be signed by the three parties to the contract at the time the boy enters the shop.

The satisfactory fulfilment of the conditions of this contract leads to a diploma, to be conferred upon the apprentice by the School Board of Fishburg upon his graduation, which diploma shall bear the signature of an officer of the company with which he served his apprenticeship.

AGREEMENT BETWEEN THREE PARTIES.

THIS AGREEMENT made the _____ day of _____, A.D., 191 _____, by and between _____ (employer) of _____ party of the first part; and _____ (apprentice) of _____ party of the second part; and _____ (bondsmen) of _____ party of the third part.

Witnesseth, that the party of the second part shall from the date hereof for the term of three years (4,950 hours, divided into three periods of 1,650

Form of Indenture for Fitchburg Co-operative Course. 57

hours a year, as stated in the "Rules and Conditions"), and so much longer as may be necessary to make up lost time, become and be the apprentice of the party of the first part to the art or trade of _____ and that said parties of the first and second parts will well and truly do and perform all things required to be done and performed by them in and by said Rules and Conditions of the Co-operative Industrial Course.

And said party of the third part, in consideration that the parties of the first and second part have with his knowledge executed this agreement, covenants and agrees for himself and his assigns, heirs, executors or administrators, with the party of the first part that the party of the second part shall well and truly do and perform all things required to be done by him, in and by this agreement, and in case said party of the second part shall in anywise violate any of the terms and provisions of this agreement or said Rules and Conditions, to pay to the party of the first part the sum of One Hundred Dollars, as ascertained and liquidated damages for such breach of contract.

In witness whereof, said party of the first part has caused these presents to be signed and sealed by _____ its _____ for this purpose authorized, and said parties of the second and third parts have hereto set their hands and seals this day and year first above written.

Signed and sealed _____ (employer).

In presence of _____.

Signed and sealed _____ (apprentice).

In presence of _____.

Signed and sealed _____ (bondsman).

In presence of _____.

AGREEMENT OF RELATIVE OR GUARDIAN.

I, _____ (relative or guardian) of the above named _____ (apprentice) do hereby give my consent to his entering the employ of the said _____ (employer) upon the terms named in the above articles of agreement; and I further agree that in consideration of such employment the wages or earnings of my said _____ (son or ward) shall be paid directly to him, and I hereby release all claim that I now have or may have hereafter thereto.

Dated at _____ this _____ day of _____, 191 .

Signed and sealed _____ (relative or guardian)

In presence of _____

This is to certify that the within named _____ completed his term of apprenticeship.



C.—FITCHBURG HIGH SCHOOL.—CO-OPERATIVE INDUSTRIAL COURSE. SUMMARY OF WORK.

MACHINIST, DRAFTSMAN, PATTERN MAKER, AND IRON MOLDER.

(Figures at right indicate periods per week of 45-minute duration.)

FIRST YEAR.—ALL SCHOOL WORK (common to all courses):—	THIRD YEAR.—SCHOOL AND SHOP WORK:—
English 4	English 5
Arithmetic, tables and simple shop problems 5	Shop Mathematics, Trigonometry 5
Civics and American History 4	Chemistry 4
Algebra 5	Physics 4
Freehand and Mechanical Drawing and Bench Work 10	Mechanism of Machines (one-half year) 4
	Business Methods (one-half year) 4
	First Aid to injured 1
	Freehand and Mechanical Drawing 6
SECOND YEAR.—SCHOOL AND SHOP WORK:—	FOURTH YEAR.—SCHOOL AND SHOP WORK:—
English 5	English 5
Shop Mathematics, Algebra and Geometry 5	Shop Mathematics 5
Physics 4	Economics (one-half year) 5
Industrial History and Commercial Geography 5	Mechanism of Machines and Jig Design (one-half year) 5
Mechanism of Machines 5	Physics, Electricity and Heat 4
Freehand and Mechanical Drawing 5	Chemistry 6
	Freehand and Mechanical Drawing 4

NOTE.—Text books for each trade in Mechanism deal with individual trade, and class work discusses his particular problems.

COTTON MANUFACTURE.

SECOND YEAR.—SCHOOL AND SHOP WORK:—	FOURTH YEAR.—SCHOOL AND SHOP WORK:—
English 5	English 5
Shop Mathematics, Algebra, and Geometry 5	Economics (one-half year) 5
Physics 4	Mathematics 5
Industrial History and Commercial Geography 5	Cotton Machinery (one-half year) 4
Cotton, sources and preparation 5	Physics, Electricity and Heat 4
Freehand and Mechanical Drawing 5	Chemistry of dyeing, soaps, sizing 6
	Pattern Design 4
THIRD YEAR.—SCHOOL AND SHOP WORK:—	
English 5	
Mathematics 5	
Chemistry 4	
Physics 4	
Cotton Machinery 4	
First Aid 1	
Freehand and Mechanical Drawing 6	

Some idea of mill end of work may be gleaned from following schedule of course in yarn manufacture at Grant Yarn Co. :—

CARD ROOM.—1 AND 1-2 YEARS.

Picker Room	2 weeks.
Card Room	2 months.
Doublers	6 weeks.
Combers	7 months.
Drawing	1 month.
Speeders	6 months.

SPINNING ROOM—6 MONTHS.

FINISHING ROOM—10 MONTHS.	
Spooling	2 months.
Twisting	4 months.
Warpers	2 months.
Winders	2 months.

MACHINE SHOP—2 MONTHS.

PRINTING.

SECOND YEAR.—SCHOOL AND SHOP WORK:—

English	5
Mathematics, Algebra and Geometry	5
Physics	4
Industrial History and Commercial Geography	5
Printing Machinery and Appliances, type forms	5
Drawing (lettering, designs)	5

THIRD YEAR.—SCHOOL AND SHOP WORK:—

English	5
Mathematics (estimating)	5
Chemistry	4
Physics	4
Printing Machinery (one-half year)	4
Business Methods (one-half year)	4
First Aid to Injured	1
Laying Out and Design	6

FOURTH YEAR.—SCHOOL AND SHOP WORK:—

English	5
Mathematics (cost estimating)	5
Economics (one-half year)	5
Machinery and Setting (one-half year)	5
Physics, Electricity and Heat	4
Chemistry (inks, gelatines, cleaners, &c.)	6
Drawing and Design	4

OFFICE WORK LEADING TO SALESMANSHIP.

SECOND YEAR.—SCHOOL AND SHOP WORK:—

English and Penmanship	5
Office Organisation	5
Mathematics (Business Arithmetic)	5
Physics	4
Industrial History and Commercial Geography	5
Freehand and Mechanical Drawing	5

THIRD YEAR.—SCHOOL AND SHOP WORK:—

English	5
Mathematics (estimating, book-keeping systems)	5
Chemistry	4
Physics	4
Salesmanship (one-half year)	4
Business Methods, Study of Shop Manufacture (one-half year)	4
First Aid	1
Freehand and Mechanical Drawing	6

FOURTH YEAR.—SCHOOL AND SHOP WORK:—

English	5
Mathematics	5
Economics (one-half year)	5
Psychology of Salesmanship (one-half year)	5
Typewriting	4
Chemistry	6
Freehand and Mechanical Drawing	4

D.—INDUSTRIAL SCHOOL, CITY OF BEVERLY, MASS.

(1) COURSE AND METHODS OF CLASS-ROOM INSTRUCTION.*

The work of our teachers tends more and more to individual teaching or to meeting small groups. The aim of this work is to keep the topic closely related to the work last handled by the boy in the shop. The shop work being all piece work on various machine parts, involves the taking up of topics as they are needed, without regard to a strict traditional pedagogical order. We expect to give each student as much as he can use and needs to use now.

In grouping the topics those for "Elementary Work" are considered as absolutely essential; other topics, a knowledge of which is desirable are to be taken up only when the essentials have been covered, the test for admission of any topic are:—

1. Has this topic definite and vital connection with the boy's present shop experience?
2. Is this essential to the training of a boy machinist?
3. If, under conditions attached to the piece-work system, the boy cannot connect this topic with his present shop experience, will he likely have an opportunity to do so within a reasonable time?
4. Will this topic add to the all-round equipment of the boy as a citizen-to-be?

As far as possible topics are taken up as the boy has met them in his preceding week of shop, either in handling a blue-print, operating a machine, using tools, or filling out time cards. The blue-print will show the need of, say, reducing sixty-fourths to equivalent decimals, adding or subtracting fractions, representing a machine part by three views, figuring allowance for finish, &c. Mathematics is taught almost entirely from the blue-print in the hands of the pupil, the blue-prints (of which the school has nearly one thousand copies in sets) are in actual use in the machine shop.

Most of the class-room instruction is individual; this does not prevent gathering small groups to discuss and solve common difficulties. Note-books of operations performed in the shop are discussed by the instructor and used as a basis for instruction.

MECHANICAL DRAWING.

The aim is to develop the ability to read a blue-print and to make a simple working drawing. At every stage models or parts of machines are used. The method in general consists in handing the boy a machine part which he is to measure and study, then make a working drawing fully dimensioned. After criticism by the instructor the drawing is corrected and compared with the shop blue-print of the same object.

Geometrical problems are not given, this being left to geometry proper.

Elementary Work.—Introductory lessons are given covering the tools and their uses, lettering and lay out of sheets. By the use of simple objects, similar to what he has handled in the shop-work, instruction is given in projections, simple intersections, drawing to scale, sectional views, developments, detail, and assembly. These topics are treated in as regular a sequence as possible; a large variety of parts of machines provides for instruction meeting the boys' immediate need as well as furnishing work suited to his ability.

Advanced Work.—While more difficult subjects are taken up, greatest stress is laid upon the making of freehand drawings, and the design of jigs

* From the Annual Report of the School for 1911.

and tools. Although not required of machinists, the fact that in the United Shoe Machinery Company, the machinist who can show skill and resourcefulness in drawing may be promoted either to the drafting-room or to experimental tool work makes this training of intense practical value.

ELEMENTARY ARITHMETIC.

Addition, Subtraction, Multiplication, and Division of whole numbers used mainly in connection with fractions as below.

Common fractions used are principally $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, and $\frac{1}{64}$, as practical measurements come to machinists expressed in these denominations: reduction to lowest terms is used only as applied to these fractions. Blueprint reading and laying out work require fractions in the following operations: finding all-over length when parts are given, figuring tapers, finding sizes of set or distance blocks, laying out drill holes. Problems brought in by the student from his work are used.

Decimal fractions are limited to no more than four places, or ten thousandths. Dimensions come from the drafting-room commonly expressed in common fractions, but must be reduced to decimals for exact work by the machinist, work correct to one ten thousandth is common. Change halves, fourths, fifths, sixths, thirty-seconds and sixty-fourths to equivalent decimals of four places so as to familiarise the boy with these common measurements.

Reducing decimal fractions of four places to common fractions in terms of the nearest sixty-fourth. A common and necessary use of these is in setting over the footstock for tapering.

Fundamental Operations applied to both Common and Decimal Fractions.—The shop work of the student calls constantly for facility in correct handling of fractions. Some processes mentioned above as involving only one operation call for many others. Problems for this division are taken from straddle milling (addition of common fractions), gauging cutters (addition of common and decimal), milling slots (subtraction required in getting distance between slots and depth and width of same), reading micrometer and vernier (using operations in decimals), figuring tapers (all operations), finding sizes of set or distance blocks (involving whole numbers as well as fractions), laying out plan of drill holes (whole numbers and fractions), selecting sizes of counter-bores, drill holes for tapping, figuring clearance holes for bolts, &c.

Problems from Students' Shop Time Cards.—All of the fundamental operations using whole numbers and fractions are used. Data given: total pieces handed in to inspector, rate per piece or per hundred hours on the job, idle hours; to find total wages, student's share (half total), rate per working hour, rate per hour for entire time consumed whether working or idle, amount lost by idleness. Several time-cards received by the student add to the problems. The use of these cards covers the following operations: addition, subtraction, multiplication, and division of common and decimal fractions.

ADVANCED MATHEMATICS.

Ratio and Proportion.—Simple uses as required in crowning pulleys, figuring speed of machines and gearing.

Use of Mechanics' Tables.—Throughout the course the student will be provided with a handbook of machinists' tables, and will be drilled in their use. Elementary work in these will handle decimal equivalents of common fractions in sixty-fourths, size of taps and drills for U.S.V. and Sharp V threads, gear teeth parts, squares and square roots, circumference and areas of small circles, surfaces and volumes of spheres.

Formulas and Graphs.—The function and formation of graphs is taught to assist the student to place facts in concrete shape, to find approximate answers and problems, and to understand graphs as used constantly in industrial periodicals. Sheets are provided for plotting facts of his work such as wages earned each week, rate per working hours, &c.

Formulas are given to beginners in the simplest manner possible.

Area and volume of solids are given only when needed in actual operations.

Geometry.—Taught (1) to assist in thinking in three dimensions (2) to visualise surfaces and solids with the idea of developing quick apprehension of working drawings, and (3) to give facility in using simple problems in tool and jig design and in laying out the work. Ten problems to be worked out, but not demonstrated, are given; the development of surfaces and solids is taught as leading up to the measurement of these quantities.

Trigonometry.—Advanced pupils use logarithms and learn the solution of triangles, using the natural functions. Every problem used in this work is taken directly from the blue-print. Such training is invaluable for machinists in, say, laying out angles from a base line by linear measurements when the blue-print gives angles expressed only in degrees—a common drafting-room practice. The use of the sine bar is thus made clear.

ENGLISH.

A better descriptive term for this study is Expression, as this is the objective of the instruction. Pupils are required to describe and explain—both orally and in writing—shop operations in detail, the use of the various tools, shop troubles and their remedy, technical articles read, &c.

The collection, arrangement, and presentation of material are discussed and illustrated so that the boy may learn to express himself clearly and forcibly. Actual letters are written and mailed, and the answers are followed up if need be.

Spelling lessons are based upon the words used in the written matter of the daily exercises; new words are introduced only when met in the course of the work.

The following topics are treated as growing out of shop experience—Simple business transactions as actually used or desirable by shopmen, such as: Depositing money in bank, drawing money, making and endorsing cheques, making out and collecting money orders, registering letters, ordering goods, making receipts and bills, routine of sending packages by express, handling shop organization funds (such as belong to social groups and athletic teams), disbursing organization funds, making out notices for shop meetings, &c., keeping records of such meetings.

From the view point of culture, definite instruction is given by suggestions regarding reading and by questions on the same, designed to cultivate desirable reading habits. Hints are given regarding the use of public library and reading rooms, catalogs, indexes and files. The school reading-room is very profitable in this study.

GENERAL TOPICS.

Personal Hygiene.—As related to the care of the body, sleep, food, exercise, bathing and special needs of growing boys.

First Aid.—In accidental emergencies how given; common materials used. How to guard against accidents, especially from contact with moving machines.

Civics.—From the practical stand-point as developed by an examination and discussion of the local form of government, the state and federal governments. A plain impartial statement of the great questions now before citizens is used, not so much to give the boy information as to awaken his interest and thought.

SCIENCE.

The course is not yet fully developed in so far as relating it closely to shop experience, but the following list of topics is the basis for the course which is being worked out and arranged in groups as A, essential; B desirable, yet not necessary for actual work in the trade.

Work: measured, common units, levers, as instruments for developing power, illustrated by wrenches, belt shippers, handles of machines, treadles, cranks, vises, handling files, hammers; applied to belting and gears.

Screw and Wedge: illustrated by cutting tools, cams, vises, drill screws, keys, chutes, reamers, drills and files.

Friction: as a hindrance in securing power, varying with metals; how lessened and why.

Heat: how generated and used as energy; effects on tools and measurements; relative expansion with different metals; use in hardening, annealing, tempering.

Steam and Air: applied to trip hammers, drop forge, steam and pneumatic drills.

Hydraulics: law of water pressure, how used in hydraulic pumps and presses.

Specific Energy: definition and use as applied to weight of iron, steel, brass, lead, aluminium—figures for each of these metals.

Inertia: how overcome, relation to starting and stopping machines, and to strength of hammer blows.

Centrifugal Force: illustrated by familiar occurrences in the factory.

ELECTRICITY.

As little theory as is necessary to understand operation of electrical motors, bells, lighting. More stress is laid on the application of electricity than on theory. Ohm's law is discussed.

Common electrical units are explained and illustrated.

CHEMISTRY.

Oxygen: rusting, tarnishing. Relation to health.

Hydrogen: simple treatment.

Water: composition, "hard," "soft."

Nitrogen and its compounds: elementary study of the composition and production of steel, iron and coke; illuminating gas, kinds of iron and steel, with combinations for tool work; high carbon, low carbon.

Sulphur and its compounds.

Lead: tested with blowpipe; studied as regards composition, production, and distribution; analysis by the wet process; analysis of Babbitt metal and solder.

The foregoing treatment of lead is used with other metals handled by machinists, such as steel, iron, copper, brass, aluminium.

The method of teaching chemistry is by technical work in the laboratory. No text-book is used, instruction being given entirely by the teacher and incorporated in the note-books. So far as possible, beginning studies cover the essentials of the above topics. Advanced pupils carry further analysis of above topics and are given much laboratory work, especially in its bearing upon iron and steel. This is preparatory for probable openings in other departments of the United Shoe Machinery Company.

(2)—BEVERLY INDEPENDENT INDUSTRIAL SCHOOL.
SHOP REPORT CARD. [size 6" x 4"]

Independent Industrial School Beverly, Mass.		SHOP REPORT			Name	Shop No.			
		Week ending.....191							
	Machine.	Part No.	Original Number of Pieces.	Opera- tion.	Price.	Number of Pieces.			Nature of Work and Distribution of Hours. (<i>See</i> Key below.)
						Com- pleted.	Passed.	Junked.	
Mon.									
Tu.									
Wed.									
Th.									
Fr.									
Sat.									
x Returned for Correction,									
Total - -									

Total net earnings _____ Rate per hour _____ Per cent. junked _____

A—Absent. B—Bench. C—Cam cutter. CE—Centering machine. CH—Chucking. DU—Drill (Upright). DS—Drill (Sensitive). G—Gear cutter. GS—Grinder (Surface). GU—Grinder (Univ.). LE—Lathe (Engine). LS—Lathe (Speed). MH—Miller (Horizontal). MV—Miller (Vertical). P—Planer. S—Screw machine. SH—Shaper. T—Tardy. TL—Time Lost.—Tools, &c., not ready.

E.—QUINCY INDEPENDENT INDUSTRIAL SCHOOL.

(1) COURSE OF STUDY.*

The course of study outlined for the first year is as follows:—

	Hours per Week.
Shop mathematics - - - -	10
Drawing (freehand and instrumental) - - - -	10
English - - - - -	4
Geography - - - - -	2
Civics - - - - -	4
Physiology and first aid - - - -	1
Total - - - -	31

The instruction is individual as far as possible to make the work more effective, and to enable each pupil to get just the sort of information that he needs without being dependent upon the progress of others.

In a class of 20 boys it is found necessary to use seven different text books in mathematics; also many type-written sheets and blue prints. The range of work covered in a single period would be about as follows:—Common and decimal fractions, board measure, ratio and proportion, square root, mensuration, formulas, simple and compound gearing for lathe and milling machine, simple and compound and differential indexing for the milling machine, cutting speeds and feeds, solution of right and oblique triangles, gearing, logarithms, graphs, &c. The instruction is so given

* From the Annual Report of the School Department of the City of Quincy, Mass., for the year ending December 31st, 1912.

that it will develop in the pupil's mind a quick and accurate method of solving problems which he is meeting or will meet in the shop.

The instruction gives him the method used to get results, not a method of proving and developing formulas.

Drawing is principally freehand, aiming to develop in the pupil an ability to make freehand detail sketches of broken machine parts which, in his future experience, he is sure to encounter. The sketches made in school are taken from the work, machine or machine parts, which the pupil is working on in the shop. Each sketch is dimensioned and explained. This method serves to develop an idea of proper proportion, and forms an excellent English lesson showing up the points in the boy's work which he does not understand, and makes him more observing. It teaches him also to be accurate and neat in the execution of his work.

English is also taught by using technical magazines, public library books, books from school library, and by making sketches on the blackboard and explaining them to the class orally.

Spelling lessons are given by taking the misspelled words from the papers, and having the class write the word and give the correct meaning which they look up in the dictionary.

Geography is taught to give the pupils a more thorough knowledge of the physical and industrial surroundings in the United States. The work is started by studying the local surroundings, broadening out to the entire State of Massachusetts, then the New England States, comparing the advantages and disadvantages of other States with those of Massachusetts.

Civics is taught to give each pupil a good knowledge of the government of the city or town in which he lives. A text book is used as a base of ideas which are to be developed by means of the pupils' investigating the various branches of the local government, and comparing the local conditions with those of the surrounding towns and cities. For example, the boys from Rockland, Weymouth, Braintree, Abington, Cohasset, and Boston report on the government conditions of their communities, and the boys from Quincy make their report. The various points are talked over and made clear to all.

First aid is taught to warn pupils against accidents, and show them how to take care of an injury should an accident occur.

Flexible Promotion.

On account of the extremely wide range of education and experience which pupils have when they come to us, it would seem advisable to maintain a very flexible method of promotion that will allow each pupil to advance as rapidly as he is capable without any thought of the remainder of the class. This method of promotion would enable the good, bright, energetic boy to complete the three years' course in much less time; this condition would place him in the shop full time, with full credit from the school and the same degree of supervision by the school as the boys in the school would receive. It is the intention of the school to watch over its pupils for eight years, three years in the school and five years in the shop.

The work in the shop is arranged so as to give the boy the best experience. The course of training in the Fore River Shipbuilding Company would be about as follows:—

Machine shop: Office work, tool-room, bolt-cutting machine, hand milling machine, key seater, cold and hand saws, erecting, outside work, drill work, engine lathe, planer, slotter and shaper, boring mill, milling machine.

Wood shop: Glue room, mill work, bench work, outside shop work, drawing room, mill work, laying out work, special form work.

The time allotted for each branch depends somewhat upon its importance; that is, on the more important machines a pupil would get about three months' actual shop experience and three months of relative work in the school. On the less important machines the actual shop experience would be about six weeks, with six weeks of relative work in the school.

(2) CARDS USED.

(a) SHOP RECORD CARD.

[Size 5" x 3"]

City of Quincy.		SHOP RECORD CARD.		Industrial Department.	
Name.	Number.	Name.	Number.		
Machine.	Kind of Work.	Machine.	Kind of Work.		
Monday - -					
Tuesday - -					
Wednesday - -					
Thursday - -					
Friday - -					
Saturday - -					
Week ending	191 .	Week ending	191 .		
Class A.		Class B.			

(b) TIME REPORT.

[Size 8½" x 11"].

[This form is used for reporting school attendance to employers.]

QUINCY INDUSTRIAL SCHOOL. TIME REPORT.

Week ending _____ 191 .

Name of Pupil.	Hours in School.	Hours Late.	Hours Absent.	Remarks.

(c) APPRENTICE WORK REPORT.

[Size 8½" x 10½"]

APPRENTICE WORK REPORT.

Form 490.

 Regular _____
 Part time _____
 Special _____

 Apprentice _____ No. _____ M _____
 Mate _____ No. _____ M _____

Year.	Began.	Hours Absent.	Remarks.
1			
2			
3			
4			

(c) APPRENTICE WORK REPORT—cont.

CLASSIFICATION OF WORK.

Class.	Routine.	Time Allowed. Hours.	Time Employed.		Leading Man.	Remarks.
	Work.		Started.	Finished.		
A	Office - - -					
	Tool room - - -					
	Bolt cutting machine.					
	Hand milling machine.					
	Key seater - - -					
	Cold and hand saws					
	Screw machine - -					
	Punch press - - -					
	Miscellaneous machines.					
	Helping on large machines.					
B	Floor-bench work -					
	Erecting - - -					
	Outside work - -					
C	Drill { Upright -					
	{ Radial - -					
D	Engine lathe - -					
E	Planer - - - -					
	Slotter - - - -					
	Shaper - - - -					
F	Boring mill { Horizontal					
	{ Vertical - -					
G	Milling machine { Horizontal					
	{ Vertical					

General Remarks.

(d) SCHOOL HISTORY CARD. [Size 7" x 9 1/4"]

[For record of boy who has entered the School.]

QUINCY INDUSTRIAL SCHOOL. SCHOOL HISTORY CARD.

School. Assigned No. _____

Name _____ Age _____

1. School _____ Grade _____

Good standing on leaving grade? _____ If not, what was the trouble? _____

2. School history card on file? _____

3. Date of admission? _____ Training desired? _____

So stated by pupil? _____ So stated by parents? _____

Admitted after conference? _____ Both parents present? _____

Which, if only one? _____

Name of Father? _____ Occupation? _____

Name of Mother? _____ Occupation? _____

Address? _____

4. Date of admission to the school? _____

Date of promotion? (1) 1st to 2nd year _____

(2) 2nd to 3rd year _____

(3) 3rd to 4th year _____

(4) Graduation _____

5. Date of leaving? _____ Cause? _____

Where did he go? _____

In what occupation? _____ At what age? _____

6. History after leaving:

1st year report: _____

2nd year report: _____

5th year report: _____

7. If he is a non-resident pupil: _____ Town: _____

Date of application _____

Date of approval by State Board of Education _____

On the reverse side.

8. Local objections, if any? _____

9. Transfer to other departments:

From _____ To _____

Date _____ Cause _____

From _____ To _____

Date _____ Cause _____

From _____ To _____

Date _____ Cause _____

10. Rating by faculty opinion:

1st Year. 2nd Year. 3rd Year. 4th Year.

- a. Manual work - -
- b. Shop intelligence - -
- c. Technical work - -
- d. General education - -

Remarks: _____

Address _____

No. _____ Street _____ Town _____ State _____

Form No. 7.300.

CHAPTER VI.—TRADE SCHOOLS AND CLASSES FOR GIRLS.

VOCATIONAL TRAINING FOR GIRLS.

The trade education of girls is hardly so advanced in the United States as that of boys. It is easier to determine a boy's vocation than a girl's, for with the girls there is the ever present problem whether the girl shall be educated for the five to seven years of wage-earning life, which is all the majority follow, or for her later probable vocation of home-maker. There are two parties interested in girls' education in the United States, the one urging that girls from the elementary schools be given training which shall make them efficient producers, the other that it is of far greater importance that they become enlightened and intelligent consumers.

Domestic science teaching and the various studies correlated with the art of home-making have received far more attention in the United States than in England. A domestic arts course is found in most high schools to correspond with the manual arts course for boys, and in several universities the completion of a domestic arts course for women ranks as equivalent to an arts or science degree. In the elementary school and in evening classes the vocation of home-maker is that most universally recognised. In the prevocational schools the girls are given cooking, sewing for home use, and laundry work, while the boys take one or more actual manual trades. In the evening classes and compulsory continuation classes which the writer visited, dressmaking and millinery were always taught for home use, while classes for women were arranged in cooking and home needlework. It is only in the few definite trade schools that real trade teaching is offered.

It has already been pointed out that the prospects of the girls of 14 who leave the elementary schools to seek immediate earnings are as uncertain of advancement as those of boys. Employers will not trouble to train these thoughtless, irresponsible, and immature girls, and more and more of the better-class firms and those in skilled trades are refusing to take on any under 16 years of age. An inquiry made by the Women's Educational and Industrial Union of Boston into the need of a Trade School for Girls at Worcester, Massachusetts, showed that of 727 girls leaving school in the year 1909 to 1910, 22 per cent. went into mercantile establishments (this includes the little cash and floor girls of the department stores), 35 per cent. entered unskilled industries, 39 per cent. medium skilled industries, and only 1 per cent. skilled industries. Moreover, a huge amount of shifting and instability was revealed amongst the 14 to 16 year

old girls in the textile mills and corset and other factories of Worcester.* It is only the girl with perseverance, ambition and steadiness who can pass through the demoralising and disheartening two years of unskilled work unaided and unharmed, and can afterwards, when she is of sufficient age, fit herself for responsible and well-paid employment. There is a large and unsatisfied demand for those who can undertake skilled work, but a great unwillingness on the part of employers to undertake the work of training employees for it. Hence, as with the boys, philanthropic effort first stepped in and endeavoured to bridge over the entry into skilled work by offering girls a technical training which would make them of immediate use to employers in skilled trades. Under the new laws for industrial training certain of these girls' trade schools have been taken over by the public authorities and new schools have been started, and with the increasing interest in industrial education technical courses for girls have been started in some high schools. These last are described in the next chapter.

HISTORY OF THE TRADE SCHOOL MOVEMENT.

The first philanthropic institute to offer definite trade training to white girls was the Pratt Institute, Brooklyn, New York, which in 1891 opened courses in millinery and dressmaking for women and girls over 17 years of age, and which are still continued. The trades are taught by trained teachers in the earlier stages, and by practical trade experts at the end, and are arranged in one-year courses. A dressmaking establishment is maintained by the Institute, in which customers' work is taken, and where the girls spend their last three months in order to gain trade experience. Since the age of entry must be at least 17, this school is attended by those girls who have been able to spend two or three years at high school, or by those who find themselves obliged to turn to wage-earning unexpectedly, but can nevertheless afford a time of training first.

Between 1890 and 1900 two or three philanthropic trade schools, mainly for Jewish girls, were founded in New York. In 1902 the Manhattan Trade School for girls was founded by philanthropic effort, and in 1904 the Boston Trade School for girls, by the Women's Educational and Industrial Union.

This last school was taken over by the public school authorities in 1909, and in the same year, under the Wisconsin Law of 1907, the Milwaukee School of Trades for Girls was founded by the public school authorities. In 1910 the School Board for New York City took over, and has since maintained, the Manhattan Trade School for Girls. These three schools are the

* *A Trade School for Girls, a Preliminary Investigation in a Typical Manufacturing City, Worcester, Mass., by the Research Department of the Women's Educational and Industrial Union of Boston. United States Bureau of Education. Bulletin 1913. No. 17. Washington. Government Printing Office, 1913.*

principal girls' trade schools in the United States. In addition, there are departments for girls in which trade dressmaking and sometimes millinery are taught in the trade schools for boys at the following cities:—Bridgeport and New Britain, Connecticut; Lawrence, Lowell and New Bedford, Massachusetts; Albany, Rochester, Syracuse, Yonkers and the Iroquois Indian School, New York State; and the Vocational School and Armstrong Manual Training School for negroes in Washington, D.C. There are separate schools for girls in dressmaking and millinery at Springfield and Worcester, Mass., power machining being also taught at the latter school. These schools, it will be noticed, are all in Eastern States. Except at Milwaukee, nothing has been done by the public authorities in the middle west to start trade schools for girls. Technical High Schools have been opened at Chicago and Cincinnati, but trade schools are maintained only by two Settlements in Chicago, and by the Young Women's Christian Association in Cleveland. These philanthropic schools have shown that there is a demand there, too, for training which will enable girls to be of immediate use to employers on entering the workrooms.

TYPE OF INSTRUCTION.

The condition of entry to a technical high school is completion of the grammar grades—for a trade school no high standard of schooling is necessary, a girl must only have reached 14 years of age.

The three large public trade schools were visited by the writer. It was curious to find that, in all these, girls are admitted who have had no previous practice in sewing, since sewing is not taught in many of the parochial elementary schools. Hence in each school some preliminary hand-sewing must be given to girls taking a needle trade, and many enter who prove to have no aptitude for the needle. Such girls must drop out or change to some other trade.

The newer boys' vocational schools do not aim at giving a complete training in a trade, but only such a grounding in theory and practice as will enable a boy to profit quickly by further instruction and practice when he enters an employer's workshop. The girls' trade schools of New York and Boston, on the other hand, offer such training in technical processes as enables girls to be of immediate use to employers, and to earn good salaries as soon as they become accustomed to workroom speed and life. Experience is given in the technical processes only pertaining to that branch of the trade which the girl intends to enter. Each of these schools has a two-year course, but scarcely any girls attend for more than one year, many dropping out to seek employment before even that is finished. The schools therefore are training the rank and file of workers rather than those for the higher positions which would more probably be filled by pupils from the technical high schools.

The upkeep of the schools is costly,* and since the workers are being trained for certain trades only, a direct subsidy is being offered to employers in these trades at the cost of the city. That the training offered is approved by employers is shown by the readiness with which the trade school girls obtain positions.

MANHATTAN TRADE SCHOOL FOR GIRLS, NEW YORK CITY.

This school has accommodation for about 500 girls and offers instruction in seven occupations, viz., dressmaking, millinery (with lampshade making), clothing machine operating, straw machine operating, embroidery machine operating, sample mounting and novelty work. Girls may enter on the Monday of each week and may choose the trade they wish to learn. The school is in session throughout the year except in August. The school day extends from 9 a.m. to 5 p.m. every day except Saturdays. Five hours each day are given to trade practice and two hours to English and other subjects taught in relation to the trade work. A little practice in cooking is given. An hour each day is given for luncheon and recreation. Materials for the training are supplied free and the school is free to girls of any nationality from all the boroughs under the New York City Board of Education.

The official notice of the school is given in Appendix A (1) to this chapter, page 75 below.

On going through the school, the writer learned that the courses are planned with a view to the girls' work as wage earners under existing conditions rather than to their wider education. For instance, the girls taking power machine operating do not learn hand sewing at all. They start at once

* Per capita cost of New York Schools. Year ending July 31st, 1912.
(From the Annual Report of the City Superintendent of Schools. New York City, 1912).

	Elementary Schools, Estimated on		High Schools, Estimated on		All Public Day Schools, Estimated on		Manhattan Trade School for Girls, Estimated on		Vocational School for Boys, Estimated on	
	Average Monthly Register.	Average Daily Attendance.	Average Monthly Register.	Average Daily Attendance.	Average Monthly Register.	Average Daily Attendance.	Average Monthly Register.	Average Daily Attendance.	Average Monthly Register.	Average Daily Attendance.
Salaries -	\$ 28.54	\$ 35.86	\$ 78.45	\$ 91.08	\$ 35.58	\$ 29.33	\$ 102.86	\$ 132.24	\$ 64.16	\$ 70.56
Supplies, Janitors, &c. -	-	-	7.37	8.51	7.93†	8.77†	0.18	.68	5.15	5.66
Totals -	-	-	85.82	99.59	43.51†	48.10†	103.04	132.92	69.31	76.22

† These figures are really too high, as some share of the supplies was for evening schools, recreation centres, &c.

with practice work on the machines, since as machine operators in a factory they would have no call for hand sewing. Neither are they taught cutting and fitting; garments are cut out by the teachers and given to the girls ready to put together.

The largest number of girls choose dressmaking for their trade. They do not learn cutting, fitting or draping. This is done by the teachers, but the girls can watch the draping done in the work rooms. It has recently been arranged that for one month in the year, girls if they wish can buy their own materials and make garments for themselves. In this case they cut out the garments and fit each other.

Millinery is discouraged as a trade owing to its very seasonal nature. Very few girls take this, and all who do learn lampshade-making as a subsidiary trade, for this is busy when millinery is slack. Lampshade-making is not taught as a trade by itself, but one or two girls who have done well at it, have followed the trade later.

Straw machine operating is a trade at which machinists can earn very high wages during the busy season, but girls taking this work have to face long slack times.

Sample mounting, the making up of manufacturers' samples, is taken by girls who wish to earn good wages immediately after a short training. It is not a highly skilled trade, and does not offer very good prospects. Girls who can are urged to stay on after completing the course of six months' sample mounting, and take novelty making. This is the making of fancy goods, such as the best cardboard and linen boxes, trays, blotters, frames, &c., of linen, leather, or other fancy material.

The teaching of drawing has been abandoned, and design in relation to the particular trade is taught. Thus girls on the embroidery machines make and stamp out their own designs.

When the school does not have sufficient suitable orders from customers, work is made for stock and sold in the school sales room. Trade prices are charged unless an article is in any way faulty. The school produces about 2,000l. worth of garments and other articles in a year. The receipts from garments, &c., disposed of in 1911-12 was \$2,344.72 (=about 470l.) in excess of the value of supplies and materials used. It is obvious that the organisation of the school is a tremendous piece of work and necessitates a large staff of teachers and clerical assistants.

The teachers in charge of the beginners are trained teachers who may have had some practical experience at their trades. It is found that these give the girls a better grounding in essentials. Teachers drawn from the trades are in charge of all the advanced work so that workroom methods may be inculcated.

Numbers of girls leave without completing their courses. The figures for 1911 to 1912 are as follows:—"During the year, 1,050 different girls have been in attendance at the school, 383 of these began their course the year before, and 667

new girls have entered. Of these, 435 girls have dropped out without completing their course, some remaining a few weeks only, because they found themselves not adapted to the work of the school, but by far the larger number being unable to remain because of absolute necessity of partial self-support. 188 have completed their course during the year."^{*}

Girls who complete the course are given a diploma and have the services of the placement secretary in securing a situation, the time of one lady being entirely devoted to investigating the applications of employers for girls, placing the girls who have completed the course, and keeping in touch with those placed.† Girls who have left their work through slackness of the trade or other unavoidable reason are replaced. Those who are failing to make progress or who have left the school for temporary financial needs are encouraged to return to the school for further work whenever possible.

BOSTON TRADE SCHOOL FOR GIRLS.

This school offers training in the same trades as the Manhattan Trade School for girls, and is worked much on the same lines. It is not quite so large, having an equipment sufficient only for 300 girls. A second-year course is offered here, but only about five girls in 100 remain beyond one year. A careful placement agency is maintained in connection with the school, and a new feature recently introduced is that a certificate for completion of the course is not awarded until the girl has worked on probation with an employer for six months after leaving the school and has received his approval.

PUBLIC SCHOOL OF TRADES FOR GIRLS AT MILWAUKEE.

This school was established in 1909. It offers courses in two trades only, dressmaking and millinery. At the time of the writer's visit in January, 1913, about 275 girls were attending. The course is arranged for two years, but many girls leave without completing it. Unlike the schools at Boston and New York, where the time is almost entirely devoted to the girl's education as a wage-earner, the course at this school is planned to give a training which shall also make the girls useful in the home. Hence, in addition to the trade work, the girls learn to renovate and to use up pieces of material and give some time to cooking and household work. In a model flat the girls learn the care of the home, and the planning of school luncheons affords practice in estimating cost and preparing food in quantity.

* Annual report on the school, July 2nd, 1912, included in the 14th Annual Report of the City Superintendent of Schools 1911-12, p. 346. (City of New York.)

† For two forms used see Appendix A (2) to this chapter, page 78.

Work is undertaken for customers and for themselves, the girls learning to cut and fit in this way. A charge of 50 cents a month is made for material. Three-fifths of the school day of seven hours (8.30 to 12 and 1 to 4.30) is given to trade work and two-fifths to general instruction applied to the trades, to household work and to drawing and design.

The girls attending this school appeared to be from somewhat better-class homes than those at the other two trade schools. Some, after completing the course, are remaining at home, apparently not having to take up paid work, but the school had not been in existence sufficiently long to show to what extent the girls who complete the course would follow their trades. It is hoped that many will later undertake work on their own account. The teaching of the trade work is of a wider nature than at Boston and New York.*

EVENING CLASSES.

Evening classes have recently been opened in connection with the schools at New York, Boston, and Milwaukee. At Boston, however, the writer was told that quite one-third of the women attending are only taking the work for their own home use. A certain number of others came to learn a trade, especially straw hat machining. Others, including some old pupils, have come to better themselves in their trades.

APPENDICES TO CHAPTER VI.

A.—MANHATTAN TRADE SCHOOL FOR GIRLS.

209-213 EAST 23rd STREET.

DEPARTMENT OF EDUCATION, NEW YORK CITY.

(1) COURSE OF STUDY.

The course of study at the Manhattan Trade School is planned for girls who enter some line of employment as soon as possible after the compulsory school attendance is completed. The majority of these girls cannot spend more than one year for training, and provision is therefore made for a course of study extending over one year, with opportunity for advanced work in any subject if girls can remain a longer time.

While the course is planned to extend over one year only, the daily sessions are longer, and the vacations shorter than the high school, so that the actual number of hours of instruction given in a year at the trade school is 1,575 as compared with 965 given in the high schools, or nearly one and two thirds of the high school year.

About five hours each day or twenty-five hours per week are devoted to trade practice, and the remaining time to related subjects.

The instruction is individual, and girls are promoted as rapidly as their work will permit. The approximate amount of time required to complete each grade is indicated in the table.

* For outlines of the courses see Appendix B to this Chapter, pp.79-84.

A diploma is given to girls who complete the year's course in any one of the following trades:—

COURSES OFFERED.

A.—TRADE. (Each trade course requires one year to complete.)

1. *Dressmaking* :

GRADE I.—Elementary Sewing :

Hand and machine practice ; making school work aprons, gymnasium bloomers, and worker's equipment, six weeks.

GRADE II.—Intermediate Sewing :

Children's dresses, rompers, guimps, and undergarments, six weeks.

GRADE III.—White Work :

Fine hand and machine made underwear, using embroideries, laces, insertions, &c., six weeks.

GRADE IV.—Wholesale Class :

Speed tests in hand and machine made garments, three weeks.

GRADE V.—Elementary Dressmaking :

Shirt waists, linen, cotton, and lawn unlined dresses, three months.

GRADE VI.—Advanced Dressmaking :

Lingerie blouses, street and evening gowns of silk, chiffon, velvet, &c., three months.

2. *Millinery* :

GRADE I.—Elementary Sewing, as for dressmaking, with special emphasis on hand sewing, four weeks.

GRADE II.—Elementary Millinery :

Lidings, bands, frames, facings, covering frames, &c., three months.

GRADES III.—IV.—Making hats in spring and winter materials: straws, silks, velvets, malines, &c., five months.

3. *Lamp Shade Making* :

A short course in candle and lamp shade making is included in the millinery course, as the seasons in millinery are short, and girls who learn both lines of work can secure steadier employment, three months.

4. *Clothing Machine Operating*.—(Electric power machine ; no hand sewing.)

GRADE I.—Elementary Operating :

General control of machine ; making operator's equipment, work apron, sleeves, &c., practice stitching on simple garments, two months.

GRADE II.—Children's dresses and simple undergarments ; use of attachments for hemming, tucking, ruffling, &c., two months.

GRADE III.—Shirt waists, negligés, petticoats, and women's underwear. Special emphasis on speed practice, three months.

GRADE IV.—Filling of special orders on women's and children's dresses and other garments, made in a large variety of materials, three months.

GRADE V.—Special machine practice on hemstitching, scalloping, and embroidery machines, two months.

5. *Embroidery Machine Operating Course* :

GRADES I.—II.—III.—In clothing machine operating (as above), seven months.

GRADE IV.—Special Machines for:—

(1) Bonnaz embroidery, including chainstitch, cording, braiding, on pillow covers, dress trimmings, &c.

(2) Singer embroidery on shirt waists, fancy collars, dresses, table linen, underwear, &c.

(3) Stamping and preparing of patterns for embroidery, five months.

6. *Straw Machine Operating Course :*

GRADES I.—II.—Same as clothing machine operating (*see above*), four months.

GRADE III.—Hats made of coarse, inexpensive straw, in simple shapes, such as sailors, sun hats, &c., two months.

GRADE IV. Hats made in every variety of expensive straw, chip, hemp, milan, fancy straws, &c., in difficult shapes, six months.

7. *Sample Mounting :*

Consists of the mounting of cotton, woolen, and silk dress goods samples, all kinds of novelties, such as braids, embroideries, laces, ribbons, &c. Also the making of sample book covers, cases, &c., one year.

NOTE.—While the entire course in sample mounting requires one year, a course in plain mounting may be completed in six months.

8. *Novelty Case Making :*

GRADE I.—Elementary sample mounting, two months.

GRADE II.—Making of fancy cases, desk sets, scrap baskets, and a large variety of novelties in cretonnes, brocades, and other materials, ten months.

B.—OTHER SUBJECTS.—(Taught in relation to trades.)

Unless otherwise stated, the following subjects are required for a certain number of periods only, depending upon previous preparation of girls.

1. *Business Arithmetic :*

Practice problems in measurements, in buying and estimating materials, in keeping accounts, &c.

2. *English :*

Spelling, and use of terms, applications for positions, bills, receipts, descriptions of garments, &c.

3. *Design.*

Embroidery design, perforating and stamping of patterns, color and costume design.

