

DYNAMIC ECONOMICS

THEORETICAL AND STATISTICAL STUDIES OF DEMAND,
PRODUCTION AND PRICES

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DYNAMIC ECONOMICS

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TO MY MOTHER,
MARY ISABEL HOLDSWORTH ROOS,
WHOSE THRIFT MADE MY EDUCATION POSSIBLE AND
TO MY WIFE
MARY MAE BARKULOO ROOS,
WHOSE ENCOURAGEMENT AND DEVOTION HAVE CON-
STANTLY INSPIRED ME IN MY WORK

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PREFACE

In 1929 I decided to refrain from publishing research papers in economics until I had a sufficient number ready to justify the publication of a book which would relate these researches to each other and to my earlier papers. The basic reason for this decision was the fact that each paper required so much of the background of previous papers that an essay of great length was required to explain a simple new idea.

I originally wrote up ideas associated with my paper "A Mathematical Theory of Price and Production Fluctuations and Economic Crises"* in a sixty-five page paper that explained a few of the concepts. I was at a loss to know where to send the paper for publication. Finally, I sent it to a mathematical journal. The editor wrote me that he would be glad to publish the mathematical parts of the paper if I would rewrite it and delete the economics. Instead of following this suggestion I sent the paper to a statistical journal and was told that the journal would publish the parts dealing with statistics and economics if I deleted the mathematics! I might have complied with this suggestion and the one of the mathematical journal and published two papers instead of one, but I still felt that the paper ought to be published in full in one journal. I therefore sent the manuscript to the *Journal of Political Economy*. It was accepted but publication was delayed. After waiting for a year an editor suggested that I write an abstract about a dozen pages in length. I did this and as a consequence produced a paper that I myself have great difficulty in reading now.

Following my experience with the paper on crises I set about to write a treatise on dynamic economics. While holding a *Social Science Research Council Fellowship* in the summer of 1930, I wrote Chapters I, II, IX and X almost in the form in which they appear in this book. In fact, the sections on the relation of maximum profits to employment (Chapter IX) and on technological unemployment (Chapter X) are the chief additions to the work done on this fellowship. Parts of Chapters II and V were published recently in my paper "Theoretical Studies of Demand."†

* *Journal of Political Economy*, Vol. XXXVIII, October, 1930.

† *Econometrica*, January, 1934.

After I became permanent secretary of the American Association for the Advancement of Science in February, 1931, I was unable to carry forward the job of writing a book. In the spring of 1933 the award of a *Guggenheim Memorial Foundation Fellowship* made it possible for me to take up my work where I had laid it aside two years previously. During the three months that I spent in London on this fellowship I was able to write most of Chapter V, the section on technological unemployment in Chapter IX, most of Chapter XI, Sections 3, 4 and 5 of Chapter XII, and most of Chapter XIII.

On July 10, I received a cablegram from Alexander Sachs, then Director of the Division of Research and Planning of the National Recovery Administration, requesting me to return to the United States to do econometric research work for the Recovery Administration. I owe a large debt of gratitude to the Guggenheim Foundation for granting me indefinite leave so that I could accept this position. One of my earliest assignments at the National Recovery Administration called for an analysis of Roy Wenzlick's data on factors influencing residential building activity in Greater St. Louis. While this study was in progress, the Recovery Administration asked for a theory of joint demand and loss leaders. The next request was for "an equation of exchange in a capitalistic economy by tomorrow." By drawing on the unfinished work of Chapter XIII, Max Sasuly and I were able to produce "an equation of exchange in a capitalistic economy" in the time allowed. The Administration next suggested a study of automotive demand for gasoline in order to determine whether public works expenditures for highways could be expected to bring increased demand for gasoline, automobiles, steel, etc., so that acceleration of recovery could be attained by such expenditures.

Several of the above-mentioned studies were presented as papers at the 1933 winter meetings of the Econometric Society. Professor Harold T. Davis of Indiana University suggested that the papers be published together in a monograph. I countered with a suggestion that they be made part of a book on *Dynamic Economics*. I felt that the new material together with what I had prepared at Cornell University and in London could be put together very quickly in a book which in some respects would go beyond what I had intended to accomplish on my fellowship. I felt that even though chapters on banking, wages, value and foreign ex-

change, originally intended to be included in my book *Dynamic Economics*, were omitted, the material that I had was sufficiently cohesive and complete to be brought out at this time, especially if certain additions—Chapters IV, VII* and the first part of XII—were made. Accordingly, I am able at this time to publish the first part of what I hope to be a three volume presentation of *Dynamic Economics*.