Required—three hours per week, throughout the course.

4. *Textiles :*

Study of cotton, wool, flax and silk; tests for different fabrics; study of weavers, prints, dyes; names of common fabrics with their uses, prices, width, durability, &c.

5. *Industrial Problems :*

Factory system, division of labor; sweatshop work; work of Consumers League, trade unions, child labor committee; labor laws relating to women and girls; factory inspection, sanitary requirements, &c.

6. *Physical Training :*

Physical examinations and "follow up" work to see that each girl is physically fit to enter her chosen trade. Special attention to personal cleanliness, care of eyes, teeth, &c. Gymnastic exercises for body building, standing and sitting postures, &c.

Required—Throughout the course.

7. *Cooking :*

Preparation of food for the school luncheon served to pupils and teachers each day, including also, lessons in serving, acting as cashier, &c.

(2) PLACEMENT BUREAU.

(a) CARD USED FOR THE INVESTIGATION OF EMPLOYERS. [Size 8" x 5"].

NAME OF FIRM.		ADDRESS.		TRADE.		DEPT. INVESTIGATED.		Remarks
Kind of Work for Women.	No. Women.		General Sectional Team.	Weekly Wages.		Position at Work.		
	Total.	14-15 Years.		Time.	Piece.	Sit.	Stand.	

LEARNERS _____ Min. Age _____ By whom trained _____
 Length of Training _____
 Trade School Training Desirable _____
 Preliminary or Continuation _____

WORKERS DESIRED—Age _____ Nationality _____
 Other Qualifications _____ Nationality Excluded _____
 New Employees Taken On. Learners _____ Experienced _____
 Workers most in demand _____

SEASONS—Rush _____ Max. Force _____ Dull _____
 Min. Force _____ Closed _____

HOURS—Begin _____ A.M. End _____ P.M. Sat. _____ P.M. Noon _____ Hr.
 Total Daily _____ Total Weekly _____ Variation _____

OVERTIME—No. times per week _____ Closing Hour _____ P.M.
 Supper _____ Hr. No. weeks per year _____ Rate of Pay _____
 Fines _____ Charges _____ Vacation With pay _____
 Without pay _____ Home Work _____

PHYSICAL CONDITIONS—Type of Bldg _____ Floors used _____
 Elevator _____ Stairways _____ Workroom—Light _____
 Ventilation _____ Space _____ Cleanliness _____
 Lunching Facilities _____ Toilets _____
 Cloak Rooms _____ Fire Protection _____

How reached _____ Person to Address _____
 Date _____ Investigator _____
 Source of Information _____

(b) NOTICE HANDED TO GIRLS SECURING SITUATIONS THROUGH THE SCHOOL.

The Manhattan Trade School requires *no fee* for placing you.
 It intends to see that you get as fair a chance to *earn* and to *learn* as your trade offers and your ability permits.

In return it asks you two things :

I. In case your position proves unsatisfactory,—

Do not "walk out," Instead.

Report your complaint to us, and

Wait for our Reply before leaving.

After Receiving that reply you are entirely free to follow its advice or not as you see fit.

II. Prompt reports

- a. A postal on the day you are sent to a place, saying whether or not you have taken the position.
- b. A postal one week later, reporting hours, wage, and general conditions.
- c. If "laid off" (no matter what the reason) report to us immediately; in person if possible, by mail if not.
- d. Prompt return, fully filled out, of any blanks sent you.

A failure to comply with these requirements means loss of opportunity for you, with great inconvenience to us, and if persisted in results finally in an inability on our part to assist you as we would like to do.

B.—MILWAUKEE PUBLIC SCHOOL OF TRADES FOR GIRLS.

OUTLINE OF COURSES.*

(1) DRESSMAKING COURSE.

ELEMENTARY SEWING.

1. *Pincushion*.—Measuring; basting; overhanding; back-stitching.
2. *Bag*.—Hemming; overhanding; basting; running.
3. *Towel*.—French hem; overhanding.
4. *Belt*.—Mitering of corners; turning points; basting; machine stitching.
5. *Cooking Apron*.—Straight kitchen apron with bib. Plain seam; overcasting; turning of hem; cutting of bias band; application of tailored belt in putting bib and apron skirt together; button hole; sewing on of button (2-holed); blindstitching; basting.
6. *Child's Petticoat*.—Problem in flat-fell; use of tucker; gathering of ruffle; finishing of placket; common placket; inserting ruffle under tuck; button-holes; sewing on four-holed buttons; cutting and piecing true bias; feather stitching.
7. *Princess Apron*.—Joining of bias seams: flat-fell; another application of tailored belt; another application of pocket; turning of circular hem; blindstitching; cutting and piecing true bias; button-hole; sewing on button.
8. *Baby's Petticoat*.—French seam; overhanding of lace; stitching of lace; sewing-on of buttons; button-holes; cutting true bias.

UNDERWEAR DEPARTMENT.

1. *The Use of Paper Patterns*.
2. *The Making of Two Corset Covers*.—Peplin; band. One for experience in plain construction. One for responsibility and speed (plain or fancy).
3. *Two Pairs of Drawers*.—A new placket; sewing on lace and insertion by machine. One for experience in construction. One for responsibility and speed.
4. *Four Petticoats*.—The gathering of flounce and stitching to skirt with one stitching. One plain tucked. One with bias ruffles. One net ruffle with bias bands, made without basting. One fancy white, embroidery and lace.
5. *One Night Gown or Night Shirt*.—Facing; fly; collar, sleeves, &c.
6. *Kimono*.

CHILDREN'S DEPARTMENT.

1. *Child's Rompers*.—First lesson in putting button-holes on bias bands.
2. *Little Girl's Apron*.
3. *Little Girl's Gingham Dress*.—Plaited skirt; making and placing of plait.

4. *Little Girl's Dress*.—Full skirt.
5. *Little Girl's Guimpe*.
6. *Little Boy's Suit*.—Blouse and bloomers.
7. *Child's Lingerie Dress* with inserts of lace and tucks.
8. *Child's Cotton Coat*, with sailor collar trimmed with lace, embroidery or braid.
9. *Baby's Fine White Dress*, with original embroidered design, first lessons in eyelet embroidery.

WAIST AND COTTON DRESS DEPARTMENT.

1. *Taking of Measurements*.
2. *Elementary Drafting*.
3. *Making of Shirt waists (blouses)*.
4. *Making of Lingerie Waists*.
5. *Making of Simple Cotton Dresses*.

DRESSMAKING DEPARTMENT.

1. *Measurements*.
2. *Drafting*.
3. *Cutting*.
4. *Fitting*.
5. *Finishing of Linings*.—(a) Finishing of seams. (b) Boning. (c) Sewing on of hooks and eyes. (d) Making, fitting, finishing of sleeve lining. (e) Fitting, placing and finishing of collars.
6. *Making of Dresses and Suits (coats and skirts)*.
7. *Alteration of Dresses and Suits*.
8. *Developing Costumes and Draping in Tissue Paper*.
9. *Making of Dress Trimmings*.—(a) Embroideries; (b) Ornaments; (c) Fancy buttons, &c.

CUSTOM WORK DEPARTMENT.

1. *The Gaining of Experience by Three Months' Work on Gowns*.—(a) Silk gowns. (b) Chiffon gowns. (c) Velvet gowns. (d) Fine woolens. (e) Lingerie gowns.
2. *Training for Waist Finishers*.
3. *Training for Skirt Finishers*.
4. *Training for Sleeve Makers*.

(2) MILLINERY COURSE.

ELEMENTARY DEPARTMENT.

Practice in hand and machine sewing and the special stitches used in millinery.

1. *Pincushion*; 2. *Work Bag*; 3. *Milliners' Stitches*; 4. *Bandeau*; 5. *Hats*; 6. *Buckles*.
7. *Large Flat Buckram Hat*: Pattern of brim; wiring; binding; facing; covering; whipping a cord. Pattern of crown; wiring; binding; covering; folds (piping fold and milliner's fold); lining of hat. Fastening crown to brim; necktie bow.
8. *Wire Frame* (mushroom shape). Making of frame; covering of frame.
9. *Straw Sewing of*: (a) Brim. (b) Facing. (c) Crown.
10. *Wire Frame of Toque or Turban*. According to given measurements; straw; sewing of same; quills in straw sewing; buckle or rosette in straw sewing.
11. *Straw Hat over Wire Frame*. Tucked or shirred facing.
12. *Chiffon Hat*. Wire frame; covering of chiffon (cheese cloth) folds.

13. *Shirred Hat.* Buckram frame (mushroom style); covering of shirred cambric, for crown, brim and facing; velvet binding, velvet binding with cord piping.

14. *Shirred Net Hat.* Shirring done on wire frame.

CHILDREN'S MILLINERY.

15. *Infant's Bonnet.* Foundation of silk, covering of lace, insertion and straw; lining of bonnet, hemming of ties; trimming of bonnet (a) baby ribbon rosettes; (b) braided ribbon.

16. *Child's Bonnet.* Wire frame; covering of lace; insertion of straw, and silk shirrings; line bonnet; ribbon rosettes or flower trimming.

17. *Other Children's Bonnets.* As style demands.

18. *Little Girl's Hat.* Wire frame (mushroom style): covering of straw; facing of shirred silk; trimming (a) drapery of silk, in which cords are shirred; (b) tucked and shirred rosettes.

19. *Several Other Fancy Children's Hats.* Copied from fashion plates or models.

Note.—Above lessons in children's millinery may be modified as styles or seasons demand, using silks, velvets, felts, &c., instead of laces and straws.

EMBROIDERY LESSONS.

Embroidery of such designs as were prepared by students in the art room for children's bonnets, and lingerie hats.

20. *Children's Bonnets* of washable material. Cut according to pattern desired; embroidered design; hemming ties; lining the bonnet; trimming with little rosettes.

21. *Lingerie Hat.* Wire frame made as individual desires; cover frame with mull; embroider the design on washable material; facing of shirred net or lace ruffles, trimmed as desired.

TRIMMINGS.

22. *Rosettes.* Shirred cord rosette; shirred rosette of lawn; plaited rosette of tarlatan over cambric, velvet centre; box plaited rosette and wing of cambric; shirred rosette with whipped edge, black velvet centre; leaf rosette; shirred net leaf with milliner's folds; chiffon, tulle or net rosette; different ornaments for children's bonnets copied from models and fashion plates.

23. *Ruchings* (a) Side-plaited ruching; (b) Box plaited ruching of tarlatan with lace edge.

VELVET TRIMMINGS.

24. *Hemming of Velvet.* Plain hem, rolled hem.

25. *Velvet Ear.*

26. *Finish End and Velvet Band.*

27. *Folds.* Piping fold; plain fold; milliner's or French fold.

28. *Rosettes of Circular Piece of Velvet.*

RIBBON LESSONS.

29. *Bows.* Tied bow: butterfly bow; Alsatian bow; cockade bow and bows of other design.

30. *Ribbon Rosettes.* Loop rosette, with or without foundation; rosette of gathered ribbon; wired rosette of knotted satin ribbon; rosettes of clipped ends; rosette of loops and ends.

31. *Knots of Velvets and Silks.* (a) Single knot; (b) Double knot.

32. *Plume of Ribbon.*

33. *Flowers Made of Ribbon.* Daisies; roses; violets.

34. *Wiring of Ribbon Down Centres and Along Edges.*

35. *Wiring Lace for:* Fans; wings; quills.

36. *Finishings of Laces and Net.* (a) Single box plaited; (b) Double box plaited.
37. *Joining Ends of Laces.*
38. *Sewing Laces Together* with fancy stitches.
39. *Various Lace Rosettes* with flower or ribbon centres.
40. *Lace Sewed on Maline* for pompons and various lace trimmings.

ADVANCED DEPARTMENT.

OLD LADIES' MILLINERY.

1. *Making Bonnets.* Wire frames; embroidering bands on tulle or chiffon for front and crown; ties of ribbon, lace, tulle, or velvet.

Note.—For winter the making of silk, velvet and felt bonnets; also embroidering velvet or tulle bands with chenille, fancy silks, &c.

MOURNING MILLINERY.

2. *Widow's Bonnet.* Marie Stuart style with plain cords of nun's veiling around front.
3. *Draping of Veil.*
4. *Face Veils.*
5. *Plain Hem and Extension Hem.*
6. *Mitering a Corner.*
7. *Mourning Toques and Hats* of shirred or plaited taffeta silk.
8. *Trimmings of quills, rosettes and bows* of mourning material.
9. *Mourning Straw Hats* of pyroxyline braid; trimmed as desired; adjusting of veils.

RENOVATING LESSONS.

10. *Renovating and Cleaning of Crepe, nun's veiling, maline, black laces, white laces, ribbons, silks, netting.*
11. *Steaming of Velvet, flowers, foliage, plumes.*
12. *Curling Plumes.*
13. *Dyeing of Laces, flowers, feathers.*
14. *Furs.* Sewing furs, stretching furs.
15. *Cleaning.* Felt hats; straw hats; chenille braids; felt braids; straw braids.

DRAPING.

16. *Draping of Soft crowns; straw crowns.*
17. *Draping of Silk and Velvet* for toques and large hats.
18. *Sewing and Placing on Different Shapes.* Flowers; fancy feathers; wings; tips; plumes; imitation aigrettes; ribbon bows and rosettes.
19. *Fur, Flower and Foliage Toques and Hats.*
20. *Copying Hats.*

SUBJECTS FOR TALKS AND MILLINERY NOTES.

- | | |
|-------------------------|-------------------------|
| (a) Silks. | (k) Fancy feathers. |
| (b) Velvets. | (l) Quills and wings. |
| (c) Chiffon. | (m) Straws. |
| (d) Maline. | (n) Braids. |
| (e) Laces. | (o) Willow. |
| (f) Ribbons. | (p) Felts. |
| (g) Mourning materials. | (q) Furs. |
| (h) Flowers. | (r) Ornaments. |
| (i) Foliage. | (s) Blending of colors. |
| (j) Ostrich feathers. | |

- (t) Materials used in making frames, such as; buckram, willow esparta, rice net, French cape net, wires (sizes and names of, &c.).
- (u) Hats to be worn on different occasions.
- (v) The evolution and history of female headgear.
- (w) Terms used in millinery.

(3) APPLIED ART AND DESIGN.

The aim of the Course in Applied Art and Design is to correlate with the trades and home decoration,

1. *Use of the Ruler.*—Measurements; accuracy; proportion.
Application: 1. Tucks. 2. Miters. 3. Bands (bias).
2. *Freehand Drawing.*—Parallel lines, spacing; development of straight lines; development of curved lines.
Application: 1. Striped patterns. 2. Plaids. 3. Simple borders.
Study of historic ornament. Discussion of the use of these forms.
3. *Principles of Design.*—
Application:
 1. Braiding.
 2. Embroidery.
 3. Block printing.
 4. Stencilling.
 5. Accessories of millinery: (a) Frame making; (b) Bands; (c) Ornaments; (d) Bows; (e) Hats.
 6. Accessories of dressmaking: (a) Tucking; Ruffles; underwear; shirt-waists. (b) Underwear embroideries: Hand, machine. (c) Shirt-waists: Design of shirt-waist embroideries. (d) Costume: Design of house dresses; design of afternoon dresses; design of evening gowns.
 7. Discussion of hats and gowns *à la mode* and their relation to figures, *ensemble* of gloves, bags, neckwear, jewellery, &c.
4. *Color.*—Theory, nature; Japanese prints; textiles.
Application: 1. Millinery. 2. Dressmaking. 3. Household decoration.
5. *Household Decoration.*—Side elevations; rooms in perspective.
Accessories: Ornaments, picture hanging, sofa cushions, doilies, rugs, table decorations.

(4) ACADEMIC BRANCHES.

1. *Aims:* To improve the general education of the pupil through problems relating to her trade and her home life. To make the girl an intelligent and efficient trade worker.

2. *Course of Study.*—1. Arithmetic. 2. English. 3. Civics. 4. Industrial History. 5. Commercial Geography. 6. Textiles.

Arithmetic.—Sewing: Use of tape line, fractions of; cutting of gauges for hems and tucks, cutting of materials for ruffles with tucks, finding the amounts of materials required for garments, economy of different widths; making out of bills, receipts, checks, &c.; discounts, economy of buying in large and small amounts. *Millinery:* Use of tape line, fractions of; measuring of plaitings; quantity of material necessary for hats; making out of bills, receipts, checks, &c.; discounts, economy of buying in large and small amounts.

English.—*Aim:* To develop oral and written expression.

Spelling: Technical words and terms used in the trades; business terms, meaning and use of words.

Letters: Business and social forms of application; ordering goods; wording of telegrams and cablegrams, use of code words; preparation of money orders.

Civics.—Ethical: Duties of citizens as members of society.

Political: Administration of home city, libraries, museums, parks, playgrounds, &c.; state government; national government.

Industrial conditions: Industrial evolution in the United States; local industries; factory legislation; working papers; ethics of business; cost of living; minimum expenditures for maximum results.

Industrial History.—Economy and industrial aspect of the changes of fashions.

Fabrics: weaving, &c.

History of Dress: dress as a protection; dress as an ornament; development of dress ornamentation.

History of Head Dress: dress of the hair; hats as a protection; hats as a dress accessory.

Commercial Geography.—Raw materials: Production; transportation; exports and imports.

Manufactured Products: Location of factory centers; transportation and its effect on costs; exports and imports.

Textiles.—Fibers: Animal, vegetable; geography, growth and process of manufacture of fibers; judging kinds and qualities of materials; learning uses, widths, prices, &c.

(5) DEPARTMENT OF HOUSEHOLD SCIENCE.

Cooking as a Course Supplemental to the Trades.—*Aim:* To stimulate an interest in wholesome, economical food; to teach the food principles; to teach principles of planning, cooking and serving in a practical manner; to raise the standard of living; to give a general knowledge of house-keeping to develop responsibility, punctuality, cleanliness of person and neatness in work; to make a better home maker out of a trade worker.

Luncheon Period.—Work planned to give: Practical experience in following of recipes; compiling and use of cook book; planning of menus; buying and cost of materials; preparation of foods in large and small amounts; practice in preparation of bread, cakes, cereals, cheese, desserts, eggs, entrées, fish, meats, pastry, poultry, preserves, salads, sauces, soups, and vegetables.

Serving.—Attractive table setting, table decorations for special occasions, daily serving of luncheons to large and small numbers, practice in home and cafeteria serving.

General Housework.—Practical instruction in: (a) Dish-washing; (b) Sweeping; (c) Scrubbing; (d) Dusting; (e) Order and cleanliness of china and utensil closets; (f) Care of refrigerator; (g) Care of sinks; (h) Cleaning of cloths, dusters, and towels; (i) Laundry work; (j) Care of kitchen tables; (k) Care and management of coal and gas ranges; (l) Use of fireless cooker; (m) Care of bathroom; (n) Care of rugs; (o) Care of floors; (p) Making of beds; (q) Airing and ventilation of rooms.

(6) PHYSICAL TRAINING.

1. Physical exercises are given to develop the best physical condition of each pupil. A girl cannot do her best work if her physical condition is poor.

2. Each pupil is examined by a woman physician and an effort made to assign such exercises as will aid in correcting any physical defect found to exist. The pupils also receive lectures relating to their physical welfare and personal hygiene.

3. Emphasis is laid on the care of: (a) The body; (b) The necessity of proper food; (c) Sleep; (d) Exercise; (e) Correct positions; (f) The need of fresh air.

CHAPTER VII.—TECHNICAL HIGH SCHOOLS FOR GIRLS.

DESCRIPTION OF THE SCHOOLS.

The Technical High School or High School of Practical Arts for girls offers training in some selected occupation together with a continuation of general education. These schools are a new development. They differ from the trade schools for girls in the first place in that more attention is given to the general education of the girl than to her technical proficiency, the amount of time given to trade work being comparatively small. Secondly, the completion of the elementary school course is a condition of entry, and thirdly the class of girl who is able to complete the three or four years' course must come from a wealthier and better-class home than the girl who takes the one-year course at the trade school with a view to immediate earnings. From some figures kindly shown to the writer at the High School of Practical Arts, Boston, of the occupations of girls who had completed the four-year course (since the school was only founded six years ago these are not many) she judged that roughly one-third had passed on to other educational institutions, the majority with a view to training as teachers of domestic subjects, one-third were at home and unoccupied, and only one-third had taken up paid work at once, of whom half only were following a trade taught in the school. It appears from this that the girls who complete the course come from a somewhat leisured class. On the other hand a number of girls attend these schools for a time and probably leave without completing the course to enter some trade. Since the general instruction is correlated with the trade work, such girls would receive more direct benefit from attendance at one of these schools than at a general high school. A complete course in a trade subject would give a good grounding for eventual promotion to one of the higher positions of the trade.

WASHINGTON-IRVING HIGH SCHOOL FOR GIRLS, NEW YORK CITY.

This school is the counterpart for girls of the Stuyvesant High School for boys. The whole school takes over 5,000 girls. The writer was unfortunately unable to see it under the best conditions, since there had been a delay in the completion of the new building, and at the time of her visit to New York the school was occupying somewhat crowded and scattered quarters. Now that it is housed in its new building it must have magnificent accommodation. The school offers an academic or cultural four-year course, similar to the other high schools of the city. The technical or vocational courses offered are a librarian's course (for four years) and three-year courses in commercial work, dressmaking, millinery, design, printing and book-binding. The first-year course is preliminary and

common to all girls in the technical courses and is so designed that a girl who has entered for the academic course at a high school and wishes to change to technical work may do so during the first year without difficulty. The girls specialise in technical work in the second and third years. The official programme is as follows:—

COURSE OF STUDY OF THREE YEARS.

FIRST YEAR.		SECOND YEAR.	
<i>Required.</i>		<i>Required.</i>	
	Periods.		Periods.
English	5	English	5
Commercial Arithmetic	5	Physical Training	2
German, or French, or Spanish, or Group II., or Group III., of second year, and additional drawing	5	Music	1
Drawing	2	Declamation and Voice Training	1
Domestic Science and Art	5	Drawing	2
Physical Training, including Physiology and Hygiene	2	<hr/>	
Music	1	11	
Declamation and Voice Training	1	<hr/>	
<hr/>		5	
26		The modern language chosen in the first year may be continued during the second and third years as an alternative for music, declama- tion and drawing in the case of those who select Group I., Group IV., Group V., and Group VI.	
<hr/>		<hr/>	

ELECTIVES. FOR THE SECOND YEAR.

	Periods.
GROUP I.—(Stenographers and Typewriters).	
Stenography, Typewriting, Book-keeping, Penmanship, Spelling, Office Practice	19
GROUP II.—(Dressmakers and Embroiderers).	
Sewing, Draughting, Fitting, Study of Materials, Textiles, Color, Form and Design	19
GROUP III.—(Milliners).	
Millinery, Study of Materials, Trimming, Frame-making, Color, Form and Design	19
GROUP IV.—(Designers).	
Design, Still Life, Plants, Casts, Composition, Picture Study, History of Ornament, Textiles, Interior Decoration, Draperies, Costume Design, Book Covers, Illustrating, Lettering	19
GROUP V.—(Printers).	
Typesetting, Printing	19
GROUP VI.—(Bookbinders).	
Bookbinding, Design	19
GROUP VII.—(Library Assistants).	
English	5
Ancient History	3
Geometry	4
German, French, Latin, or Spanish (Elect. 2) (5)	10
Physical Training	2
Music	1
Declamation and Voice Training	1
<hr/>	
26	
<hr/>	

THIRD YEAR.	
<i>Required.</i>	
	Periods.
English - - - -	5
Physical Training - - - -	2
Music - - - -	1
Elocution - - - -	1
	9
	—

<i>Electives.</i>	
Group I, as in Second Year, with Commercial Law and Civics -	21
Group II, as in Second Year -	21
Group III, as in Second Year -	21
Group IV., as in Second Year -	21
Group V., as in Second Year -	21
Group VI., as in Second Year -	21
Group VII.—Course for Library Assistants.	
English - - - -	5
A Foreign Language - - - -	4
History of England - - - -	3
Physical Training - - - -	2
Music - - - -	1
Library Economy - - - -	15
	—
	90
	—

FOURTH YEAR.	
<i>(Library Assistants.)</i>	
Group VII.:	
	Periods.
English - - - -	5
A Foreign Language - - - -	4
American History and Civics -	4
Library Economy - - - -	12
Physical Training - - - -	2
Medieval and Modern History, or Additional Library Work	3
	—
	30
	—

GENERAL PROVISIONS.

1. A period shall not exceed fifty minutes.
2. Physical culture and music shall not be regarded as subjects requiring preparation.
3. Certificates will be awarded to those who satisfactorily complete this course.

The commercial course prepares girls for office work and resembles the courses offered in the commercial sections of many high schools. The course in design trains girls as (a) commercial designers for illustration work, advertisement agencies, &c. or (b) costume designers. The second year work, the study of freehand and object drawing, the principles of design and the application of colour is common to both classes; in the third year the girls specialise in the branch of design they wish to follow. The course for costume design includes some study of anatomy, but no sewing beyond the simple preliminary work taken in the first year. The methods of construction of costumes are learnt by observation. In the dressmaking and millinery courses the girls work for themselves or their friends only. No customers' work is taken by the school. The girls provide their own materials and learn to cut out and to fit each other. Beginning with undergarments, each girl makes during the two years a plain sailor blouse, a fine linen baby frock, and finally two whole dresses, one of woollen and the other of fine material. A little costume design is taken. The other courses at the school were not seen by the writer. There is no separate department for household arts, but each girl during her first year course gives five hours a week to this work.

LUCY FLOWER TECHNICAL HIGH SCHOOL FOR GIRLS, CHICAGO.

This school offers a two-year course and a four-year course to the girls of the city in household science and arts with the opportunity of devoting extra time to millinery and dress-making if desired.

The two-year course is intended for girls who cannot afford more than two years at high school and offers more specialised work in the second year. The school was only opened in September 1911, and the training given is more of the home than the trade type.

HIGH SCHOOL OF PRACTICAL ARTS FOR GIRLS, BOSTON.

The course at this school lasts for four years, the first year work being preliminary and common to all and including a course in simple cooking and sewing. In the last three years the girls specialise in either millinery, dressmaking or household arts, giving ten hours out of the 30 hours week to the practical work. A general training is secured by the arrangement that each girl gives the time for practical work to her chosen trade on three days a week and to the other two branches on the other days: thus a girl who has chosen millinery for her trade subject gives her time on three days a week to this, and on one day to cooking and one day to dressmaking, while a girl in the household arts course takes millinery on one day and dressmaking on another. Dressmaking and millinery as subsidiary subjects are taught from the home point of view only. When taken as chief subjects the girls work mainly for themselves, but in the last year they undertake work for customers. A sale of spring models was in progress on the day the writer called at the school and the work appeared to be of good workmanship, while some elaborate and stylish dresses made for customers by the fourth year girls were also seen.

A feature of this school is that under the vocational counsellor, who advises the girls on their after careers, arrangements have been made for some of the girls in the dressmaking and millinery courses to obtain a little practical experience before the termination of their course by entering selected workrooms in the city for a few weeks during the busy season. The girls in the household arts course also undertake outside work.*

This school appeared to offer a training which would enable a girl to attain to a high place in a trade eventually should she be willing first to devote the necessary time to acquiring a

* For cards used see Appendix A. to this chapter, pp. 96-8.

knowledge of workroom methods and to gaining speed in work, but so far the majority of girls attending apparently have not had this object in view.

The school programme is as follows:—

First Year.

Required.	Periods per Week.	Elective.	Periods per Week.
English - - -	5	After the first year each pupil is required to select one of three industrial courses.	
History - - -	3		
Mathematics: Applied Arithmetic, Algebra.	3		
Elementary Science -	2		
Art - - - -	4		
Sewing - - - -	6		
Cooking and Housewifery.	4		
Choral Practice - -	1		
Physical Training -	2		
	90		

Second Year.

English - - -	4	Dressmaking Course -	10
History - - -	2	Millinery Course - -	10
Mathematics: Plane Geometry.	2	Household Science Course.	10
Chemistry - - -	4		
Art - - - -	5		
Choral Practice - -	1		
Physical Training -	2		
	20		

Third Year.

English - - -	4	Dressmaking Course -	10
History, Civil Government.	4	Millinery Course - -	10
Physics - - -	4	Household Science Course.	10
Art - - - -	5		
Choral Practice - -	1		
Physical Training -	2		
	20		

Fourth Year.

Required.	Periods per Week.	Elective.	Periods per Week.
English - - - -	5	Dressmaking Course -	10
Household Arts, 1 to 2 Years. } Home Nursing, 1 to 2 years. }	2	Millinery Course - -	10
Economics - - - -	2	Household Science Course.	10
Physiology, Hygiene, Sanitation.	3		
Art - - - - -	5		
Choral Practice - -	1		
Physical Training -	2		
	20		

WOODWARD HIGH SCHOOL, CINCINNATI.

This school is such an interesting example of a large co-educational high school offering a variety of courses that a description of it is included. It is, the writer believes, one of the most advanced in the country. There are now nine different courses in the school. Boys and girls share equally in the general, classical, commercial, art and music courses, and are taught together. The manual training course is a special course for boys intending to enter college but preferring some manual work, and has its counterpart in the domestic science course for girls. The technical co-operative courses are intended for boys and girls who cannot afford a four years' school course and must early follow a wage-earning career. After a two years' preliminary course, which can be taken either at this school or at the Hughes High School, they spend half time at work and half at school, on the week about or co-operative plan. The preliminary training includes all the manual work taken in the four-year manual course for boys or domestic science course for girls. Double time is therefore given to manual work, the school hours being slightly lengthened to allow for this. During the last two years manual work is not taken in school by the boys, since they are doing shop work in alternate weeks, and the entire time is devoted to theoretical work allied to the trade chosen. These courses are intended to enable both boys and girls to discover the occupations for which they have aptitude, and to give a sound preliminary training suitable for various trades during the first two years and combined education and practical experience afterwards.

The courses in art and music have been arranged as half-time courses in order to secure that boys and girls who have decided talent may continue their general education along with

their preparation for an artistic or musical career. The mornings are spent in school, and the afternoons at the Art Academy, or the College of Music in Cincinnati, or with an approved private teacher.

The following is the programme of all the courses:—

FIRST GROUP.—Academic Courses for General Culture and leading to Colleges and Professional Schools.

SECOND GROUP.—Technical Courses, leading to vocations.

ACADEMIC COURSES.

Elocution 1, Music 1, Gymnasium 2 periods a week, throughout the four years.

First Year.

1.		2.		3.		4.	
General.		Classical.		Domestic Science.		Manual Training.	
Periods.		Periods.		Periods.		Periods.	
English -	4	English -	4	English -	4	English -	4
Algebra -	4	Algebra -	4	Algebra -	4	Algebra -	4
Latin or German (Adv.) or German (Beg.)	5	Latin -	5	Latin or German (Adv.) or German (Beg.)	5	Latin or German (Adv.) or German (Beg.)	5
Botany or Zoology (1 double)	5	Botany or Zoology (1 double)	4	Domestic Science -	8	Manual Training -	8
Drawing -	2	Drawing -	2	Applied Art -	2	Drawing (Mech.) -	2

*Second Year.**

English -	4	English -	4	English -	4	English -	4
Geometry -	5	Geometry -	5	Geometry -	5	Geometry -	5
Latin or German (Adv.) or German (Beg.)	5	Latin -	5	Latin or German (Adv.) or German (Beg.)	5	Latin or German (Adv.) or German (Beg.)	5
History (Ancient) -	4	Greek -	5	Domestic Science -	8	Manual Training -	8
Drawing (Opt.) -	2	Drawing (Opt.) -	2	Drawing and Applied Art -	2	Drawing (Mech.) -	2

* French, 4; English Composition, 1; an option in the General Course for English for those preparing for colleges requiring three years of French.

Third Year.

1.	2.	3.	4.
General.	Classical.	Domestic Science.	Manual Training.
English - Periods. 4	English - Periods. 4	English - Periods. 4	English - Periods. 4
3rd year Latin or 1st year German or German (Adv.) or German (Beg.) } 5	Latin - 5 One from :— Physics (1 double) 6 German - 5 French - 4	One from :— 3rd year Latin or 1st year German or German (Adv.) or German (Beg.) } 5	One from :— 3rd year Latin or 1st year German or German (Adv.) or German (Beg.) } 5
Two from :— French or Spanish - 4		French - 4	French - 4
Algebra, $\frac{1}{2}$ year Trig'try, $\frac{1}{2}$ year Physics (1 double) 6 History (Med.) History (Mod.) } 4	Algebra, $\frac{1}{2}$ year - 4	Spanish - 4 History (Ancient) 4	Spanish - 4 Algebra, $\frac{1}{2}$ year Trig'try, $\frac{1}{2}$ year } 4
Drawing (Opt.) - 2		Chemistry (1 double) 6 Domestic Science - 8 Applied Art 2	Physics (1 double) 6 Manual Training - 8 Mechanical Drawing - 2
	Greek - 5 Drawing (Opt.) - 2		

Elocution, Music, and Gymnastics optional first half-year in the Classical Course.

Fourth Year.

1.	2.	3.	4.
General.	Classical.	Domestic Science.	Manual Training.
English - 4	English - 4	English - 4	English - 4
4th Year Latin or 2nd Year German or German (Adv.) or German (Beg.) } 5	Latin - 5	Two from:— 4th year Latin or 2nd year German or German (Adv.) or German (Beg.) } 5	One from:— 2nd year German or 4th year Latin } 5
Two from:— French or Spanish - 4	One from:— Chemistry (1 double) 6	French - 4	German (Adv.) - 5 German (Beg.) - 5 French - 4
Chemistry (1 double) 6	Sol. Geom., ½ year } 5	Spanish - 4	Spanish - 4
Sol. Geom., ½ year } 5	Advanced Math., ½ year } 5	Physics (1 double) 6	Chemistry (1 double) 6
Adv. Math., ½ year } 5	German - 5	History (Am.) } 5	Sol. Geom., ½ year } 5
Astron., ½ year } 4	French - 4	Civics } 5	Advanced Math., ½ year } 5
Geology, ½ year } 4			
History (Am.) } 5		Domestic Science - 8	History (Am.) } 5
Civics } 5		Physiol. and Hygiene - 2	Civics } 5
Drawing (Opt.) - 2	History (Ancient) 4	Applied Art 2	Manual Training - 8
	Greek - 5		
	Drawing (Opt.) - 2		Mechanical Drawing - 2

One year of Science required in third or fourth years in General Course.
 One year of History required in Domestic Science Course.
 Each period is 45 minutes.

TECHNICAL COURSES.

First Year.

5.	6.	7.	8.	9.
Commercial.	Technical Co-operative Course for Boys.	Technical Co-operative Course for Girls.	Art.	Music.
Periods.	Periods.	Periods.	Periods.	Periods.
English - 5 Algebra - 4 German or Spanish 5 Bookkeeping and Com. Arithmetic - 5 Penmanship and Applied Art - 5 Physical Training 2 Music - 1	English - 4 Arithmetic and Algebra 4 Industrial Geog. raphy - 4 Drawing 4 Turning, Pattern and Cabinet Making 16 Physical Training (Opt.) - 2	English - 4 Arithmetic and Algebra 5 Applied Art - 5 Cooking - 4 Sewing - 8 Physical Training 2 Music - 1	English - 4 Algebra - 4 Botany or Zoology 5 Physical Training 2 Music - 1 Elocution 1 Art at the Art Academy - 10	English - 4 Elocution 1 German - 5 Algebra - 4 Physical Training 2 Chorus - 1 Music Dictation 1 Instrumental Art outside of School - 10

Second Year.

English - 4 Arithmetic and Geometry 4 German or Spanish 4 Stenography and Typewriting 5 Commercial Geography 4 History (Mod.) - 4 Drawing (Opt.) - 2 Physical Training 2 Music - 1	English - 4 App. Mathematics - 4 Physics - 4 Drawing 4 Foundry, Forge and Machine 16 Physical Training (Opt.) - 2	English - 4 Geometry and Arithmetic - 3 Applied Art - 4 Cooking and Household Arts - 5 Millinery and Dress-making 10 Physical Training 2 Music - 1	English - 4 Geometry 5 French - 4 History (Ancient) 4 Physical Training 2 Music - 1 Elocution 1 Art at the Art Academy - 10	English - 4 Elocution 1 German - 5 Geometry or Ancient History 5 Physical Training 2 Chorus - 1 Intervals and Chords 1 Instrumental Art outside of School - 10
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Third Year.

5.	6.	7.	8.	9.
Commercial.	Technical Co-operative Course for Boys.	Technical Co-operative Course for Girls.	Art.	Music.
Periods. English - 4 German or Spanish Correspondence - 4 or Physics 6 History (Am.) - 4 Stenography and Typewriting 8 Commercial Law - 3 Bookkeeping - 5 Physical Training 2 Music - 1	Periods. Chemistry 8 English - 4 Drawing 10 App. Mathematics, Shop Problems and Practice 10 Co-operative Plan; alternate weeks in Shop and School.	Periods. English - 4 Physiology 4 Applied Art - 5 Elect Specialty 20 Millinery, &c. Dress-making, Tailoring and Art Needlework; Home Economics; Office Training; Salesmanship.	Periods. English - 4 French - 4 History (Med. and Mod.) - 4 History of Art - 2 Physical Training 2 Music - 1 Elocution 1 Art at the Art Academy - 10	Periods. English - 4 Elocution 1 German or French 5 Physics - 6 or Med. and Mod. History - 4 Physical Training (Opt.) Chorus - 1 Harmony 2 Instrumental Art or Vocal - 10 Music Appreciation.

Fourth Year.

English - 4 German or Spanish Correspondence - 4 or Chemistry 6 Economics, } 1/4 year } 5 Civics, } 1/4 year } Bookkeeping and Accounting 8 Stenography and Typewriting 5 Drawing (Opt.) - 3 Physical Training 2 Music - 1	History (Industrial of U.S.) and Civics 5 Shop Science and Shop Practice 10 Drawing 10 App. Mathematics and Shop Problems 10 Co-operative Plan; alternate weeks in shop and school.	American History and Civics 5 English - 4 Applied Art - 5 Elect Specialty 20	English - 4 French - 4 History of Art and Art Criticism - 4 Physiology and Hygiene - 4 Art at Art Academy 10	English - 4 Elocution 1 German or French 5 American History and Civics 5 Physical Training (Opt.) Chorus - 1 Harmony or Counterpoint - 2 Vocal and Instrumental Art - 10 Music Appreciation. History of Music. Art.
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The Commercial Course and the Co-operative Courses for boys and girls are given only at Woodward and Hughes High Schools.

APPENDICES TO CHAPTER VII.

A.—CARDS USED AT THE HIGH SCHOOL OF PRACTICAL
ARTS FOR GIRLS, BOSTON.

(1) VOCATIONAL RECORD CARD FOR EACH PUPIL. [Size 10" x 8".]

VOCATIONAL RECORD.

183 BOSTON PUBLIC SCHOOLS. HIGH SCHOOL OF PRACTICAL ARTS.

Name _____ Address _____

Date of entering _____ Whence _____ Date of Birth _____

Date of graduating or leaving _____ Age _____ Course _____

—	First Year.	Second Year.	Third Year.	Fourth Year.
Personal appearance.				
Health - -	Ab. Tar. Ex.	Ab. Tar. Ex.	Ab. Tar. Ex.	Ab. Tar. Ex.
Attendance -				
Reliability - -				
Application -				
Initiative - -				
Academic standing.				
Industrial standing.				
Record of outside work before date of graduating or leaving.				

(1) VOCATIONAL RECORD CARD—*cont.*

RECORD OF VOCATIONAL ASSISTANT'S VISITS TO HOME.

Date.	Reason for Visit.	Result.

On reverse side.

INDUSTRIAL RECORD.

—	First Position.	Second Position.	Third Position.
Date employed			
Employer's name			
Address			
Position (how found)			
Kind of work			
Hours			
Initial wage			
Increase			
Date of leaving			
Reason for leaving			
Remarks			

RECORD OF HIGHER EDUCATION.

(2) EMPLOYERS' CARD. [Size 6" x 4".]

182 BOSTON PUBLIC SCHOOLS.*

HIGH SCHOOL OF PRACTICAL ARTS. EMPLOYMENT RECORD.

Name _____

Address _____

Business _____

No. of workers _____

Class of workers _____

Condition of workroom _____

Hours _____

Wages _____

Remarks _____

On reverse side.

RECORD OF GIRLS EMPLOYED.

Name.	Wage.	Date employed.	Date of Leaving.

**B.—SIMMONS COLLEGE.—PROGRAMME OF COURSES
IN INDUSTRIAL TEACHING, 1912-13.***

INDUSTRIAL NEEDLE ARTS TEACHING.

A one-year programme has been arranged for the training of teachers of the needlework industries and millinery in public or industrial schools. The programme is open to students who have had at least two years of educational training subsequent to the high school, either in college, or in a technical institute or normal school. Candidates must have had some elementary training in plain sewing. The subjects include plain hand and

* It should be noted that the courses for training industrial teachers form but a small part of the activities of Simmons College, which is a woman's college of university rank, offering four-year programmes of study in the following divisions:— A. Household Economics; B. Secretarial Studies; C. Library Science; D. General Science; E. Social Work; F. Industrial Teaching. For students who have had their collegiate training elsewhere, one-year or two-year technical programmes are provided.

machine sewing, drafting, cutting, the general principles of design and their application to costumes, and the general principles of millinery.

Outline of programme.

First Term Hours.			Second Term Hours.	
Class.	Study.		Class.	Study.
6	2	Sewing	6	2
5	2	Sewing and	5	2
		Design		
3	2	Textiles	—	—
—	—	Costume Design	4	1
6	2	Dressmaking	6	2
3	1	Millinery	3	1
3	0	Sewing or	3	0
6	0	Industrial Sewing	—	—
—	—	Sewing	2	3

TRADE SCHOOL TEACHING.

By an arrangement with the Women's Educational and Industrial Union two one-year programmes have been established to train teachers and directors for trade and industrial schools. Programme I. offers courses in the teaching of salesmanship; Programme II., in the teaching of the trades which demand a knowledge of the Needle Arts.

The purpose of these programmes is to afford a knowledge of trade standards and requirements through carefully directed practice and observation in trade shops, and to provide opportunity for practice teaching in preparation for teaching in trade or industrial schools.

Instruction is given both at the Women's Educational and Industrial Union and at Simmons College. The Union provides the actual practice in shop work, salesmanship, and teaching, and arranges for the required observation. This part of the programme requires the full time of the student except for a period from November 4 to December 19 when the students spend their entire time at Simmons College for academic instruction related to the practice of the year, or in observation of trade work. During the past year students following this course have been permitted to practise in the Boston Trade School for Girls.

The programmes are open only to women whose maturity and experience give assurance of success in this new field. At least two years of training in college, in a technical institute, or in a normal school are essential, and preference is given to candidates who have had experience in teaching or in administrative work, or who give promise of executive ability. Skill in the Needle Arts is an essential for Programme II.

The tuition fee for each programme is one hundred dollars. The number of students is restricted, therefore application for either programme should be made at an early date. A personal interview is desirable.

Programme I.—The Teaching of Salesmanship.

The work in this programme is divided into four parts:—

1. Actual practice in department stores, with reports and discussions.
2. Observation of salesmanship in department stores and in the Union School of Salesmanship, together with discussion of store systems, the relation of employer and employée, and the work of the pupils.

3. Observation and practice in teaching at the Union School, with discussion of such subjects as salesmanship, business arithmetic, textiles, color and design, and hygiene; together with a teachers' course in textiles and in the principles of teaching.
4. Class work at Simmons College in the following courses:—
Accounts A., Design A., Economics A., Education A.

Programme II.—The Teaching of the Needle Arts Trades.

The work in this programme is divided into four parts:—

1. Practice in various shops of the Women's Educational and Industrial Union. The work includes making sample garments, filling orders, buying materials, caring for stock, workroom management, bookkeeping, selling, and taking orders.
2. Observation and practice in other shops representing trades employing girls.
3. Practice teaching in trade schools or classes.
4. Class work at Simmon's College in the following courses:—
Accounts A, Design A, Economics A, Education A.

Accounts A.

The purpose of this course is to teach the fundamental principles of double entry bookkeeping in their application to retail business; to familiarize students with the forms and uses of business papers pertaining to the accounting department of a retail organisation; and to develop an understanding of the workings of the accounting department, particularly in its relations with the selling department.

Five exercises a week from November 4 to December 19.

Design A.

This course is intended to give the student a practical knowledge of the elementary principles of design and color. Composition, proportion, and harmony are taught through practice in simple designing and color arrangements.

Five exercises a week from November 4 to December 19.

Economics A.

The Economics of Industry.—Discussions, lectures, and field work. A study of economic laws in their application to industry. In order to meet the requirements of trade-school teachers, this course presents a general survey of modern industrial organisation. Special attention is given to the industries in which women are employed, and for which girls are being prepared in the trade schools.

Five times a week from November 4 to December 19.

Education A.

Industrial Education.—Lectures and discussions. Among the topics considered are the following:—The changes which have made industrial education essential; the types of schools already organised; the details of an ideal school flexible enough to meet local conditions; equipment, methods, courses of study, teachers, organisation.

Five exercises a week from November 4 to December 19.

CHAPTER VIII.—VOLUNTARY PART-TIME CLASSES.

ATTITUDE OF EMPLOYERS TO PART-TIME INSTRUCTION.

A visitor to the United States cannot fail to be struck by the value given by manufacturers and other employers to part-time instruction during the day. The importance of some continuance of education for skilled workers appears to be very widely realised by the large number of enlightened employers in whose hands lie many of the larger enterprises in the country. That such education may be given because of the return brought to the employer rather than the worker is no reason against it. The large railroads of the country provide some day theoretical instruction for their apprentices; so do such concerns as the General Electric Co., Lynn, and the International Harvester Co., Chicago. The Report of the Commissioner of Labor for 1910 gives 9 railroads and 12 manufacturing firms which maintain such part-time day schools for their apprentices, and which are described in the report as "Apprenticeship Schools," as against 3 railroads and 3 manufacturing firms holding classes in the evening.

Some of the large department stores maintain schools for their own juvenile employees. The writer visited one such school, that at Wanamaker's, Philadelphia, where boys and girls under 16 receive instruction from 7.30 a.m. to 9.30 a.m. on two mornings a week in such English subjects as will help to make them efficient salesmen and saleswomen. Arithmetic is taught by the making out of order sheets; English, that they may write and spell correctly; and commercial geography, that an intelligent interest may be taken in the goods passing through their hands. The school is under a schoolmaster who takes a personal interest in the boys and girls, knows their positions in the shop, and endeavours to teach them by any means in his power the value of steady, faithful, and honest service to their employers. Such a school is of equal benefit to employer and employee.

In certain cities this day part-time instruction has been provided by an outside institution. At Cleveland, for instance, the Young Men's Christian Association is giving instruction in the theory applied to their trade to apprentices from 15 firms for four hours a week for 40 weeks in the year. Five classes with about 20 apprentices in each have been formed. The employers not only pay the wages of the boys for the time they attend but also the fee of \$20 a year for each boy asked by the Young Men's Christian Association. A similar school is maintained by the Young Men's Christian Association at Bridgeport, Conn. At Milwaukee similar instruction has been given by the Extension Department of the University of Wisconsin to engineering apprentices in the city, who attended for two hours once or twice a week in working hours. These classes have been used by the State since 1912 as apprentice classes under the new compulsory law.*

* See page 120.

Perhaps the greatest value of these classes lies in the fact that they pave the way for the introduction of compulsory day continuation classes.

The writer had the opportunity of visiting certain classes maintained by education authorities as part-time classes for employees. Where these classes are maintained to give theoretical instruction applied to the trade followed by the workers at their firms, they are spoken of as co-operative part-time classes.

CINCINNATI CO-OPERATIVE CONTINUATION SCHOOL.

Of all the classes seen, those at Cincinnati are among the most interesting. The school was originated by its present principal, Mr. Renshaw, who some years ago started a private evening school for mathematics. About six years ago he began to get day classes in the shops together. In this he received the hearty support of the trade unions. His triumph came when an employer regretfully lost an old employee to whom a certain difficult job was always assigned. "Try one of the apprentices," suggested Mr. Renshaw, and the apprentice was so successful that the employer allowed he was now getting his money back from the school.

Three years ago this school was taken over and developed by the Board of Education. There were in the winter of 1912-13 208 boys, 16 to 21 years of age, attending from the engineering, pattern-making and printing firms of the city.* Since all the boys are at different stages as regards school work, there are two teachers for each class in order that individual help may be given.

The boys attend in classes not exceeding 24 in number, less for beginners, for 3½ hours a week, *i.e.*, from 7.30 to 11 or from 1.30 to 5. Their wages are paid for this, returns of attendance being sent weekly to the employers. The school is in session the whole year. Mr. Renshaw is still in charge with two assistants, and these take their holidays in turn. Each teacher has two mornings a week off, during which he visits the works and makes himself acquainted with the environment of the apprentice, consults the foreman and gathers material for the course of study.

These boys though termed apprentices are not bound, and migration from one trade to another was common. The school has decreased this migration by about 50 per cent. each year. Mr. Renshaw is an enthusiast on part-time schools. Beyond the mere instruction given he believes greatly in the personal influence which can be exerted by the individual attention paid to the needs and surroundings of the lads.

* The following figures of the number of pupils enrolled are taken from the Annual Report on Cincinnati Schools for the year ending August 31st, 1912:—Number of pupils enrolled, 275; number of pupils withdrawn, 94; number of pupils remaining, 181; average number belonging, 151; average daily attendance, 143; average daily absence, 8.

Though many of them are working under some of the worst conditions he claims to turn them out fine men industrially and morally.

So important is this aspect of the work of continuation classes that the following extract from a paper by Mr. Dyer, the former Superintendent of Schools at Cincinnati is worth quoting :—

“The employers and foremen say there is no loss of output by the boys being out one half-day a week. They more than make up for the absence by their diligence and zeal when they are at work. When the boys start to school they are as a rule depressed, indifferent, disgruntled. They look upon their employer as an aristocrat, their foreman as a slave-driver, their machine as a treadmill, and the world at large as against them. Their faces are frozen in a perpetual frown. The path to advancement seems long and uncertain. As they feel their mind and body settling in a groove they become rebellious and ready to quit. The school comes as a new interest in their lives. They can scarcely realise at first that anybody cares, but they soon thaw out and a new light shines in their eyes. They see for the first time the purpose of instruction which bored them in their school days. They have a motive. They can put their knowledge to use. They become interested and intellectually awakened. Their attitude changes toward their employer, their foreman, their machine, the world. They are no longer mere hands, cubs, operatives. They are becoming masters of an honourable craft. As they are induced to go from one shop to another they have been known to make it a condition that they be permitted to attend the continuation school.”*

It was evident in talking to Mr. Renshaw that the success of the school has been largely due to his personal enthusiasm and influence.

A summary of the course for engineer's machinists from the Annual report of the Cincinnati Public Schools for 1911 is given in Appendix A. to this chapter, pp. 107-8. The classes for printing had been too recently introduced to allow of a definite course having been planned.

It is certain that the benefits obtained through this school led to the agitation of the Cincinnati School Board for compulsory part-time continuation classes for children between 14 and 16 years of age, and to their subsequent introduction in the State of Ohio under a permissive law.

Voluntary part-time classes are also maintained in Cincinnati for saleswomen, and in household arts and domestic economy in connection with mothers' clubs.

VOLUNTARY PART-TIME CLASSES IN CHICAGO.

No description of voluntary part-time classes in the United States would be complete without mention of those held by the Board of Education of Chicago at the request of two trade unions in the city.

School for Carpenters' Apprentices.

All apprentices to this trade in union firms must be indentured through the trade union. They are bound for four

* National Society for the Promotion of Industrial Education. Bulletin No. 13, Part III. Part-time and Evening Schools, p. 127. Issued by the Society. New York City, January 1911.

years and during this term must spend January, February, and March of each year in school. Special classes for these boys are arranged at the two technical high schools of Chicago, the Lane and the Crane Schools, and are under the supervision of the joint arbitration board of five masters and five men which acts as an advisory committee for the classes. Such boys as wish may attend instead the day school in the building trades supported by the Young Men's Christian Association.

The hours in school are 9 to 4.10, with 30 minutes' interval for lunch. The boys have two periods of shop work, two of architectural and mechanical drawing, one of English, one of history and civics, and one of mathematics each day, the small remaining time being given to gymnastics, ethics, and freehand drawing. No home-work is done.

There are over 400 carpenters' apprentices in the city of Chicago, so that the organisation of these schools for only three months in the year entails a heavy piece of work. The classes have now been held for three years. They were instigated at the request of the trade union, the training supplied by the masters to apprentices being found to be most incomplete. The representatives of masters and men seen both seemed entirely satisfied with the schools. It should be added that the boys receive their wages as usual during the time they are in school, a report of attendance for each boy being supplied to the joint arbitration board.*

School for Electrical Workers.

The Electricians' Union requested part-time classes for their apprentices in 1912. The boys attend for 3½ hours one day a week, in the mornings, from 8 to 11.30. They are accommodated at two schools. Two periods are given to shop work (including drawing) and one to mathematics. These are taken by the same teacher and are correlated one to the other. The fourth period is given to English subjects. The teacher and the president of the Electrical Workers' Trade Union both spoke enthusiastically of the classes. The boys used to attend evening classes, but they were too sleepy to profit by them.

* The following figures for the School for Carpenters' Apprentices are given in the Annual Report of the Chicago Board of Education for the year ending June 30, 1912:—

Enrolment	-	-	-	255
Average daily membership for the three months' session	-	-	-	224
Average daily attendance	-	-	-	206.5
Percentage of attendance	-	-	-	92.2
Cost of Maintenance of the School.				
		Total.		Cost per pupil.
		\$		\$
Teachers' salaries	-	3,330.10		14.87
Fuel, light, janitors, supplies and other expenses	-	448.00		2.00
Total cost	-	3,778.10		16.87

During 1912-13 about 200 boys attended the day classes, *i.e.*, all apprentices to the installation work. The trade union hoped to include the electrical instrument makers also in the following year. The boys' wages are paid for attendance at school. For non-attendance wages would be docked, but with this there has been little trouble. Returns of attendance are sent regularly to the masters.

The Painters' District Council maintains its own school which painters' apprentices attend one day a week, and the Bricklayers and Stonemasons hold theirs on Saturday afternoons and in the evenings. Both these are entirely supported by the unions concerned and have no connection with the Chicago Board of Education.

A small class for young employees from department stores has been held at the Lucy Flower Technical High School, Chicago, from 9 to 11.30 on two mornings a week. The Crane Company, one of the largest contractors in the city, sends some of its younger boys who have not completed their elementary school work to one of the day schools for two or three hours a week, for instruction in arithmetic, history and English. An enquiry carried out by the Department of Vocational Supervision of the Board of Education during the summer of 1913 showed 260 employers willing to try the plan of part-time classes. Arrangements are being made for carrying these out.

VOLUNTARY PART-TIME CLASSES IN MASSACHUSETTS.

Voluntary part-time continuation classes have been developed further in Massachusetts than in any other State. The report of the State Board of Education shows that such classes were held in 1911-1912 at Boston, Lowell, New Bedford, and Worcester, in addition to the half-time schools previously described. The classes at Boston were visited.

Boston Continuation School.

In 1909 the Boston School Committee approached merchants and manufacturers, inviting their co-operation for the purpose of establishing continuation schools where part-time instruction related to their daily work might be given to employees. In 1910 courses were established for the employees, in the shoe and leather industry and the wholesale drapers' warehouses, and also, under the title of "preparatory salesmanship," for employees under 18 years of age from the retail drapers' shops.

An advisory committee for each class, composed of representatives from the firms sending employees, gave useful aid in preparing the courses by their recommendations and advice. Besides the regular teachers of the school, experts are obtained from the various industries to give special lectures on particular

processes. The pupils, too, are taken to visit various works and factories in order that they may better understand the processes of manufacture of the goods with which they have to deal.

Attendance on the part of the students is voluntary. As all classes are held during working hours, the employer's consent is necessary and he pays wages during attendance. So successful have the classes been that they have grown rapidly since they were started. In March 1913 over 900 students were attending in classes averaging 15 to 20 in membership. The following courses were held:—

Courses for a period of 30 weeks.

1. Preparatory Salesmanship. Tuesday and Thursday or Wednesday and Friday, 8.30 to 11 a.m., or 3 to 5.30 p.m. Each class attends twice a week. This course is for the young employees of the department stores.
2. Department Store Salesmanship. Tuesday and Thursday or Wednesday and Friday, 8.30 to 10.30 a.m., or 3.30 to 5.30 p.m. This course is a continuation of the previous course.

Courses for a period of 12 weeks only.

1. Shoe and Leather. Monday and Friday, 3.30 to 5.30 p.m.
2. Dry Goods (Wholesale Drapery). Monday and Friday, 3.30 to 5.30 p.m.
3. Banking. Tuesday and Thursday, 4 to 6 p.m. (for employees from the banks and trust companies of the city).
4. Clothing. Monday and Thursday, 3.30 to 5.30 p.m.
5. Retail Shoe Salesmanship. Tuesday and Wednesday, 8.30 to 10.30 a.m.
6. Business Organisation and Salesmanship. Friday, 3.30 to 5.30 p.m.

One set of pupils attends the course the first half and another set the second half of the year.

The teaching is of a practical nature. When the writer visited the school in an afternoon a class of employees from shoe and leather warehouses had before them samples of all kinds of leather in order to discuss their manufacture and qualities.

In the 30-weeks courses for the younger employees attention is given to the practical part also. On the morning the school was visited, a class of girls were having a lesson on flax and its manufacture into linen. Samples of flax in all stages of the processes of its manufacture were before them. The lesson ended with a discussion on linen, for what articles it is most suitable, what are its properties, under what conditions should it be stored, &c.

Besides this, instruction of a more general character is given. The girls learn arithmetic by the making out of order

books and bills, and they have a short course of design with a view to understanding the nature of design and to gain some taste in colours. Talks on personal hygiene are also given, and on the principles of salesmanship.

A summary of the courses is given in Appendix B. to this chapter, pp. 108-9.

Household Arts Continuation School for Girls.

In the autumn of 1911 part-time classes for girls employed in the candy and paper box factories of the North End of Boston were started in a suite of simply furnished model rooms, kitchen, bedroom, and dining and living rooms, provided by the North Bennet Street Industrial School. Seventy girls are allowed by various firms to attend for two hours twice a week without loss of pay. They come in classes of 12 to 15 in number. The instruction is not planned to increase their industrial skill, but to give them fresh interests, raise their standard of living and relieve the monotony of factory work. It consists of plain cooking, marketing, home furnishing and decoration, care of the home, household economy, selection and care of clothing, personal and home hygiene and general efficiency, and is given by simple talks and practice work under one teacher who devotes her time to this work.

The classes have been very successful. Two of the girls married during the course and asked leave to continue it, also requesting advice from the teacher on the furnishing of their homes. Employers say they notice an increase of steadiness in behaviour and work on the part of the girls.

It is obvious that here, again, success is due to the personality of the teacher as much as to the subjects taught. "Those girls," said the superintendent of continuation classes, "are getting infinitely more from their teacher than merely how to cook a meal or buy a chair."

APPENDICES TO CHAPTER VIII.

A.—CINCINNATI CO-OPERATIVE CONTINUATION SCHOOL.

MACHINE APPRENTICE.—COURSE OF STUDY.*

The students are classified as closely as possible into four groups, according to their year of apprenticeship. The more immature come the early part of the week, and the advanced students the latter part of the week. The course is four years long, corresponding to the term of apprenticeship.

First Year Subjects.—Shop Arithmetic, Spelling, Reading, Composition, Reading Blue prints, Drawing, Geographical Relations of the Shop Materials, and Civics.

* From the 82nd Annual Report of the Public Schools of Cincinnati for the year ending August 31st, 1911.

Second Year.—Objective Geometry, Science, Iron and its Manufacture and Founding, Blue prints, Mechanical and Freehand Drawing, Shop Practice, Shop Conventionalities and Necessities, Civics and the reading of the lives of the world's improvers.

Third Year.—Geometry and Algebra, Physics, Shop Practice—Foreman's Question Box, Drawing, Civics and Economic History and Literature.

Fourth Year.—Trigonometry and Applied Mathematics, Shop Chemistry, Shop Practice—visiting of industrial plants and discussing observations, especially of economy and waste; Culture—the man as a wage-earner and citizen; debates.

No machine work is done in the school. The boys get that in the shop and in the night school machine shop, which many of them attend voluntarily.

The work done in the school is closely related to the work in the shop. For example, one of the first machines the boy becomes acquainted with is the drill press. In the school the set of catalogs containing illustrations and descriptions of this machine is read by the class, the technical names of the parts learned, and then other catalogs with other types of drill press are examined. A set of blue prints containing details of the feed-box is next studied, and the boys are encouraged to make freehand drawings showing what the various lines mean to them. The discussion leads on to the various merits of the different types of machines, to scientific principles involved, and to the mathematical calculation of the speed of different spindles. This involves a problem in complex fractions which have been wholly forgotten by the boys. A lesson follows in arithmetic, and they, for the first time in their lives, take an intelligent interest in complex fractions. They are then given a carefully prepared sheet of problems and questions on the drill press, which they work out and preserve in their course of study envelope.

The last period each day is given to general culture. The school is supplied with a piano, a stereopticon and slides, sets of books on civics and industries, maps, pictures, &c.

At the end of the course the boy is given a diploma, and can attend the Department of Engineering of the University of Cincinnati, if he desires. The school has been given six rooms at the school building on Ninth Street.

B.—BOSTON CONTINUATION SCHOOL PROGRAMME.

SHOE AND LEATHER.

The production and distribution of leather; tanning processes; leather manufacture; recognition of kinds, grades and comparative values of leather; manufacture and classification of shoes; salesmanship; efficiency training; visits to industrial plants.

DRY GOODS.

Fibres; cotton goods; wool, worsteds and woolens; silk and silk fabrics; linen and linen fabrics; recognition and comparison of mixed fabrics; simple tests for determining quality; coloring materials and color preservation; shrinking; mercerization; non-inflammable fabrics; care of stock; salesmanship; efficiency training.

BANKING.

Brief history of banking; different classes of banks and their relation to each other; department work; correspondence; notes, usury, protests, discount; currency; foreign monetary systems; circulation; credit; clearing houses; stocks and bonds; brokers; the Stock Exchange; foreign and domestic exchange; funds and funding systems; efficiency training.

CLOTHING.

Fibres; processes in manufacture of cloth; kinds and values of cloth used in clothing; process in the manufacture of clothing; principles underlying cutting and fitting; quality of material and workmanship; design; style; store system; salesmanship; fitting; alterations; efficiency training.

RETAIL SHOE SALESMANSHIP.

Leather; source of, tanning processes, kinds, comparative values. Leather substitutes. Boots and shoes; processes of manufacture; quality of material and workmanship; shape, style, lining, trimming. The human foot; anatomy and hygiene of, variations in shape. Fitting of shoes; store system; salesmanship; efficiency training.

BUSINESS ORGANISATION AND SALESMANSHIP.

Merchandise, handling of, care of; the sale; demonstrations with discussion and criticism thereof; store and office organization; efficiency.

DEPARTMENT STORE SALESMANSHIP.*

Brief review of subjects found under preparatory salesmanship; store system; the psychology of the sale; demonstrations; hygiene; efficiency training.

PREPARATORY SALESMANSHIP.*

Commercial correspondence; facility in oral and written expression; store arithmetic; sales slip practice; sources of merchandise and its distribution; raw materials; textiles; penmanship; color and design; hygiene; talks on the fundamental principles of success; salesmanship.

HOUSEHOLD ARTS.

Plain cooking; marketing; home furnishing and decoration; care of the home; household economy; selection and care of clothing; personal and home hygiene; general efficiency.

* These classes owe their inception to the Women's Educational and Industrial Union, which has maintained a School of Salesmanship since 1905. The leading stores of Boston allow certain of their women and girl employees to attend the school from 8.30 to 11.30 each morning for a certain period without loss of pay. This school has trained many teachers, who are now teaching similar classes in various parts of the States.

CHAPTER IX.—COMPULSORY CONTINUATION CLASSES.

The desirability of instituting compulsory continuation classes is now much under discussion in the United States. Attention is being directed to the numbers of children leaving school at the earliest legal limit, many of them without completing the elementary grades, and entering work which is entirely uneducational. The success of the voluntary part-time classes where they have been held has shown the advantage which would be gained if such classes were more universal.

The presence of so many foreign-speaking people in the country first brought about a certain amount of compulsory evening schooling. For instance, in Massachusetts, anyone who cannot get a certificate of literacy (in English) and who is between the ages of sixteen and twenty-one, must be in regular attendance at an evening school or a part-time class if he wishes to be employed. Such an evening school attendant must furnish his employer with a record of his school attendance each week while the evening school is in session. The employer becomes liable if attendance is irregular.

COMPULSORY EVENING CLASSES.

In *New York State* this idea was carried further. There any boy who has not passed through the eighth grade of the elementary schools must attend evening classes until he is either 16 or has completed the grade work, if he does not choose to stay on in day school.

The law, as amended in 1910, reads as follows:—

Section 622.—"When a boy is required to attend evening school.—Every boy between fourteen and sixteen years of age, in a city of the first class or a city of the second class* in possession of an employment certificate duly issued under the provisions of the labor law, who has not completed such course of study as is required for graduation from the elementary public schools of such city, and who does not hold either a certificate of graduation from the public elementary schools or the preacademic certificate issued by the Regents or the certificate of the completion of an elementary course issued by the education department, shall attend the public evening schools of such city, or other evening schools offering an equivalent course of instruction, for not less than six hours each week, for a period of not less than sixteen weeks or upon a trade school a period of eight hours per week for sixteen weeks in each school or calendar year."

Section 627. "Employer must display record certificate and evening certificate.—The employer of any child between fourteen and sixteen years of age in a city of the first class or a city of the second class shall keep and shall display in the place where such child is employed, the employment certificate and also his evening school certificate issued by the school authorities of said city or by an authorized representative of such school authorities, certifying that the said boy is regularly in attendance at an evening school of said city as provided in section six hundred and thirty-one of this chapter."

* *i.e.*, in cities of 5,000 or more inhabitants.

Section 631. "Evening school certificate.—The school authorities of a city of the first class or a city of the second class, or officers designated by them, are hereby required to issue to a boy lawfully in attendance at an evening school, an evening school certificate at least once in each month during the months said evening school is in session and at the close of the term of said evening school, provided that said boy has been in attendance upon said evening school for not less than six hours each week for such number of weeks as will, when taken in connection with the number of weeks such evening school shall be in session during the remainder of the current or calendar year, make up a total attendance on the part of said boy in said evening school of not less than six hours per week for a period of not less than sixteen weeks or attendance upon a trade school for at least eight hours per week for not less than sixteen weeks. Such certificate shall state fully the period of time which the boy to whom it is issued was in attendance upon such evening school or trade school."—(Education Law, 1910, Chap. 140—Laws of 1910.)

In *New York City* the evening schools are in session for four evenings a week, Monday, Tuesday, Wednesday and Friday, from 8 to 10 p.m., and attendance is expected on these four days.

The instruction given consists of the grammar school subjects with a bearing, wherever possible, in the direction of the boy's employment or interest in work.

The hours of work are from about 8 a.m. to 5 p.m., the hours for all young people under 16 being limited, in New York State, to eight a day or 48 a week in factories, and to nine a day (54 a week) in other occupations.*

The law for compulsory school attendance is not, however, supported by either public or educational opinion, and since no special machinery has been created for its enforcement, it is, as Dr. Maxwell, Superintendent of Schools, said, "more honoured in the breach than in the observance."

The report of the District Superintendent of Evening Schools for the year ending July 31st, 1912, shows that of approximately 21,000 boys who should have attended evening schools during some part of the session, there were exactly 7,085 who attended one night or more in spite of efforts made to get into touch with each boy discharged from the day school. Had there been a special corps of attendance officers to deal with the question the number of registered pupils could have been increased, but Superintendent Maxwell told the writer that he considered the boys could not at that hour profit from the schools—they wanted to go to sleep.

The law is referred to by the Superintendent of Evening Schools in his report as one approved "neither by popular sentiment, nor by the wisest educational considerations," and Dr. Maxwell himself says in his report for the school year 1911-1912:—

"The most important recommendation I have to make regarding evening schools is one which I have made in two former Annual Reports and which is strongly supported in District Superintendent Shiel's report—that the

* See Summary of Labour Legislation affecting Children and Young Persons in Certain States, pp. 181-3 below.

compulsory education law be so amended that it shall no longer be required of boys who go to work at fourteen years of age to attend evening school, but that it shall be compulsory for them to attend continuation schools, forenoon or afternoon, six or eight hours a week."*

It appears that these compulsory evening classes are of use to those boys only who are anxious to learn, and who would have attended voluntary evening classes equally well. That the law is not approved of was told the writer by many social workers and others interested in education in New York City. So little is the idea of instituting compulsory evening classes for children under sixteen considered in the United States, that during the six months spent in the country, when there were many opportunities of discussing and hearing discussions on the changes needed in the American educational system, the writer never heard a suggestion made that such classes should be started; nor in any of the many reports and pamphlets presented to her did she find any reference to them, except as to their undesirability. Where compulsory continuation classes for children are suggested, it is always during the day-time, and Dr. Maxwell said that this is the trend throughout the country. Even the National Association of Manufacturers has declared in favour of compulsory day classes (*see* pp. 179-80). Surprise was expressed whenever it was mentioned that in England, the country looked to by Americans as so much in advance of them in social legislation, compulsory continuation classes are sometimes contemplated by certain people for the evening hours. It is a striking thing that the American people, so noted for their energy and powers of work, should so strongly realise the value of evening recreation for their young workers and of study only during the fresher hours of the day.

COMPULSORY DAY CONTINUATION CLASSES.

The Indiana Commission on Industrial and Agricultural Education, and the Board of Education of Massachusetts reporting on the "Needs and Possibilities of Part-time Education," in January 1913, both recommend that the local authority should have power to institute day continuation classes and compel attendance, and in June, 1913, a law was passed in Massachusetts to take effect from September, 1913, giving this power and authorising the Commonwealth to provide one-half the cost of maintenance of the classes.† Two States have

* Since the writer's visit to New York, two Bills have been introduced into the New York Assembly, one increasing the amount of State aid given to trade schools, and extending it to part-time, continuation and evening schools, the other authorising local school authorities in all cities and districts to establish part-time and continuation schools, and require the attendance upon them of all children between 14 and 16 who have not completed the elementary school course for from 4 to 8 hours a week between 8 a.m. and 5 p.m. In cities of the first and second class, such boys may be required to attend such schools in place of the evening attendance now required. Information has been received after going to press that this second bill has already been passed.

† See Appendix B. to this chapter, pp. 121-2.

already instituted such classes for some time, viz., Ohio and Wisconsin, both in the middle west. Connecticut is considering the matter.

Compulsory Continuation Classes at Cincinnati, Ohio.

In 1910 Ohio* passed the following law authorising compulsory attendance at day classes :

"In case the Board of Education of any school district establishes part-time day schools for the instruction of youth over 14 years of age who are engaged in regular employment, such Board of Education is authorized to require all youth who have not satisfactorily completed the eighth grade of the elementary schools to continue their schooling until they are 16 years of age ; provided, however, that such youth, if they have been granted Age and Schooling Certificates and are regularly employed, shall be required to attend school not to exceed eight hours a week between the hours of 8 a.m. and 5 p.m. during the school term." (Sec. 7767, G.C of Ohio.)

Cincinnati, an industrial city of 363,591 inhabitants, and the second largest in the State, whose predominant manufactures are foundry and machine shop products, boots and shoes, men's clothing, and printing and publishing, was the first to establish these classes.

The following quotations from the Annual Report of the Public Schools of Cincinnati for the school year ending August 31st, 1911, describe the reasons for their establishment and the results :—

"Reasons for.—By our attendance department it was shown that more than 2,000 go to work yearly between the ages of 14 and 15, and that most of these have completed only the fifth grade of the elementary school. These immature and uneducated youths drift, as a rule, from one little position to another, not increasing appreciably in efficiency. The parents feel that they must now be given freedom, and employers do not usually feel responsible for their conduct, so that many of them in their leisure acquire habits that undermine their characters. Thus, in the critical period of adolescence, when they are most susceptible to evil influence and most in need of moral and vocational guidance, they are overlooked by everybody. The State provides liberally for those who go to school until they are 21 or older, but for the unfortunate ones who must go to work nothing whatever is done. Recognizing this, and finding from our experience in voluntary continuation schools the immense influence the school exercises in the few hours a week it has the pupils, in giving them a right attitude toward work and life, and in stimulating their ambition and guiding their energies in wholesome channels, our Board was instrumental in securing legislation for continuing the schooling of those who go to work under 16 years of age.

"In January 1911, the Board of Education adopted a resolution to provide compulsory continuation schools to meet the provisions of the law in September 1911, and to notify employers that children would be required after that date to attend school four hours a week.

"No child is issued a certificate to work until he brings from an employer an agreement to co-operate with the public school authorities in obtaining the attendance of said child at the continuation school to which it may be assigned, as long as it shall be under 16 years of age and in the employ of the subscriber, provided the child has not passed the eighth grade.

* Ohio, in the U.S. Census for 1910, is given as fourth State in the size of population, and 35th in area. It is largely a manufacturing State, occupying fifth place among the 49 States and Territories in the value of its products. Its chief industries are foundry and machine shop products, iron and steel mills, and cars and general shop construction.

"Employer will please indicate on back of card* the school and hours preferred for the child."

"Method of operation.—When a certificate to work is issued, the officer notifies by card the continuation school to which the child is assigned, with full information, so that if the child does not report as directed, he may at once be looked up. Usually, all that is required is to call the employer by telephone, and co-operation is at once secured. Absence is treated in the same way. The resulting attendance is remarkable. Eleven hundred enrolled the first month, and this is steadily increasing, now reaching a total of 1,843, and 1,524 for the month of January. Twelve centers have been opened at school buildings convenient to the industrial sections of the city. The employer chooses the school and hours preferred. Only four hours a week are required, one hour a day for four consecutive days, or four consecutive hours in one day, as may be preferred. Three teachers are employed their entire time, and 40 teachers are employed part time in this work. One building operates from 8.0 a.m. to 5.0 p.m. all week. Seven centers operate from 4.0 to 5.0 p.m., and six centers on Saturday afternoons. At Woodward High School about 500 are collected on Saturdays, making extensive use of the shops and commercial equipment.

"Classification and Course of Study.—The students group naturally in three classes according as they left school in the sixth, seventh, or eighth grade, our law requiring that children complete the fifth grade before going to work. The sixth grade group is much the largest, and is so different in attainments that, where possible, a preparatory group is formed in addition to the three regular grades. The sexes are separated where the number justifies it. The classes vary in size from 20 to 35. At one center in some hours of the day classes are as low as 12. The average is 25.

"It was soon found that many of the children had gone to work, partly, at least, because they wanted to escape school. The ordinary routine of school duties was distasteful to them. This is taken into account in preparing their course of study. The beaten path is left wherever possible, and while attention is given chiefly to arithmetic and English, these are presented in ways as different from the ordinary and in as objective, picturesque, and interesting forms as the ingenuity of the most capable teachers of our city can devise. These subjects are connected up with the daily life and work of the children and applied to problems and situations which they feel worth while. Daily drills are given in spelling, correct English, and rapid calculation. The teachers are called in weekly meetings, and the course is worked out in detail as they go along.

"About two-thirds of the time is given to studies, and one-third to industrial work. The course of study comprises English, including spelling, reading, and correct speech; arithmetic, including much practice in the fundamental operations, fractions, percentage, pay-rolls, business forms, the keeping of accounts, and simple bookkeeping in the eighth grade.

"Hygiene, civics, moral instruction and personal guidance. This is given more serious consideration by the teachers than any other part of their work, and is brought in incidentally in all subjects in addition to its place on the program. Handicraft.—One third time is given to shop work, with tools, for boys, including work with wood and iron; and in applied art and design, sewing and garment making, millinery, paste and glue work, and cooking for girls. This work is applied as closely as possible to the vocation.

"Difficulties.—The chief difficulty encountered is in adapting the strictly vocational work to the needs of the children. But few are in the vocation in which they expect to remain. They have been thinking of the few dollars a week, and not of a life occupation. To compensate for this, however, they are all immensely interested in all forms of handicraft and enter into such work with enthusiasm. They need expert counsel and wise vocational

* On the back of the employer's card is a complete list of the schools. See Appendix B. (3) to Chapter XL, page 157.

guidance until they discover themselves. Conditions in the industrial and commercial world are such that it is not easy to advise. Another difficulty is that some of the children are in such distant places that it is too great a hardship to require their attendance. Their wages are too pitifully small to enable them to pay car fare. Some employers even dock them for their time besides. We have not as yet required attendance of those who would have to pay car fare. Most of the difficulties we had expected have not developed. Our children come from 1,100 employers. Scarcely one in a hundred has tried to evade the law or offered any obstruction other than in some cases to deduct from the children's pay for the time. The latter is only temporary, as the children quickly make themselves more valuable, and discerning employers will give them more. Our greatest difficulty in attendance is where least expected—with children who work at home. Some of these parents begrudge their own children four hours a week in school. They demand their entire time, and that usually without paying them anything. Such cases are by no means rare. But from employers almost uniformly we meet with cordial co-operation and encouragement. We had expected great difficulty in maintaining attendance and interest, but since we have introduced manual work for boys, and domestic work for girls, the interest is as great and the attendance much more regular than in the night schools. Many children who have reached the age of sixteen request the privilege of continuing.

"One thing above all else impresses one with these children—they are sacrificing themselves for such a pittance. Probably they are not worth more to their employer than they are getting, but it ought to be the business of the State to see that they are worth more, and that is one of the purposes of these schools."

The writer visited Cincinnati at the end of January, 1913. Some of the classes seen are worth describing. On a Thursday between 12.30 and 1.30 two classes were found meeting at the buildings of an elementary school. Both come for this hour four days a week, the children getting their dinner either just before or just after the class. A class of 25 boys, nearly all errand-boys, were taking the history of the United States, having a talk with the teacher about life in the pioneer days. A class of 20 girls from a large department store near by were that day having a lesson in salesmanship, *i.e.*, what makes a good saleswoman, how to treat a customer, &c. The girls seemed to enjoy the lesson, they readily answered questions, and their behaviour while in class and on entering and leaving was excellent. These girls were mostly packers and messenger girls who had occasionally served in the store and who hope in time to become saleswomen. A few were from the workrooms, and it seemed a pity that these should be taking a subject which would not be helpful to their trade. The organisation of these classes has however been a tremendous piece of work, and the Superintendent of Schools said that they had not yet arranged all the classes as they could wish.

In the large building of the Woodward High School great variety was found in the arrangement of classes on a Saturday afternoon from 1.30 to 4.30, the hours having been reduced by half an hour when the half-hour recess was given up. Freedom of choice is allowed the children as far as possible. Some girls who have reached the eighth grade take domestic subjects only, for instance a class of girls, some over sixteen, were

learning millinery, but the work was broken in the middle by a spelling bee, others devoted themselves to dressmaking, while yet others spent half the afternoon at cooking and half at needlework. Those in lower grades spent half time at sewing or cooking, and half at English subjects. The girls make garments for themselves, paying for the material at cost price.

In the commercial section there were boys and girls. Some of the most advanced took the whole afternoon for commercial subjects only, others spent half the afternoon at general work and half at bookkeeping. These were all either in clerical work, or anxious to enter it. Some girls had chosen to devote their whole time to drawing and design, others spent half the period at this and half at the ordinary grade work.

The manual work for the boys comprised woodwork, forging, electrical wiring, and wood-turning, the boys usually spending half the time at one of these and half at general subjects. Two classes were found taking history, another was having a talk on methods of transportation, with lantern views of canal-locks, and another was watching experiments in electricity. Certain boys had chosen drawing as their chief subject, and these, 20 in number, were spending all the afternoon under one master. He said that for the first hour they took drawing, and here the greatest variety was found, one, an architect's boy, making a plan of a country bungalow, while others were doing plain lettering, one of these bending with the greatest absorption over the words "dressmaking done here." He seems to have chosen this class in order to make a sign for his sister's use. After an hour at drawing their master reads Kipling to them for a while and then takes arithmetic and spelling, ending the afternoon with a talk on some topic of general interest, which the boys themselves often suggest. One week they had asked for an explanation of the new parcel post. Four boys in this class were over sixteen and had asked to stay on.

Every few weeks each class is taken to visit some exhibition or museum, the girls sometimes going to one of the large shops at a time of special display, and inspecting furniture, costumes, &c., the things seen being discussed the next week. It is hoped to influence the girls' taste in this way.

Attendance is secured as follows. If a child is absent once a postcard is sent to him; if twice following, notice is sent to the truant officer who finds out the reason. The employer is held responsible for attendance on Saturdays and other days and is notified if the child is absent without good reason. The work certificate is broken and the child returned to regular day school if he persists in truancy without excuse. This was done in 20 cases in the first year—in the year 1911-12 it was not once necessary.

In the year 1911-12, 457 children were given certificates for completing the eighth grade and were excused from further

attendance. More than one-third returned voluntarily to their classes.

Of 2,686 attending during the year, 25·5 per cent. went to the all-day school, 16 per cent. attended classes between 4 and 5 p.m., and 58·5 per cent. the classes on Saturday afternoons.

The most striking thing perhaps about the schools was the freedom and elasticity of the system, and it is to this that they owe their success. It was an uphill fight, for the children, especially those who lost wages through it, resented being returned to school, and came determined to make trouble. That they now realise the good of the schools is evident; they bring their difficulties to the teachers for help in solving them, and their behaviour in class and on entering and leaving school is excellent. Employers co-operate now by making suggestions as to subjects which might be helpful to the children, sometimes sending lists of technical terms for use in spelling lessons.

Before leaving Cincinnati the writer had the opportunity of discussing the classes with two of the leading trade unionists of the city. Organised labour, they said, thoroughly approved of them. In the Annual Report for 1911-12 the following words of one of them are quoted:—

“I think the continuation school of Cincinnati was little short of an inspiration. Next to the principle itself of public and universal education, is the idea of prolonging it and making it effective through the study period of young men and women. . . . Of the continuation school there can be nothing said but praise. . . . Not the least of its benefits is what it teaches the employer. He learns that the workman must not be considered a mere ‘hand,’ but a mind and a soul as well. He learns that this boy he has employed is a ward of the public as well as a servant of his, and he also learns that his work is going to be done better and his product carry a higher value by reason of the more intelligent labor bestowed upon it. In fact, reasons immediate and reasons that stretch away to the farthest markets of the world, are reasons for the continuation school from the employers’ standpoint.”

The attitude of employers was probably well summed up in the words of one of the heads of a large department store seen by the writer, whose girls attend the classes for one hour in the middle of the day, four times a week. “I’m one of the largest taxpayers in the city,” he said. “If the classes were no good I’d be making a fuss. My girls that go are improving all the time.”

The city of *Cleveland* had also begun to organise classes under the State law by the autumn of 1912. These were for six hours in the week, but organisation was proceeding slowly, no attempt being made to draw in all children at first. *Columbus, Toledo, and Dayton* were also beginning to organise classes during the winter of 1912-13. These, with Cincinnati, are the five largest cities of the State.

Compulsory Continuation Classes in Wisconsin.

The legislature of the State of *Wisconsin* passed many laws dealing with child labour and the establishment of trade, continuation, and evening schools in 1911. Under the new boards of industrial education described on pages 23 and 24, the organisation of part-time continuation schools for children between fourteen and sixteen, whether the eighth grade had been passed or not, began in 1912. The law authorising the formation of the continuation schools is as follows:—

“Employment of persons between fourteen and sixteen years of age in certain cases.

“Section 1728 c—1. 1. Whenever any evening school, continuation classes, industrial school or commercial school, shall be established in any town, village or city in this state for minors between the ages of fourteen and sixteen, working under permit as now provided by law, every such child, residing within any town, village, or city in which any such school is established, shall attend such school not less than five hours per week for six months in each year, until such child becomes sixteen years of age, and every employer shall allow all minor employes over fourteen and under sixteen years of age a reduction in hours of work of not less than the number of hours the minor is by this section required to attend school (1911, c. 660).

“Section 1728, c—1. 2. The total number of hours spent by such minors at work, and in the before-mentioned schools shall together not exceed the total number of hours of work for which minors over fourteen and under sixteen years of age may by law be employed,* except when the minor shall attend school a greater number of hours than is required by law, in which case the total number of hours may be increased by the excess of the hours of school attendance over the minimum prescribed by law.

“3. Employers shall allow the reduction in hours of work at the time when the classes which the minor is by law required to attend, are held whenever the working time and the class time coincide. (1911, c. 505.)

“Section 553, p—5. 1. The courses of study in these schools shall be approved by the State superintendent of education and the State board of industrial education, and shall include English, citizenship, sanitation and hygiene and the use of safety devices, and such other branches as the State superintendent and the State board of industrial education shall approve. (1911, c. 616.)”

The employer and parent are held responsible for the attendance of the children, failure to ensure it being punishable by a fine to the employer of not less than 10 nor more than 100 dollars for each offence, and to the parent of not less than 5 nor more than 25 dollars.

The classes in two cities in *Wisconsin* were visited. At *Milwaukee*, a city of 373,857 inhabitants, the schools had not then been completely instituted, but special rooms had been taken and classes of girls were already attending from some of the department stores. They came on one day a week from 8 a.m. to 12 p.m. or from 1 p.m. to 5 p.m., taking sewing and general subjects, and were under one teacher, who devoted her time to the work. It was intended later to introduce cooking. The boys' classes had not then been organised.†

* Eight hours a day and forty-eight hours a week. See *Summary of Labour Legislation affecting Children and Young Persons in certain States*, pp. 181–183 below.

† For a recent account of these classes, see Appendix E., p. 184.

At *Racine*, a small city of 38,000 inhabitants, situated on the shores of Lake Michigan to the north of Chicago, there are 300 children between the ages of 14 and 16 employed during the day. Classes had been organized for all these children. One was held at a factory employing 15 girls of the necessary age. A room was lent by the firm and the teacher went three days a week from 1 to 2.40 p.m. The teacher spoke of this as one of the best classes she had, though in other places there is a tendency to suspicion at the desire of employers to have the classes held on their premises. At Cleveland this had been altogether refused.

Between 3.30 and 5.30 on a Thursday the writer found a class of 12 errand-boys from factories taking civics and spelling, followed by some manual work, and 12 girls from factories were in another room discussing the use of fire-alarms and a visit paid by some of the class to a fire station. They were later to take spelling, and then sewing.

By the description of the classes seen and subjects studied the work may appear very elementary both at Cincinnati and in Wisconsin. Undoubtedly it is so, but it must be remembered that the teachers are dealing with children, of whom the majority in Wisconsin, and all in Cincinnati, have not completed the grades of the elementary school and are not anxious to learn. The teachers must arouse the children's interest, and, in the little time available, can only hope to keep up their grounding in elementary subjects, show them something of the duties of citizenship and help them better to understand the little problems they meet with in their work. Probably few of us realize how difficult the first steps in work are to children unaccustomed even to many of the articles made in the factories in which they run the errands, and how helpful it is to them to have explained and see in writing the names of the things which they have to handle. This the teachers in Cincinnati have realized, and they have taken the trouble to learn the children's needs. No great educational advance can be expected from these classes; no training for a trade can possibly be given; but great good will be, and, indeed, has been, effected where the children, who are all, it must be remembered, in uneducational, unprofitable work, become steadier and more responsible, and where they learn to look on the teachers as their friends. It is, again, by personal influence rather than matter taught that these short-time compulsory schools will most succeed.

The effect of the schools at Cincinnati is summed up in the Annual Report for the year ending 1912.

" We feel that two years' effort in continuation school work has only served to show us the problem and to point us the direction of its solution. Of the great need of more systematic and more intensive and more diversified work for these needy children we all feel convinced. That it

even now satisfies a great need for most of these children, they themselves continually give testimony. The work has meant for many of them the taking up again of a thread lost somewhere in the course of educational progress through the regular elementary school. It spells new hope for the discouraged.