Concomitant with the writing of this book much important economic legislation has been enacted in the United States. The National Industrial Recovery Act, which was passed by Congress, June 16, 1933, has led to important fundamental changes in methods of doing business. It has been my privilege to follow closely the course of these changes as Director of Research for the National Recovery Administration, since July 27, 1933.

On September 15, 1934, I shall become Director of Research for the Cowles Commission for Research in Economics and will then begin to write an economic and statistical appraisal of this New Deal legislation. I hope to publish this Volume some time in 1935. The third part of my work will deal with such subjects as banking, wages, foreign exchange, and value.

While research material developed at the N.R.A. appears in this work, theories presented are my own personal ones and are in no sense to be taken to represent the official views of the National Recovery Administration. This work is presented as research material and not as something to uphold or criticize the economic policies of the present Administration. I have tried to be an econometrist; that is, I have tried to present my material without political or nationalistic bias. I make no claim of having proved the various theses offered. In a science that is developing as rapidly as is econometrics it must be expected that there will be false leads which will have to be corrected. If my effort is stimulating to economic thought and if it is provocative of further research, I shall feel happy even though some of the theses are disproved.

I have not attempted to provide an extensive bibliography of econometric material pertaining to the work presented here, nor have I attempted to acknowledge all ideas that I might have borrowed from others, or had independently of them. I have naturally done a large amount of reading of economic works by Pareto, Cournot,

*Chapter VII grew out of a suggestion of Victor von Szeliski that I ought to include something on the economic structure of an expanding economy.

Marshall, Bowley, Fisher, Edgeworth, Wicksell, Amoroso, Divisia, Schumpeter, Davenport, Moore, Young, Keynes, Pigou, Mitchell, Frisch, Evans, Schultz, Hotelling and others.

I am deeply indebted to my secretary, Mrs. Mildred Chisholm, who spent much time at night after working hours typing the manuscript. Her painstaking care in checking statements and equations has made it possible for me to present an accurate manuscript. I am also greatly indebted to Andrew Court for reading the manuscript and suggesting pertinent changes. I, of course, assume full responsibility for any errors that may appear.

Emily Pixley, Victor Perlo, Max Sasuly, Clement Winston, Jack Biscoe, Goldie Back, Anne Golden and Elizabeth Wilcox helped with the computations for Chapters III and VI. Other acknowledgments to Mrs. Pixley, Victor Perlo and Max Sasuly appear in the text.

Professor H. T. Davis of Indiana University and the Principia Press offered valuable suggestions and arranged for the printing of the manuscript. Alfred Cowles III, Director of the Cowles Commission for Research in Economics, had the charts redrawn so that lettering etc., would be uniform, and offered helpful advice as to arrangement and format.

The Dentan Printing Company has patiently incorporated changes in the text and in general extended every possible courtesy to me.

To all who have helped in the preparation of this volume, I am deeply grateful.

C. F. Roos.

APPENDIX I

CORRELATION OF TIME SERIES

Some workers, inspired by G. Udney Yule's paper, "Why Do We Sometimes Get Nonsense Correlations Between Time Series," *Journal Royal Statistical Society*, Vol. 89, 1926, pp. 1-84, in which certain nonsense correlations are exhibited, have fallen into the natural error of maintaining that one should always "eliminate trend" and correlate the residuals. A reference to a physical problem is probably the quickest way of showing how ill-founded this opinion really is. Let V = voltage, R = resistance, and I = current, and for definiteness suppose $R = 2$ so that $V = 2I$. Now, it is possible to vary the voltage V so that $V = 6t +$ random errors of measurement, where t is time measured from some fixed time t_0 . In a physical experiment of this type, it is possible to read meters, etc., to a high degree of accuracy. Small random errors would occur in V and I due to a variety of uncontrolled causes, such as differences in temperature of the atmosphere surrounding the meters, etc. To remove the trend $6t$ from V and $3t$ from I and correlate the residuals would be to correlate random errors. A zero correlation would be found between voltage and current. This is, of course, much greater nonsense than the nonsense mentioned by Yule.

In the case of economic series, the relative size of the error is much larger than in the physical example just given. Thus, it may happen that the removal of a linear trend or a quadratic trend will leave little more than random errors. The trend in this case is the significant movement. Jordan has shown that if orthogonal functions are used to fit trends to two series and the residuals are correlated, the correlation coefficient is a function of the parameters of the trend functions, and has derived formulas describing the relationship.