It has borne direct result in a more considerate spirit toward fellow workmen, until there has come to be a real community spirit which is helpful to all. A very decided cultural uplift is seen in the rapidly improving manner of address to their fellows, in the less general use of slang and the factory vernacular, and in the frequent expression of little courtesies to their teachers and classmates. The growth in cleanliness and in a desire for saner methods of dress and living is decided. There is a growing interest in good books, as is proved by constant requests for information regarding them. There is, in general, an awakening to the interesting things about the child in every-day life, as is shown by the interest of the child in what he sees and hears and by his desire to make his information of service. In many cases there have been confidential conferences, which have given the teachers rare opportunities for personal counsel and guidance towards the highest ideals. The school has placed in these lives a new ideal, working through a greater respect for authority, a more willing obedience, an increased responsibility, which will end for many in a desire to be of fullest service to themselves and their fellows."

SCHOOLS FOR APPRENTICES.

Under the apprenticeship law of Wisconsin for 1911, every indenture must contain besides other necessary clauses :—

- (a) An agreement stating the number of hours to be spent in work and the number of hours to be spent in instruction. The total of such number of hours shall not exceed 55 in any one week.
- (b) An agreement that the whole trade, as carried on by the employer shall be taught, and an agreement as to the time to be spent at each process or machine.
- (c) An agreement between the employer and the apprentice that not less than five hours per week of the aforementioned 55 hours per week shall be devoted to instruction.

This instruction must include two hours a week given to the following subjects: English, citizenship, business practice, physiology, hygiene and the use of safety devices. The instruction may include other subjects and may be given at a public school or such other place as the local board of industrial education may approve.

As no boy is apprenticed until he is 16 years of age, apprentices do not attend the ordinary compulsory part-time classes. At Milwaukee the Extension Department of the University of Madison had for some time maintained a voluntary class for apprentices attending for two hours once or twice a week during work hours. This has been constituted a school for apprentices under the new law, and the boys now come one afternoon a week from 1.30 to 5. There were 67 in attendance on the various afternoons, about 12 in a class, in January 1913. Individual instruction is given according to each one's needs.

These boys say that they are being taught more of their trade than they were before the Act was passed, but it was then of too recent a date to know whether this Act would lessen the number of indentured apprentices taken or not. The evidence given to the writer at the time of her visit was conflicting, but information has been received since that the Act is meeting with little opposition.

APPENDICES TO CHAPTER IX.

A.—STATISTICS OF THE COMPULSORY CONTINUATION SCHOOLS TAKEN FROM THE ANNUAL REPORT ON THE CINCINNATI PUBLIC SCHOOLS FOR THE YEAR ENDING AUGUST 31st, 1912.

	Male.	Female.	Total.
Number of different Teachers.	27	42	69
Number of Teachers employed entire year	15	25	40
Number of Teachers necessary to supply school.	27	42	69
Total number of Pupils enrolled in 12 buildings.	1,227	1,541	2,768
Number withdrawn	366	511	877
Number remaining	862	1,029	1,891
Average number belonging	859	968	1,827
Average daily attendance	741	891	1,632
Average daily absence	118	77	195

B.—COMMONWEALTH OF MASSACHUSETTS BOARD OF EDUCATION. EXTRACTS FROM THE REPORT ON "THE NEEDS AND POSSIBILITIES OF PART-TIME EDUCATION." (JANUARY 1913.)

CONCLUSIONS.

As a result of the present inquiry into the needs of part-time education, the following conclusions have been reached:—

1. There is a distinct need for the further development of part-time education in Massachusetts, evidenced by the fact that upwards of 40,000 young persons from fourteen to seventeen years of age are constantly employed in wage-earning pursuits, most of whom have not completed an elementary education, and nearly all of whom have little or no opportunity for systematic vocational training toward occupations suitable to adults.

2. Part-time education as a means of giving more adequate liberal and vocational education to young persons already employed is, in large measure at least, practicable, both from the standpoint of the employing industries and from the standpoint of the schools, as a means of efficient instruction for young persons.

3. The further development of part-time education in Massachusetts requires but little legislation additional to that now existing. There is need, however, for the further development of existing agencies for the conduct of vocational education, and also for the elaboration and testing of practical plans for part-time training.



6. The experience of the last four years in dealing with the problem, and the facts collected during this investigation, justify the conclusion that the general Massachusetts policy of moving forward by voluntary schemes of experimentation and investigation should be continued, with the ultimate view of further educating in some degree every boy and girl between the ages of fourteen and seventeen.

7. Legislation already enacted relative to vocational and part-time education has been sound. Chapter 471, Acts of 1911, is fully adequate for the further establishment of voluntary part-time courses, and for their supervision on an efficient basis. It is the belief of the Board that part-time schooling should be made compulsory throughout the State at some time in the future, when all children employed between the ages of fourteen and seventeen might be required by law to attend part-time schools, at the rate of not less than eight hours weekly. When this is done these schools should offer courses of general training for citizenship, and, in the case of the girls' trades, where sufficient related and theoretical training cannot be given, training in the household arts should be substituted.

RECOMMENDATIONS.

In obedience to the resolve, the Board of Education respectfully submits the following recommendations:—

I. Legislation should be enacted requiring the attendance upon some school of every child, not specifically exempted for satisfactory cause, between fourteen and sixteen years of age, who is not regularly employed; in addition to which, such changes in the present laws regarding compulsory education should be enacted as will enable school committees effectively to control and require attendance at approved schools of boys and girls of such ages.

II. Legislation should be enacted which will enable cities and towns, through their school committees, to require part-time school attendance of all boys and girls between fourteen and sixteen years of age who are regularly employed, at the rate of not less than four hours per week upon an approved school during the time when such schools are in session. Such attendance should be made between the hours of 7 a.m. and 6 p.m. of any working day or days.

C.—QUOTATIONS FROM "AN ACT RELATIVE TO THE ESTABLISHMENT AND MAINTENANCE OF CONTINUATION SCHOOLS AND COURSES OF INSTRUCTION FOR WORKING CHILDREN." APPROVED JUNE 16, 1913. [COMMONWEALTH OF MASSACHUSETTS.]

Section I.—When the school committee of any city or town shall have established continuation schools or courses of instruction for the education of minors between 14 and 16 years of age who are regularly employed in such city or town not less than six hours per day, such school committee may, with the consent of the board of education, require the attendance in such continuation schools or on such courses of instruction of every such minor thereafter receiving an employment certificate and who is not otherwise receiving instruction approved by the school committee as equivalent to that provided in schools established under the provisions of this act. The required attendance provided for in this act shall be at the rate of not less than four hours per week and shall be between the hours of eight o'clock in the morning and six o'clock in the afternoon of any working day or days. The time spent by a child in a continuation school or class shall be reckoned as a part of the time or number of hours that minors are permitted by law to work.

Section III.—The Commonwealth, in order to aid in the maintenance of approved continuation schools or courses, shall as provided in this act pay annually from the treasury to cities and towns maintaining such schools or

courses an amount equal to one-half of the sum to be known as the net maintenance sum. Such net maintenance sum shall consist of the total sum raised by local taxation and expended for the maintenance of such a school, less the amount, for the same period, of tuition claims paid or unpaid and receipts from the work of pupils or the sale of products.

Section VI.—The employer of any minor between fourteen and sixteen years of age who is compelled by the provisions and regulations either of the school committee in the city or town in which such minor resides or of the school committee in the city or town in which such minor is employed to attend a continuation school or courses of instruction as defined in section one of this act, shall cease forthwith to employ such minor when notified in writing by the superintendent or his representative duly authorized in writing, having jurisdiction over such minor's school attendance, that such minor is not attending school in accordance with the compulsory attendance regulations as defined in section one of this act. Any employer who fails to comply with the provisions of this section shall be punished by a fine of not less than ten nor more than one hundred dollars for each offence.

Section VII.—The superintendent of schools having jurisdiction, or a person authorized by him in writing, may revoke the age and schooling, or employment certificate of any minor who is required by the provisions of this act to attend a continuation school or courses, if such minor fails to attend such school or courses as provided by this act.*

**D.—RECOMMENDATIONS ON CONTINUATION CLASSES OF
THE INDIANA COMMISSION ON INDUSTRIAL AND
AGRICULTURAL EDUCATION, REPORTING
DECEMBER 1912.**

(a) That when the school authorities have established vocational courses approved by the State Board for boys and girls from 14 to 16, and have formally accepted the provisions of the Act permitting it, they may require every boy and girl from fourteen to sixteen who is employed, to return to the day school for instruction for at least five hours per week, employers to be required to allow time off for such instruction.

(b) That contracts of apprenticeship shall specify that the child shall be given at least five hours per week of instruction in the day schools in English, citizenship, physiology and hygiene, use of safety devices, and such other subjects as may be approved by the State Board of Education.

* This Act came into force in September 1913. Boston at once began to take steps in order to carry out the provisions in September 1914.

CHAPTER X.—PREVOCATIONAL AND INDUSTRIAL COURSES IN THE ELEMENTARY SCHOOLS.

RETARDATION IN THE GRADE SCHOOLS.

The average child should be completing the grades of the elementary school at 14 years of age, the most common compulsory limit in the north. A few complete the grades before that age and pass on to the high school earlier than customary. It has already been stated that numbers leave as soon as is legally possible without completing the elementary school work. Many returns have been published showing the lack of schooling among those going out to work at 14 years of age.

The following figures from three cities indicate the position. In Chicago for the year 1st July 1911 to 30th June 1912, 8,466 boys and 4,847 girls were granted work certificates.*

	Boys.	Girls.
	Per cent.	Per cent.
There left from grade 5 or below - - -	21·4	22·3
" " " 6 " - - -	19·0	18·9
" " " 7 " - - -	19·5	20·6
" " " 8 " - - -	23·0	25·9
" " high schools - - -	6·8	5·8
" " evening schools or unclassified	10·3	6·5

At Cincinnati during the year ending 31st August 1912 work certificates were given to 2,366 children.†

	Boys.	Girls.
	Per cent.	Per cent.
From grade 5 - - - - -	29·3	23·9
" " 6 - - - - -	26·7	27·0
" " 7 - - - - -	23·6	26·6
" " 8 - - - - -	18·0	20·2
" high schools - - - - -	2·4	2·3

72·8 per cent. of the boys and 72·7 per cent. of the girls were from 14 to 15 years of age. The remainder were between 15 and 16.

* Fifty-eighth Annual Report of the Board of Education of Chicago for year ending 30th June 1912, p. 248.

† Cincinnati Public Schools—83rd Annual Report for year ending 31st August 1912, p. 39.

The inquiry into the 1,000 boys attending evening schools in New York City showed that they had left the grades as follows* :—

From grades 5A-5B	-	-	-	-	141
" "	6A-6B	-	-	-	287
" "	7A-7B	-	-	-	403
" "	8A-8B	-	-	-	169

The investigation into the conditions under which children leave school to go to work made for the United States Senate showed a very similar dropping out from the grades.

The following reasons for which children were repeating the grade a second time were given by the teachers for 1,468 children enquired into in four northern cities.†

	Boys.	Girls.
Change of school system	35	27
Moving	18	10
Irregular attendance or non-attendance	202	185
Lack of interest	177	94
Lack of ability	35	41
Slow or dull	66	38
Immature	66	48
Mentally defective	29	23
Defective health or physical defects	112	99
Lack of English	63	64
Miscellaneous	56	37
Total	859	666

From this it is seen that in these schools 19·3 per cent. fell behind in their grade work through either lack of ability or immaturity (excluding mental defect) and 18·4 per cent. through lack of interest. This does not include those whose attendance may be irregular or altogether wanting through dislike of school. It is the children who have fallen behind in their grade work who tend to leave as soon as possible and try a life of work which appears so very much more interesting than the discouraging school routine.

Prevocational or Industrial Courses have lately been started in many cities for selected children who have failed to "make good" in their ordinary school work, and these courses have already proved very successful in holding the children in school. In these courses a large part of the school time is devoted to practical work, and the teaching of the general subjects is given

* Fourteenth Annual Report of the City Superintendent of Schools. Report on Evening Schools for the year ending July 31, 1912, p. 79. (City of New York.)

† Report on Condition of Women and Child Wage-earners in the United States, Vol. VII., pp. 274-279. Washington, 1910.

as close a bearing as possible on the manual work being performed by the children in the workshops or kitchens. The children selected have been those under 14 years of age who were likely to leave school as soon as they reached that age, while still far behind the average in their school work. Such children often reveal abilities which lie dormant till they find expression in some form of manual work. Their minds and desires are often of an intensely practical turn, and school subjects are of no interest until applied to actual work. Latent powers have been drawn out by the appeal to hand and brain, and children who but for these courses would have completed the school stage of their lives under a sense of failure and discouragement, have been given one of the most helpful of all starts in the industrial world, for they have left school with the knowledge that in certain subjects they can not only compete with but even excel their fellows, they have been given the opportunity of "trying themselves out" at trade work, and they have acquired habits of application and steadiness which will be of service to them in whatever employment they may have to follow later.

In a few cities the opportunity to take up some practical trade or household arts work is offered to all classes of children in the elementary school. The writer had the opportunity of seeing both kinds of prevocational work. That for retarded children is described first.

BOSTON PREVOCATIONAL CENTRES.

The first industrial class in Boston was introduced in 1904 with a teacher paid from private sources. There are now six schools or "prevocational centres" maintained by the School Committee in which special industrial classes for boys have been formed. Boys from the fifth, sixth and seventh grades have been selected to enter these courses; invariably those are chosen who are not getting on with their school work and will probably drop out of school at once on reaching 14. The work is arranged as much as possible on industrial lines, jobs being undertaken for the Boston School Committee and executed by the boys in working gangs.

The objects and methods of instruction of the schools as stated in the Annual Report of the School Committee for 1912 are as follows:—

1. To influence the boys to remain in school after they have reached the age of 14.

2. To enable them to graduate earlier than they would under ordinary circumstances.

3. To awaken in them a desire for an industrial career, and offer them a definite opportunity for vocational guidance therein.

4. To point to the Boston Industrial School for Boys or to the Mechanic Arts High School after graduation from the elementary school in preference to the street or some occupation that holds out no promise of future advancement.

5. To give some definite preparation to boys who actually do go to work at 14 years of age."

"The classes in these prevocational centers are small, and the academic work is restricted to reading, spoken and written English, arithmetic, drawing, geography, history and hygiene; each of these subjects being closely connected with the shop work. The shop work itself occupies almost one-half of the school day and concerns itself with the production, under shop conditions, of commercial products to be used in the schools or in other city departments. The shop instructors are practical people. Time cards and job tickets are kept, and attention is given to the ability to economize time and material and to work under, over and with other people.

"Among the things produced in these centers are drawing models, looms, loose leaf covers, portfolios and boxes for a wide variety of uses made of wood and pasteboard. The furniture-making centre has made stools, test-tube racks, tables, chairs, desks and cabinets. The machine shop centre has sent out adjustable desk fittings, brass tags, tree guards, angle irons, sheet metal working tools, &c. The printing plant issues "The Workmaster" a little monthly which gives the news of all the centers. It has also printed addresses on envelopes and door cards for school use. The bindery has produced loose leaf covers and notebook-holders, and has bound and rebound books. The sheet metal shop has made water cups, trays, a cover for a motor, &c.

"The boys make visits to factories, shops and mills, to industrial exhibitions, and to wharves, and bonded warehouses."

Each trade is taught in a separate school. The list of centres for 1913 is as follows:—

Bookbinding	-	Ulysses S. Grant School	-	East Boston.
Machine shop practice.		Quincy School	-	City proper.
Sheet metal work	-	Sherwin School	-	Roxbury.
Printing	-	Lewis School	-	Roxbury.
Boxmaking and woodworking.		Eliot School	-	Jamaica Plain.
Woodworking	-	Oliver Wendell Holmes School.		Dorchester.

Quincy School.

Of these, the class in machine shop practice at the Quincy School was visited by the writer. A large room in the basement has been set aside as the machine shop, a small portion being screened off to form the master's office and tool room as in any works. A class room upstairs is set apart for the sole use of the two sections of the industrial class. Repairs and new work are undertaken for the School Committee, any castings being done outside and finished off by the boys.

The following extract from the Annual Report for 1910 of the Superintendent of the Boston Public Schools describes the scheme of work.

"An industrial class in iron work, consisting of 44 boys, was organized by the Quincy School, September 8, 1909. The boys belonged in the fifth, sixth, and seventh grades, and were selected for these reasons; they were

fourteen years of age or would be during the year, they had shown particular ability in manual work, and their conduct and interest in school was unsatisfactory. It was hoped that they would be induced to remain in school until they had received more academic knowledge, and some industrial training.

In October they set up their shop, constructing wall benches from second-hand manual training material. Work commenced with a meagre equipment of vices and files. Later a second-hand drill press was added, also a new one, a second-hand grindstone, and a few chisels, left by students at the Mechanic Arts High School. In spite of this small outfit only two boys left the class during the year.

The regular program was altered to provide 10 hours a week of shop practice, and three periods a week of mechanical drawing. While one-half of the class was in the class-room, it was arranged that the other half should be in the shop, each under efficient instructors—one a teacher of mature experience and the other a machinist by trade, and previously a manual training instructor.

A comparison of the regular and modified programs is here given:—

	Regular (Minutes).	Modified (Minutes).
Manual Training	120	600 (Iron working)
Drawing	90	135
Arithmetic	236	220
Elementary Science	45	—
Geography	150	60
History	120	60
Music	60	30
Opening Exercises	30	25
Physical Training	80	—
Recesses	160	75
Reading and Literature	165	45
Physiology	30	—
English—spoken and written	280	250
	1,500	1,500

The loss of time for the regular work is largely and perhaps wholly compensated for by the greater amount of individual instruction possible with a class of 20 as compared with that in a class of 44.*

A lady teacher has charge of the boys for all academic work. The teacher of the machine work is a Scandinavian of great enthusiasm both as a teacher and a mechanic, and to this the success of the class is evidently due. The boys are divided into gangs which work under foremen appointed as the best and most experienced of the class. These foremen form a club, and once a fortnight take part with the two teachers in a co-operative lunch in the machine shop. Some outside guest is usually invited. It was the good fortune of the writer to visit the school by chance on a day on which this lunch was held and to have the pleasure of partaking of it. That same day the master had asked the boys to write down what they intended to

* For the outline of work in 1910, see Appendix A to this chapter, pp. 137-8.

do on leaving school. It was real evidence of the success of the work that a large proportion had said they intended to go on to the Mechanic Arts or some other High School—two or three intended to be apprenticed to engineering, only two or three had other occupations in view.

North Bennet Street Prevocational School.

A special course for backward and difficult boys has been maintained by the North Bennet Street Industrial School, Boston, a privately supported craft school and settlement, in co-operation with the Public School Committee. Boys who have reached 13 years of age at least, and the fifth grade of the elementary school, are selected from the neighbouring Hancock Elementary School, and are given their entire work in the North Bennet Street School. Six hours a week are given to woodwork and two to printing. All other lessons are specially based on industrial conditions and needs; for instance, geography is developed from the shop work to a discussion on the sources of the materials used, methods of transportation, &c.; history includes a sketch of industrial history; and present-day labour legislation, wages, hours of labour, workmen's compensation, and trade unions are explained and discussed. The boys who enter must agree to remain for two years.

The only course for backward girls in the city of Boston is maintained by the North Bennet Street Industrial School. Girls from the Hancock School are chosen similarly to the boys. Six hours a week are spent on sewing and three on cooking and housekeeping, actual practice in the latter being obtained by opening a refreshment room where light lunches are served to teachers and other workers in the neighbourhood. In sewing the girls make garments for themselves, articles for school use, and a certain number of simple goods for sale. Efforts are made to raise the girls' interests above their daily experiences by visits to shops and factories, other schools, and the Art Museum. Good pictures are studied and simple design is taught and applied in the fancy work which is done at home by those who wish.

It is probable that these classes, which are now supported by the North Bennet Street Industrial School, will be taken over by the Public School Authorities. Their object is to show the kind of training best adapted to those who have to enter industrial pursuits, a training which aims at giving "manual dexterity, industrial intelligence, adaptability and power to advance, so that the children will be more alert for every opportunity and more capable of acquiring the skill which industry demands."

WILLARD SCHOOL, WEST QUINCY.

At the Willard School, a grade school at West Quincy, Mass., a very similar experiment is being made with 20 boys

and 20 girls who have failed in their school work. Here again the afternoons of the girls are devoted to sewing and domestic work, the mornings by the boys to their trade work. This is taken in a formerly unused attic and the boys as part of their work have partitioned off a room for the girls, made the benches for cooking, and their own carpentry bench, and other useful things. In printing they also undertake a good deal of work for the school. Each class is under one teacher for the whole day.

CHICAGO INDUSTRIAL COURSES.

In Chicago "Industrial Courses" have been introduced in three schools.* The writer visited two of these schools, but unfortunately had the opportunity at only one of them of seeing the children actually at work. At the *Von Humboldt Elementary School* children in the sixth, seventh, and eighth grades, in certain rooms, were given the opportunity of devoting half the school hours to practical work. The girls spend three mornings a week at cooking and household science, and two mornings at sewing. Household science includes practice and instruction in house cleaning and furnishing. At sewing the girls learn to cut and baste white garments for themselves, to machine, and to make simple household draperies. Two rooms opening off the school kitchen and at present not in use are being fitted up by the girls as a model dining-room and bedroom in which actual practice in housework may be obtained. The boys selected devote half of their time to woodwork in the manual workshop of the school and are helping in the furnishing of the rooms.

At the other school visited, the *Jackson School*, which is in a very poor neighbourhood, the girls selected for the special course spend two afternoons a week of 2½ hours at cooking, including instruction in household service. On the afternoon of the writer's visit a class of girls was having a lesson in laying the table, the difficulty with which many of them followed instructions showing that a table laid for dinner was not customary at home. On the other three afternoons the girls take sewing, including knitting, crochet and a little fancy work besides plain sewing, from the learning of simple stitches up to the making of a white dress for the commencement exercises at the time of graduation. Machining is learnt on treadle machines. They provide their own materials and take the garments home when finished. The boys have a greater variety of work here than at the *Von Humboldt School*—woodwork is taken in the sixth grade, working drawings being made of the articles upstairs and then carried out in the workshop. These are of the usual type, photo frames, brackets, thermometer

* For official suggestive programme, see Appendix B. to this chapter, page 138-141.

holders, &c., a little carving being introduced. Later on the boys may undertake larger pieces of work for the school, or for themselves if they provide the material.

In the seventh grade printing is taken, viz., compositors' work with a small hand press for the taking of impressions. In the eighth grade most attention is given to mechanical drawing and to elementary experimental physics. The boys undertake simple experiments themselves such as can be done without any elaborate laboratory equipment. This work is designed to help them understand the why and wherefore of the work of an electrical engineer and hence to give them some grounding for that trade.

In the autumn of 1912 a further experiment with backward children was made by drafting numbers of boys who were nearly 14 from the neighbouring elementary schools to the *Lane* and the *Crane Technical High Schools*, where they were given half-time for practical work in the workshops of the schools, and half-time for finishing their grade school work. At the time of the writer's visit to Chicago these classes had been too short a time in operation to allow of any definite results, but the principals of the schools expressed themselves as satisfied so far.

Similarly a set of girls had been sent to the *Lucy Flower Technical High School for Girls*, and were encouraged to complete their elementary school work while learning sewing and machining from the trade point of view.

An experiment made in some Chicago elementary schools through the gift of one of the wealthiest citizens is worth mentioning here. Money was given for five years to provide special handicraft classes in five schools, and instruction was given in leather work, modelling and weaving, with some surprising results. Specimens of the work were seen at the *Cooper School*. At the end of the five years the Board of Education refused to take up the work owing to its expense and it was therefore necessarily dropped. The looms have remained in this school and are used by the head teacher for instruction to a certain number of boys who wish to do the work. The time given to it comes mainly from the dinner hour or before or after school. The *Cooper School* is in the Bohemian district of Chicago, almost all the children attending being of Bohemian parentage. Hand-weaving strikes one as an obsolete handicraft, particularly in America, but the teacher believes in it as an educational subject. She told the writer that stupid boys take more interest in their school work after a course at weaving: one incorrigible boy was kept at it for three months and became a changed character. He came early and stayed late to work at his loom, and now he has left school he writes to say how much he would like to be back again in order to do weaving. A distinct improvement in taste and an appreciation of colour is noticed among the boys after a time of working a loom. The recollection of the beautiful

objects made in the school, and their contrast with the grey skies and monotonous streets of Chicago on a winter's day suggests the question whether the children in this school may not have gained as much or more from their craft work as those who take the more strictly practical and utilitarian work of the industrial centres.

New York has no special industrial courses in the elementary schools. At *Cleveland* an Elementary Industrial School was opened with 136 pupils, specially selected because the teachers believed they would not continue in school beyond the compulsory school age; 67 of these returned in the second year for 8th grade work, and 55 completed the course. With few exceptions they were then beyond the school leaving age; 20 selected to go on to high school.* *Cincinnati* has introduced courses of an industrial nature for all children in certain of the elementary schools.

An interesting experiment was conducted at the *Trade School of Bridgeport, Connecticut*, during the summer of 1912. In July and August boys from the elementary schools of the city were allowed to attend and take some manual work, thus getting some opportunity of trying trade work. That the offer met a real demand is shown by the fact that 378 boys wished to attend, when only 150 could be accommodated.

FITCHBURG STATE NORMAL SCHOOL.

At the grade school in connection with the Fitchburg State Normal School, Massachusetts, special forms of practical arts work have been introduced into grades 5 and 6. These are:—

1. Assembling toys.—The separate parts ready for immediate use, but painted or unpainted as desired, are bought from the manufacturer. A completed toy, such as a wooden acrobat, is shown the child, who, after a few experiments, soon finds how to put it together. It is said that doing this so that the toy will work is a splendid exercise. The toys are sold before Christmas.

2. Maps are mounted for use in the geography department; cardboard boxes are made from cardboard forms properly scored, and purchased from a box factory. Simple portfolios and envelopes are made.

3. Color-work.—Half-tone prints are colored by the children, a colored original being shown as a copy. Borders of Christmas and Easter cards, mottoes, initial letters, &c., are also hand-colored. Questions of color scheme are made a subject for discussion and study.

4. *Passé-partouting*.—Inexpensive half-tone prints and those colored by the children are framed in this way. Glass is used after the children have developed sufficient skill.

5. Knitting and crocheting of simple articles are undertaken by the girls. The work is well started at home and finished at school.

Much general work is turned over to the children in connection with the school and boarding hall for students, such as cutting leaves of periodicals and new books, attaching tags, examining and dusting library shelves, setting tables, and scouring boarding-hall silver, folding leaflets

* Annual Report of the Cleveland School Board, 1910-11, p. 15.

and tying folders and booklets for the printing department, dusting and caring for schoolrooms, and enameling water cups and paint pans.

Three new courses have been introduced for children in the 7th and 8th grades, a literary, a commercial, and a practical arts course for boys, with a household arts course for girls.* The basis of all manual work is utility for the school community. Thus in the household arts course, cooking, sewing and laundry-work are undertaken for the school and hall of residence for students. The girls learn such purely household work as altering and mending sheets. The boys, as is always the case, have a greater variety of occupations. Under the direction of skilled artisans they have made and varnished picture frames, have made and painted several greenhouse frames, have laid the floor of an attic, put up a partition, made cooking benches, painted the basement rooms, caned chairs, resealed others, heeled shoes, and made several footpaths.

The children's parents determine, in consultation with the school authorities, for which of the three courses the children shall enter.

THE SCHOOLS OF GARY, INDIANA.

Another example of a school in which the needs of the building and the school life of its inhabitants are used to give both boys and girls practice and experience in some real work is to be found at Gary, Indiana. There is no hard and fast line at Gary between the elementary, the high, and the trade school, but an account of the work seems best included here. This school at Gary being remarkable for its enterprise, and freedom from the conventions occurring in many of the school systems of the United States, deserves a full description.

Gary is a rapidly growing town. Seven years ago only prairie and sand-dunes were to be found on the shores of Lake Michigan, where now stands the town. The United States Steel Corporation selected the site as suitable for the erection of works, and immense steel mills now lie between the railway and the shore. Foreign immigrants of at least 15 nationalities flocked to the town for the unskilled work of the mills. The town has grown very rapidly, the population being nil in 1906, 16,000 in 1910, and 26,000 in 1912; and is therefore estimated at at least 30,000 in 1913. By the laws of Indiana, the school revenue must be obtained from an assessment of property made two years before, and hence the grants in such a rapidly developing town were totally inadequate. This led to innovations in school arrangements which have proved very successful. The steel mills offer no educational employment to boys of 14 and 15, and it was therefore necessary to offer such work

* For fundamental principles and courses of study, see Appendix C to this chapter, pp. 141-2.

as would keep them in school over 14 years of age, and encourage them later to look higher than the heavy unskilled work prevailing there.

Instead of putting up cheap buildings, omitting playgrounds, laboratories, workrooms, &c., and employing cheap teachers, the authorities at Gary decided to have thoroughly good school buildings and teaching staff, and then to save by getting the maximum amount of service from them. The school hours were lengthened from 8.30 to 4, and the curriculum was divided into regular and special subjects. The regular subjects are taken in the classrooms under class teachers, and consist of arithmetic, history, geography, reading, writing, spelling, and composition. The special subjects are nature study, drawing, manual work, literature, music, and gymnastics, and are taken under special teachers in the laboratories, workshops, auditorium (the large hall where the whole school can be accommodated if needed), and the gymnasium. Each class gives 3 hours to the regular subjects, $1\frac{1}{2}$ in the morning and $1\frac{1}{2}$ in the afternoon. The time can be divided between the different subjects as the teacher thinks fit, but half of it must be devoted to study, during which time the teacher is supposed to get his own work done. During the remainder of the school hours the teacher can take another class or give instruction in one of the special subjects, in which he may be an adept. Each class has an interval of 1 hour for lunch at varying times, spending $\frac{1}{2}$ an hour in the lunch room and $\frac{1}{2}$ an hour in the playground, and each class also has $\frac{3}{4}$ of an hour off during the day for free play.

By this arrangement one classroom is sufficient for two classes, since each class is in its room for half the school day only, and in other parts of the building for the remainder of the time, every part, including the playground, being thus constantly in use, while by employing specialists in so many subjects, the most efficient teaching is obtained. The children benefit by the large amount of freedom that can be allowed. If a child is backward at some one of the regular subjects he can give double time to that, and for a time omit some special study. If he is delicate he can so long as necessary omit all regular subjects and devote himself entirely to manual work and outdoor activities, for in summer the school garden and gardening play a large part in the "nature study" work. It is also often possible to promote a child in those subjects only in which he shows special ability, thus avoiding the hard and fast grading of the ordinary elementary school.

Gary at the time of the writer's visit had three elementary schools in which the children remain until they have passed the 4th grade, and one other school, the Emerson School, in which, contrary to the custom of the States, all grades from kindergarten to the second year of high school are taught, with the result that the older children care for and feel responsible for

the little ones, and the latter look forward to the time when they too may enter the laboratories, the workshops, or the commercial rooms, where the big ones go. A second similar school, the Froebel School, has since been opened, which includes all ages from the kindergarten to the first two years of college work.

On reaching the 8th grade the boy or girl passes naturally into the 9th and 10th grades, the only change being that in these upper grades he can, if he wishes, give half of his time to vocational (*i.e.*, technical or commercial) work.

Manual work is a feature of the schools. In the first grades folded paper books, paper and cardboard boxes, portfolios, &c., are made, and clay modelling and mat weaving are taken up, while incidentally something is learnt of the methods of construction of textile materials and of industrial history. Girls in the 4th, 6th, 8th and 10th grades take cooking; in the 5th, 7th and 9th grades sewing. Those in the upper grades who wish for more work in cooking are encouraged to give help during school hours in the kitchen attached to the school lunch-room, preparing the food which is sold for lunch at cost price, while others assist in the clearing and washing up.

The boys have more choice of trades than the girls. The school has its forge, foundry, cabinet-making and engineering, sheet metal, painting and printing shops. The latter trade can also be taken by girls. Each shop is in charge of a competent artisan, who is a member of his trade union and can teach, and who in spare time does work on the school buildings. Each boy before he completes the 8th grade can follow one of the above trades during his period of manual study. If he has chosen one he does not like he may turn to another. In practice he is encouraged to try two or three before settling at that for which he is most suited. No money is spent on material for exercises, and nearly all work done is for the school. The following are examples: The painters had recently painted the gymnasium, the cabinet-makers had made special desks for the manual training room of the little children, and domino blocks for teaching counting; the printers were just then printing forms for use by the teachers, and the engineers were about to engage on altering the locks of the school lockers. It might be a year before the whole set were completed, but that did not matter, said the teachers, for meanwhile the boys would be doing work of educational value with a motive for it, and incidentally be saving money for the school. The woodworking and printing shops have shown profits in the last year. In the commercial section the pupils learn to purchase supplies for the school under the supervision of a teacher, they keep the stock, type the school correspondence, and manage the school savings bank. All the rooms have glass doors and are intermixed, for instance, the kindergarten is next the machine shop, so that the younger ones

in passing to their classrooms can see the various occupations followed, and therefore when it comes to their turn to choose a vocation they usually quickly know what they want to do. The children love their work, and therefore no strict discipline is needed, the freedom allowed in the classrooms and corridors being truly remarkable. A farm is maintained in connection with the school a few miles into the country, to which boys desiring agricultural work or those with whom there is any difficulty may be sent for a time. The results of the manual work, as given by the director, are as follows:—

“Our printing and cabinet departments have been better than self-supporting. Our painting department could have been so, but over half the painter's time was given to regular class work, in lettering and sign painting, work which produced no value for the school plant, but was very valuable and with which we are perfectly satisfied.

In the printing department thirty-five boys and girls received training in the trade. Several boys profited greatly through our cabinet shop. The training was such that we are pleased to employ two of them during their summer vacation at good pay.

One boy who had training on our school farm is taking care of the school horse and delivery work for the summer, and is making good. Quite a number of boys have had a valuable year's experience with our painter and are ahead in their apprenticeship, if they decide to follow the work as a trade. And what is just as important, a number have found it is not the calling in which they would be apt to make a success. A large class of boys and girls have been receiving very valuable training in our commercial department, and a number have been well located. Others, by having been kept in school a little longer, have become ambitious for a higher education and will continue their school work.”*

The introduction of trade work into the schools would seem admirable so long as it is clearly understood that no attempt can be made to teach a trade under the age of 14 and in the time that can be devoted to it in an elementary school. The work must be undertaken for its educational value only. Where the method of working through large orders in gangs is introduced, the boys receive an education which will be of lasting benefit to them. When each one undertakes one process in the completion of a number of articles, he learns to take his place as a part of an industrial machine, to be responsible for his bit of work and to understand that the bad work of one worker means the spoiling of the whole article. He learns, too, to work under the orders of a foreman, while those boys who act as foremen learn to use their judgment and discretion. All these are invaluable assets to the boy who is going out to work in factory or workshop, as one among a number of employees. It is obvious that these practical courses enable both the children and those who have opportunities of advising them to determine readily in what direction their aptitudes lie, a most essential thing in the development of vocational guidance work.

* “Can the Administration Department of a school system serve as a laboratory for the Vocational Training of children!” by G. E. Wulfsing, Director, Industrial Education, Gary, Ind. School Board Journal, August, 1912.

Little is offered to girls, as will have been seen, but sewing and domestic work. Where classes are arranged for backward pupils, there must be many who have no aptitude either for books or sewing—paste and novelty work would give these a training in accuracy and neatness, while the girls would benefit as much as the boys by the development of the spirit of co-operation and corporate responsibility in the production of large quantities of work.

The courses for backward pupils have proved their success by the retention of the children in school. The school authorities have spared no expense in affording these children the individual instruction, the practical work and the altered curriculum suited to their needs. The best teachers have been put at their disposal and a freedom from regulations has been allowed which has enabled the teachers to do their utmost for the children. There is one danger that must be avoided. If these children pass from the schools into skilled trades, care should be taken that the dull boy is not thrust upon an industry in which he can never hope to be more than an inefficient worker. The choice of composition work in printing for such a course appears questionable since for success a compositor needs a quick brain and high scholarship.

Vocational education is in the air at present. With their usual enthusiasm, Americans are regarding it as the salvation of the children. In certain quarters one hears the demand that industrial training shall form a regular part of the elementary school curriculum, a course opposed by the American Federation of Labor. However good the introduction of trade work instead of the ordinary Sloyd or manual training is, both as a "try out" and for its inculcation of responsibility and right methods of work, it would be the greatest pity if the education given under 14 years of age were to become too narrow and too vocational.

APPENDICES TO CHAPTER X.

A.—QUINCY SCHOOL, BOSTON—OUTLINE OF WORK.*

MECHANICAL DRAWING.

First Year.

1. Outlines.—Drawing in two dimensions, using rectangles, circles, and combinations.
2. Solids.—Drawing in three dimensions, using cubes, cylinders, pyramids, cones, &c.
3. Work from models involving invisible parts indicated by dotted lines.—Knowledge of full dotted lines, centre lines, and simple dimensions.
4. Developments for sheet metal work; simple problems; use of the section and its development.
5. Working drawings of wrenches, irons (hinged) for step ladders, development of brass (door) card receivers.
6. More individual work adapted to the grades of boys.

* Boston Public Schools: Annual Report of the Superintendent. Boston, 1910.

(Proposed) Second Year.

1. Drawing of first year in quick review.
2. Shop drawings.
3. Elementary mechanics and the necessary drawing, *e.g.*, lay out of room for machinery speeds needed, sizes of pulleys, &c.

SHOP WORK.

First Year.

1. Snagging castings—cleaning, filing.
2. Chipping, filing, bending, counter sinking wrought iron.
3. Laying out work, using chalk or blue vitriol and prick punch, also drilling holes.
4. Laying out sheet brass, bending over forms.
5. Wrought-iron wrenches, laying out, drilling and filing to lines.
6. Care of tool room.
7. Fitting and painting castings.
8. Assembly work, laying out, fitting, drilling, riveting and painting.
9. Visiting shops.

(Proposed) Second Year.

1. Higher grade vice work in filing and fitting keys, hinges and brackets.
2. Lathe work—turning, boring, facing, cutting threads, &c.
3. Planing simple forms and fitting same.
4. Boring and counter boring on upright drill.
5. Forging, bending, forming hot metal, tempering and brazing.
6. Repairing and constructing such simple machinery as might be sent to a school job shop.

B.—CHICAGO PUBLIC SCHOOLS.

AN INDUSTRIAL COURSE OF STUDY.*

For the sixth, seventh, and eighth grades of the Elementary Schools.

The Industrial course of study is not offered for general use in the elementary schools at the present time. Principals who are satisfied that the conditions in their districts are such that at least four divisions in the upper three grades can be organized and maintained in the industrial courses, should confer with the Superintendent of Schools. No divisions should begin the work without special permission from the Superintendent.

Suggestive Program.

- 9.0-9.30—Mathematics.
 9.30-9.40—Music.
 9.40-10.20—English.
 Penmanship.
 Physiology.
 10.20-10.35—Recess.
 10.35-11.05—History, Geography.
 Civics.
 Chicago Course.
 11.05-11.30—Study.

* Course of Study for the Elementary Schools, Chicago, 1912.

<i>Boys.</i>	<i>Girls.</i>
1.0-2.20—Shop.	Textiles.
	Drawing and Art.
2.30-2.35—Recess.	Bookbinding.
2.35-3.30—Drawing.	Recess.
Design and Art.	Cooking.
Printing.	Laundering.
	Printing.

Industrial Course in Household Arts.

(Grades VI., VII., VIII.)

Sixth Grade.

Household Science three times a week (two hours). The kitchen.—Method in kitchen arrangement; in dish washing; the range, its care and control; the ice-box, its structure and care; vegetables, their structure and value; cereals, at least four, with and without fruit; eggs, in at least five ways; milk and eggs in combination; butter-making and comparison of butter with substitutes; milk, at least five ways of using as the principal ingredient; flour mixtures, batter and doughs; meat, structure and methods in cooking; beverages, cocoa, chocolate, coffee, tea. Infant diet. The responsibility of citizens, including children, for the cleanliness and beauty of home yards, of alleys, and streets, the school premises as a center of influence; the inspection of milk, meats, fish, fruits and other supplies. Food storage in plants; experiments with plant foods; animal foods; simple experiments in digestion; amount of water in food stuffs.

Sewing twice a week; hand towel, sewing apron, sewing bag, pillow slip, crocheting (lace), knitting (lace), corset cover, damask hem, hem-stitched collars, cuffs and doiley.

All pieces to be carefully laundered.

Study (and culture, if possible) of flax, hemp and cotton.

Study of wool, washing, carding, spinning and weaving. Comparison of various fibers; testing with chemicals and microscope.

Seventh Grade.

Household Science twice a week.

Household Science.—Preservation of fruit; flour mixtures, general principles controlling them; specific and general rules; study and general understanding of leavens; experimentation with yeast; deep fat frying; meats, experiments for understanding various conditions of solubility and coagulability of muscle juice; methods of cooking reviewed and practised; judgment of fish, methods of cooking; soups, stocks; salads, salad dressings; combinations of fruits and greens; desserts, economical puddings and frozen desserts.

Laundrywork.—Washing and ironing towels, table linen and simple garments; study of various bluinga, making of soap; starching simple articles.

Housewifery.—Sweeping, dusting, cleaning, care of linens, furs, and woollens; disinfection, prevention of vermin; emergency nursing, prompt aid in accidents; sick room arrangement; personal and household hygiene; some understanding of municipal housekeeping.

Experimental study of bacteria, yeasts and molds in relation to man's food and health; preservation of food by drying, heating, sweetening, pickling and freezing; vinegar from cider, sour milk from sweet milk.

Simple experimental study of the effects of heat, expansion of liquids, gases and solids; explanation of common applications; of the freezing and thawing of a vessel full of water, noting effects. Change of state: liquids from solids, melting; gases from liquids, evaporation; liquids from gases,

condensation; solids from liquids, freezing; the study and explanation of common applications; making and using simple distillation apparatus; chemical change as illustrated in charring wood, bone or food.

Binding two books.

Sewing three times a week.—Machine practice, dish towels, gingham apron (salable), Kimono dress or Russian blouse (four-year child, salable), night-dress, petticoat, gymnasium suit (salable), weaving.

Type-setting, once a week for six weeks, and twice a week for remainder of year.

Sewing, beginning with seventh week and continuing remainder of year, three times a week except for the spring millinery.

Thin dresses.

Waists of various kinds and any garment in vogue possible to skill of pupils.

Embroidery.

Millinery, three times a week for six weeks, and three times a week beginning middle of March, eight weeks.

Eighth Grade.

Household Science, once a week for whole year. Work in large quantities and training in waitress work.

Simple experimental studies of liquids; the tendency of liquids to seek their level, illustrated by water in communicating vessels and in water gauges on steam boilers; water pressure due to height of column; application in stand pipes and water-tank systems; pressure transmitted through liquids; application in force pump, pumping stations, circulation of the blood in man and animals; simple hot water circulation apparatus.

Industrial Course in Manual Training.

(Grades VI, VII, VIII.)

Sixth Grade.

Benchwork.—Kite, tip-cat and bat, book-rack, gained joint, castanets, flower-box for school window, trellis and model board.

Drawing.—Sketch of kite, sketch of tip-cat and bat, sketch of castanets, sketch of flower-box for window.

Supplementary.—Bird-house, water-wheel, sun-dial, bread-cutting board, whisk broom holder, salt-box, chair caning.

Clay work—tile, bowl, tray, pot, flower-box, jar, vase.

Venetian ironwork.

Printing.

Bookbinding.

The principle of the lever, determined by means of weights suspended from the beam of the balance; the application of the simple lever in machines of different kinds; in the human body; experiments to discover the mechanical advantages of pulleys, by lifting weights with them; in the application of pulleys to simple machines; in determining the mechanical advantage of the wheel and axle by lifting weights; the making and connecting belted gearing with windmill or water-wheel; studying applications of wheel and axle to belted gearing in simple machines.