Suppose that $y(t)$ and $u(t)$ represent two observed time series which it is desired to correlate. Suppose that Tchebycheff polynomial trends

$$Y = C_0 + C_1\varphi_1 + C_2\varphi_2 + \dots + C_n\varphi_n$$

and $U = K_0 + K_1\varphi_1 + K_2\varphi_2 + \dots + K_q\varphi_q$,

where C_0, \dots, C_q and K_0, \dots, K_q are constants, are fitted to the series $y(t)$ and $u(t)$ respectively. Let $z = y - Y$ and $x = u - U$. Then, as Jordan has shown, the simple correlation between z and x is given by the formula

$$r = \frac{\frac{1}{n} \sum yu - C_0K_0 - C_1K_1 \dots C_qK_q}{\sqrt{\sigma_y^2 - C_1^2 - C_2^2 \dots C_q^2} \sqrt{\sigma_u^2 - K_1^2 - K_2^2 \dots K_q^2}}$$

where σ_y is the standard deviation of y , and n is the number of observations in the series.

If orthogonal functions different from Tchebycheff polynomials are used for example, $\sin mx$, $m = 1, 2, 3, \dots$, a different correlation coefficient will evidently be obtained. A single term $\sin mx$ might conceivably represent a trend better than a fourth degree polynomial, and of course there is nothing sacred about a polynomial trend.

The analysis of Jordan's paper* may be carried through for general orthogonal functions as well as for the special orthogonal functions which he uses. The form of the correlation coefficient will be the same, but its value depends on the particular orthogonal functions used and on the number of terms employed in the expansions. Consequently it is not at all clear what a coefficient of correlation of deviations from trend means. In fact, it is simple to show that by using one set of orthogonal functions a correlation of almost -1.00 can be obtained between two series, whereas by using different orthogonal functions a coefficient of correlation of almost $+1.00$ can be obtained.

Let $w(x)$ and $u(x)$ be two functions defined on (a, b) with approximating parabolas y_n and η_n respectively, y_n and η_n being of degree n .† The correlation coefficient of the deviations $(w - y_n)$ and $(u - \eta_n)$ is defined to be

*Charles Jordan, "Sur la Détermination de la Tendance Séculaire des Grands Statistiques par la Méthode des Moindres Carrés". Tirage à Part de *Journal de la Société Hongroise de Statistique*, Année 1929, No. 4.

†This is a report of a study undertaken by Emily C. Fixley and Clement Winston at the suggestion of the author.

$$r = \frac{\int_a^b (w - y_n)(u - \eta_n) dx}{\sqrt{\int_a^b (w - y_n)^2 dx \int_a^b (u - \eta_n)^2 dx}}$$

If $w(x)$ and $u(x)$ are developed as series of orthogonal functions,

$$\begin{aligned} w(x) &\infty y_n = a_0 + a_1\varphi_1 + \dots + a_n\varphi_n \\ u(x) &\infty \eta_n = b_0 + b_1\varphi_1 + \dots + b_n\varphi_n, \end{aligned}$$

$[\varphi_i]$ being any complete set of orthogonal and normal functions on (a, b) , (that is

$$\int_a^b \varphi_n(x)\varphi_m(x) dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases},$$

the correlation coefficient may be expressed in terms of the coefficients of the expansion as follows:

$$r = \frac{\int_a^b w(x) \cdot u(x) dx - \sum_{i=0}^n a_i b_i}{\sqrt{\left[\int_a^b w^2(x) dx - \sum_{i=0}^n a_i^2 \right] \left[\int_a^b u^2(x) dx - \sum_{i=0}^n b_i^2 \right]}}$$

Although the coefficients a_i, b_i , vary for different sets of orthogonal functions, the form of r remains invariant. It is thus seen that in general nothing very definite can be said about the value of r obtained by this method, since it does not seem possible to obtain a relation between the different sets of coefficients.

Some examples which show considerable difference in the value of the correlation coefficients obtained by using different sets of orthogonal functions, on the same interval, were next set up.

1. Consider the interval $(0,1)$, and the functions $w(x) = x$, and $u(x) = 2x - x^2$ on this interval. Using the first three terms of the expansion in Legendre polynomials it can be shown that

$$w(x) \infty 1/4 P_0 - 9/20 P_1 + 1/4 P_2$$

$$u(x) \infty 3/4 P_0 - 11/20 P_1 - 1/4 P_2$$

where $P_0 = 1, P_1 = (1-2x), P_2 = (1-6x+6x^2)$.

In this case $a_0 = 1/4, a_1 = -9/20, a_2 = 1/4$

$$b_0 = 3/4, b_1 = -11/20, b_2 = -1/4.$$

Substituting these values in the formula for r it is found that $r = -1.00$.

On repeating the process above, representing the same functions by Fourier coefficients, a strikingly different value of the correlation coefficient is obtained. In fact, here

$$\begin{aligned}w(x) &\infty .25 + 3/2\pi^2 \cos 2\pi x + (3/2\pi^2 - 1/\pi) \sin 2\pi x \\ &\quad + 3/8\pi^2 \cos 4\pi x - (3/16\pi^2 - 1/2\pi) \sin 4\pi x \\ u(x) &\infty .75 - 3/2\pi^2 \cos 2\pi x - (3/2\pi^2 + 1/\pi) \sin 2\pi x \\ &\quad - 3/8\pi^2 \cos 4\pi x - (3/16\pi^2 + 1/2\pi) \sin 4\pi x.\end{aligned}$$

and using the values

$$\begin{aligned}a_0 &= .25, & a_1 &= 3/2\pi^2, & a_2 &= 3/2\pi^2 - 1/\pi, & a_3 &= 3/8\pi^2 \\ & & & & a_4 &= 3/16\pi^2 - 1/2\pi \\ b_0 &= .75, & b_1 &= -3/2\pi^2, & b_2 &= -(3/2\pi^2 + 1/\pi), \\ & & & & b_3 &= -3/8\pi^2, & b_4 &= -(3/16\pi^2 + 1/2\pi)\end{aligned}$$

the correlation coefficient turns out to be $r = +.976$.

2. For the interval (0,1) consider the functions $\bar{w}(x) = x^3$, $\bar{u}(x) = x^4$.

As above, it can be shown that, using Legendre polynomials, $r = 0.9921$, while using Fourier series, $r = 0.176$.

With reference to the first example, it can easily be demonstrated that for the interval (0,1), the coefficient of correlation is always -1 if w and u are expanded in a series of Legendre polynomials and $w = 2x - u$. In this case, the number of terms in the expansion of w and u is arbitrary.

If the coefficient of correlation r is to be used to indicate a linear relationship between the two functions considered, then the results presented here would tend to show that such a notion is entirely erroneous. The literature, however, seems to give methods for determining the coefficient only, with no discussion of its use or significance.

It may be of interest to study the effect of using the polynomials given by Jordan, Fisher and Sasuly to observe whether the same results are obtainable when sums instead of integrals are used. One would expect similar results.

In examining time series, it is of paramount importance to take all major factors into account. A rational, theoretical analysis based on broad acquaintance with the economic situation to be analyzed is fundamental. If all important factors are taken into account, the influence of the many neglected but minor factors (all economic quantities are related) should give a net constant effect ("systematic error", in the theory of the adjustment of observations). If the analysis is complete, there will be only random residuals left. The use of a trend in correlation analysis is therefore a confession of ignorance of fundamental factors involved. If a time-trend must be used owing to lack of observations for certain neglected factors, it can be eliminated implicitly, as in Ezekiel *Methods of Correlation Analysis*. Ragnar Frisch and Frederick V. Waugh, *Econometrica*, October 1933, show that for additive linear trends the implicit trends are identical with the linearly fitted trends. See also Ragnar Frisch, *Pitfalls in the Statistical Construction of Demand and Supply Curves*. Or, time-trends should be treated as systematic variations appearing in the residuals left after effects of the known factors have been eliminated. It is hardly necessary to refer to the high (but fallacious) correlations of data obtained by manipulating trends. All correlations of residuals from trends are therefore to be looked upon with suspicion.

APPENDIX II

ANALYSIS OF RANDOM ERRORS IN TIME SERIES*

The statistical theory of time series has usually been based on the assumptions that the observations are the resultant of two main types of variation:

(a) Systematic variation: for instance, (1) secular variation (production series), (2) periodic variation, such as in the theory of the tides, and the seasonal variation in economic phenomena, (3) cyclic variation (business cycles).

(b) Random variation, chance: for instance, the residuals left after graduation of the observations by seasonal analysis and a good smoothing formula. The theory and the treatment alike have assumed that the random errors are the same kind of thing as the random errors of the classical theory of observations, developed for measurements on precision instruments.