Correlation.

Geography.—Nature of trees, habitat, size, growth and transportation.

Cedar, gum, chestnut, ash, cherry.

Technique.—Names of tools and woods used. Water power, nails, glue, hammer, use of chisels, elementary principles of construction. Belt joints.

Art.—Original design for book-rack, bird-house and flower-box and applied ornament. Design for whisk broom holder.

Seventh Grade.

Benchwork.—Scouring board, flower-stand, stationery holder inkstand, broom-rack, footstool, kneading board, window screens and lathe work.

Supplementary.—Teapot-stand, child's furniture of simple design, sleeve-board, mail-box, towel-rack.

Clay and pottery.

Sheet metal work—tin, sheet iron, and copper work, and plumbing work.

Printing.

Simple experimental studies of the atmosphere; the elasticity of air and transmission of pressure through gases; application in pop-guns, air-guns, air brakes, hollow rubber tires, pneumatic cushions.

Drawing.—An instrumental, mechanical dimensioned drawing of each article made, with sketches of details.

Correlation.

Geography.—Latitude, winds, wind pressure, water power.

Habitat of spruce, hemlock, birch, maple, oak, walnut, cyprus, sycamore.

History.—Development of cutting tools.

Art.—Radial designs, interlacing, Greek, Saracenic.

Eighth Grade.

Benchwork.—Nail-box—lap or miter joints. Blacking case—dovetail or mortise and tenon. Magazine rack. Picture frame—review of miter joint. Taboret and furniture.

Supplementary.—Any mechanical problem, such as work-bench and vise, planetarium, aeroplane, pump.

In this grade the elementary principles of construction in concrete may be given.

Drawing.—Mechanical working drawings of all articles made in shop properly dimensioned and lettered.

Supplementary.—Camp-stool, folding chair, screen.

Printing.

Sheet metal work in tin, iron and copper, and plumbing.

Elementary electrical construction.

Photography.

Simple experimental study of the effects of heat; expansion of liquids, gases and solids; explanation of common applications; the making and use of air and liquid thermometers. Change of state: liquids from solids, melting; gases from liquids, evaporation; liquids from gases, condensation; solids from liquids, freezing; the study and explanation of common applications; making and using simple distillation apparatus; chemical change as illustrated in charring wood, bone or food.

Correlation.

Geography.—Habitat of mahogany, ebony.

Science.—Development of aeroplanes. Development of pumps. Development of motors.

Technique.—Joinery. Sharpening tools. Mill processes. Physical and chemical properties of timber.

C.—FITCHBURG STATE NORMAL SCHOOL.

PRACTICAL ARTS (Grades V. and VI.), 1913.

Fundamental Principles.

1. Work must meet a need—a school, home, group, or individual need.
2. The needs of the largest social groups should be met first. The school is the largest social unit, realised by a fifth or sixth grade child, then the home and play group. Meeting these needs teaches the necessity of social service and develops a love for such service.
3. Very fine workmanship ought not to be expected, and therefore should not be insisted upon, but the article when completed must adequately meet the need in response to which it is made. It must "work."

4. Projects should usually be selected from the world's work, from articles of common manufacture, such as toys, boxes, calendars, bookmarkers, pictures, mottoes, quotation cards, doilies, face-cloths, slippers, &c.

Included in the "world's work" is the work of the school and of the home. Select from this work that young children can do, *e.g.*, cutting pages of magazines and books, mounting pictures and maps, folding leaflets, passe-partouting series of pictures to illustrate stories, and countries studied in geography, and similar projects.

5. In general, work must not be too difficult for the pupil. Structure and design, however, will often, if not usually, be beyond the ability of pupils who will perform only the simpler parts of the work, such as assembling toys from parts too difficult in structure for young children to plan and make, coloring designs of a higher character than they can themselves originate, or coloring photographs, which they do not take, develop, or print.

6. Introduce the pupil to the world's work as it is actually done. If "education is world-building," then help the child to build up in his mind by his experience a real world. Therefore as a regular procedure, pupils should not be required to perform, entirely by hand, processes which are now always done by machinery in the real business world. For instance, the cardboard frames of the boxes are not cut out by the children, but are bought from the box-shop already cut and scored. Toy-assembling illustrates the same principle.

7. Nothing should be done simply for the sake of "keeping children busy." A higher motive should govern the teacher. There is plenty of work which appeals to these higher motives.

8. Work, once begun, must be completed before new projects are attempted. Exceptions to this only "prove the rule."

**COURSES OF STUDY FOR PUPILS IN THE SCHOOLS OF OBSERVATION
AND PRACTICE.* GRADES VII. AND VIII.**

Literary Course—30 Hours per Week.

12½ hours to Literature, Composition, Spelling, Penmanship, Mathematics, Geography, History and Science.

7½ hours to Physical Training, Music, General Exercises and Recesses.

5 hours to a Modern Language.

5 hours to Drawing, Designing, Printing, Making and Repairing (Household Arts for Girls).

Commercial Course—30 Hours per Week.

12½ hours to Literature, Composition, Spelling, Penmanship, Mathematics, Geography, History, Science.

7½ hours to Physical Training, Music, General Exercises and Recesses.

10 hours to Typewriting, Shorthand, Bookkeeping, Business Arithmetic and Related Design.

Practical Arts Course for Boys—30 Hours per Week.

12½ hours to Literature, Composition, Spelling, Penmanship, Mathematics, Geography, History and Science.

7½ hours to Physical Training, Music, General Exercises and Recesses.

10 hours to Drawing, Designing, Printing, Making and Repairing.

Household Arts Course for Girls—30 Hours per Week.

12½ hours to Literature, Composition, Spelling, Penmanship, Mathematics, Geography, History and Science.

7½ hours to Physical Training, Music, General Exercises and Recesses.

10 hours to Drawing, Designing and Household Arts.

Grammar School Course—25 Hours per Week.

The same as in other grammar schools in the city.

Two hours of home work per week are required in Grade 7, and 5 hours in Grade 8.

* From the Annual Report of the Fitchburg Normal School, 1911-12.

CHAPTER XI.—EMPLOYMENT CERTIFICATES AND AFTER SUPERVISION BY EDUCATION AUTHORITIES.

SCHOOL OR EMPLOYMENT.

With the exception of Rhode Island, where the age is only thirteen, school attendance is compulsory in the north till fourteen years of age. The entire responsibility of the education authorities does not always cease when the child leaves school to work. The marked deterioration in the morals, manners, intelligence and appearance of unemployed children is realised, and fifteen States have a law that between the ages of fourteen and sixteen a child shall be at school if he is not employed. In two States the age is fourteen to fifteen, in New Jersey it has lately been raised to seventeen.

Unfortunately, adequate means of enforcing this law have not always been created. Thus, in Illinois, there is as yet no power to return children over fourteen to the Parental (or truant) School should they refuse to obey the law. An inquiry made in Chicago for the school census * revealed the following figures: total population between fourteen and sixteen years of age, 80,466.

—	Total.	Percentage.
In School—Public - - - - -	43,415	53·9
" Private - - - - -	13,636	16·9
Not in School—Idle - - - - -	11,750	14·6
" " At work - - - - -	11,665	14·4

Thus 70·8 per cent. were in school, 29 per cent. had left, and half of these were idle. In an investigation of 16,672 children holding work certificates, 2,947 or 17 per cent. were in school, compelled to return by the attendance officers, who found the children neither in school nor at work.

Attention has been directed to the question of securing attendance, and in certain States definite arrangements are being made for the enforcement of this part of the Compulsory Education Law.

In most States a system of work or employment certificates is in force. A child under the required age (usually sixteen) who wishes to work must produce certain evidence, and therewith obtain a certificate, without which no employer may legally employ him. These certificates are usually issued by the local

* School Census Report, 1910. Quoted in "A Report on Vocational Training in Chicago and in other Cities, p. 34." City Club of Chicago, 1912.

school authorities or by a judge; but in New York State they are granted by the Commissioner of Health or local executive officer of the Department of Health, in Wisconsin by the Industrial Commission, and in Connecticut and Rhode Island by the State Board of Education. The certificate must be kept on file by the employer during the employment of the child, and must be produced when required by the factory inspector or other appointed official.

In *New York State* the following information must be supplied by any child between the ages of fourteen and sixteen years who desires to work:—

(a) A birth certificate, or, failing this, a passport or a duly attested transcript of baptism, a certificate of graduation from an elementary school showing that the child must be fourteen years, or, failing other documentary evidence, a certificate of age from two physicians, in addition to an affidavit by the parent.

(b) A certificate from the principal of the school last attended showing that the child is able to read and write simple sentences in the English language, has received during such period instruction in reading, spelling, writing, English grammar and geography, and is familiar with the fundamental operations of arithmetic up to and including fractions, and also that he has attended the school for not less than 130 days during the 12 months next preceding his fourteenth birthday or date of application for such school record.

The child must appear in person before the officer issuing the certificate and undergo a simple examination in reading, writing and English, and show that he has reached the normal development of a child of his age, and is in sound health and physically able to perform the work which he intends to do. In doubtful cases such physical fitness shall be determined by a medical officer of the board or department of health.*

Should the child be in a state of health sufficiently bad to prevent his working, the certificate may be withheld and the child be obliged to continue in school until he is sixteen or has improved in health. This does not appear to be much enforced.

The employment certificate becomes the property of the employer during the employment of the child. He must return it to the child on dismissal who can then use it again in obtaining a fresh place.

In *Connecticut* the work of issuing employment certificates is undertaken by the State Board of Education, at whose office copies of all certificates granted are filed, the applications for certificates being received by six inspectors under the Board who visit the various localities at certain specified times. In addition to the information required by New York State, the child must show a statement on a given form † by an employer certifying that if the child obtains a legal certificate before a

* Summarised from the Labor Law, Chap. 36, Laws of 1909, as given in the Manual of the Board of Education of the City of New York, published by the Board, November 1911.

† For forms in use by the State Board of Education, see Appendix A to this chapter, pp. 149-155.

certain date, he intends to employ him ; this, therefore, prevents the child from leaving school and being without work. The certificate to work is issued in triplicate, one part being kept by the Board of Education, the other by the parent, and the third is posted to the employer, who returns a notice of the date on which the employment of the child commences. The name of the employer is entered on the portions held by him and by the Board, and the certificate is thus made good for that employer only. On dismissing the child the employer sends notice to the Board on a given form. On obtaining fresh work the child must apply to the Board for a fresh certificate, which is then issued in the same way for the new employer ; a new certificate is granted as often as required for each employer until the child is sixteen and beyond the scope of the law. In order that the child may not lose work by waiting for the issue of a new certificate, the portion held by the parent may be used as certificate for one week's employment only ; thus a child may take temporary work for one week or less without a new certificate. Should notice have been received from an employer that a child has been dismissed, and no application have been made for a fresh certificate within a few days, an officer of the Board is instructed to look the child up and enforce the law that he be either at school or at work. A child once at work does not often return to the elementary school, the usual course being that fresh employment is found.

If, when application is first made for a certificate, the child's health is found to be bad or his schooling insufficient, he is refused the certificate, and must continue in day school. Notice is sent to the school which he decides to attend, in order that a report may at once be made should he fail to appear. Should the parents wish for the child's help at home either in domestic or other work, application must be made for a certificate by them as employers, and it is granted if the other conditions have been fulfilled.

Should the employer employ a child without a proper certificate, he is liable to a fine not exceeding \$100, and if he fails to send notice of the commencement or termination of the employment to a fine of not more than \$10.

This law came into force on 1st September 1911. At the time of the writer's visit to Hartford in March 1913, *i.e.*, during $1\frac{1}{2}$ years of working, she was told that 15,000 children had been granted working permits. Since the Board of Education hold records of all their employers and of their dates of beginning and ending work, an immense amount of information is being collected as to child work. The returns, for instance, showed that during this year and a half one child had held 12 jobs, two had held 11, and quite 4,000 had held three places. During this period only about 15 children had been refused certificates for health reasons.

The State of Connecticut has enforced an admirable system for supervising its working children, but at the time of the writer's visit no attempt had been made to build on to this any constructive work. A Bill giving local authorities power to compel attendance at day continuation classes has since been before the Legislature with every likelihood of passing.

At *Cincinnati, Ohio*, the work of certificate granting has been given over to a Child Labor Bureau partly supported by private munificence. Evidence of having passed the fifth grade at school and of having secured employment must be shown, in addition to evidence of age and a record of health after examination by a physician of the Board of Health. The employer intending to give the child work must state the exact occupation to which he is going. Thus, the city of Cincinnati is obtaining detailed information of all the work taken up by children on leaving school, a card index being kept in which for each child particulars of all work held are entered.* This information shows almost invariably that children under sixteen can only get work which gives no training for any skilled employment. Supervision is kept over the children by the rule that the employer must return the employment certificate direct to the work certificate office when a child under sixteen leaves his employment, and must state the reason for dismissal. If the child does not claim a certificate for a new employment within two weeks, notice is sent to a truant officer, who visits with a view to securing attendance at a day school; the result is usually that the child secures fresh work. The children are also directed to send notice to the office themselves when they lose their work.

In order to secure attendance at the compulsory continuation classes, the card which the employer signs when he promises employment, has on its reverse side a list of the continuation schools and an extract from the law. He is asked to specify the building and the hours preferred for the child's attendance. Notice is sent by the work certificate office of the Child Labor Bureau to the Superintendent of Continuation Classes, of every child to whom a work certificate is granted, with particulars of employment and school chosen for attendance.

In addition to the information collected on every child, a more detailed inquiry, with tests of mental and physical development, is being conducted on about 800 children to whom certificates for work have been granted and whose progress is being watched for at least five years. No full results have as yet been published. This inquiry is supported privately and is in charge of a trained psychologist.

In *Wisconsin* also, under the industrial laws of 1911, any child between fourteen and sixteen years of age who is not regularly employed must attend school. A letter from an employer stating his intention to give employment must be

* For cards used at Cincinnati, see Appendix B. to this chapter, pp. 155-158.

shown before a work certificate is granted. Co-operation between the officer granting the certificates and the Board of Industrial Education maintaining the compulsory continuation classes exists as at Cincinnati in order that attendance at the classes may be secured. Provision has also been made for the establishment of special schools to which the children are referred when out of work after having left the elementary schools.

In *Massachusetts* supervision will be kept over young people till the age of twenty-one under the new law of June 1913. The 4th grade at school must have been completed if the child wishes to work between the ages of fourteen and sixteen. If he does not work he must remain in school. An employment ticket signed by an employer, stating the nature of employment and the hours of work, must be shown when obtaining a work certificate. The reverse of this card must bear a statement signed by a school, family, or appointed physician, saying that he has thoroughly examined the child, and that in his opinion he or she is in sufficiently sound health, and physically able to perform the work indicated on the other side.* The employment certificate is granted for this work only, and must be returned to the office of the Superintendent of Schools within two days after the employment has terminated. When the child reaches sixteen, the employment certificate must be exchanged for an "educational certificate," which is issued afresh for each employment, and shows that the young person is exempt from attendance at the evening classes obligatory for illiterates.

Certain advantages in the system of employment certificates are obvious. In the five States† where a promise of employment must be shown before the certificate is granted, and the child permitted to leave school, the custom of leaving on the 14th birthday solely to assert a newly gained freedom is prevented. The child stays on at school as a matter of course until employment is found, and the period of idleness often indulged in through failure to find work, or desire to wait till the right opening turns up, is avoided. Dr. Maxwell, in his report on New York Schools for 1912, says:

"There is no question that the placing of employment certificates in the hands of children who have no trade and who are not employed is a fertile cause of truancy. . . . School records, without which employment certificates cannot be granted, should be withheld in each case until (a) the child has reached at least the 6 B grade in his school work (the present limit is 5 A), and (b) he produces evidence from an employer that, in case the employment certificate is granted, he will have regular employment."

The law compelling return to day school when not at work is excellent, but great difficulties lie in its administration. The teachers naturally dislike the appearance in their classes of

* For this card, see Appendix C. to this chapter, page 159.

† California, Connecticut, Massachusetts, Ohio, and Wisconsin.

children who have been at work and may tend to upset the discipline of the school. A return to school is naturally disliked by most of the children, and the fear of being compelled to go back may lead to their hastily obtaining unsuitable work, when delay and more careful choice might have enabled them to find something better. The law can only be satisfactorily administered when, as in Wisconsin, special classes are formed for these children. At Racine, the only city visited by the writer where these classes were then in existence, two teachers had charge respectively of a few out-of-work boys and girls from 7.45 a.m. to 12 noon. The children are free in the afternoon to look for work. They attend school because they know a new work certificate might otherwise be withheld. These classes are a difficult problem, for the teachers have to deal with a constantly shifting body of pupils, whose standards of education, interests, and former occupations are very varied. All teaching must be left to the initiative of the teacher, who must necessarily be one of the best and most adaptable. Granting, however, that the actual education can be little, it is certainly of the greatest value that the children are under the discipline and care of the school authorities during their idle time and that the responsibility of the State for its children up to 16 years of age is thus definitely shown.

The system of granting employment certificates to all children who desire to work, and by which records of all are filed, is admirable, both as a means of securing information on young people's work and as a basis for securing attendance at compulsory continuation classes. At Cincinnati the system has been used, too, for research work, as has been pointed out, and for a certain amount of voluntary aid. If sufficient help could be secured, it would be easy to use this system for the development of much after-care work, in which at times of sickness or other difficulty special help could be rendered when knowledge of the dismissal from employment is obtained.

APPENDICES TO CHAPTER XI.

A.—FORMS IN USE BY THE STATE BOARD OF EDUCATION,
CONNECTICUT.

[Those for the record of birth are not given here.]

(1) INFORMATION CARD. [Filled and filed at Education Office.]
[Size 6 $\frac{3}{8}$ " x 3 $\frac{7}{8}$ ".]

Employment form 4.

INFORMATION CARD.

Town _____ date _____
 Name of child _____
 Post Office address _____ street no _____
 Place of birth _____ date of birth _____
 Name of father _____ name of mother _____
 Evidence of age _____
 Evidence of education _____
 School _____ grade _____ teacher _____
 Physical condition _____
 Hair _____ eyes _____ complexion _____ height _____
 Certificate _____ series _____ no _____ notice to attend _____ no _____
 Name of employer _____ address _____

(2) TRANSCRIPT FROM REGISTER. [Filled up by Teacher.]
[Size 7" x 4".]

Employment form 8.

TRANSCRIPT FROM REGISTER.

From the register of the _____ school

IT APPEARS THAT

1. attended the _____ school from _____ 19
 to _____ 19 .
 2. has completed the studies required in the said school for the _____
 grade
 3. was born in _____ on the _____ day of _____
 4. the father's name is _____ and resides
 at _____ (street) _____ (no) _____
 Dated at _____ 19

 superintendent, principal or teacher

(3) FORM sent by EMPLOYER to STATE BOARD OF EDUCATION after engaging CHILD before CERTIFICATE is granted. [Size 5½" x 3¼".]

Employment form 16.

Conn 191

To the
Secretary
State board of education

Dear sir

If _____ obtains
a legal certificate before _____ I intend
to employ him (date)
her

Signature of person or company intending to employ

(4) WORK CERTIFICATE in 3 parts (perforated)—one kept by PARENT, one by EMPLOYER and one FILED at OFFICE. An abstract of the law for the Employment of Children is printed on the back.
[Size of each part 7" x 4".]

[First Part]

Employment form 1.

For Parent; not good for employer longer than one week

EMPLOYMENT CERTIFICATE
Chapter 119 public acts of 1911
Office of State Board of Education
Room 42 Capitol Hartford

19

Series No.

THIS CERTIFIES that

(name)

of _____ (town) _____ (state)

1. was born at _____ on the _____ day of _____ and is over fourteen years of age

2. can read with facility, write simple sentences legibly and perform the operations of fundamental rules of arithmetic with relation both to whole numbers and fractions

3. does not appear to be physically unfit for employment

4. the address of father is _____ (if no father, mother or guardian)

Signed _____ in triplicate (agent)

[Second Part]

Employment form 1.

The child named below can be lawfully employed only by the employer named in the certificate

For employer; good only for _____ (employer)

EMPLOYMENT CERTIFICATE

Chapter 119 public acts of 1911

Office of State Board of Education

Room 42 Capitol Hartford _____ 19__

Series No.

THIS CERTIFIES that

_____ (name)
of _____ (town) _____ (state)

1. was born at _____ on the _____ day of _____ and is over fourteen years of age

2. can read with facility, write simple sentences legibly and perform the operations of fundamental rules of arithmetic with relation both to whole numbers and fractions

3. does not appear to be physically unfit for employment

4. the address of father is _____ (if no father, mother or guardian)

Signed _____ (agent)
in triplicate

[Third Part]

Has obtained employment at _____ Employment form 1.
For the State board of education

EMPLOYMENT CERTIFICATE

Chapter 119 public acts of 1911

Office of State Board of Education

Room 42 Capitol Hartford _____ 19__

Series No.

THIS CERTIFIES that

_____ (name)
of _____ (town) _____ (state)

1. was born at _____ on the _____ day of _____ and is over fourteen years of age

2. can read with facility, write simple sentences legibly and perform the operations of the fundamental rules of arithmetic with relation both to whole numbers and fractions

3. does not appear to be physically unfit for employment

4. the address of father is _____ (if no father, mother or guardian)

Signed _____ (agent)
in triplicate

[On reverse side of each part.]

Public Acts of 1911.

CHAPTER 119.

AN ACT CONCERNING THE EMPLOYMENT OF CHILDREN.

SECTION 1. No child under fourteen years of age shall be employed in any mechanical, mercantile, or manufacturing establishment. Every person, whether acting for himself or as agent for another, who shall employ or authorize or permit to be employed any child in violation of the provisions of this section shall be fined not more than one hundred dollars.

SECTION 2. No child under sixteen years of age shall be employed in any mechanical, mercantile, or manufacturing establishment unless the employer of such child shall have first obtained a certificate, signed by the secretary or an agent of the state board of education, or by a school supervisor, school superintendent, supervising principal, or acting school visitor designated by said board, stating the date of the birth of such child, showing that such child is over 14 years of age, and stating that such child is able to read with facility, to legibly write simple sentences, and to perform the operations of the fundamental rules of arithmetic with relation both to whole numbers and to fractions, and does not appear to be physically unfit for employment. Such certificate shall be in the form prescribed and upon a blank furnished by the state board of education, and shall be issued in triplicate; and one copy thereof shall be delivered to the parent or guardian of such child, one copy shall be delivered to the employer, and one copy shall be deposited in the office of the state board of education. Copies of such certificate shall be obtainable from the state board of education, upon application, at any time. The copy of such certificate delivered to the parent or guardian of the child may be accepted by the employer as a temporary certificate, good for one week, after which time it shall be returned to the parent or guardian of such child. Every person, whether acting for himself or as agent for another, who shall employ or shall authorize or permit to be employed any child in violation of the provisions of this section, shall be fined not more than one hundred dollars. The secretary or the agent of the state board of education or the school supervisor, school superintendent, supervising principal, or acting school visitor to whom application shall be made for a certificate as provided in this section, shall have power to require all statements of fact offered in support of such application to be made under oath, and such oath may be administered by said secretary, or such agent, school supervisor, school superintendent, supervising principal, or acting school visitor, and said secretary, or any such agent, school supervisor, school superintendent, supervising principal, or acting school visitor may cause any child to be examined by a reputable physician, for the purpose of aiding him in determining whether such child is physically fit for employment, and may charge the expense of such physical examination against the state as a part of his expenses.

SECTION 3. Every employer receiving a certificate issued under the provisions of this act shall promptly notify the state board of education, in writing, in the form prescribed and upon a blank furnished by said board, of the time of commencement of the employment of any child thereunder and, whenever such employment terminates before such child attains the age of sixteen years, of the time of the termination of such employment. Every person violating any provision of this section shall be fined not more than 10 dollars.

SECTION 4. The provisions of sections two and three of this act shall not apply to employers of children over fourteen years of age in cases in which the employment commenced prior to the date on which this act shall take effect and in which the employer has also complied with the requirements of the statutes in force at the time of the commencement of such employment.

SECTION 5. Every employer or other person having control of any establishment or premises where children under sixteen years of age are employed who shall neglect to have and keep on file the certificate described in section two of this act or to show the same, with a list of the names of such children so employed, to the secretary or an agent of the state board of education, when demanded during the usual business hours, shall be fined not more than 100 dollars.

SECTION 6. The provisions of section 4707 of the general statutes shall be applicable to sections one, two, and three of this act. Sections 4704, 4705, and 4706 of the general statutes, chapter 75 of the Public Acts of 1903, chapter 115 of the Public Acts of 1905, and chapter 123 of the Public Acts of 1909 are hereby repealed.

SECTION 7. The Act shall take effect September 1, 1911.

(5) FORM sent by EMPLOYER after receiving CHILD'S WORK CERTIFICATE to notify the Board of Education when employment commences. [Size 6 1/4" x 4".]

Employment form 2.

Notice of

COMMENCEMENT OF EMPLOYMENT.

Section 3 chapter 119 public acts 1911.

19

(Town)

(Date)

THIS NOTIFIES the State board of education that the employment of _____ whose certificate number is _____ signed by _____ commenced on the _____

Signed _____

(firm name)

(6) FORM sent by EMPLOYER to notify the BOARD OF EDUCATION when he dismisses the child. [Size 7" x 4".]

Employment form 3.

Notice of

TERMINATION OF EMPLOYMENT.

Section 3 chapter 119 public acts 1911.

19

(Town)

(Date)

THIS NOTIFIES the State board of education that the employment of _____ whose certificate number is _____ signed by _____ terminated on the _____

Signed _____

(firm name)

(7) This FORM is used when it is considered that the child has not sufficient education. The outer piece is given to the parent, and the other two to the principal of the school which it is decided the child shall attend. One piece is returned to the State Board of Education for filing. It is one long form, the total length being 14 inches, the three sections being joined to each other by means of perforated lines. The size of the two first portions is 3½" x 4" each, the longer third piece being 7" x 4". For convenience of arrangement the third portion has been printed underneath the other two, instead of alongside.

D No 7
 Date of notice _____
 Town _____
 Name of child _____
 Age _____
 Name of parent or guardian _____

 Address _____
 Place of birth of child _____
 Date of birth " " _____
 Investigated by _____
 Sent to _____ school _____ town
 _____ Agent
 Final action _____

D No 7
 Date of notice _____
 Town _____
 Name of child _____
 Age _____
 Name of parent or guardian _____

 Address _____
 Place of birth of child _____
 Date of birth " " _____
 Investigated by _____
 Sent to _____ school _____ town
 _____ Agent
 Final action _____

RETURN TO STATE BOARD OF EDUCATION HARTFORD

D No. 7.

Attendance blank 11

ATTENDANCE NOTICE.

To _____ parent guardian

Office of State Board of Education
 Room 42 Capitol Hartford

Under the provisions of chapter 36 of the public acts of 1905 the state board of education has ascertained that your child _____ whose age is said to be _____ has not sufficient education to warrant leaving school to be employed

And you are hereby notified to cause said _____ to attend _____ school regularly in the town of _____ until you have obtained a certificate from the state board of education that the education of said _____ is satisfactory to said board

Dated _____ 190 _____ by _____ its agent

[Law on reverse]

[On reverse side of parent's portion of form.]

LAW.

Chapter 36 Public Acts 1905.

Whenever the school visitors, town school committee, or board of education of any town or district shall by vote decide, or whenever the state board of education shall ascertain that a child over fourteen and under sixteen years of age has not schooling sufficient to warrant his leaving school to be employed, and shall so notify the parent or guardian of said child in writing, the parent or guardian of said child shall cause him to attend school regularly during the days and hours that the public school in the district in which said parent or guardian resides is in session, and until the parent or guardian of said child has obtained from said board of school visitors, town school committee, or board of education, or from the state board of education, if the notice shall have been given by the said state board of education, a leaving certificate stating that the education of said child is satisfactory to said visitors, town school committee, or board of education, or to said state board of education, as the case may be; *provided*, that said parent or guardian shall not be required to cause his child to attend school after the child is sixteen years of age. Each week's failure on the part of a person to comply with the provisions of this section shall be a distinct offense, punishable with a fine not exceeding five dollars, and the provisions of section 2117 shall be applicable to all proceedings under this act.

**B.—FORMS IN USE AT THE CINCINNATI WORK
CERTIFICATE OFFICE.**

Cards 1, 2 and 3 are obtained by the child who desires employment from the Principal of his School.

(1) SCHOOL RECORD OF APPLICANT FOR WORK CERTIFICATE.
[Size 6" x 4".]

November 1912. 5 M.

Cert. No. _____ Take this and employer's card, when filled out, to Old
Hughes High School, Fifth and Mound Streets.

SCHOOL RECORD OF APPLICANT FOR WORK CERTIFICATE.

(To be filled out by the Principal of the School last attended.)

Name _____ Address _____
Date of Birth _____ 189 _____ School _____ Grade satisfactorily
completed _____ with the following marks: Reading _____ Spelling _____
Writing _____ English Language _____ Geography _____ Arithmetic _____
Number of weeks actually attended during 191 _____ to 191 _____
Conduct _____ If troublesome, state reason _____
Application _____
Promptness _____
How long has this child had Manual Training? _____ Domestic Science? _____
Date _____ 191 _____

A complete report is required.

Must be filled out and signed in ink.

Signature of Principal.

See that the other side is correctly filled out before going for the Certificate.

[On reverse side.]

This is to be filled out by the pastor of the church at which the child was christened or confirmed or by the Bureau of Vital Statistics at the City Hall. In case neither of these is available, the parent should accompany the child when it comes for a work certificate and make affidavit before the officer in charge in the office at Fifth and Mound Streets.

BIRTH RECORD.

This is to certify that, according to the records of this { Church
Office

[Full name of child.]
was born on the _____ day of _____ 189____
in _____ County of _____
State of _____

(Signed) _____

FILL OUT AND SIGN _____ Pastor of _____ Church.
IN INK. Clerk of Bureau of Vital Statistics.

See that the other side is correctly filled out before going for the Certificate.

(2) HEALTH RECORD OF APPLICANT FOR WORK CERTIFICATE.
[Size 6" x 4".]

HEALTH RECORD OF APPLICANT FOR WORK CERTIFICATE

(To be filled out by the District Physician of the Board of Health)

November 1912. 5 M.

Cert. No. _____

Name _____ School _____ Grade _____

Family History _____

Personal History _____

Weight _____ Heart _____ Vision, R. _____

Height _____ Lungs _____ L. _____

Analysis of Urine _____ Hearing, R. _____

L. _____

Remarks _____

This applicant is approved by Dr. _____

Board of Health

(3) CONTRACT. [Size 6" x 4".]

(Filled in by employer.)

Sep. 1912. 15 M.

Take this when filled out, together with birth and school records to The Work Certificate Office, Old Hughes High School, Fifth and Mound Streets.

CONTRACT.

Cert. No. _____ Cincinnati, Ohio, _____ 191 _____
 The undersigned agree to employ legally _____ in

_____ (Name of Firm) _____ (Kind of Business)
 as _____ and to co-operate with the public school

(Child's Exact Occupation)
 authorities in obtaining the attendance of said child at Continuation School so long as it shall be under sixteen years of age, and in the employ of the subscriber—PROVIDED such child has not passed the Eighth Grade—and to return its Working Certificate to the Work Certificate Office within two days of the child's withdrawal or dismissal from _____ employ, GIVING THE REASON FOR SUCH WITHDRAWAL OR DISMISSAL.

Individual Signature of member, manager or superintendent.

EMPLOYER will please indicate on the back of this card the school and the hours preferred for the child.

I, E. D. Roberts, Acting Superintendent of Schools, Cincinnati School District, hereby certify that I have duly authorized Helen T. Woolley to approve contracts and issue Age and Schooling Certificates.

Signed, E. D. Roberts.

[On reverse side.]

CONTINUATION SCHOOLS.

Employer will please underscore the school and hours preferred for the child.

- | | |
|-------------------------------------------|-------------------------------------------|
| Four to Five p.m. | 8 a.m. to 4 p.m. Any Hour of Day. |
| Guilford, Sycamore, South of Fifth Ave. | Guilford, Sycamore, South of Fifth Ave. |
| 1st Int., Gest and Baymiller. | One to Five p.m. Saturday. |
| 4th Int., Baymiller, North of Dayton. | 4th Int., Baymiller, North of Dayton. |
| 5th Dist., Third, between Elm and Plum. | 25th Dist., Pinetree and Tremont. |
| 6th Dist., Elm and Odeon. | Oyler, Storrs and Burns. |
| 25th Dist., Pinetree and Tremont. | Washington School, Hopple St., Camp Wash. |
| Washington School, Hopple St., Camp Wash. | Winton Place, Winton Road and Hand Ave. |
| Kirby, Bruce and Innes, Northside. | Woodward, Woodward and Sycamore. |

CONTINUATION SCHOOL LAW.

Sec. 7767, G. C. of Ohio: "In case the Board of Education of any school district establishes part time day schools for the instruction of youth over fourteen years of age who are engaged in regular employment, such Board of Education is authorized to require all youth who have not satisfactorily completed the Eighth Grade of the elementary schools to continue their schooling until they are sixteen years of age; provided, however, that such youth, if they have been granted Age and Schooling Certificates and are regularly employed, shall be required to attend school not to exceed 8 hours a week between the hours of 8 a.m. and 5 p.m. during the school term."

Note:—Attendance will be obligatory during the school year of 1911-12 four (4) hours a week; hours to be arranged, as far as possible, to suit the convenience of employers.

(4) RECORD OF AGE AND SCHOOLING CERTIFICATE. [Size 6" x 4".]

[Kept as record of the child in the office.]

RECORD OF AGE AND SCHOOLING CERTIFICATE.

3 M. Mar. 1912.

No. M _____

Cincinnati, O. _____ 191_____

Name _____ Born on _____

In _____

Height _____ Weight _____ Complexion _____

Hair _____ Eyes _____ School _____

Grade _____ Parent or Guardian _____

Residence _____ Occupation _____

Age Proof _____ Schooling Proof _____

Employer _____

Work _____ Wages _____

Left School _____

(5) FILLED IN AND FILED IN OFFICE FOR EACH EMPLOYMENT THE CHILD HOLDS TILL 16 YEARS OF AGE. [Size 6" x 4".]

Mar. 1912. 15 M.

Employer _____

Certificate No. _____ Name _____ Date of Birth _____

Date of taking _____ Date of leaving _____

Cert. sent _____ Cert. returned _____

Child's Occupations _____

Wages. Initial _____ Subsequent _____

Child's reason for leaving _____

Employer's reason _____

C.—BOSTON EMPLOYER'S TICKET [Size 4½" × 3".]

PROMISE OF EMPLOYMENT.					
Required by Acts of 1913, Chap. 779, Sect. 16, for Children 14 to 16 years of age.					
<div style="border: 1px solid black; border-radius: 50%; width: 30px; height: 30px; display: flex; align-items: center; justify-content: center;"> Seal </div>	<table border="1" style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 70%;">Name of child whose employment is promised</td> <td style="width: 30%;">Factory Number</td> </tr> <tr> <td colspan="2" style="text-align: center;">Firm Name of Employer</td> </tr> </table>	Name of child whose employment is promised	Factory Number	Firm Name of Employer	
Name of child whose employment is promised	Factory Number				
Firm Name of Employer					
Address of Employer.					
City or Town	Street and Number				
Nature of Employment to which child named is to be assigned	Number of Hours per Day during which the child named hereon is to be regularly employed				
<p>The undersigned hereby agrees to employ the above-named child in accordance with the facts as above set forth and in compliance with the provisions of Chaps. 779 and 831 of the Acts of 1913 as soon as said child presents to him an Employment Certificate authorizing the employment of said child by him. The undersigned further agrees to return the said Employment Certificate to the office of the Superintendent of Schools from which it is issued within two days after the employment of said child terminates) (Penalty for failure to return Certificate \$10 to \$100.)</p>					
Date	Signed by				
	Employer, Authorized Mgr., Superintendent				

[On reverse side.]

PHYSICIAN'S CERTIFICATE OF HEALTH.

(Required by Chap. 779, Acts of 1913)

The undersigned hereby certifies that he has thoroughly examined the following-named child

and that in his opinion said child is in sufficiently sound health and physically able to perform the work indicated on the reverse side of this Certificate in the Section giving the nature of employment to which the child named is to be assigned.

Date _____ Signed _____

School, Family or appointed Physician

CHAPTER XII.—VOCATIONAL GUIDANCE AND PLACEMENT WORK.

In this section two distinct movements for establishing a connection between the child's school and the work which he afterwards takes up are described. These movements have different supporters, and somewhat of a controversy is carried on between those who believe that the extension of "vocational guidance" work is alone required, and those who urge the establishment of "placement" agencies or employment bureaux which shall help the children who are obliged to go out to work to find as suitable places as is possible under the present conditions.

In England, during the last few years, attention has been largely devoted to a development of the machinery for placing children in work, and for watching their after careers, guiding them where possible to the best positions available for suitable applicants, but in any case hoping to secure, through the placement work, information which will help eventually to a bettering of industrial conditions. For the time, conditions in school, home and factory have been accepted as they are.

In the United States during the same period the question under discussion has been mainly what can be done in the schools to prevent the children taking up unskilled work, and to promote prolonged school attendance. As has been shown, the result has been the introduction of prevocational courses, and the establishment of trade schools. For those who have entered unskilled work, the trend in the direction of compulsory day continuation classes offers hope of improved prospects. The system of free schooling during adolescence, the belief of the American nation as a whole in the value of extended education, and the fact that the majority of parents can afford to keep their children in school over fourteen years of age, have caused any suggestion for the establishment of official agencies to find employment to be looked at rather suspiciously. Much interest is shown in the work of the Juvenile Employment Bureaux, the Juvenile Advisory Committees, and the voluntary agencies in England, but it is thought that the introduction of such work into the United States on a large scale might encourage children to leave school earlier than necessary, and might or would promote their exploitation, since work with permanent prospects cannot be obtained at the age of fourteen.

The little placement work already done has been carried on by philanthropic agencies, and, though elsewhere it has the co-operation of the school authorities, it is only in Boston and Chicago that it has received any official recognition. The vocational guidance movement is regarded with greater favour.

VOCATIONAL GUIDANCE.

The word vocation is in very common use in the United States to-day. One hears constantly of vocational conferences, vocational schools, the necessity of choosing a vocation and so on. The need for the careful consideration of vocations is being realised by all classes of educational institutions.

The movement for vocational guidance originated in *Boston* through the work of the late Mr. Frank Parsons, who pointed out the evil results of thoughtless drifting into occupation, and the "Vocation Bureau" of Boston, founded as the result of his work, has done much to call the attention of the whole United States to the question.

"Vocational Guidance," as defined in the Report of the Commissioner of Labor for 1910 "does not mean selecting a pursuit for a child, nor finding a place for him. It means rather leading him and his parents to consider the matter themselves, to study the child's tastes and possibilities, to decide for what he is best fitted, and to take definite steps towards securing for him the necessary preparation or training."

To this end, the *Vocation Bureau* at *Boston* is investigating and collecting information on the trades and professions open to boys from the elementary and high schools, and is publishing bulletins giving comprehensive descriptions of these various occupations. So far, those dealt with have been suitable for high school rather than for elementary school boys. The Girls' Trade Education League, another voluntary agency, has undertaken similar publications on trades and occupations for girls, and the Women's Municipal League of Boston has published charts showing all the public and private educational institutions in the city which offer opportunities for vocational and continued education. All these publications are available for use in the schools.

A distinct result of the movement has been the formation of a committee on vocational direction under the Boston School Committee. In each elementary and high school in Boston a "vocational counsellor" has been appointed. This counsellor, who is almost always one of the teachers, voluntarily devotes his or her time to advising the pupils on their after careers, noting their tastes and discussing the matter with the parents. Lectures on the various possible occupations open to them have been given to the older children in the schools, sometimes by trade experts, and those who are able to remain in school have been advised as to the high school or trade school suitable for the particular vocation the child may have in view. The teachers acting as vocational counsellors appear remarkably enthusiastic about their work, and it is satisfactory to find a decided increase in the number of children wishing to remain at school after fourteen years of age. The "guidance" has therefore, so far, been mainly in the direction of telling the children of schools or other educational institutions where

they may qualify themselves for the work they intend to follow. The counsellors meet once a fortnight to discuss common difficulties, and to hear an address on some trade or other occupation, in order that they may learn the conditions of work, and be the better able to advise the children in the choice of a career.* A system of cards has been introduced into the elementary schools in which a record of each pupil's progress is entered. These cards are passed on to the high school which the child may enter.†

A vocational assistant has been appointed by the School Committee to work under one of the Assistant Superintendents of Schools, and assist in the development of the work.

The importance of bringing home to teachers the responsibility they have in influencing the children's choice of after careers is being realised. This is shown by the fact that at the Harvard Summer School a distinct course of lectures for teachers on Vocational Guidance was maintained by Harvard University in co-operation with the Vocation Bureau. A similar course of lectures was given at Columbia University, New York City, on Saturday mornings in the Lent term of 1913, and in the summer session, and another course was held at the University of Missouri. The syllabus of the lectures at Columbia University is given in Appendix E. to this chapter, pp. 173-5.

Two developments of this interest in vocational guidance in Boston are interesting. One, that at the instance of the Vocation Bureau, a union of managers or other employees of large firms with whom rests the engagement of the young workers, has recently been formed, with a view to discussing, amongst other things, the qualities necessary for workers in the various occupations, their training and efficiency, and questions connected with the most effective employment of young people. The direction of attention to this should do much to prevent the haphazard choice often only too common among employers, and should also enable those with practical experience to place this at the disposal of those interested in wisely guiding the children in their careers. The other is that students from the Universities of Harvard and Radcliffe have taken part in the investigations into trades for the Vocation Bureau and the Girls' Trade Education League, and the college authorities have recognised this investigation work as a part of a course for a degree. The development of placement work in Boston is no doubt also due to the activities of the Bureau.

At *Cincinnati* a course for teachers on vocational guidance was held in the winter of 1912-13 by one of the teachers of the Woodward High School, the subjects discussed being "the underlying principles of vocational guidance, a study of the conditions of employment, a study of the child in relation to his vocation and the adaptation of the course of study and

* For list of subjects discussed, see Appendix A. to this chapter, page 165.

† For specimen cards, see Appendix B. to this chapter, pp. 166-7.

instruction to the life career motive." A system of cards for each pupil has been introduced into the Woodward High School, on which is entered each year a record of the pupil's ability, character, intentions, &c. This information should form a useful guide to the teachers in advising the boy or girl as to his future.*

At *New York* a "Vocational Guidance Association" has been formed, but little appears to have been done beyond the course of lectures at Columbia University already referred to, and the efforts of individual teachers. As has already been said, the inquiry made by the Vocational Guidance Survey for the Public Education Association has shown that the children leaving *New York* schools at 14 can only obtain work which offers no training and no prospects, and that the majority are not obliged by financial reasons to leave school at the earliest legal age. The Committee, therefore, conclude that a system of finding jobs for children under 16 would be not only futile but dangerously near exploitation. They have begun a thorough investigation into trades with a view to facilitating vocational training.

PLACEMENT WORK.

Some States and some cities maintain free employment offices or exchanges in which employers wanting workpeople and workpeople wanting work, are brought together; children are allowed to register and apparently use these offices a good deal. In *Boston* special officials have been appointed to deal with the boys and girls, and to advise them so far as possible on the best work available, but neither here nor elsewhere, so far as could be gathered, has any system been established for a systematic dealing with juvenile applicants coming for their first places at work, or for following up and giving advice to those whom the employment office has placed. The free employment offices do not, therefore, take any appreciable part in the direction of children from school to work. The children are usually registered along with adults, and separate returns for the young people are not published.

Such other employment work and trade investigation as is done for children leaving school has been carried on by voluntary agencies. At *Boston* a small "Placement Bureau" was started in 1911 by the Women's Municipal League of Boston and the Children's Welfare League of Roxbury. The bureau is granted the use of a room in the Roxbury High School by the School Committee of Boston, but salaries and office expenses are found by subscription. The bureau began work in the Roxbury district only. It maintains a close watch over the children placed. By a special Act of the Massachusetts Legislature in March 1913, the Boston School Committee has been given power to establish and maintain a free employment office for minors; it has not yet taken action.

* See Appendix C. to this chapter, page 169.

In *New York* the Alliance Employment Bureau, affiliated with various Settlements and social agencies, undertakes the placing of young boys and girls, and makes inquiry into their progress; in *Philadelphia* and *Cleveland* some work is done among young girls by the employment bureaux for women and girls maintained by the Consumers' Leagues, and in *Cincinnati* by the Schmidlap Bureau for women and girls. A loan fund in connection with this latter bureau aids girls without training to obtain it before going to work. The Young Men's Christian Association and Young Women's Christian Association maintain employment bureaux for their own members in many cities.

At *Chicago* a small employment bureau for children was started in connection with the School of Civics and Philanthropy, and was developed through financial assistance provided by the Commercial Club and a joint "Vocational Committee" of the Women's Clubs of Chicago. This work, which was begun in 1911, was in January 1913 adopted officially by the Chicago Board of Education, under the title "Bureau of Vocational Supervision." Office accommodation is provided by the Board at its central offices and at a number of schools, but financial support is still found by the Commercial and other clubs. The aims of the bureau, as stated in the official circular of the Department of Vocational Supervision in May 1913, are:—

1. To encourage boys and girls to remain in school and to continue their education after leaving the elementary school.

2. To refrain from suggesting to the child the possibilities of going to work before it is absolutely necessary.

3. In case a child cannot be persuaded to continue in school, to see that the child enters as far as possible the trade or occupation for which he seems best fitted.

4. To suggest to those children who enter unskilled employment to attend evening schools and classes to qualify themselves to undertake other work of a more skilled nature.

5. To keep in touch with children who have been interviewed and advise them after they have been placed, whether again in school or at work.

There is no official system of keeping in touch with all children obtaining work in Illinois as in Connecticut.

The workers of the bureau interview the children before they are given work certificates and try to persuade them to continue in school. Thus in one week in May 1913, of 50 children applying, 26 were returned to school. The workers of the bureau have also been occupied in securing attendance at part-time classes for those children who have to work, and in investigating the after careers of children who have attended the elementary industrial courses in Chicago.*

In comparison with England, little has been done therefore either by official or voluntary agencies in promoting placement work. A great deal has been said and written about the vocational guidance movement, much of which is the work of idealists, for in practice it is only too obvious that in a large city advice without some definite help in finding the right opening

* The Bureau is inquiring into the extension of part-time classes, see p. 165.

is of little value. Moreover, industrial conditions must often prevent the following of the guidance. Teachers, though they should be able to give all necessary information and advice about the opportunities for continuing education and securing training in educational institutions, can never have the time or opportunity to obtain such a close acquaintance with industrial conditions that they can give effective advice in the choice of an actual working position. This requires the expert who can devote his whole time to the matter.

The attention which is being given to the question by teachers is, however, excellent, in that it leads them to note all characteristics, mental, moral, and physical, of their pupils, and to discuss in what way these throw light on the children's probable success in after life.

The movement has also led to a discussion of the changes necessary in the elementary school curriculum to give the children opportunities of revealing their tastes and to lighten the difficulties of those who wish to aid them in their choice of work. The introduction of the extended manual or pre-vocational courses in the elementary schools already mentioned has been one result.

APPENDICES TO CHAPTER XII.

A.—OCCUPATIONS AND SCHOOLS DISCUSSED AT MEETINGS PREVIOUS TO MARCH 1913, BY THE BOSTON VOCATIONAL COUNSELLORS.

I.—OCCUPATIONS.

1. Shoe Industry.
2. Civil Engineering.
3. Mechanical Engineering.
4. Electrical Engineering.
5. Architecture.
6. Building Trades.
7. Metal Trades.
8. Advertising.
9. Department Store Work.
10. Nursing.
11. Dressmaking.
12. Clothing Machine Operating.
13. Millinery.
14. Restaurant and Lunch Room Work.
15. Candy Manufacturing.

II.—SCHOOLS.

1. Conducted by city of Boston:—
 - (a) Industrial Schools:
 - Trade School for Girls.
 - Industrial School for Boys.
 - (b) Continuation School.
 - (c) High Schools of Boston.
2. Conducted by private organizations:—
 - (a) School of Salesmanship at Women's Educational and Industrial Union.
 - (b) Wentworth Institute.
 - (c) Massachusetts General Hospital Training School for Nurses.

B.—CARDS IN USE FOR PUPILS' RECORDS IN BOSTON PUBLIC SCHOOLS.

(1) ELEMENTARY SCHOOL VOCATIONAL RECORD—GRADUATES.
 [Size 5½" × 3½".]

154 (1913) BOSTON PUBLIC SCHOOLS. ELEMENTARY SCHOOL VOCATIONAL RECORD.—GRADUATES.

Last name		First name and initial	School	Year
Residence				
Date of birth		Country of birth		
Parent's name		Occupation		
Parent's plans for pupil				
Pupil likes what subjects				
Pupil dislikes what subjects				
Pupil's plans: a trade, profession, business				
Physical defects		Height	ft.	in.
What school next year?		What work next year?		
Years of probable attendance		Employer's name and address		
After High School?	College Technical	Further plans		
	Normal Trade			
(Check thus:—)	Other special school			

[On reverse side.]

Last name	First name and initial	From what Elementary School?
High School entered		Date
Change in plans at the end of <u>first year</u> in High School		(Record below) June 191
Chief aptitude		
Change in plans at the beginning of <u>third year</u> in High School		Oct. 191
Date of leaving this High School		Destination
Reason		

C.—WOODWARD HIGH SCHOOL, CINCINNATI VOCATIONAL GUIDANCE CARD. [Size 11½" × 9".]

Name _____ Date of Birth _____ Course of Study _____
 Address _____ From what School _____ Date of Entrance _____

YEAR.	PLAN FOR FUTURE. Vocation, College, Employment, &c.	ABILITY. Special aptitude, initiative, skill, capacity, efficiency, &c. Subject in which greatest efficiency is shown. Subject in which least efficiency is shown.	PERSONAL HISTORY. Home, health, travel, employment, use of leisure time. Vocation of parents.	CHARACTER. Honesty, perseverance, promptness, stability, habits, &c.	ENGLISH TEACHER.	HOME ROOM TEACHER.
1						
2						
3						
4						

D.—SCHEDULES USED BY THE DEPARTMENT OF VOCATIONAL EDUCATION AND GUIDANCE OF THE BOARD OF EDUCATION OF THE CITY OF PHILADELPHIA.*

(1) STUDY OF POSITIONS AND ECONOMIC CONDITION OF THOSE WHO HAVE LEFT SCHOOL AT DIFFERENT TIMES.

1. Male _____ Female _____ Age _____
2. Place of birth—
Father _____ Mother _____ Self _____
3. Age at leaving day school, _____
4. Grade at leaving day school _____ School year _____
5. Why did you leave school?
 - a. Because you would rather work?
 - b. Because your family needed your wages?
 - c. Because you felt that you ought to be earning money and getting ahead?
 - d. Because you had trouble in keeping up with your school work?
 - e. Because you were not promoted?
 - f. Because you disliked school?
 - g. Other reasons?
6. Additional training
 - a. Have you attended any other school since leaving day school?
 - b. If so, what? _____ How long? _____
7. Did you have manual training in school?
Wood _____ Metal _____ Bookbinding _____
Sewing _____ Domestic Science _____
8. Your employer.
 - a. Name of firm _____
 - b. Kind of business _____
9. Your present position
 - a. What is your present position?
 - b. Just what kind of work do you do?
 - c. How long have you been in this position?
 - d. How long have you been in this trade?
10. Present wage.
 - a. What is your present wage?
 - b. If piece-work, what is the amount usually earned?
 - c. What promotion have you had in your present position?
11. Is your work steady?
 - a. How many weeks were you out of work during the last 12 months?
 - b. Why were you out?
Sickness _____ Slack work _____
Dissatisfied with position _____ Discharged _____
Other reasons _____

* This department entered upon its work on January 1st, 1914, with a staff consisting of a Director and two clerical assistants. The Director will have general supervision over all vocational training for boys in the elementary schools, will institute a definite line for this work, and correlate it with the manual training now existing in the high schools for boys, and will also organize and direct a Bureau of Vocational Guidance for all pupils leaving the public schools. The Director is subordinate to the Superintendent in the same manner as are the other Directors of special branches.

12. Positions held since leaving day school:—

—	Kind of Work.	Time in Position.	Wage.	
			First.	Last.
1st position -				
2nd " -				
3rd " -				
4th " -				

(2) OCCUPATIONAL OUTLOOK OF THOSE WHO HAVE WORKING PAPERS.

1. Name _____ Age _____ Sex _____
2. Address _____
3. Place of birth
 Father _____ Mother _____ Self _____
4. School last attended _____
5. Grade last attended _____ School year _____
6. Date of leaving _____
7. Reason for leaving
 - a. Necessity
 1. Earnings needed for family support.
 2. Help needed at home.
 3. Self-support necessary.
 - b. Child's help desired though not necessary—
 1. For family support.
 2. In home work.
 3. To earn money for education of self or relatives.
 - c. Child's preference for work—
 1. Work preferred to school.
 2. Spending money wanted.
 3. Association desired with friends who worked.
 - d. Child's dissatisfaction with school.
 1. Tired of school.
 2. Disliked teacher.
 3. Not promoted.
 4. Other difficulty with studies.
 5. Too big for class.
 - e. Other causes.
8. Pupil excels in or likes what subjects.
 " fails in or dislikes what subjects.
9. Physique of pupil—
 - a. Height _____
 - b. Weight _____
 - c. Grip of hand—
 Right _____ Left _____
 - d. Chest—
 Pigeon _____ Droop _____
 - e. Posture.
 - f. Breathing
 Adenoids. _____
 - g. Hearing.
 - h. Sight.

10. Vocation—
 - a. Life work desired by child.
 - b. Reason for choice.
 - c. Have you talked over your life work with parents, teachers, others?
 - d. Did you make the choice?
 - e. Obstacles—
 1. Financial.
 2. Educational.
 - f. Preparation.
11. Job in view—
 - a. Name of firm.
 - b. Address.
 - c. Trade.
 - d. Kind of work.
 - e. Wages promised.
 1. Increase promised.
 2. How to secure increase.
 - f. How job was found.
 - g. Reason for choice.
 - h. Chances for advancement.
12. Family—
 - a. Number in family.
 - b. Number of children at work.
 - c. Mother's occupation.
 - d. Father's "
 - e. Number of rooms.

(3) A STUDY OF INDUSTRIAL OPPORTUNITIES FROM THE STANDPOINT OF THE EMPLOYER.

1. Name of firm
2. Address
3. Trade
 - a. Department
4. Learners
 - a. Minimum age
 - b. Wages
 - c. By whom trained
 - d. Kind of work
 - e. Length of training to learn trade _____
To become skilled _____
 - f. Age preferred
 - g. Do you require applicants to have furnished any given amount of schooling?
5. To what extent are learners selected with the idea of training them to be permanent workers?
 - a. What opportunity does the trade offer for promotion?
 - b. What qualities in the worker make for promotion?
6. Do the present methods of training produce enough operatives and executives (foremen and superintendents)?
 - a. How are workers secured?
By advertisement _____
Through apprenticeship _____
Through employment agents _____
Graduates of schools _____
 - b. What per cent. of the group that apply are selected, and what is the basis for selection?

- c. What special care and training are given to employees between 14 and 16 ?
- d. What changes, if any, should be made in the present system to make young people better serve the needs ?
- e. Do you prefer high or manual training school graduates ?
- f. What kind of work is given graduates of those schools at first ?
- g. Would it be possible for the schools to help your industry—
 1. By change in the present curriculum.
 2. By adding day continuation (part-time) schools.
7. Experienced workers—
 a. How are experienced workers secured ?
 b. Is the method satisfactory ?
 c. Is the supply of skilled workers sufficient ?
8. Wages—
- | | | |
|------------------|-------|--------|
| a. Weekly wage : | Time. | Piece. |
| 1. Maximum | - | |
| 2. Minimum | - | |
| b. Yearly wage : | | |
| 1. Maximum | - | |
| 2. Minimum | - | |
9. Seasonable work :
- | | | |
|-----------------------|---------|-----------------------|
| | Months. | Number of Operatives. |
| a. Maximum employment | - | |
| b. Minimum employment | - | |

(4) INFORMATION ON EDUCATIONAL INSTITUTIONS IN PHILADELPHIA.

Courses in Elementary and Advanced Industrial Training.

1. Institution
 a. Name _____
 b. Location _____ Telephone number _____
 c. Age of school _____
 d. Incorporated _____ When _____ Where _____
 e. Purpose _____
 f. Denominational and settlement affiliations _____
2. Officers _____
3. Equipment—
 a. Buildings _____
 b. Machinery _____
 c. Tools _____
 d. Material (wood, cloth, &c.) _____
 e. Power (electric) _____
 f. Books, charts, &c. _____
4. Finances
 a. Endowment _____
 b. Annual expense _____
 c. Is the product commercialized ? _____
5. Tuition
 a. By year _____ Term _____ Subject _____
 b. Charge for materials _____
6. Requirements for admission
 a. Age _____ Sex _____ Previous training _____
 b. Are grammar school graduates accepted ? _____
 c. Other requirements
 Coloured _____ Foreign _____

7. Session
Morning _____ Afternoon _____
Evening _____ Combination of any two _____
8. Curriculum—
 - a. Time required for graduation
 - b. Season of giving course
 - c. When may pupils enter?
 - d. Probationary period
 - e. Individual subjects
 1. Length of course
 2. Hours per week
 3. Outside preparation
9. Number of students _____ Graduates per year _____
10. Number of teachers
 - a. On full time _____ On part time _____
 - b. Paid _____ Volunteer _____
11. Methods of advertisement for students
Catalogues _____ Circulars _____
Distribution of literature _____ Field agents _____
12. Credit by examinations
Written _____ Oral _____ Industrial _____
13. Diploma—
 - a. Is subject certificate or diploma granted?
 - b. Is diploma accepted by higher institutions?
14. Employment service—
 - a. Positions guaranteed?
 - b. Follow-up methods?
15. Is there state inspection?
16. Reference to business houses.

**E.—SPECIAL COURSE IN VOCATIONAL GUIDANCE
OFFERED BY THE DEPARTMENT OF INDUSTRIAL
EDUCATION, TEACHERS COLLEGE, COLUMBIA UNIVERSITY.**

SUMMER SESSION, JULY 7—AUGUST 15, 1913.

VOCATIONAL GUIDANCE. — Lectures, readings, field work, reports, and discussions. Professor Bonser and special lecturers. Daily at 3.30, Room 300, Teachers College.

This course will treat of the following topics: problems and methods in the study of vocations, the making of educational surveys, and the study of vocational aptitudes and tendencies; vocational analysis: sources, kinds, values and uses of vocational information; functions of general school work in vocational guidance; the work of vocational bureaus and of vocational advisers and counsellors; placement and follow-up; opportunities and methods of vocational education and training in New York City; history, methods, and status of vocational guidance work in the various centers of its development in the United States. For a detailed outline of the course, see below.

Professor Bonser will have charge of the course and will have the cooperation of special lecturers.

OUTLINE OF THE COURSE.

A.—Introductory.

- I. Underlying causes of the movement for vocational guidance.
- II. Fundamental problems.
- III. Methods of procedure or types of work in the course :—
 - (1) Lectures stating the problems, principles, methods and results of investigations accomplished or in progress.
 - (2) Practical training in investigation or survey work under the direction of expert investigators in various phases of the larger problem.
 - (3) Reports of investigations and discussions of lectures, methods, problems and results.

B.—Vocational educational survey.

- I. Studies of the vocations with reference to the qualities and training demanded :—
 - (1) Sex of the employees.
 - (2) Qualities needed for success—physical, mental.
 - (3) Age of possible entrance to various vocations.
 - (4) Minimum training for successful entrance.
 - (5) Opportunities for training within the vocation.
 - (6) Training needed not provided within the vocation.
 - (7) Weaknesses in education apparent in beginners—
 - (a) In general education.
 - (b) In special training.
 - (c) In attitude, responsiveness, adaptability, &c.
- II. Studies of children and young people with reference to the discovery of vocational aptitudes and fitness, vocational analysis. Possibilities of a vocational index :—
 - (1) Through the regular school subjects.
 - (2) By specialized tests.

C.—Meeting the situation revealed by the survey.

- I. Functions and possibilities of the school with reference to vocational direction and training :—
 - (1) Prevocational activities :—
 - (a) Vocational analysis.
 - (b) Vocational information—(1) the data included under B1 above; (2) other factors such as demands for workers, wages, length of day, length of season, opportunities for advancement, risks of life, limb and health, social conditions and opportunities of workers, &c.
 - (c) Differentiated work bearing upon a group of prospective vocations, familiarising pupils with related subject matter, underlying principles and appreciable technical and practical elements.
 - (d) Specific vocational training for students sufficiently mature. Opportunities for vocational training in New York City as a type study.
 - (2) Continuation work after employment is begun :—
 - (a) Supplementary education and training.
 - (b) Follow-up studies.

II. Agencies auxiliary to the school in adjusting vocational aptitudes and fitness with vocational opportunities.

- (1) Vocation bureaus. Placement.
- (2) Vocational advisers and counsellors.

III. The economic factor in securing adequate prevocational education for prospective workers and further education for youthful employees. Provision for the removal of economic necessity of premature withdrawal from school to enter vocations. Budget studies, scholarships, pensions, part-time school plans.

D.—The unification of possibilities and methods for a constructive attack in solving the problems in vocational guidance. Conclusions and Summary.

PRACTICAL FIELD WORK.

Field work providing practical training in the investigation of special problems in vocational guidance is given under the direction of expert investigators engaged in the work in New York City. Students may elect a line of investigation of special interest and arrange to devote as many hours to this field work as time will permit. The work of one of the best known settlement houses in New York and of an employment bureau which places each year over twenty-five hundred young people between the ages of fourteen and twenty is also open to students for study.

The work of the Manhattan Trade School for Girls and the New York City Vocational School for Boys may be observed as both these schools are in full operation during the Summer Session.

The Vocational Education Survey of the Public Education Association of New York City has carried on for over a year very important investigations relative to vocational guidance. The results of this study are at the disposal of the students.

Much of the material shown at the exhibit of the Second National Conference on Vocational Guidance held in New York in October, 1912, is still in New York and is available for use.

AVAILABLE SUPPLEMENTARY COURSES.

Many courses are offered in the Summer Session of Teachers College which have a strong supplementary bearing upon vocational guidance. Among these are the following:—

Industrial Arts.

Education. — Theory and practice of teaching industrial arts in secondary and trade schools. Discussions, required reading, investigations, observation, reports and practical work. Half course.

Distinction between elementary industrial arts and vocational training; function of technical courses in high schools and of trade schools; the history and development of industrial and technical instruction in technical courses and trade schools.

Organization of courses of study, and special methods in the teaching of each of the several practical and technical subjects.

Education.—Practicum in industrial education. Full course.

Industries. — Typical modern industries. Lectures, demonstrations, visits and reports.

CHAPTER XIII.—SUMMARY.

What position do the States, having the laws and arrangements described in the preceding pages, hold in the whole country with its 49 States and Territories and its population of 91 millions?

The following table shows the order of the leading States, and the action they have taken on behalf of their young workers:—

State.	Rank.*			Laws.
	By Popula- tion.	By Value of Pro- ducts.†	By No. of Wage- Earners.	
New York - - -	1	1	1	System of State grants for developing trade schools. Compulsory evening elementary classes for boys. Has just established compulsory day classes by local option. Hours of work under 16 are 48 a week in factories, and 54 a week in other occupations.
Pennsylvania - - -	2	2	2	Bill to promote industrial education passed in April 1913.
Illinois - - - -	3	3	4	No State action yet. Chicago, the second largest city in the States, has introduced schemes of education with a vocational bias in some schools. Hours of work under 16 are 48 a week.
Ohio - - - - -	4	5	5	System of compulsory continuation classes by local option, and careful plan of granting certificates for work, and supervision up to 16 years of age. Hours of work under 16 are 48 a week.
Texas - - - - -	5	17	25	A Southern State.
Massachusetts - - -	6	4	3	Well developed system of trade education. Has established compulsory part-time classes by local option, and system for supervising all workers under 21. Hours of work under 16 are 48 a week.
Missouri - - - -	7	10	11	No action yet. Hours of work under 16 are 48 a week.
Michigan - - - -	8	7	7	No State action yet.

* According to the 1910 Census.

† For the first seven States, the rank is the same for value of products added by manufacture. These States together produce three-fifths of the total manufactured products of the United States.

State.	Rank.*			Laws.
	By Population.	By Value of Products.	By No. of Wage-earners.	
Indiana - -	9	9	9	Has had commission on industrial and agricultural education and passed Act to institute a State system. Hours of work under 16 are 48 a week.
Georgia - -	10	24	17	Southern State.
New Jersey - -	11	6	6	Has legislation on industrial education.
California - -	12	11	13	Employment prohibited under 15 with exceptions. Is considering fresh legislation.
Wisconsin - -	13	8	10	Has comprehensive laws for compulsory part-time classes, for granting employment certificates and for supervision over workers under 16. Two trade schools in only big city. Hours of work under 16 are 48 a week.
Connecticut - -	31	12	8	Two State trade schools. Elaborate system of supervision of workers under 16 and granting of employment certificates. Likely to have day continuation classes by local option.

* See note on previous page.

As will have been seen from this table, New York State ranks first among all the States by population, value of manufactured and other products, and greatest number of wage-earners. It has encouraged the development of trade schools by a system of State grants, and, having already got compulsory evening continuation classes for boys in elementary school work, has just established day classes by local option. Pennsylvania, which ranks second in all points and contains Philadelphia, the third largest city in the Union, passed an Act to promote industrial education in April 1913. Illinois, the third in population and value of products, but fourth in number of wage-earners, lags behind; but in Chicago, which contains 39 per cent. of the population of the State and is the second largest city in the United States, there is much discussion of the matter, and schemes of education with a vocational bias have been, as has been shown, introduced in certain of the schools. Ohio, the fourth State in population and fifth in other points, already has a system of compulsory continuation classes by local option together with a careful plan of granting certificates for work. Texas, a Southern State, is

fifth in population. Massachusetts, sixth State in population, third in number of wage-earners and fourth in value of products, has the most developed system for trade education, and has recently established compulsory part-time classes by local option. Missouri, seventh by population, but tenth in products, has not yet acted. In Michigan, the next, no State action has yet been taken. Indiana, the ninth in all points, has had its commission on industrial and agricultural education, and has instituted a State system of industrial and agricultural education. Georgia in the south is tenth in population. New Jersey is eleventh, but ranks sixth in number of wage-earners and value of products. It has already passed legislation on industrial education. California is twelfth; here the age for employment is 15, with exceptions. Legislation is being considered. Wisconsin, with its two city trade schools and its comprehensive laws for compulsory part-time classes and supervision over working children, &c., is thirteenth in population, but eighth in the value of its products. Connecticut is small in population and area, but it is eighth in its number of wage-earners and twelfth in the value of its products. It has two State trade schools and an elaborate State system of supervision up to 16 years of age. It is likely to have compulsory day continuation classes by local option. In seven of these States, the hours of work of boys and girls under 16 are limited to 8 a day or 48 a week.

These fourteen States hold more than half the population of the United States, and include all the most important manufacturing districts. In almost all, therefore, some steps are being taken to promote the well-being of the children who leave school to work. The fact that all was not well with these children came as a revelation after the findings of the Massachusetts Commission, and the last few years have shown a rapid development in the recognition of public responsibility for children from 14 to 16 years of age. Though much has been done, a very great deal yet remains to be accomplished. Many of the schools described in the preceding pages are not yet part of a system, but the experiments of localities, a fact so far good in that their very success depends upon their freedom from conventions and adaptation to local conditions. These experiments, and the new laws in any States, are watched and discussed, and much fresh legislation will undoubtedly be forthcoming.

Certain points in the attitude which Americans hold to the subject attract attention particularly. One is the readiness to spend money and put the best the schools can offer at the service of the backward children; another, the recognition of the value of study during the day and not at night for their young people; a third, and most important of all, the constant insistence, in every scheme for helping the children, on the value of a completed elementary education and of prolonged schooling, be it only for a few months or a year, or in classes for a few hours of the week.

APPENDICES.

A.—RESOLUTIONS ADOPTED BY THE NATIONAL ASSOCIATION
OF MANUFACTURERS OF THE UNITED STATES OF
AMERICA AT ITS SEVENTEENTH ANNUAL CONVENTION,
NEW YORK CITY, MAY 21, 1912.*

Whereas, one half of the children in the common schools of the United States leave school by the end of the sixth grade, with no substantial educational acquirements beyond reading, writing, and arithmetic in their simpler forms, the essentials of education and citizenship coming, if at all, after the sixth grade; and

Whereas, this half of the children soon forget much of what they learned in their brief school experience; and

Whereas, truancy and absence are so prevalent that less than three-fourths of the children are in school as much as three-fourths of the time, the enrollment being 17,000,000, and the average attendance being under 12,000,000, 1,600,000 being permanently absent from and unacquainted with school life; and

Whereas, illiteracy in the United States is 50 times that of several continental countries, and is four times greater among the children of native whites than among the native-born children of immigrants†; and

Whereas, in many schools and many cities educators are finding great cultural and educational value in the development of the motor activities, the practical and creative desires of the youth, in highly-developed practical and extended courses, in manual and prevocational training, and such courses are developing, in an unexpected degree, an appreciation of the dignity of labor of all kinds, and such moral qualities as diligence, concentration, perseverance, and respect, and causing many to successfully continue in school who otherwise would leave discouraged early in the course; and

Whereas, a majority of the children who leave school prematurely do so from no economic need, and in fact are idle about half the time between their 14th and 16th years, being the first two years out of school, and that they average for the first two years little over \$2.00 per week in earnings, leaving school principally because their interest in practical and creative effort is not provided for; and

Whereas, the loss to the schools of 50 per cent. of the children in the middle of the elementary school courses is an incalculable waste of the human resources of the Nation, these human resources being estimated by Professor Fisher as of the economic value of \$250,000,000,000, and five times the value of all our other natural resources combined.

Therefore, for these and other reasons, the National Association of Manufacturers by resolution pledges its earnest support to the following principles of educational betterment as essential to society and to the spiritual, social and physical welfare of the youth:

1. Continuation schools for that half of the children who leaving school at fourteen years of age, and mostly in the fifth and sixth grades, these continuation schools to be liberally cultural and at the same time to be extremely practical and related as directly as possible to the occupations in which the several students are engaged.

* Proceedings of the Seventeenth Annual Convention of the National Association of Manufacturers of the United States of America, held at New York, N.Y., May 20, 21 and 22, 1912, pp. 290 ff. (New York City, Secretary's Office, 30, Church Street.)

† Percentage of illiterates among the Population of 10 years and upwards, from the U.S. Census for 1910:—

Whole White population, 5 per cent.; Native-born, 3 per cent. (including (i) of Native parentage, 3.7 per cent.; (ii) of foreign or mixed parentage, 1.1 per cent.); Foreign-born, 12.7 per cent.; Negro, 30.4 per cent.

2. The development of a modern apprenticeship system wherein by contact the respective and equal rights of employer and employee are fully recognized, the entire trade is taught, and such other subjects as are essential to good citizenship.

3. The development of secondary continuation or trade schools, by which from the more efficient of the great army of boys and girls who will enter the continuation schools many may progress from these lower continuation schools, as in some other countries, to the foremost places in industry and commerce.

4. Compulsory education through adolescence, being until the 17th or 18th year, attendance being in all-day school until the 14th year, and there after in either the all-day schools or in the continuation schools for not less than one-half day per week, without loss of wage.

5. The strengthening of all truancy laws and the development of public sentiment in support thereof.

6. The training of teachers in thorough-going methods of industrial practice, including as part of such training extended experience in actual shop work.

7. The establishment of independent State and local boards of industrial education consisting of one-third each, professional educators, employers and employees, thereby insuring, as in the more successful European countries, the proper correlation of the schools and the industries.

8. The development of the vocational and creative desires of the concrete or hand-minded children now in the grades, discouraged, anxious to quit, and often called backward, only because the education now tendered them is abstract and misfit.

9. The establishment of shop schools and part-time schools whenever practicable.

10. The establishment of departments or centers of vocational guidance so that the great majority of the children who now enter industry at fourteen with no direction, 85 per cent falling into the "blind alley" occupations, may reverse the figures as abroad, and enter, under advice, intelligently and properly upon progressive occupations.

Resolved by the National Association of Manufacturers, that it is the imperative need of the industrial workers and employers of the country that thorough-going systems of industrial education be everywhere established, so that our factories may be more constantly and better employed; that standards of skill and of output may increasingly be improved, and that foreign and domestic markets may be better held and extended.

B.—PERCENTAGE OF WAGE-EARNERS UNDER 16 OF THE TOTAL NUMBER OF WAGE-EARNERS IN CERTAIN STATES.*

New England :—

- Maine, 1·7.
- New Hampshire, 1·5.
- Vermont, 0·6.
- Massachusetts, 3·5.
- Rhode Island, 4·1.
- Connecticut, 2·6.

Mid-Atlantic States :—

- New York, 0·8.
- New Jersey, 2·3.
- Pennsylvania, 3·3.

* From the Abstract of the United States Census for 1910. Boys and girls are not differentiated.

East North Central :—

Ohio, 1·2.

Indiana, 1·9.

Illinois, 1·5.

Michigan, 1·1.

Wisconsin, 2·3.

For the whole United States, 2·4.

C.—SUMMARY OF LABOUR LEGISLATION AFFECTING CHILDREN AND YOUNG PERSONS IN CERTAIN STATES.*

1. California.

(1) Prohibited Employments.

(a) to 15 years in any manufacturing or mercantile establishment, office, hotel, apartment, house, or messenger service, laundry, restaurant, place of amusement, distribution of merchandise.

(b) to 16 years in occupations injurious to health or morals, or dangerous to life and limb, and as mendicants or acrobats.

(c) to 18 years for messengers to any immoral place—saloon, gambling house, house of prostitution.

(2) Hours of Work.

(a) For all females (except in agriculture and canning) and all workers in mines, smelters, etc., 8 hours a day or 48 hours a week.

(b) To 18 years in any employment, 9 hours a day, or 54 hours a week.

2. Illinois.

(1) Prohibited Employments.

(a) under 14 in factories, stores, offices, laundries, hotels, theatres, bowling alleys, messenger service, acrobatic and mendicant occupations, and in any occupation during school hours.

(b) under 16 in certain dangerous employments, and where health may be injured or morals depraved.

(2) Hours of Work.

All under 16 must not work more than 8 hours a day, nor more than 48 hours a week, nor between 7 p.m. and 7 a.m. in any gainful occupation.

3. Massachusetts.

(1) Prohibited Employments.

(a) under 14, in or about or in connection with any factory, workshop, manufacturing, mechanical, or mercantile establishment, barber shop, boot-black stand or establishment, public stable, garage, brick or lumber yard, telephone exchange, telegraph or messenger office, or in the construction or repair of buildings, or in any contract or wage-earning industry carried on in tenement or other houses. No minor under 14 shall be employed in any occupation for wage or other compensation to whomsoever payable during the hours the public schools are in session, nor before 6.30 a.m. nor after 6 p.m.

(b) under 16, on certain specified machines, in oiling and cleaning hazardous machinery, in proximity to unguarded machinery or belts, on scaffolding, in heavy work in the building trades, in tobacco works, in any tunnel or public bowling alley, or in a pool or billiard room, etc.

* It is impossible to give complete lists of prohibited occupations. A summary of the labour laws for all States will be found in *The Child Labor Bulletin* for August, 1913, published by the National Child Labor Committee, 105, E. 22nd Street, New York City. Full details of the recent laws for Massachusetts are given in *Bulletin of the Board of Education*, 1913, No. 8, whole number 27, "Educational Legislation of 1913," State Printers, Boston, 1913. and for Wisconsin in *Bulletin of the Wisconsin State of Education*, No. 1, Laws of Wisconsin published by the Board, Madison, 1912.

(c) under 18, in or about breweries or distilleries, blast furnaces, dynamite works, oiling and cleaning moving machinery, in places of amusement where alcoholic liquors are sold, in operating motor vehicles, and in certain work connected with railroads, etc.

(d) under 21, in connection with any saloon or bar room where alcoholic liquors are sold, or in connection with any immoral place, resort or amusement.

The State Board of Labor and Industries may prohibit the employment of minors either under 16 or 18 years of age in any occupation sufficiently dangerous or injurious to health or morals as to justify their exclusion from it.

2. *Hours of Work.*—In or about or in connection with any of the establishments and occupations given in paragraph (1. (a)).

(a) For minors under 16: not more than 48 a week, nor more than 8 a day, nor on more than 6 days in the week, nor before 6.30 a.m. nor after 6 p.m.*

(b) For minors under 18, not more than 54 in a week, nor more than 10 a day, nor more than 6 days a week, nor before 5 a.m., nor after 10 p.m., nor in textile works after 6 p.m.

No person under 21 shall be employed in the delivery of messages or goods for any telegraph, telephone or messenger company, except in connection with the publication of a newspaper, before 5 a.m. or after 10 p.m.

(4) *Street Trading.*

(a) Girls under 18 and boys under 12 may not sell or exercise any trade in a street or public place.

(b) Boys between 12 and 16 must obtain a badge before engaging in any form of street trading, and must not work before 5 a.m. nor after 9 p.m., nor during hours when the schools are open unless provided with an employment certificate.

4. *New York State.*

(1) *Prohibited Employments.*

(a) under 14, in factories, mercantile establishments, laundries, hotels, restaurants, places of amusement, barbers' shops, etc., and in all employments during school hours.

(b) under 16, in mines, quarries, oiling and cleaning machinery, and operating specified machines, in use of acids, paints, red or white lead, dry colors, making and packing of matches or explosives; or malt or alcoholic liquors; in acrobatic or mendicant occupations, and for a female, any occupation requiring constant standing.

(2) *Hours of Work.*

(a) under 16. In all factories: not more than 48 a week, nor more than 6 days in the week, nor more than 8 hours a day, nor before 8 a.m., nor after 5 p.m. In any mercantile establishment, office, restaurant, hotel, apartment house, theatre or other place of amusement, bowling alley, barber shop, shoe-shining establishment or messenger service: not more than 54 hours a week, nor more than 9 hours a day, nor more than 6 days in the week, nor before 8 a.m. nor after 7 p.m.

(b) 16 to 18 in factories (and all women). Not more than 60 hours in the week, nor more than 10 a day.

(c) Girls under 21 may not work in factories before 6 a.m. nor after 9 p.m. Nor in stores before 7 a.m. nor after 10 p.m.

(d) Minors under 21 may not work in night messenger service after 10 p.m. nor before 5 a.m.

* Until June 1913, the hours of work for all under 18 were 54 a week in manufacturing and mechanical establishments and 58 a week in mercantile establishments and workshops connected with stores.

(3) *Street Trading.*

Sale of newspapers and magazines is forbidden for girls under 16 and boys under 10. Boys under 14 must obtain a badge. No sale may take place after 10 p.m. nor before 6 a.m.

5. *Ohio.*(1) *Prohibited Employments.*

(a) under 14, commonly specified gainful, and acrobatic and mendicant occupations.

(b) under 15, work in mines during school terms.

(c) under 16, dangerous occupations.

(d) under 18, messenger work between 9 p.m. and 6 a.m.

(2) *Hours of Work.*

For boys under 16 and girls under 18 in commonly specified gainful occupations, not more than 8 hours a day, nor 48 hours a week, and not before 7 a.m. nor after 6 p.m. Thirty minutes meal time is not included.

6. *Wisconsin.*(1) *Prohibited Employments.*

(a) under 14 in any gainful occupation or employment, except that during the school vacations any child between 12 and 14 years of age may be employed in any store, offices, mercantile establishment, warehouse, telegraph, telephone or messenger service, if a permit has been obtained in the same manner as an employment certificate.

(b) under 16 in many specified occupations, very similar to those prohibited in Massachusetts, or in any employment dangerous to life or limb, injurious to the health, or depraving to the morals of the child. No female under 16 shall be employed in any occupation requiring her to stand constantly. No child under 16 shall take part in any public performance in theatre, circus, etc, without a special permit.

(c) under 18 in mines, quarries, and many occupations as in Massachusetts.

(2) *Hours of Work.*

(a) under 16, not more than 8 hours a day, nor more than 48 hours a week, nor more than 6 days a week, nor before 7 a.m. nor after 6 p.m. at any gainful occupation. A lunch period of 30 minutes shall be allowed during each day.

(b) for all females, not more than 10 hours a day, nor more than 55 hours a week.

The hours of work for boys and girls under 16 years of age are limited to 8 a day or 48 a week in nine other states, viz., Arizona, Colorado, Indiana, Kansas, Minnesota, Mississippi, Missouri, Nebraska, Oklahoma, and in the District of Columbia. In Washington State, the hours are also limited to 8 a day, but there is no limit per week.

D.—LEGISLATION ON INDUSTRIAL EDUCATION PASSED IN 1913.

1. *Indiana.*—State aid is offered to local authorities which establish schools in which the instruction is of less than college grade and designed to meet the vocational needs of children of fourteen years of age who seek preparation for their work. Such schools may be day, part-time or evening. Attendance at day and part-time classes is restricted to persons over fourteen and under twenty-five years of age, and at evening classes to persons over seventeen years of age. These schools are to be under the administration of the State Board of Education, but separate from the regular school system. One representative of employers and one of employees must be upon the Board. Local authorities are to be reimbursed

to two-thirds the salary of each teacher giving instruction in vocational or technical subjects. Local advisory committees are to be appointed in connection with the schools.

2. *Pennsylvania*.—A very similar Act has been passed. Local authorities are required to establish day, part-time, and evening classes, giving training in agriculture, trades, industries and home economics. Reimbursement is made equal to two-thirds the salary of the instructors. The Act carried with it an appropriation of \$135,000 for two years.

3. *New Jersey* passed an Act extending its original legislation and creating a State system similar to that of Massachusetts and Indiana.

**E.—THE MILWAUKEE CONTINUATION SCHOOLS,
DECEMBER 1913. EXTRACTS FROM A REPORT BY
MR. R. L. COOLEY, SUPERVISOR.**

"The Milwaukee Continuation School entered upon its work on November 1st, 1912, in a small room on the eighth floor of the Manufacturers Home Building at the foot of Mason Street. There were in attendance two hundred girls under sixteen years of age from the department stores of the city. These girls were working on "permits" or work certificates* issued by the State Industrial Commission through the office of the Factory Inspector of this city.

"The school has grown rapidly from its small beginnings. The demands of both the compulsory and voluntary attendance have led to the acquisition of additional space, and a considerable enlargement of the teaching corps. There are now 1,600 permit girls and 2,000 permit boys attending school one-half day a week in the Manufacturers Home and Stroh Buildings. Thirty-one class rooms are now occupied and seventeen women and twenty-one men comprise the day teaching corps. The hours of school attendance are from 8 a.m. to 12 noon, and from 1 p.m. to 5 p.m.

"The school aims to teach the "permit" girls to cook, to prepare tasty economical dishes with a knowledge of food values and simple dietetics, to sew, to judge qualities of goods, to assist them to acquire a taste in color and design, to select suitable qualities of goods both with reference to personal dress and the furnishing, arrangement and decoration of the home, and to review and advance the pupils in the academic branches usually taught in the elementary school. There is also a definite effort made to connect each girl with the public library and to direct her reading, to the end that a taste for good books may be formed at this critical time in the girl's life, when she is breaking away from school, also, in a measure, from home and is entering upon a wage-earning career in shop or factory.

"It may be interesting to know that 900 books were drawn from the library during the month of November by these young girls, practically all of whom have not been in the habit of reading at all.

"In connection with the girl's work there has been given a brief course in the purchasing of beds and bedding, airing and making of beds, bedroom ventilation and arrangement, and the care of the sick in bed. This work has been very much appreciated by the young people, many of whom have had responsibilities thrust upon them far in advance of their years.

"Classes in typewriting and stenography are offered to young people working in offices and to those who particularly request the opportunity.

"The boys are classified broadly according to their expressed ambitions and talents. Those who clearly intend to follow commercial careers are permitted to put most stress upon the academic work in connection with typewriting, stenography, book-keeping, business practice, &c.

"Those who manifest a special ability, as for instance, talent for and appreciation of color and design, coupled with ability in free-hand drawing, have occupations called to their attention in which such ability would be a distinct asset. Such pupils we aim to segregate and afford the kind of special training they need.

* See page 146 (Wisconsin).

"The great majority of the 2,000 boys who attend our continuation school are not working at jobs which will afford them a living as men, or which, in the main, lead to work that will ever afford a man's pay.

"Some of the boys have ill-formed ambitions to become one or another kind of a journeyman, at some remote time. On the other hand, many have no idea of what is before them, are not thinking ahead, and have no conception of the various trades or occupations available in this community which might afford them a way up. They shift without thought or reason from one job to another and frequently their latter condition is worse than the former.

"To such boys we aim to give a diversified shop experience which includes wood-working, metal working and simple work with electrical appliances. One and one-half hours each week are spent by small groups in the shops under competent instructors. In addition, these same young people spend one and one-half hours each week in the mechanical drawing, and one hour in the academic work. All classes are kept small.

"The mechanical drawing continues for thirty-two weeks in each school year and is calculated to enable the boys to read blue prints and to become as proficient in mechanical drawing as the time and ability of the boys will permit. This training it is expected will be of use in any of the various mechanical lines which the boys may ultimately follow.

"The work with both boys and girls is further intended to guide them in their selection of vocations, to preserve by means of reviews what academic knowledge they may possess upon leaving the elementary school, and to advance them in those branches, to assist and advise them in securing work, to give them greater ability to do things with their hands, and to train them in the duties and obligations of good citizenship. This work is obviously very important during the lean years while these young people are confined to jobs which inherently have little or no educative influence, and, in many instances, may be said to be "de-educative," if one may be permitted to coin the word.

"From fourteen to sixteen years is a period of retrogression for the great majority of boys and girls who go out to work. This is specially true of those who have not finished the elementary school. Instances may be cited to the contrary, but it is only by viewing the young people in large numbers that the rule may be ascertained.

"It has been demonstrated that in the brief half-day a week at the continuation school much of the work done in the elementary school can be clinched and made a permanent possession of the boy or girl. Likewise in this time a valuable practical training can be given which will be found immediately useful.

"The function of the continuation school, however, is much broader in scope than merely to provide for the "permit" pupils, of whom there are approximately 5,000 in the city. The task of growing to meet the needs of this particular feature of the work has been a large one and has led to the work of the past year being somewhat limited to that phase.

"Day classes for apprentices have been established at which 200 are in attendance. Voluntary day classes have been conducted, as in the case of the druggists clerks, who attend twice each week. Other voluntary day classes are being formed as rapidly as the need and desire for the same is ascertained.

"The work conducted under the Local Board of Industrial Education is scarcely more than a year from its inception. Preparing part-time school facilities for thousands of "permit" boys and girls has largely absorbed the attention of the authorities. This work, however, as stated before, does not measure the scope of the work intended to be carried on under the provisions of the law. It is the duty of the Local Board of Industrial Education to analyze the educational needs and desires of the employed men and women of the community and to afford reasonable opportunity for their gratification."

F.—SOME RECENT AMERICAN REPORTS DEALING WITH THE RELATION OF THE SCHOOLS TO EMPLOYMENT.***UNITED STATES BUREAU OF EDUCATION.**

Manual, Industrial, and Technical Education in the United States
By Calvin M. Woodward. (Chapter in the Report of the Commissioner of Education for 1903, Vol. 1.)

Washington: Government Printing Office, 1905.

Recent Reports of the Commissioner of Education.

Washington: Government Printing Office.

Bulletins.

Washington: Government Printing Office.

1907. No. 1. Whole No. 367. **The Continuation School in the United States.** By Arthur J. Jones.

1908. No. 6. Whole No. 389. **The Apprenticeship System in its Relation to Industrial Education.** By Carroll D. Wright.

1913. No. 17. Whole No. 525. **A Trade School for Girls. A Preliminary Investigation in a Typical Manufacturing City—Worcester, Mass.** By the Research Department of the Women's Educational and Industrial Union of Boston.

1913. No. 19. Whole No. 529. **German Industrial Education and its lessons for the United States.** By Holmes Beckwith.

1913. No. 25. Whole No. 535. **Industrial Education in Columbus, Ga.** By Roland B. Daniel.

1913. No. 39. Whole No. 549. **The Elementary Industrial School of Cleveland, Ohio.** By W. N.-Hailmann.

1913. No. 50. Whole No. 561. **The Fitchburg Plan of Cooperative Industrial Education.** By Matthew R. McCann.

UNITED STATES DEPARTMENT OF COMMERCE AND LABOR.

Seventeenth Annual Report of the Commissioner of Labor, 1902. Trade and Technical Education.

Washington: Government Printing Office, 1902.

Twenty-fifth Annual Report of the Commissioner of Labor, 1910. Industrial Education. Washington: Government Printing Office, 1911.

[The section on "Vocational Guidance" has been separately printed.]

UNITED STATES SENATE.

Report on condition of women and child wage-earners in the United States. Prepared under the direction of Charles P. Neil, Commissioner of Labor. (61st Congress, 2nd Session, Document No. 645.)

Washington: Government Printing Office.

Vol. VII. Conditions under which children leave school to go to work. 1910.

Vol. VIII. Juvenile Delinquency and its relation to employment, 1911.

Industrial Education. Report of the Committee on Industrial Education of the American Federation of Labor, compiled and edited by Charles H. Winslow. (62nd Congress, 2nd Session, Document No. 963.)

Washington: Government Printing Office, 1912.

AMERICAN FEDERATION OF LABOR.

Industrial Education. Consisting of an investigation and Report by a Competent Special Committee. Reports of officers and committees; action of A.F. of L. Convention; the attitude of organized labor and others towards the problem; a glossary of definitions, &c.; labor's Bill for congressional enactment.

Washington: American Federation of Labor, 1910.

* Can be seen at the Library of the Board of Education, Whitehall, London, S.W.

CONNECTICUT COMMISSION ON WAGE-EARNING WOMEN AND MINORS.

Report of the Commission to investigate the conditions of Wage-earning women and minors, appointed by the Governor, 1911. Report presented to the General Assembly of 1913.

Hartford: State Printers, 1913.

INDIANA COMMISSION ON INDUSTRIAL AND AGRICULTURAL EDUCATION.

Report of the Commission on Industrial and Agricultural Education, December 1912.

Indianapolis, Ind.: State Printers, 1912

STATE OF MAINE COMMITTEE ON INDUSTRIAL EDUCATION.

Report of the Committee on Industrial Education.

Augusta: Kennebec Journal Print, 1910.

MASSACHUSETTS COMMISSION ON INDUSTRIAL AND TECHNICAL EDUCATION.

Report of the Commission on Industrial and Technical Education. April 1906.

Boston: Wright and Potter Printing Co., 1906.

MASSACHUSETTS BOARD OF EDUCATION.

The needs and possibilities of Part-Time Education. A Special Report submitted to the Legislature, January 1913.

Boston: Wright and Potter Printing Co., 1913.

Educational Legislation of 1913. Enacted by the General Court, giving Acts and Resolves with Annotations. (Bulletin of the Board of Education, 1913. No. 8. Whole Number 27.) Issued August 1st, 1913.

Boston: Wright and Potter Printing Co., 1913.

NATIONAL ASSOCIATION OF MANUFACTURERS OF THE UNITED STATES OF AMERICA.

Proceedings of the Annual Conventions. Twelfth Annual Convention held 1907, to Seventeenth Annual Convention, held 1912.

New York City: Secretary's Office, 30, Church Street, 1907-12.

NATIONAL EDUCATION ASSOCIATION OF THE UNITED STATES OF AMERICA.

Report of the Committee on the place of Industries in Public Education. (Journal of Proceedings and Addresses of the Forty-Eighth Annual Meeting of the National Education Association of the United States of America, held at Boston, July 1910.)

Winona, Minn.: Offices of the Association, 1910.

WISCONSIN COMMISSION ON EDUCATION.

Report of the Commission upon the plans for the extension of Industrial and Agricultural Training. Submitted to the Governor January 10, 1911.

Madison, Wis.: State Printers, 1911.

CITY CLUB OF CHICAGO.

A Report on Vocational Training in Chicago and in other Cities. By a Sub-committee of the Committee on Public Education, 1910-11, of the City Club of Chicago.

Chicago: City Club, 1912.

PAMPHLETS.**BLOOMFIELD (MEYER).**

The Vocational Guidance of Youth.

Boston: Houghton Mifflin Co., 1911.

BOSTON GIRLS' TRADE EDUCATION LEAGUE.

Survey of occupations open to the girl of 14 to 16 years. By Harriet Hazen Dodge.

Boston, Mass.: Girls' Trade Education League, 1912.

VOCATION BUREAU OF BOSTON.

Bulletins on Vocations for Boys. (The Baker; Confectionery Manufacture; the Architect; the Landscape Architect; the Machinist; the Grocer; the Department Store and its opportunities for boys and young men; Banking; the Law as a Vocation.)

Vocation Bureau of Boston, 6, Beacon Street (Director, Mr. Meyer Bloomfield).

COMMERCIAL CLUB OF CHICAGO.

Report to the Educational Committee of the Commercial Club of Chicago. By Edwin G. Cooley.

Chicago: Commercial Club of Chicago, 1912.

Tentative Draft of proposed law for establishing a system of vocational schools for Illinois. Prepared and approved by the Educational Committees of the Commercial Club of Chicago, the Chicago Association of Commerce, the Hamilton Club, the Civic Federation of Chicago.

Chicago, 1912.

CHICAGO SCHOOL OF CIVICS AND PHILANTHROPY.

Finding employment for children who leave the grade schools to go to work. (Department of Social Investigation. Russell Sage Foundation.)

Chicago: School of Civics and Philanthropy, 116, South Michigan Avenue, Chicago, 1911.

NATIONAL CHILD LABOR COMMITTEE.

Child Labor Bulletin. August 1912. (Article on child labor laws in all States summarising education laws.)

New York City: Child Labor Committee, 105, E. 22nd Street, 1912.

NATIONAL SOCIETY FOR THE PROMOTION OF INDUSTRIAL EDUCATION.

Bulletins.

New York City: Secretary of the National Society, Mr. C. A. Prosser. 105, E. 22nd Street.

No. 3. A Symposium on Industrial Education. By James P. Haney. 1907.

Nos. 5 and 6. Proceedings of the 1st Annual Meeting of the National Society for the Promotion of Industrial Education, Chicago. Part I, Part II (Part I is out of print.) 1908.

No. 12. Legislation upon Industrial Education in the United States. Prepared for the American Association for Labor Legislation and the National Society for the Promotion of Industrial Education. By Edward C. Elliott and C. A. Prosser. [Out of print.] 1910.

No. 13. Proceedings of the Fourth Annual Convention. Boston, Mass., 1911. Part I. Trade Education for Girls; Part II. Apprenticeship and Corporation Schools; Part III. Part-Time and Evening Schools; Part IV. The Social Significance of Industrial Education. 1911.

No. 16. Proceedings of the 6th Annual Meeting, Philadelphia, December 5-7, 1912. 1913.

PARSONS (FRANK).

Choosing a Vocation.

Boston: Houghton Mifflin Co., 1909.

PUBLIC EDUCATION ASSOCIATION OF THE CITY OF NEW YORK.

Report of the Vocational Guidance Survey. By Alice P. Barrows. (Bulletin No. 9.) Public Education Association of the City of New York: 40, West 32nd Street. Reprinted from the Fourteenth Annual Report of the City Superintendent of Schools, New York. 1912.

WISCONSIN INDUSTRIAL COMMISSION.

Apprenticeship Law. (Bulletin of the Industrial Commission of Wisconsin, Vol. I., No. 4A.)

Madison, Wis.: The Industrial Commission, 1912.

Compulsory School Attendance Laws.

Madison, Wis.: The Industrial Commission [1913].

Child Labor Law, with Explanatory Notes.

Madison, Wis.: The Industrial Commission, 1913.

WISCONSIN STATE BOARD OF INDUSTRIAL EDUCATION.**Bulletins.**

Madison, Wis.: State Board of Industrial Education, 1912.

1. Laws of Wisconsin relating to Employment of Women and Children, Industrial Education and Truancy. Compiled by C. P. Cary.

2. Wisconsin Legislation governing Industrial and Continuation Education. By H. E. Miles.

3. Industrial Education. The Impending Step in American Educational Policy; its significance to the boy, the parent, the community, the State, the nation. By H. E. Miles.

4. The Education of the Girl. The Necessity of fitting her Education to her Life. By L. D. Harvey.

5. Industrial and Continuation Schools. Their Foundation, Organisation, and Adjustment to the Life of the Community. By Louis E. Reber.

6. Public School Manual Arts. An Agency for Vocational Education. By F. D. Crawshaw.

WOOLEY (HELEN T.).

Facts about the working children of Cincinnati and their bearing upon educational problems. (Article in the Elementary School Teacher, Vol. XIV., Nos. 2 and 3, Oct.—Nov., 1913.)

Chicago: University of Chicago Press, 1913.

PAMPHLETS GIVING INFORMATION ABOUT THE SCHOOLS OF GARY, IND.**Notes to Visitors.**

Can the Administration Department of a School System serve as a laboratory for the Vocational Training of Children? By G. E. Wulffing, Director of Industrial Education, Gary, Ind. (Published in the Journal of Proceedings and Addresses of the Fiftieth Annual Meeting of the National Education Association of the United States, held at Chicago, July, 1912. Offices of the Association, Ann Arbor, Mich. 1912.)

Utilization of the School Plant. By William Wirt, Superintendent of Schools, Gary, Ind. (Published in the Journal of Proceedings and Addresses of the Fiftieth Annual Meeting of the National Education Association of the United States, held at Chicago, July, 1912.)

Gary Public Schools. By William Wirt.

A School Year of Twelve Months. By William Wirt.

Bolstering up the Bulwarks. By H. Roberts. (Kansas Journal.)

All the above are printed at the Emerson School, Gary, and may be obtained on application to the Superintendent.

Articles about the Gary schools also appeared in the Annual Report for 1906 of the Indiana State Department of Education; in the American School Board Journal for June 1910 (Milwaukee, Wis.); The Brick-builder, March 1911 (Boston, Mass., 85, Water Street), and Hampton's Magazine for July 1911.

APPENDIX I.

THE SCHOOL SYSTEM OF GARY, INDIANA.

A.—REPORT BY DR. R. H. CROWLEY, SENIOR ASSISTANT MEDICAL OFFICER TO THE BOARD OF EDUCATION.*

I was recommended to visit the schools at Gary in order to see in particular the provision made for physical education, the playground system and the educational developments along manual lines. The Superintendent of Schools, Mr. Wirt, very kindly gave up the whole day to me, and I had accordingly a particularly favourable opportunity for learning first hand as much of the working of the school system he has planned and controls as was possible in the short time available.

Gary has sprung into existence during the last six years owing to the operations of the Steel Trust. This corporation bought some ten miles of land bordering Lake Michigan and has erected a long line of steel mills, which completely cut off the town behind from access to the lake and which emit a vast amount of dirt and smoke. Six years ago it is said that the site of the town was a dreary waste of sand dunes and scrub; there is now a population of over 30,000. Mr. Wirt was appointed Superintendent of the school system about five years ago. He was then in charge of the public schools of Bluffton, Indiana, but saw more scope for developing his ideas in a new town such as Gary.

There is in the Gary school system nothing revolutionary, nor has it been dictated by any theory of education. "We do not wish," says Mr. Wirt, "to take any chances with undigested educational philosophy." The system is based simply on the recognised needs of the child. As to what these needs are, Mr. Wirt claims educationists are agreed—

"That the twentieth century school must educate for citizenship and character, must develop individuality and efficiency, must develop the power of reflective thinking, form habits of application, and somehow get the child to find pleasure and joy in constructive effort. We are also agreed, I think, that such an education is possible only by giving the child the freedom of self-activity. We are also agreed that the civic care of the child in the twentieth century must include satisfactory public facilities for amusement, play, and work, as well as study."

In developing the argument for the Gary system Mr. Wirt emphasizes the need for reproducing in the school the opportunities for education which the child used to have in the home, formerly in large degree an industrial centre. In the country

* Dr. Crowley attended the Fourth International Congress on School Hygiene held at Buffalo, August, 1913, as delegate of the Board of Education, and afterwards took the opportunity of visiting certain cities and institutions in the States in order to learn of the progress of the School Medical Service. Incidentally he spent a day at Gary.

or village home the child learnt by actual contact with life, and its education; however limited in other respects, was never divorced from reality.

"To have learnt to *know* the right thing to do is not character, but to have *formed the habit* of doing the right thing is character. When the home lost the opportunity for the industrial training of its children, it lost its power for efficient morals and general character training, because these were inseparably linked with the industrial training."

The schools and the school curriculum at Gary are planned with a view to providing this training for life. In the case of the newer schools, each school is complete in itself, providing for children from infancy up to college life. In the evening the school building is used by adults. All the work usually associated with a kindergarten, elementary school, secondary school, trade school, evening continuation class, is carried on as a continuous whole. Three-fourths of the boys remain at the school for whole or part-time up to the age of 18. In this way caste tends to be eliminated and the school becomes a powerful agent in the development of democratic ideals.

The Emerson School.—This school, the working of which I saw in as much detail as time would permit, accommodates over 2,000 pupils. It is built in one block and does not give the impression of a school with an accommodation approaching this number. It stands upon five acres of ground, shortly to be increased to ten. The playground is considered an indispensable part of the "school plant." The newest school in Gary has a playground of 20 acres and I was informed that no school in future is to be built with less.

The school is built on an E-shaped plan, the class-rooms opening on to wide corridors for the most part well lit and ventilated. There are no class-rooms strictly speaking comparable to those usually obtaining in an ordinary school, *i.e.*, no one room belongs to any one class. Only a limited number of rooms, moreover, are fitted with ordinary desks and seats, each room being equipped for the purpose for which it is intended to serve. Further the whole school plant is in constant use, whether it be the workshops, gymnasium, playground, swimming bath, auditorium or class-room, and accordingly the number of pupils the school can accommodate is not based upon the regular class-room accommodation, but upon the total accommodation provided.

The class-rooms proper are furnished with adjustable desks and seats. The top of the desk is removable, one side being used for drawing, for example, the other for such work as clay modelling. It can be placed at varying distances from the child and the slope adjusted as required. No locker is provided as part of the desk, each child having a locker in the corridor. Other rooms are fitted with tables of different sizes and chairs, or with chairs only, either movable or fixed according to the purpose to which the room is put. There are all sorts of

rooms for special purposes—for botany, zoology, physics, chemistry; crafts rooms for electrical, metal and plumbing and forge work, moulding, joinery, upholstery, printing; kitchen, laundry, sewing room, music and book-keeping rooms, library, and stores of all kinds.

Educational use of School Building.—Much emphasis is laid upon the use of the *school plant as a laboratory*. In Mr. Wirt's words:—

“Every part of the school plant can be made an educational opportunity. There is no reason why the school furniture and much of the equipment cannot be made in school cabinet shops under the direction of a cabinet maker selected for his teaching ability as well as his mechanical skill. There is no reason why the school painting, stage scenery, plumbing, electrical work, carpentry, printing and book binding, forging, foundry and machine work cannot be done in the same way. There is no reason why the school engineer should not be selected because of his teaching ability as well as his mechanical skill. The boys can work with this engineer as apprentices during certain hours in the day, and learn to fire boilers, operate pumps, engines, generators, motors, heating and ventilating machinery. There is no reason why some of the nature study teachers should not be selected because of their practical knowledge and skill as well as for their college degrees. A practical landscape gardener can take complete charge of the school garden, lawns, shrubbery, and trees, and the children will be delighted to assist him. Even the stairways and school corridors provide the highest possible educational opportunity if we will but use them.”

The purchase, care, and distribution of supplies for the school is used as a laboratory for arithmetic, English, and instruction in clerical work. In the general store and book-keeping rooms all is conducted by the pupils. Strict business accuracy is demanded in order that the precise cost of each department may be determined. All school materials are supplied through the store and charged to the corresponding account. An expert accountant trains the pupils in work of this kind.

The coal consumed in the boilers, the materials used for building and repairs, the milk and other articles of food used in the school café are all tested by the pupils. The school lawn, shrubberies, and gardens form a laboratory for nature study. The sanitation of the school and the care of the building afford many opportunities for training.

The man in charge of the heating and ventilation apparatus is a skilled man and acts as a teacher. “Everyone employed on the school premises should be capable, clean enough, and of sufficient intelligence to be a teacher in his or her own line of work.”

Vocational Training.—The industrial work, however, offers the chief opportunity for using the school plant as a laboratory whether for manual or purely industrial training. In the *electrical* shop every advantage is taken of the use in the school of electricity for lighting, heating, and power purposes. Theory and practice are intimately associated. Construction is

taught, apparatus is tested. In the *forge* iron plates were being rivetted, about half-a-dozen boys, aged 12-15, working under a practical blacksmith; castings were being made for jumping poles used in the playground, for a pump, and for the school desks and seats. In the *sheet-metal* shop metal boxes were being made for use in the pupils' lockers and in the various other shops in the school. There is a complete plant for the teaching of *steam production*, some boys being associated with the more theoretical, others with the more practical, part. Calculations are made as to the results obtained from the fuel used, whether for the generation of steam or electricity. Sundry school printing—the school magazine, forms, circulars, &c.—was being done in the *printing* shop. The English class correct the proofs. In the *woodwork* shop various articles of school furniture were in process of construction, and of particular interest was a simplified adjustable school seat. All the fitting and furnishing of the school has been carried out by the pupils. Lack of time may necessitate assistance from outside, and help may be required too for some of the heavier and more laborious forms of manual work. There seems, however, but little limit to what the pupils undertake. All tables, desks, chairs, and cupboards are made by them. Repairs to boilers and engines are carried out by the boys, who mould and cast, for example, new boiler plates as required. If any work of this kind cannot be done for one reason or another at the school the necessary measurements and drawings are made and the work executed at the foundry direct from the pupils' drawings. Recently a wing has been added to the school, the designing of and drawings for which were done by the pupils as well as some of the internal construction.

At Gary a wide interpretation is placed upon the term "vocational" as applied to education. In that the school lends itself to a complete training it includes of necessity opportunities for trade training. To be eligible for this pupils must be in the seventh grade or above or if below the seventh grade be 14 years of age or older. The following extract will show the points considered important in determining what line of work a pupil should take up:—

"The pupil and his wishes are the first consideration. His parent's consent to the work desired must be had. It is necessary to limit the number of students in each class. The teacher's judgment as to the wisdom of the pupil's choice must have some weight. The pupil's previous work in any shop must be considered, the pupil having entered upon a definite course being given every chance possible to complete it.

"In all vocational lines of work offered there is provided a short probation course. This course is for the purpose of assisting the pupil in selecting the line of work he wishes to pursue. It is just as important for a boy or girl to find suitable life work as it is to have training after his selection is made."

Opposition on the part of organised labour to vocational education in the narrower sense of the word does not exist to any appreciable extent in Gary. Probably this is due in part to the fact that this form of education is an intimate and

organic part of the whole school system. The community centres in the school, the playground, gymnasium, swimming bath, and the school itself being used as well by the adult as by the child. The community looks to the school as the natural place for all education, training and recreation. And further all the trade instructors are members of their respective unions.

The following extract is of interest as showing Mr. Wirt's conception of the relationship which should exist between the elementary, secondary, and trade schools:—

"The subjects taught in our Emerson School are practically the same as those of the first-class manual training schools. Usually the manual training school is the high school or secondary school, where only those students who have completed the work of the common schools are admitted. Our reasons for changing this practice are the following:—

- "(1) The educational value of hand work, along with mental training, is as great in the elementary schools as it is in the secondary schools.
- "(2) The efforts of most cities to avail themselves of the educational advantages of hand work, through elementary manual training in the graded school rooms and manual training centres, has in many cases been disappointing, and is always much less efficient than the educational hand work possible in a properly constructed elementary manual training school.
- "(3) Only a small percentage of students go to the secondary school. The remainder must therefore be reached, if at all, in the elementary school.
- "(4) By having elementary manual training schools for the common school work we are able to do two years and possibly three years of the customary high school course in these elementary schools. In our opinion this plan offers the following advantages:—

"(a) Many students do not go to high school because they are timid. The parents believe that in high school the children must dress well, buy expensive books, belong to societies and live on a social plane which they cannot afford, as well as pay street car fares, buy lunches, and many other extra incidental expenses. We have personally demonstrated in three different school systems that many more students go to high school from the elementary school when the high school and elementary school are together in the same building. The children and parents are accustomed to the school, and do not think any more of going from the eighth grade to the ninth grade than they do of going from the seventh to the eighth.

"(b) The ordinary first two years of the high school usually has two-thirds of entire high school enrolment. The work of these two years is very elementary, and can be done just as successfully in the elementary schools. Now, the high school building, equipment, and faculty are much more expensive *per capita* than the elementary schools. Yet two-thirds of its capacity is usually given over to work that can, in fact, be done better in the elementary school at much less cost.

"(c) Our experience in manual training during the past 12 years has convinced us that the school work up to the approximate age of 16 years should be general in character. Manual training up to that age should be largely for a general educational training, and not so much for its definite vocational value. Most boys beginning at the approximate age of 16, however, are anxious to get at something definite, something that will count directly for a special vocation, to make a living, and get on in the world. We believe, therefore,

that the age of 14 is not the right period for a transfer from the elementary to the secondary schools. The secondary schools can hardly be adapted to the needs of students from 14 to 16 and 16 to 18. We wish to turn over to the elementary school all of the general indefinite period of instruction up to the age of 16 years approximately. Then we wish to adapt the secondary school to definite vocational training for students 16 to 20 years of age. By eliminating the first two years, 14 to 16, we can add two years, 18 to 20, without much increase in cost. Our secondary school will, in fact, be a technical college or polytechnic school, and will give the opportunity to every young man and woman in Gary to secure a practical training for life's work in any commercial, professional, or industrial line. The school will also offer courses identical with those of the traditional Liberal Arts College. It is also the intention to arrange for most of the commercial, professional, and trade courses to be taught in continuation schools and night schools. We wish to make it possible for young men and women to earn their own living at some occupation and at the same time continue their education for what they deem to be their proper life work. As far as possible, of course, we expect such a school to be a great advantage to persons who are serving an apprenticeship in their chosen field of work, who will be able much more thoroughly to master and make the most of their talent by being able to join with the practical experience in the shop or business house the scientific training of the school.

"The traditional school attempts to fit a boy for a practical everyday industrial life, by taking him out of and isolating him from the practical industrial life of the world. Then when the boy has lived several years in this separate and distinct school world the school suddenly cuts him off absolutely, and the boy is left to adjust himself as best he can to the unknown world of practical industry and business. This mistake we wish to avoid in the schools at Gary by keeping the schools as close to the industrial and business life of the people as possible, and by going with the boy into his life's work. Every successful man is a student throughout his whole life. What he learns in school is insignificant compared with what he learns out of school. With our continuation schools we hope to get many young men to form the habit of being students of their work. The continuation school will not exist so much for what it teaches and does directly for the boy, for it will justify its existence by what it gets the boy in the habit of doing himself independent of the school."

The School Curriculum.—The school curriculum is so arranged that the whole of the school plant, including the gymnasium, baths, and playground, is in operation all day. Accordingly, approximately one-half of the number only of ordinary class-rooms is required. Two hours daily are given to formal instruction in desks. English and mathematics are taught incidentally as well as formally, and special measures are taken to ensure that this is in fact done. Great elasticity is observed in the size of the classes. Thus the moulder has six pupils only at one time; in the auditorium there may be 400. For formal class work the maximum is 40 children. The playground, like the rest of the plant, is in constant use, a batch

of about 40 children being always out of doors in charge of one of the physical training teachers. Speaking generally, the teachers are specialists in their subjects, *e.g.*, botany, zoology, geography, &c. There are no "class teachers" having the sole charge of a class except in the kindergarten. There are consequently no "superintendents" to see that the ordinary teachers teach the special subjects properly. There are, however, supervisors of the teaching of English and mathematics respectively. This is necessary, since the instructors in special subjects may not be sufficiently competent to teach these other subjects in an adequate or uniform manner, and the school aims at a high academical standard. There is a supervisor of industrial education generally, but his functions are administrative. The theoretical and practical teaching of a subject are not always entrusted to the same teacher. There is, for example, a teacher of the theory of cookery and of food values and also a practical cook.

The system admits of much *flexibility*. A child who is particularly backward in one subject can by joining one or another class devote more time to this subject. An older child, according to his or her bent, can, as may seem suitable, devote more or less time as the case may be to manual work or one or another of the special subjects. A delicate child may spend more time in the playground, may rest out there if necessary or engage in light forms of manual work.

An interesting feature of the school is the presence of small boys of 8-10 years of age in the various shops and laboratories. They spend their time observing and picking up what they can, but receive no kind of instruction. This association of younger and older children Mr. Wirt thinks very helpful. The younger ones look forward to doing the work of the older pupils, and when their turn comes for entering one or another of the shops they are familiar with the kind of work which goes on in each and are able to exercise some choice as to the form of manual training they would like.

Physical Training.—There are four teachers of physical training, two men and two women who work under the general direction of the two medical officers, a man and a woman respectively. These four teachers have control of the playground, gymnasium, and swimming bath, and are responsible for the training in hygiene generally. The playground is in constant use all day, before, during and after school hours. The school is not turned out for "recreation," but the playground is occupied for an hour at a time by successive classes of about 40 children. Facilities are provided for a great variety of games, *e.g.*, baseball, football, tennis, basket ball, hand ball, &c. The outdoor gymnasium includes a variety of apparatus, and the small children have sand pits, a wading pool, &c. Part of the ground is reserved as a school garden, trees are

planted wherever possible, and several animal houses have been constructed by the pupils. The swimming bath, 40 ft. by 18 ft. with a shower bath installation adjoining, is used by both boys and girls. The former bathe twice a week and in the summer more often. The water is purified by filtration every day. Two gymnasias are provided, one for each sex; 30 minutes are given daily to physical exercises.

About 150 to 200 of the children take their midday meal at the school. After our tour of the school in the morning Mr. Wirt took me to the school café where some half dozen girls behind a long table were selling dishes in great variety, all of which had been prepared by themselves and other girls in the school kitchen. In the café adjoining, set out with small tables, teachers and scholars were taking the midday meal.

Medical Supervision. — Two medical officers, a man and a woman employed for whole time, supervise the Gary schools and children. A couple of rooms are set aside for their use and also a bacteriological laboratory. In the work attached to the latter assistance is given by one or another of the pupils. Examination is made of all children on admission to school and shortly before leaving and all children will now be examined annually. The man doctor visits the school each day for half a day, the woman doctor three afternoons each week. They treat trivial ailments, supervise the physical training, remedial exercises, the swimming pool and the school buildings generally. No nurse or dentist is at present employed, but such officers are likely to be appointed in connection with the new Froebel School.

School Attendance. — The school attendance problem is considerably simplified in Gary. As already pointed out there are no regular class teachers in the schools, but arrangements are made for each teacher to keep in close touch with one group of children. To this end each teacher meets all the children coming from one "block" for the first 15 minutes of each day. The children are of course drawn from all the grades. A plan of each "block" has been drawn by each group respectively showing the accommodation, &c., for each family occupying the block, and in this way an intensive social study of each block is rendered possible. If a child is absent the cause is probably known to one of the other children, or they can quickly find out. Should there be a case of infectious disease the medical officer is at once informed and necessary action taken. Mr. Wirt considers that this system gives an important human and social touch to the work of each teacher. But little scope is left for the attendance officer, his energies being almost entirely devoted to seeing that newcomers are placed on the books of the school and to dealing with genuine truants. Much useful social work is made possible by this system. Should it be clear, for example, that a family is unsatisfactorily housed

information can be obtained of satisfactory quarters in some other block. A teacher keeps the same block year after year. I was informed that a high average of attendance is obtained.

One might suppose that a school system such as that outlined would be costly. This does not appear, however, to be the case. By using the whole of the school plant continuously the accommodation of the school is greatly increased. The cost per head of the Emerson School for 2,000 pupils was approximately \$100. Mr. Wirt informed me that an ordinary elementary school at Chicago cost \$200 per head to build. The cost of maintenance of the Emerson School works out at considerably less than that of an ordinary school. Mr. Wirt shows that the annual cost of maintenance of the Jefferson School at Gary, the first built, and of the ordinary type, is much greater than that of the Emerson School. The cost per head of teachers and equipment, he states, is materially less. In the case of the Emerson type of school the craftsmen are no charge upon the funds. They are retained only on the understanding that the work done in their respective department pays their salary and covers the cost of material used. This obtains without difficulty in the case of all the trades.

The cost of land is high. No less than \$5,000 per acre has been paid for the five acres of ground upon which the Emerson School is built. For the new school with 20 acres of land attached \$4,000 per acre is being paid.

In considering the economy of the system it should be remembered that the whole school plant is available for the use of adults in the evening. In this connection Mr. Wirt writes:—

“ We also desire that each of these buildings be a social center for the respective school districts. The manual training, technical, commercial and industrial courses, and physical culture classes, will be especially emphasized in the night school. The gymnasias, swimming pool, shower baths, playground, auditorium, lecture rooms, outdoor gymnasium, branch station of the public library and industrial rooms, &c., are intended for the adult population, as much as for the school children. The Emerson School building includes everything belonging to the recreation buildings of the playground parks of Chicago. Even though these are controlled by the school corporation, yet the general public will have just as free use of the plant as they have in Chicago where the park boards are in control. In our opinion small towns cannot support playground centers of the Chicago type unless they are associated with the public schools.”

And again:—

“ The opinion is becoming fixed that we have not utilised the school plants completely unless they are used for recreation and social centers by adults. Fortunately a school plant that provides for the constructive play and recreation activities of children is also most admirably adapted for similar activities with adults. The playgrounds, gymnasiums, swimming pools, auditorium, club and social rooms, library, shops, laboratories, &c., make a complete social and recreation center for adults. Experience has demonstrated that the facilities for academic instruction add to the attractiveness of the plant as a social and recreation center.”

It is the desire of Mr. Wirt to have the schools in full swing all the year round, a given child attending any three of the quarters out of the four. At present this is not practicable. The school plant, however, is never closed down completely. Much of it is available for use during the summer vacation, as also all day Saturday. On Sunday the library, gymnasium, swimming bath, and playground are available.

Most of the "street and alley time" is spent on the school premises at Gary. Towards securing this the playgrounds are available and the playground teachers in charge an hour before school, at midday and for an hour or two after school. The older pupils too use the gymnasium and swimming bath out of the regular school hours.

B.—REPORT BY MISS HILDA WILSON, LATE LECTURER AT THE
MOORFIELDS TRAINING COLLEGE.

The town of Gary, Indiana, has become famous in America for its success in providing an education for its children that is exceptional in its economy, popularity, and usefulness. The town is of exceptional type, as it is completely industrial. It is, in fact, a town made to order by the Steel Trust, to accommodate the thousands of their employees in the great steel rail factory that stretches between the present town of Gary and Lake Michigan. The town grew in population as fast as it could be built, and in five years time 25,000 people were living in brick blocks on paved streets that had been mere sand dunes.

Such a development, especially as the laws of the State were not in any way adapted to it, put great difficulties in the way of the school authorities for the town, and from these very difficulties they evolved exceptional virtues. The schools of Gary had to be a great popular and administrative success, or they would have ruined the infant town.

The economy of the system was made advisable by the following facts. The school funds come from two sources, as in England: from the town taxes, like our rates, and from the State School Fund, like our taxes. The schools began in 1907; by the law, the income from the first source must be obtained from an assessment of property made two years before. Property was assessed in March, 1907 and the tax levy made in August, but this levy was to cover the cost of maintaining the schools in the year 1908-9. During this year the town attained nine times and more than nine times its size in March, 1907. This disadvantage remains, for there is no reason at present why the growth of the town should not continue rapidly. As for the second source of income, that is apportioned on the basis of the number of school children in the town in the April of any year, for the period beginning the September of that same year. In Gary there is always a large increase in the population during the late spring and summer months, because those are the

building months of the year, and also the months when immigrants, who are needed in large numbers by the Steel Works, arrive from Europe. Thus the town has only received about half its equitable share from the State Fund.

This inadequate income has had unusual demands upon it from the first. There was nothing with which to start. All funds for the first year had to be borrowed, and everything needed, from the school site to the maps, had to be purchased at once. In most towns the school plant is gradually secured and increased, and school sites can be procured at reasonable rates some years before it is necessary to build upon them. This could not be done in Gary, where the cost of land for school sites was from the first 12 to 20 per cent. of the cost of the buildings, even though such sites were far smaller than they should have been. The Superintendent strongly urged from the first that sites should be reserved in the territory that would be eventually built over, sites not merely for immediate school requirements, but for all time. These sites should also be sufficient for public playgrounds, preferably near the schools, so that the school lavatory and bath facilities should be available.

The limited resources of the School Town and the overwhelming obligations to be met led to a most careful investigation and study of economy in the provision and upkeep of the school plant. The result of this investigation was the decision to provide a school, and eventually several schools, of a type altogether superior to that of the ordinary Common School of American towns. As a town grows, it is usual in America and Canada to add year by year to the number of these Common Schools, and then, as demand arises for the secondary education, which is also provided free, to form advanced classes in one or more of them until the city is large and prosperous enough to build a separate High School. This demand would have arisen late in the life of Gary, for the population has so large a proportion of foreigners that the demand for secondary education, which generally comes from the American born, would have been poorly supported.

There is in Gary one Common School of a usual American type, accommodating 360 children, and having a small playground. I understand that this school was not built by the School Board, but by the Gary Land Company when they were first laying out and improving the estate; and that they let it at a fixed rent to the School Board. The economy of building schools of a superior type is best shown by comparing their cost with that of this small school.

The Common School, which is called the Jefferson School, had an original cost for site, building, and equipment of 50*l.* per pupil. The superior school already built and equipped, and called the Emerson School, can accommodate at least 1,800 pupils, and cost 25*l.* per head. The saving to the town on the

original cost of the eight or so large schools that will eventually be required, as compared with the first cost of the 40 small schools that would be built otherwise, is thus at least 50 per cent.

It is the annual cost of maintenance, however, including the interest on the building debt, that is of chief importance to the taxpayer. This in the two schools is actually as follows (the amounts are in full, not per head):—

	Jefferson School.	Emerson School.
	£	£
Light, water, heat and ventilation -	360	700
Janitor service - - - - -	300	800
Principal - - - - -	240	400
4 per cent. on building debt -	720	1,800
Insurance, depreciation and repairs -	500	400
	£2,120	£4,020
	5	
	£10,600	

Annual cost for 1,800 children in Jefferson School would therefore be 10,600*l.* as against 4,020*l.* for the same number in the Emerson School, a clear saving to the taxpayers of 6,580*l.* per annum.

In explanation of these figures it should be stated that the cost of assistant teachers is reckoned to be the same in either case, on a basis of 40 pupils to a teacher. The Emerson School employs a number of specialists, as would an ordinary High School, but the extra cost of these teachers is counterbalanced in ways that will be explained below. The first item, heat, &c., is more economically provided in a large building than in a smaller one. The janitor service includes that of engineer, caretakers, cleaning, &c. The salaries of principals in both schools may seem small to Englishmen, but it is not the custom in America to make a very great distinction between the salaries of the headmaster and those of his chief assistants. The Superintendent of Schools does a good deal of the work, and undertakes much of the responsibility that would be laid upon the headmaster in England. Lastly, the fact that "insurance, depreciation and repairs" would actually cost less in the more costly building is due to the superior workmanship and material and fireproof nature of the Emerson School. It is a practical demonstration of the economy of superior construction.

"The construction is fireproof except the roof. All walls are built of vitrified brick laid in cement mortar. Ground floor halls, manual training rooms, all locker and toilet and wash rooms, gymnasias, and swimming pool have white enamelled brick wainscoting seven feet high. All halls have

marble baseboard. All stairs have marble risers, stringers and newel posts. All closet stalls in toilet rooms have marble partitions, and high-grade plumbing and electrical fixtures are used throughout. The blackboards are natural slate. The interior finish is quarter sawed oak. All window stools are brick. A program master clock controls secondary clocks and the program bells in all class-rooms, hall and playground gongs. Telephone system is installed to necessary points in the building, and in the basement from Principal's office. Floors are marble, granitoid or clear maple. The most approved heating and ventilating apparatus is used, with automatic temperature and ventilation control. All air for heating and ventilating is washed before it is delivered to school rooms, which lowers the temperature about ten degrees during hot weather and brings up the humidity of the parched dry air in cold weather. This washer also removes all dust and soot, which in a short time would otherwise ruin the school room walls and ceilings, which will be oil painted.*

With all this, the Emerson School works out at a less building cost per cubic foot than the other building, 5.15 pence against 5.835 pence per cubic foot. The reason for this will be stated later in the description of the school.

The School Board expect to be able to reduce the tax levy for the town considerably by building schools only of the Emerson type in future. The tax required for their upkeep would amount to .6*d.* in the pound on real estate values. For schools of the other type the tax would be nearly 2*d.* in the pound. In one year after the opening of the Emerson School the tax was more than .7*d.* in the pound less than the preceding year.

The construction and curriculum of the Emerson School are therefore contributory to genuine economy. They try not only to decrease the school expenditure in Gary, but also to get more for their money. They do this by providing a fourfold equipment instead of a single one, and then by seeing that this equipment is all put to its proper use for as many hours as possible in the day and for as many days as possible in the year.

First, there are the ordinary class-rooms and libraries where the child studies from books and has its lessons from the teachers of the academic subjects. Secondly, there are playgrounds, gymnasias, and swimming pools, where the child can play and secure a general physical training. Thirdly, there are gardens, studios, laboratories and shops, "where the child can work and learn to do efficiently many things by doing them." Lastly, there is a large hall, where by lectures, recitals, dramatic performances, phonograph, player piano, stereopticon lantern and moving pictures, the visual and auditory education of the child may be efficiently carried on. "Four separate and distinct places are provided for each child," but the cost per child is not increased, it is actually diminished. It cost less to build and equip the swimming pool, the gymnasium auditorium, manual training rooms, and to equip the playground with material for all sorts of games, than it would have done to provide the ordinary class-rooms and libraries four times over. The economy is due

* First Annual Report of the Gary Public Schools, 1908.

to the fact that these less expensive parts of the school equipment are kept working all the time. The child spends about a fourth of his time in the expensive class-room, and the other three-fourths is divided between the less expensive rooms and physical education. The usual American plan of giving each child a school desk for his exclusive use, which is filled with his books and equipment, has been abandoned. Each child has a steel locker for his wraps and school material. These lockers, though they have a special compartment for books and keep all possessions perfectly safe, cost much less than the fitted cloak-rooms usually provided.

On this basis of full use of the building the Emerson School will actually accommodate 2,240 children, and it is only for the sake of comfort and rather small classes (less than 40 per teacher) that the figure 1,800 was taken as the basis of calculations. In addition the school can easily accommodate 1,500 evening students without at all interfering with the work of the regular day school. In fact, adults use the school building as freely as children. A branch of the Gary Public Library occupies two rooms on the ground floor, so placed that it can be entered directly from the outside without climbing steps. The school provides for the adult population what the neighbouring city of Chicago has been at great cost to provide and maintain separately—social and recreation centres. The playgrounds, gymnasium, swimming pools, auditorium, club and social rooms, library, shops and laboratories make ideal surroundings, and the fact that attendances registered at the gymnasia, for instance, are about three million a year, and more than those registered in Chicago, proves their popularity.

The popularity of the school system is its next important feature. It is shown by the attendance. As the Superintendent says, after the school has had two or three years' trial, "the children want to go to such a school every day in the year and eight to ten hours each day."

This popularity seems to be due to the variety of the work offered, its exceptionally attractive presentation, the freedom allowed to the children, and the motive supplied for their work. These reasons are in order of increasing importance. First, the variety. All the subjects usually taught in the public schools are included in the curriculum, three languages, four sciences, commercial subjects, music and art throughout, and all things taught in the first-class manual training schools. Domestic science and art are of course included for girls, though there is no intention to keep the boys out of the kitchen or the girls out of the woodwork rooms if they wish to exchange activities. As the school is not a trade school, but a vocational one in the sense of helping children to choose their vocation, the pupils are encouraged to try their hands at many kinds of work, and great

proficiency in any one task is not expected. Many of the best trade teachers in the States desire this very training in all-round manual work to be carried up to the age of 16 or more, before a special trade is chosen and learnt. The Gary authorities hope eventually to have a Trade School to take pupils from 16 upwards, but there will be no need of the ordinary High School, either with a special vocational bent or otherwise. Add to this the exceptional facilities for physical training, dramatic and other displays, and for play, organised or free, and it would seem that some, if not many, sides of the school life must attract every child.

The attractive presentation of the work is due to two factors, special teachers and special equipment. With the exception of the arithmetic, history, political geography, and English (exclusive of literature), all subjects are taught by trained specialists. Each of the elementary school classes has a class teacher, who is responsible for the four subjects named above, but who may, and generally does, interchange with the teacher of the next class for certain subjects. The most usual division is for one teacher to take the reading, writing, spelling, and formal language in two classes, while the other teacher takes their history, political geography, and arithmetic.

In America, specialist teachers of manual training, music, drawing, &c., expect and receive higher salaries than all-round teachers. The cost in the Gary schools is reduced by the elimination of supervisors. Americans do not expect one teacher to be able to teach all the academic subjects satisfactorily, and manual work, sewing, physical culture, music and art as well. But they compromise as a rule by letting the teacher train in all these subjects, and then setting supervisors over her in each of the more technical subjects until she wearies of their supervision. The teacher in an American elementary school may, after her training, have to pick up the methods of six to a dozen special supervisors, men and women, who spend their time going round and teaching her their pet devices for manual training, reading, writing, music, drawing, nature study, and physical culture. It occurred to the Gary Superintendent that while these supervisors could not make specialists out of the ordinary teacher, they would not be needed if specialists were employed. So the specialists are an economy from several points of view. It seems a mere truism to say that they do their work better; that alone is a great saving of the children's time and energy. They enjoy their work, and the children enjoy it in consequence. Perhaps the falling off of the supply of elementary teachers in England may be partly due to the fact that those who would, ten years ago, have become pupil-teachers and then trained have now a chance of some years in a secondary school. There, in the hands of specialists, they begin to take a real interest in certain of their studies; and the idea of teaching every subject

required by a class of children becomes exceedingly distasteful to them. In Gary the surroundings also of the specialist teachers are carefully prepared for their work. The manual training rooms, laboratories and gymnasia have been mentioned, but there is also a nature-study room for that special instructor, two music rooms, two drawing rooms and a literature room. The work of the special literature teachers is typical.

“ These teachers meet every child for a 30-minute period on alternate days. By means of stories and reading from sets of books furnished by the public library, the children are interested in the best literature. This literature teacher's class-room is, in reality, a library of literature for children, and the teacher is in the truest sense a children's librarian. Many of the books furnished by the public library are supplied in sets of 30 or 40 volumes of the same book, so that class work is possible in this directed reading.”*

One may mention that books circulated in this way under the control of the special teachers last ten times as long as the library books used otherwise.

The freedom allowed the children is another form of economy. They are not the best type of American child. They come of “ a five-year-old community of steel mill operatives, speaking 16 different languages.” Yet this heterogeneous body of pupils governs itself. In the basement, one room is the Council Chamber of Boyville, the corporate body of the school. This is the Mayor's office, and here he and the duly elected aldermen sit to pass laws for boy behaviour in and out of school. Recent enactments include one to prohibit the children of the town taking short cuts over vacant building lots near the school. A delegation was sent to the Gary City Council recently, asking them to enforce more strictly the law against the sale of cigarettes, and also to provide more dustbins for the town, into which the boys promised to put the waste-paper found in the streets. The boys and girls have also a school bank, run by themselves, with real money, pass and cheque books, and real interest on their money, which is deposited in one of the city banks at 3 per cent. The children take turns to officer their class, giving necessary orders for moving from room to room, preparing for drill, and so on. One pupil is also officer for the day in each class, and sees to the comfort of visitors, showing specimens of work and answering their questions. Another form of freedom much appreciated is the freedom to work at their special subjects, and to advance in the school at their own rates. The times during which the school is open after lesson

* From a reprint made at the school of a paper “ Utilization of the School Plant ” read before a meeting of the Department of Superintendence, St. Louis, Mo., by William A. Wirt, Superintendent, Gary, Ind. (Published in the Journal of Proceedings and Addresses of the 50th Annual Meeting of the National Education Association of the United States, held at Chicago, July, 1912, p. 494. Offices of the Association, Ann Arbor, Mich., 1912.)

hours have been mentioned. Good pupils can secure extra help from their teachers on Saturdays, and can advance in one or more of their subjects to a higher grade any month in the year, irrespective of their companions or of their status in other subjects. The work is specially organised to this end.

The motive given the children for all their work is, briefly, its immediate usefulness. The school provides a life for the child. Every aspect of life is catered for on the premises; "it is the purpose to provide a right educational environment where the child may live naturally and grow through the exercise of his self-activity under wisely directed supervision." The fact is that the child has opportunities for activity in so many directions that appeal to him at first, and so little of his time is claimed for classroom study of the ordinary type, that he becomes eager for work.

"The play impulse is transformed into a work impulse so that real pleasure is experienced in work. The school life creates a need and desire for the academic and cultural work of the school. There is no attempt to remove the difficulties. The supposed distasteful work of the school is not sugar-coated with sentimentalism. The wasted time and the mis-directed energy of the street and alley are utilized to awaken ambition, develop initiative and create power in the child, so that he can find real joy in the mastery of difficulties. The child is busily and actively engaged the year round educating himself."*

An example of this was the case of a boy who gave no satisfaction to any of his teachers, except the instructor in manual training. As soon as this fact was ascertained all his work in other subjects was stopped. He spends all his time on woodwork, with the best possible chance of growing into the other subjects in time as he finds the need of them in his work and as his brain develops owing to the use of his eyes and fingers. First, give the child healthy energy which will make him seek occupation of mind or body for every waking moment. Then, apply his work to practical ends throughout. The 10 year old children test foods in the domestic science laboratory to discover their natures and food values. Before this has time to pall, they are testing again for adulterants and preservatives. Their interest in this matter brought back numbers of the children after school hours with samples of the milk they were using at home, and formaldehyde was found in so many of these samples that one milkman in the city who had refrained from formaldehyde reaped an astonishing reward for his virtue. The manual training boys make things the school needs for their exercises, the domestic economy girls cook, and serve, and sell every day luncheons that are bought and eaten by the pupils, the teachers, and the visitors who come daily to the school. "The item of dealing with real things and feeling the responsibility of doing things right would add

* "Utilization of the School Plant," p. 493.

the very desirable element of seriousness to a child's work,"* and also the no less desirable element of contentment, satisfaction, real interest.

The idea of using every part of the school plant as an educational opportunity† has been worked out with great success and considerable economy. The upper corridors of the school, for instance, are beautifully lighted (*see* plan) and are used as museums and picture gallery. The printing and cabinet-making departments have been better than self-supporting. Two boys became so useful that they were employed in the cabinet shop throughout the summer holiday, of three months, at good pay. Other work suggested, and probably in practice by now, includes photography and photo engraving in connection with the printing, the handling of school supplies, under a teacher, who would show how to use catalogues, compare prices, judge goods, estimate costs, and so on. Then there would be valuable practice in keeping the stock in good order and tabulated. Lastly, the charge of a room, to be kept clean and sanitary by the pupils, who should be paid for such work when it is done properly.

To sum up in the words of the Superintendent:—

"The twentieth century school is planned to secure the highest possible efficiency from buildings, grounds, and equipment, and the time and energy of teachers and pupils. It gives the child, not a shop, not a playground, but a life."‡

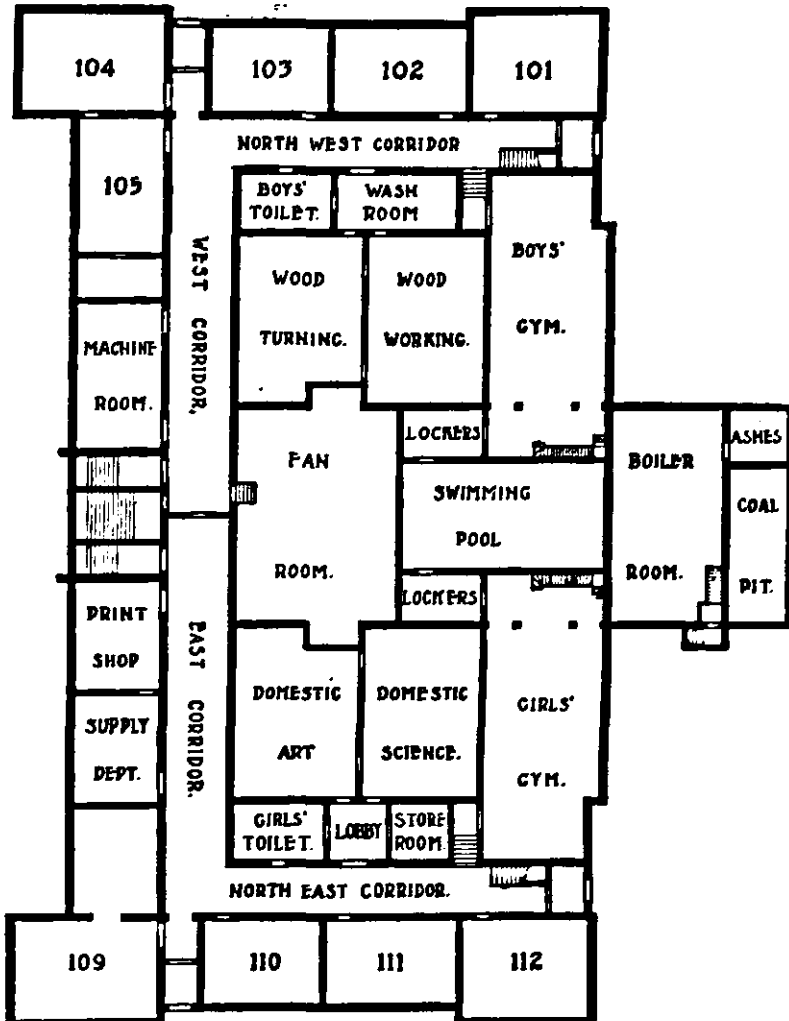
* From a reprint made at the school of a paper "Can the Administration Department of a School System serve as a laboratory for the Vocational Training of Children?" By G. E. Wulfing, Director of Industrial Education, Gary, Ind. (Published in the Journal of Proceedings and Addresses of the 50th Annual Meeting of the National Education Association of the United States, Chicago, 1912, p. 1225.)

† *See* p. 192, above.

‡ "Utilization of the School Plant," p. 495.

PLAN AND DESCRIPTION OF THE EMERSON SCHOOL.

Dimensions, 245 feet 9 inches by 141 feet 9 inches—not including boiler house.



GROUND OR BASEMENT FLOOR.

Six regular class-rooms.

Two kindergarten rooms, same size as class-rooms.

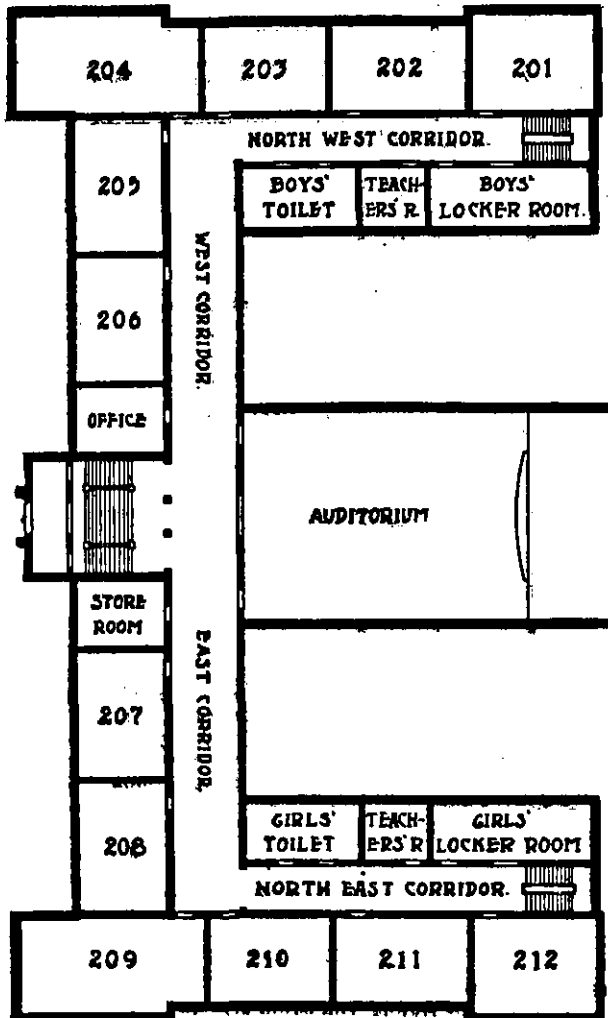
Two library rooms, same size as class-rooms.

Four manual training rooms with store-rooms, work-rooms, &c., each room as large as two class-rooms.

Two gymnasias with lockers, wash and toilet room, each the size of two class-rooms.

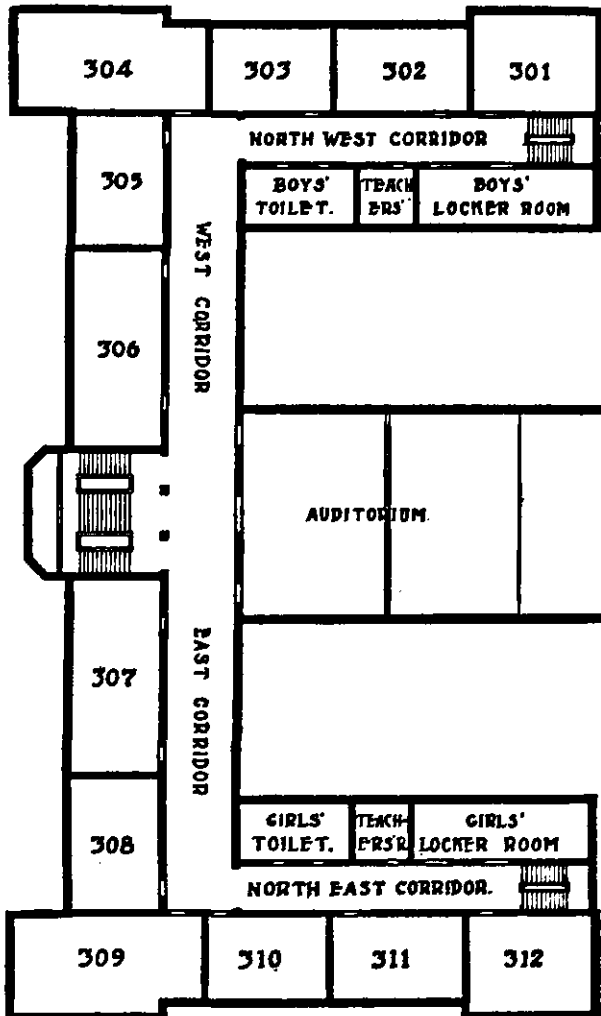
One swimming pool, with shower baths, the size of two class-rooms. This makes the total of the equivalent of 24 regular class-rooms in the basement. All class-rooms and work-rooms in the basement are entirely above ground.

In addition there are two large locker-rooms, one janitor store-room, two general toilet and wash rooms, rooms for heating plant, boiler-room, ash-room, and a coal-room which will hold one season's supply. The playground is 320 x 295 feet. In the basement is a corridor 16 x 180 feet which will be used in place of the playground during bad weather.



FIRST FLOOR.

- Twelve regular class-rooms.
- Principal's office.
- Two teachers' rooms.
- Store-room for the distribution of all school supplies.
- Two large locker-rooms.
- Two general toilet and wash rooms.
- Main floor of auditorium seating 546 adults, with stage constructed in accordance with fire ordinance in the City of Chicago.
- Large main corridor with lighting and construction planned for School Art Gallery as well as general corridor purposes.



SECOND FLOOR.

Eight regular class-rooms.
 Two manual training rooms, the same size as class-rooms.
 Gallery of auditorium seating 278 adults.
 Two rooms for infirmaries.
 Conservatory for plants.
 Two general toilet and wash rooms.
 Two teachers' rooms.
 Two locker rooms.
 Large main corridor with lighting and construction planned for school museum as well as general corridor purposes.

THIRD FLOOR.

One drawing-room equal in size to two class-rooms, above auditorium.

APPENDIX II.

A NOTE ON VOCATIONAL SCHOOLS IN THE
FAR WEST.

BY MISS HILDA WILSON, LATE LECTURER AT THE MOORFIELDS
TRAINING COLLEGE.

Los Angeles.—It is no wonder that the city of Los Angeles, California, has a school system which offers the most complete and varied vocational training in America, for the city itself is prosperous and favoured above all other American cities. The marvellous climate and the beautiful surroundings of this city, its great natural wealth in its fertile soil and abundant oil-wells, have drawn to it a population of over 300,000, and bring many tourists in the winter months. It is a city of gardens, for outside the central square mile every house or apartment-block stands in grounds, so that the ground covered exceeds that included in the County of London, and one can travel from eight to thirty miles from the heart of the city without crossing its limits.

In the curriculum of the elementary schools of Los Angeles, a very generous amount of time is allotted to manual work. In the first six years an hour and a half or more per week are given to this branch. Another 60 minutes is given to Drawing, which is generally correlated with the former. For instance, the writer was shown some of the productions of the third and fourth year children, which were toy figures cut out of flat wood, mounted on blocks to stand, and brightly painted. They looked like toys from a West-End shop.

Los Angeles has a number of intermediate schools, which take the pupils for the seventh and eighth years, especially those who intend to go on to high school, and give them a more specialised course which fits them better for high school work. This is an interesting attempt to bridge over the chasm between the common and the high school which should be compared with the scheme in operation at Gary, Indiana, where a much smaller population and area are catered for.

In these intermediate high schools there are three principal courses, within which a good deal of variation is possible. They are three year courses, as the first year of the High School proper can be passed in these schools. In several cases they share the building with the High School, and have the advantage of its splendid equipment.

The following is the programme of the courses, 1912-1913. English has five periods a week, Physical Training one or two periods a week, in all three years of all courses.

FIRST YEAR.

GENERAL.		COMMERCIAL.		VOCATIONAL.
	Periods a week.		Periods a week.	Periods a week.
Arithmetic	5	Arithmetic	5	Same as the General Course.
Geography or History	5	Bookkeeping	5	
Music	2	Stenography	5	
Drawing	2	Penmanship	2	
Penmanship	2	Geography or History.	5	
Manual Training, Girls, Cooking, Sewing, Boys, Woodwork.	4	One of the following, French, German, Spanish	5	
One of the following, French, German, Spanish, Latin, Bookkeeping, Stenography.	5	or Music and Manual Training.	6	

SECOND YEAR.

History and Civics	5	History and Civics	5	Manual Training, Girls, Cooking and Sewing, Boys, Woodwork } 10
Oral English or Music.	2	Bookkeeping	5	
		Stenography	5	
Physiology and Hygiene.	2	Penmanship	2	
Manual Training as First Year.	4	Physiology and Hygiene.	2	Drawing, Freehand or Mechanical } 5
Two of French, German, Spanish, Latin, Bookkeeping, Stenography, Algebra, Arithmetic, Freehand or Mechanical Drawing	5 each.	One of French, German, Spanish	5	General Science, with Hygiene. } 5
		or Oral English or Music with Manual Training	6	One of U.S. History, French, German, Spanish, Bookkeeping, Algebra } 5

THIRD YEAR.

GENERAL.	COMMERCIAL.	VOCATIONAL.
<p>Periods a week.</p> <p>Music or Oral English. } 2</p> <p>Three of a foreign language, Bookkeeping, Stenography, Algebra, Com. Arithmetic, Ancient History, General Science. } 5</p> <p>One of, Cooking, Sewing, Woodwork, Freehand or Mechanical Drawing. } 5</p>	<p>Periods a week.</p> <p>Com. Arithmetic - 5</p> <p>Bookkeeping - 5</p> <p>Stenography - 5</p> <p>Two of, French, German, Spanish, General Science, Algebra, Penmanship. } 5</p> <p>Music, or Oral English with Manual Training. } 6</p>	<p>Periods a week.</p> <p>Manual Training, Girls, Cooking or Sewing, Boys, Woodwork. } 10</p> <p>Drawing, Freehand or Mechanical. } 5</p> <p>Elementary Physics or Chemistry. } 5</p> <p>One of French, German, Spanish, Ancient History, Bookkeeping, Algebra. } 5</p>

There are eight high schools in the Los Angeles School District. Four large ones, Los Angeles High, Polytechnic High, Manual Arts High, and Evening High, are central, the others, Hollywood, San Pedro, Gardena and Wilmington, are in the suburbs.

"All of these schools offer courses leading to the University, and all are accredited (i.e. have the right to send students to the University without special examination). The faculties employed and the work done in these different schools are equal in rank and merit. Each has for its purpose the physical, mental and moral development of the child, and is more concerned with these than with his preparation for so-called higher academic work. While they are the same in these broad fundamental lines, each has its own personality and distinguishing features."

"The Los Angeles High School is the parent High School of the city, and is distinctively the academic, classical and cultural High School."* It is the only one offering Greek, yet "much practical and applied work, such as strong courses in food chemistry, mechanical and architectural drawing, applied art, surveying, and commerce has been instituted."* One may refer here to the wonderful "Vocational Bulletin" published by the authorities and prominent in the office of each Elementary School. This shows better than anything how each of the High Schools prepares for various forms of employment, and how the "classical and cultural" High School prepares for

* Courses of Study for the High and Intermediate Schools of the Los Angeles City School District, 1912-13.

various artistic occupations, and gives partial preparation for commercial life.)*

"The Polytechnic High School, while young in history, has had a phenomenal growth." It has well over 2,000 pupils. "It is the technical high school of the city. All but three of its fifteen or more courses are highly specialised and aim to meet the needs of those who do not intend to go directly into college. It offers strong courses in electricity, mechanical engineering, architectural drafting, assaying and commerce. As large a percentage of its graduates, however, as those of any other high school are entering higher institutions of learning." All the academic work of other high schools is offered, except Classics. This high school was visited, and as it took two hours to go through the most interesting parts of the building, its size may be imagined. It is very imposing inside and out, yet the main building is not nearly large enough for the work, and two enlargements have already encroached upon the playground. One of these is pointed out with pride, since it was designed by the young architects of the classes in that subject. It is chiefly remarkable for its fine windows. Inside this new part are the Art rooms, and one of these is fitted up for pottery making. The oven was built in the room, and the writer was offered her choice of one of the finished vases. These are designed, shaped and coloured entirely by the students, and are both strong and graceful in character. Here also sets of coloured tiles are being designed and made, which are to be set into spaces left for them on the main staircase of the new building. Near by is the architecture studio, where a girl was working at plans for a two-story house. At this school also the electrical department is very fine, and some wonderful machines are on exhibit. In the mining laboratory some boys were assaying ore for silver and gold in a small furnace. It looked very dangerous, but the director said they hardly ever have accidents. Perhaps these boys were descendants of the early prospectors who made California American. The geological museum was very fine, with exhibits of ores from all over America, and fossils and portions of prehistoric animals. A case of radium is also kept there. The courses at this school

* A study was published in 1912 of the Vocational subjects, including Commercial, offered in the high schools of California. This states that 24 of the schools offered no vocational subjects. These form 14.5 per cent. of all the high schools.

In 37 schools, 1 to 5 per cent. of the subjects are vocational.

" 21	"	5 to 10	"	"	"
" 44	"	10 to 15	"	"	"
" 23	"	15 to 20	"	"	"
" 10	"	20 to 25	"	"	"
" 6	"	25 to 46	"	"	"

The subject most frequently offered was geometrical drawing. These come in order of decreasing frequency, and, roughly, of increasing expense of equipment, mechanical drawing, book-keeping, shorthand, typewriting, commercial history, geography and law, woodwork, domestic science, forging, machine-shop work, foundry work, and architectural drawing.

can be extended for those who wish to acquire greater technical skill, and students will be able to stay there six years or longer, instead of four, if they wish to. In any case they can return to the evening classes, which are free to all residents in the city who are over 15 years old. These constitute the Evening High School, which gives all the courses of the Day High Schools as far as there is demand for them.

"The Manual Arts High School, although but two years old [in 1912] has attained remarkable success. It has already made for itself a place in the school system. It occupies a place somewhere between the classical high school and the technical high school, offering all academic subjects except Greek, in addition to the industrial work which is presented somewhat from the educational or developmental standpoint."* This high school had not fully worked out its courses in 1912, but as there are over 2,000 pupils it will probably offer as many as the Polytechnic. The different standpoint from which the work is approached is evidently set forth to the students with some emphasis, for in their yearly magazine, the Annual (they have a smaller one, appearing from time to time) which they write, illustrate, and publish themselves, a student writes very clearly and enthusiastically on "Industrial Education," but draws a clear distinction between it and manual training. He says, "We understand by manual training a thorough instruction and practice in tool and machine work, designed to improve the intellectual powers of the worker, and by industrial education the instruction necessary to produce the kinds and degrees of skill needed for the exercise of particular trades or industries."

That this intellectual ideal makes no difference to the market value of the product was evident, since numbers of useful things were being made in the metal and woodwork rooms for other schools in the city. There were some fine wooden settees, decorated with carving, destined for the corridors of a high school. All work was marked or stamped "M.A.", and there was a strong feeling evident that articles so stamped should be of superior quality and workmanship throughout. The woodwork is very successful at this school, and boys who have not had one year's course may be seen making substantial pieces of furniture, such as tables with veneered tops, large armchairs, and so on.

The Manual Arts High School has also a very beautiful building. The writer spent hours in seeing the wonderful heating and ventilating arrangements. The engine houses were not finished, but will now be used as part of the school "shops." The kitchens where the school luncheons were being prepared had a long array of machines doing the cook's work. One was mashing potatoes, another was mincing, and a third was

* Courses of Study for the High and Intermediate Schools of the Los Angeles City School District, 1912-13.

a device for washing dishes mechanically. There are several cafeterias in use at the luncheon hour. The staff have their own, but fetch their own food from the serving table. A student comes round and takes the money from each, and then they meekly pack up their trays and take them back to the appointed place. In one of the open corridors connecting three blocks of the school building there is a long counter specially built in a large recess and elaborately fitted for the serving and selling of ice-cream. This is an essential item of the menu of any school luncheon counter in America, but the writer never saw elsewhere so elaborate and deliberate a special provision made for supplying it. All the serving here and in the cafeterias is in the hands of the students. The luncheon arrangements are in the hands of the student body, who do not try to make a profit out of the sales, but generally have a considerable balance after paying expenses. The money was being counted and banked in one of the students' offices, as the writer went round after lunch to see some of their arrangements for self-government. The girls and boys organise separately for the purpose, and at two desks suitably labelled in an office near the entrance they were receiving the excuses of those who had arrived late for any lesson since the day before. Only excellent reasons were accepted, and those who could not offer them were sentenced to detention in a certain room under guard. The great school auditorium is also very striking and most lavishly equipped. The stage has three or four complete sets of scenery, all of which can be adjusted and changed by machinery, and the electric lighting is controlled by a switchboard of some 50 handles. There seem to be lights of every colour in every possible part of the stage, and the plays presented have certainly all the "atmosphere" possible.

Of the suburban high schools, the Hollywood "is distinctly cosmopolitan, comprising all of the old and new features of high school work. One of its characteristic features is the prominence given to home economics, for which work it has ideal equipment."* This study of "home economics" is not confined to cooking and sewing. It includes, with English subjects, mathematics and a foreign language, a good deal of science, general and special, home crafts such as stencilling, leatherwork and block-printing, civics and economics, household management, laundrywork, study of architecture, dietetics, sanitation, furnishing and musical appreciation. There are twelve courses at this school, including a special art course, one in agriculture, one in mechanic arts, and one in engineering.

The San Pedro High School is near the coast, and of the seven or more courses it offers the "marine" course is probably unique. The object of this course is to give preparatory training to marine engineers, boat builders, and followers of other sea

* Courses of Study for the High and Intermediate Schools of the Los Angeles City School District, 1912-13.

vocations (*see below*, Vocational Chart). The scientific work of this school is to be strongly developed with a view to marine studies.

The Gardena High School is strongest in agricultural work. It is beautifully situated in a rural district, with a large farm attached, and has a special course in agriculture, besides three other courses, and a large group of elective subjects, from a choice of which other suitable courses can be arranged. Besides the various branches of farm work, horticulture, forestry, and landscape gardening are taught. Other subjects, such as home economics, machine work and carpentry, law and the keeping of accounts, are all taught in their application to rural life.

The Wilmington High School is on a smaller scale, and only offered three courses for 1913. It prepares for commercial pursuits and provides some manual training.

The school authorities of Los Angeles are not content with this extraordinarily wide range of school work. They are anxious to open in addition a Trade School, in which the work and conditions shall be as like those of real life as possible. This school is to benefit boys who never qualify to enter a high school, and who, therefore, cannot obtain the mechanical training there offered. In almost every case where trade schools are established in the United States this would seem to be an important motive of their originators. The pre-vocational department of the Lucy Flower Technical High School in Chicago, and the Manhattan Trade School for Girls in New York, both admit girls who have not passed through the two or three last Grammar grades. In Los Angeles they wish this school to teach at least eight trades, carpentry, painting, masonry, printing, machine shop, blacksmithing, electricity, and plumbing. According to the latest accounts, such a school will be started this year or next.

Denver, Colorado.—Another western city that has a similar system of schools is Denver, Colorado. There are five High Schools in the city, and a trade school for boys. The North, South, East, and West Side High Schools offer the usual academic courses, and commercial training, and are primarily for the students of the districts in which they are built. The Manual Training High School Course is given at the North Side High School, the largest and newest of the four, for the first two years of high school life. It is not a full time course, and the work can be arranged so that more or less of manual work, and of academic work, can be taken together for these two years. The manual work will only count up to 25 per cent. of the "credits" needed for graduation. The year is divided into two terms irrespective of holidays, and in the first year boys can take joinery for one term and design for the other, spending ten periods a week on them. They have four hours drawing, free-

hand and mechanical, per week throughout the year. The girls have the same arrangement, with sewing instead of joinery, and no mechanical drawing. In the second year, the boys have wood turning and moulding for ten hours a week one term, and pattern-making the next term; also, they have mechanical drawing for two hours a week throughout. The girls have cooking, design, and mechanical drawing. The third and fourth years work in this course is to be set aside gradually in favour of the last two years of the new Technical High School Course. Thus, when a boy or girl has had two years high school with more or less manual training, he or she can choose, to complete the four years, either two years academic or commercial work, or two years technical training, with a choice of subjects, and some cultural work, or, in the case of a boy, a course at the School of Trades. The new Technical High School Course is also complete in itself, but is not all given in one building. The first year's work is given in the Longfellow School, an elementary school of which some rooms are equipped for this course. The other three years work is given at the Manual Training High School, where the manual training students from the North Side High School formerly came for the last two years of their work. At this school the hours are rather longer than in the other High Schools; from 8.45 a.m. to 3.35 p.m.

"The course is not elective during the first two years. Each student is expected to carry three academic subjects, and spend 18 periods per week in the shops or laboratories. At the end of the first two years the students of the Technical High School [Course] will have covered approximately as much shop work as is now secured in the Manual Training High School [Course] at the completion of the four years' course. At the end of the second year, students will be expected to choose some special subject for intensive work; carpentry, cabinet-making, pattern work, forge work, machine-shop work, dressmaking, millinery, catering, architecture, are among the subjects that may be chosen.

"Students will spend at least four periods per day during the third and fourth years of the course in work upon the special subject chosen. A student completing the four years course will have secured, in the first place, an excellent academic education; in the second place, a general preparation in manual training, including several varieties of manual work; in the third place, special intensive preparation for some vocation, which, while not fitting him to be immediately a skilled artisan, will have given him such training as will enable him soon to become a skilled workman. This particular training, plus the general academic and manual training, should furnish a most excellent preparation for foremen and superintendents." *

* Denver High Schools, Important Information for Parents and Students about Admission, &c., for 1912-13.

TECHNICAL HIGH SCHOOL COURSE.

English is compulsory for five periods a week throughout the Course.

FIRST YEAR.

Boys.	Periods per Week.	BOYS AND GIRLS.	Periods per Week.	GIRLS.	Periods per Week.
		Algebra - - -	5		
		Science - - -	5		
		Study - - -	5		
Mechanical Draw- ing - - -	6			Mechanical Draw- ing - - -	2
Freehand Drawing	4			Freehand Drawing	2
Modelling and De- sign - - -	2			Modelling and De- sign - - -	4
Joinery - - -	4			Domestic Science	4
Turning and Pat- tern Making -	4			Sewing - - -	8

SECOND YEAR.

		Geometry - - -	5		
		Freehand Draw- ing - - -	2		
Physics - - -	8			Chemistry - - -	7
Mechanical Draw- ing - - -	4			Mechanical Draw- ing - - -	2
Pattern Making and Foundry Work - - -	6			Design - - -	2
Forging - - -	6			Sewing - - -	6
Study - - -	4			Domestic Science	6
				Study - - -	5

THIRD YEAR.

BOYS.	Periods per Week.	GIRLS.	Periods per Week.
Two of the three :—		Two of the following :—	
Chemistry - - -	7	Physics - - -	8
Book-keeping and Solid Geometry - - -	5	German - - -	5
German - - -	5	Algebra and Solid Geo- metry - - -	5

THIRD YEAR—continued.

BOYS.		GIRLS.	
	Periods per Week.		Periods per Week.
First Term :—		One of the Following :—	
Mechanical Drawing -	4	*Household Arts Course -	} 17 or 20
Machine Shop Work -	12	*Dressmaking and Milli- nery -	
Second Term :—		Domestic Science Course -	
Technical Lecture and Laboratory Work -	4	General Fine Arts- Course -	
Selected Technical Work, with Drawing -	12	Illustration Course -	
Study (throughout the year) -	9 or 7	Design Course -	
		Architectural Course -	

* Design and Drawing are included in these Courses.

FOURTH YEAR.

American History and Civics	5	Academic Work as boys.	
German, or Mathematics, or Science - - - -	5 or 7		
Selected Technical Work	16	Selected Technical Work -	20
Study - - - -	9	Course as Fourth Year (except for Millinery, which forms a Complete Course).	

The art work done in these courses is exceptionally interesting, and most carefully adapted to its purpose. Thus, at the Manual Training High School the more advanced girls were making splendid figure studies from large casts and from life, chiefly that afterwards they might design dresses on these studies, with due attention to proportion. A good deal of clay modelling was done in the first year, and one negress student showed great aptitude for this work. In several schools the progress of coloured girls in manual arts was very noticeable. The sewing rooms were adorned with numbers of charming coloured designs for simple dresses, and the girls who have entered upon a technical course are always busy during their drawing lessons in working out some practical problem which bears upon their sewing, embroidery, millinery or household decorative work. In the North Side High School, where the work is vocational rather than technical, the younger pupils produce most charming designs that could well be applied to their own surroundings. One class will be working on Japanese prints, which they copy and adapt for decorative purposes.

Then they go on to large and simple designs for rugs in coloured chalks. These rugs are such as the children weave in the schools from strips of calico or very coarse wool. One design on show was of a fat bear looking at his reflection in the water. It was broadly drawn and would be both amusing and pretty when worked up. Then they go on to designing borders, with corners, from nursery rhymes. These were inspired by the well-known books decorated by Walter Crane and Boutet de Monvel, but the ideas were the children's own. Of course all the work was daintily coloured.

The building is nearly new and very fine. It has 16 acres of floor space, and is built round a large auditorium. The Science department is most lavishly equipped, and the Colorado Museum keeps its collections in the corridor adjoining. The Manual Arts work is done in a separate building, where also the shops of the Trade School are situated. The three trades being taught were plumbing, carpentering and printing. The plumbing shop had an "upstairs" part for the baths and pipes. This course is in three parts, bending and soldering pipes, fitting boilers, testing drains and gasfitting. The printing work is especially good, and the boys do 80 per cent. of the work for the Education Authority in a completely satisfactory way.

This Trade School differs from others in the States in requiring entrants to have passed the High School entrance standard, and also to be sixteen years of age. The headmaster wants all to have had a varied technical course for two years before entrance, but this is not practicable at present. There is no labour opposition to the school, and he anticipates no difficulty in placing all the pupils after their two years' course as advanced apprentices or journeymen. They have a rather longer school day, 8.30 a.m. to 4 p.m., and there is an extra summer term of two months' shop work, so that they can work for nearly eleven months in the year. Half the school day is given to shop work, a quarter to trade drafting, and a quarter to trade mathematics, trade history and science, and to business English.

It is noticeable that no time is allotted to physical training. Under the State law, all students are physically examined once a year. The schools have voluntary clubs for games, and there is a very active Playground Association in Denver, which does much to enliven school athletics by arranging yearly displays and competitions on a large scale.

The city of *Seattle, Washington*, is working out an interesting experiment in its elementary schools. The school law makes education compulsory till the age of fifteen, and the upper grades were sometimes blocked with boys and girls who had no real interest in the ordinary academic work; many of them had come from other parts of the country and had been earning their own living before they had to return to school. For these and

others, a "Grammar School Industrial Course" has been established, and is most successful. If the headmaster thinks that some particular pupil would be benefited by this course, he recommends it to the parents for them. The object of it is to awaken these children's interest in their daily life and useful occupations. The girls are trained to be home-makers, the boys to be industrious. The official plan is as follows* :—

"Boys and girls have English, geography, history, and arithmetic. Boys have mechanical drawing and shop work, girls have drawing and design, sewing and cookery. The periods are forty minutes each. Shop-work has three periods daily, cookery two.

"English will include reading, spelling, penmanship, letterwriting and composition. Geography will include map studies, climatic conditions and influences, industries and products, exports and imports, routes and centres of trade, the studies to be correlated as far as practicable with the work in shop and kitchen. In history there will be a review of the influential events, in the development of our country, including particular reference to the country's greatest characters and their achievements, and of the causes attributing to our present national standing. The purpose will be to give an elementary knowledge of the important facts in our history and to imbue with a patriotic desire to be serviceable. In arithmetic, the fundamental operations, including fractions applied in shop-work and in local problems; percentage and interest; applications of measurements and mensuration. The purpose will be to secure accuracy in the use of figures and practice in their application to practical affairs.

"*Industrial*.—The shop instruction will consist of work intended to give knowledge of materials and their source and use; tools, and skill in their use; methods of construction; problems in machine and handwork; acquaintance with factory and individual production; the use of preservatives, as paints, oils, &c., discussions of the various vocations; visits to work under construction, to manufacturing and commercial establishments. The work will need to be under a very practical and specially competent man.

"The industrial work for girls will consist of:

"Plain sewing, garment cutting and fitting, repairing, household linens, fabrics used in the home, sewing machine, class talks and discussions regarding clothing, hygiene, style, costs, methods of manufacture, the sweat shop, trades and vocations for women.

"Plain cooking, properties of foods, economy, table service, sanitation, laundry work, care of the home, &c. Actual conditions make it possible for the girls to purchase and prepare a simple lunch daily, and to serve the same to other pupils at noon at cost. Class talks upon related topics of home life and its obligations, domestic service, income and expenditure, &c.

"Applied design in surface decoration as affected by material and service, the use of color, problems in making designs for note book covers, belts, pillows, draperies, &c. The aesthetics of the home."

This course covers two or more years, and includes some sort of vocational guidance. The supervisor of the girls' classes had inquiries made in the autumn of 1912 as to the way in which the pupils had spent their long summer holidays. It was discovered that a good proportion had taken up some work. One had made her own clothes; the course is arranged so that they learn to make everything they wear. One had made her mother's clothes. Two girls had joined together and made twenty loaves of bread every other day and sold them.

* Grammar School Industrial Course, 1912-13.

Children of this type would not go on to the High School as a rule. There are seven large High Schools in Seattle, which is a city nearly as large as San Francisco or Los Angeles, with 280,000 inhabitants, and 38 elementary schools. One of these High Schools has 2,500 pupils. Five courses of study are offered in these schools, of which three are vocational, the Commercial, Boys' Manual Arts, and Girls' Manual Arts Courses. In the last two a good deal of choice is allowed, but English, mathematics, United States history and civics and a two years' course of gymnastics are compulsory. About two-thirds of their time is spent on the vocational training with choice of languages and other subjects, such as music. In one of the newer High Schools the Domestic Arts department included, besides the usual kitchens and laundry, a small apartment with a little kitchen, diningroom, living room and bathroom. This was not yet fully furnished, as the girls were themselves choosing the furniture for it, and arranging the decorations. When ready, it was to be used for practice. Two girls at a time were to use the kitchen, and to prepare meals there for six teachers; the girls were to do the marketing, keep the accounts, and charge for the meals. A course on Home Nursing was also to be given there. In all the High School teaching of these subjects, great care is taken that the girls know the reason for everything they do. Evening classes are also carried on at these schools in vocational subjects.

San Francisco has also similar work in certain of the High Schools, chiefly in the Polytechnic High School and the Commercial High School. In *Salt Lake City, Utah*, both the public High School and that under the direction of the Latter Day Saints offer complete courses in woodwork, metal work, and domestic arts. It should be noticed that woodwork is rather more of an educational subject in those parts, for furniture has to be built up very carefully if it is to withstand the climate. No English furniture would last a year; if a cabinet, for instance, be designed with a curved front, that front has to be carried out in five ply wood, or its curve would speedily vanish. For that reason, also, furniture is usually very heavy and solid in structure.

THE TRAINING OF VOCATIONAL TEACHERS.

The Western States do not need to train all the teachers they require, for there is a constant influx from the East. Both climate and other conditions are very attractive on the Pacific Coast. At Seattle, for instance, there is no Normal School at all. San Francisco weeds out her numerous applicants by a special competitive examination, but has also a large Normal School. In this, and in the similar one at Los Angeles, there is no provision for training teachers for vocational subjects. It is curious that when these subjects are

established as an essential part of the free school curriculum, the State should make no provision for training teachers in them. In California, as elsewhere, the authorities have relied on teachers from Europe, whose thorough technical training they admire, and teachers from the large private institutions in the East. Now, however, an institution has been taken over by the State in California which will doubtless be the first of many, with the object of training teachers expressly for the advanced technical and vocational teaching required in the newer high schools. This is the State Normal School of Manual Arts and Home Economics, Santa Barbara, California.

The School began in 1891 as the "Anna S. C. Blake Manual Training School," founded by a lady philanthropist. To it went the lady who is now President, Miss Ednah A. Rich. After her training in the school she spent some years in Europe, studying various handicrafts under their most famous exponents. On her return to Santa Barbara she became Supervisor of the School and of Manual Training in the public schools. Five years ago, the State wanted to make some provision for training teachers in vocational subjects, and as this school was already famous for the quality of its work, granted it a subsidy of 1,000*l.* a year on condition that it should take and train students. Miss Rich had many applicants, but she was determined to make the training a great success, even with the limited funds allowed, so she accepted only 25 students, and chose them with very high qualifications indeed. As a result, they all got posts before their training was over. Next year she took 30 students, and had to select them from hundreds of applicants. Then the State made the school a large grant, and built them a very beautiful new building, into which they entered last year. A year ago, there were 150 students working in the little old school, and the new buildings will accommodate 300 or more.

The students are generally university graduates, and often teachers of experience as well. They must have some technical skill in addition. Miss Rich said they did not want the High School graduate of 18, and did not incline to the student who had been through the longer courses at such schools as the Los Angeles Polytechnic High School. She seems to have trained numbers of teachers who, having already taught the academic subjects in the high schools, have become convinced of the high educative value of vocational and technical work, if done and taught in the right spirit. Over 90 per cent. of the students are women. This may be because there is a woman President, but is more probably because there is so great a demand for men teachers that they need not give up time to special training in order to secure posts. The enthusiasm shown by the students is remarkable. Directly they arrive to train as teachers, they are informed that they are teachers already. If they remark that they do not know how to make

anything, they are reminded that the children know less. They begin promptly to make things, and directly they are made, to teach the children how to make them. They are supervised by all the manual training teachers in the city, who are in their turn supervised by Miss Rich, and who all understand and sympathise with her methods and ideals. All the work turned out by the students is thoroughly professional, and that accomplished by the children under their care is not far behind it. All kinds of woodwork and metal work are taught, from heavy forging and casting to jewellery in silver and other metals. Applied art includes the finest woodcarving, leather carving and staining, bookbinding, design and embroidery. Some work in architecture and house building is being developed, and the President designed the building in use a year ago. An elementary course leading to plumbing and gasfitting is also arranged. Pottery and printing are in great favour. On the domestic side, the usual subjects are studied with special reference to teaching them, and to the bringing out of their educational values. Miss Rich also intends the students to apply themselves specially to food problems in connection with the lunch rooms which will be an important part of the new building, and in which only the best and purest food will be served, after it has been thoroughly tested, cooked and prepared in the best way. She intends the students to make a study of economics in their home surroundings, and, as the State does not provide residences for them, to eschew boarding-houses and to take bungalows, two or more together, and work out healthy ways of living on 3*l.* a month, or less. Santa Barbara is not a cheap place.

In most of the State Normal Schools elsewhere in rural districts some training is given in Household Arts and Manual Work. The Colorado State Teachers' College at Greeley is an example of this, and teachers are trained there for High School teaching in the High School attached to the College, as well as in the Elementary School. But the Santa Barbara School is unique as a training college for teachers of high academic qualifications and designed solely for advanced work, above the High School standard if necessary. The period of training varies according to the number of subjects which the student wishes to complete. Some become so adept that they set up for themselves as producers of art metal work or other such things; one accepted a post as designer at the greatest jewellery house in San Francisco. This is not because they could not get work as teachers, or because they underestimate the importance of teaching. Miss Rich is very emphatic in setting forth the mental value of perfectly performed manual work, and unifies all the training by following out the ideal of her master, Mr. Cobden-Sanderson, that man and woman should learn to work together at what will build and adorn the home.

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MARINE VOCATIONS

MERCHANT MARINE - - -	Officers and engineers of coasting and sea going vessels.
NAVAL ARCHITECT - - -	Drawing and construction of vessels - -
<i>MARINE BIOLOGY.</i>	
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2. FISH EXPERT - - -	- - - - -
3. FISH PROPAGATOR - - -	- - - - -
And similar. Vocations - - -	- - - - -

THE POLYTECHNIC HIGH SCHOOL devotes itself to courses leading to all of them. All of the high schools offer

In almost all the vocations here listed, the student needs

TECHNICAL and SEMI-TECHNICAL VOCATIONS—TRADES.

ASSAYER - - - - -	Testing and rating Ores - - - - -
BLACKSMITH - - - - -	Working metals by heat, by hand or power -
CABINET MAKING - - - - -	Making furniture, interior fittings, &c. - -
CHEMISTRY - - - - -	Work in analytical laboratories, or drug or chemical supply houses.
DRAFTSMAN, ARCHITECTURAL - - - - -	Preparing plans for buildings, &c. - -
DRAFTSMAN, MECHANICAL - - - - -	Making working drawings for machines, machine parts, &c.
FOUNDRY - - - - -	Making metal castings in sand from patterns
<i>ELECTRICAL WORK.</i>	
CENTRAL STATION SUB-STATION. - - - - -	Care of dynamos, transformers, &c. - -
TELEPHONE WORK - - - - -	Wiring, station work, line work, &c. - -
ELECTRIC LIGHT WORK. - - - - -	
ELECTRICIAN - - - - -	General electrical work, contracting, estimating, &c.
MACHINE SHOP WORK - - - - -	Making machines, parts, &c., by means of lathes, planes, &c.
PATTERN MAKING - - - - -	Making wood models of machine parts from which to make castings. Requires great skill and mathematical accuracy.
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