Now, in a time series consisting of the successive readings of a precision instrument, it is assumed, and in general correctly, that the residuals, after elimination of constant and systematic error, are mutually independent of one another. Their assumed origin guarantees this independence. Thus, the residual v_1 in the observation y_1 at time t_1 , and the error e_1 in that observation, are conceived to be the net result of a multitude of individually small shocks, etc., to the instrument, and defects in the sensory apparatus of the observer. Each such minute shock, twist, strain, etc., is an "elementary error," the summation of all of which is the actual error. From the hypothesis of elementary errors the well known normal law

$$f = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

may be deduced. Similarly with v_2 and e_2 in the observation y_2 . But there are no shocks common to e_1 and e_2 , and therefore e_1 and e_2 are wholly independent of one another. Or, rather, shocks or strains which persist from one observation to the next with the same sign,

*This appendix was prepared by Victor von Szeliski.

are constant errors and systematic errors, and may be eliminated by an approximate method of graduation. There may be elementary errors and shocks common to v_1 and v_2 due to inadequate elimination of systematic error.

But in many economic time series, especially historical frequency series, the observed residuals from careful graduations do not conform to the normal law of error, and consideration of how they arise will show that they differ from the random errors and random residuals of classical theory.

True, the residuals do doubtless arise in part from causes of the same type as the random variations of theory, but in addition two successive errors may have the same "elementary error" in each, but *opposite in sign*. Thus, in building contracts a particular contract may be due to be signed on May 31 and reported in that month; but "by chance" it is recorded on June 1 and goes into that month. The May observation is of the form $B_1 - e$ and the June contract is of the form $B_2 + e$, one being increased, the other diminished, by the same amount. Or several contracts may be lumped together and assigned to one month or the other, whereas actually some were in one month and some in the other. Or where consumption of a commodity is determined by adding production and decrease in inventory, errors in the inventory figure through failure to measure it correctly or to measure it at the proper time, will enter the succeeding determination with opposite sign.

Price series may also have this type of variation; thus there is some tendency for violent changes in prices of speculative commodities and stocks on one day to be followed by reverse changes the next.

We may call errors and residuals of this type *alternating errors and alternating residuals*. A series containing them could be symbolized this way:

(1)

$$\begin{aligned}
 y_1 &= x_1 + a_0 - a_1 + e_1 \\
 y_2 &= x_2 + a_1 - a_2 + e_2 \\
 y_3 &= x_3 + a_2 - a_3 + e_3 \\
 y_4 &= x_4 + a_3 - a_4 + e_4 \\
 y_5 &= x_5 + a_4 - a_5 + e_5 \\
 y_6 &= x_6 + a_5 - a_6 + e_6 \\
 &\dots \dots \dots \dots \dots \dots \dots \\
 y_n &= x_n + a_{n-1} - a_n + e_n ,
 \end{aligned}$$

where y_1, \dots, y_n are the observed values; x_1, \dots, x_n are the true values; $a_0, a_1, a_2, \dots, a_n$ are the alternating errors; and e_1, \dots, e_n are the random errors. If these are summed, it follows that

$$(2) \quad \sum_1^n y = \sum x + a_0 - a_n + \sum e.$$

Since, in general, a_0 is independent of a_n ,

$$(3) \quad \sum y = \sum x + \bar{a}\sqrt{2} + \bar{e}\sqrt{n},$$

where \bar{a} and \bar{e} are the average absolute values of the alternating and the systematic errors to be feared. Thus if monthly observations are summed over a year, the alternating errors cancel out with the exception of the errors in the terminal months coming in from the "outside." For example, while a daily series of building contracts awarded appears to be mostly chance variation, with little observable regularity, a series of annual totals exhibits comparatively smooth cycles, with very little of the random. While alternating errors still exist at each end of the larger time interval, the size of the error is reduced relative to the size of the observation.

This analysis suggests several problems:

(1) Has a series containing alternating errors distinctive statistical characteristics, such as functions of successive differences, which would identify it as such?

(2) What is the best way of graduating the observations y ?

(3) Can estimates be made of the alternating residuals, or at least of the sum of successive pairs, $a_{i-1} - a_i$, so that the series can be analyzed into:

(a) Graduated values x_i ?

(b) Alternating residuals a^1_i or $(a^1_{i-1} - a^1_i)$?

(c) Random residuals e^1_i ?

(4) What are the probable errors of the graduated values?

(5) What are the probable errors of constants determined by fitting curves to the observations?

(6) How should curves be fitted to such series, so as to give the best fit?

(7) Does this theory throw any light on how to calculate the standard error and reliability of coefficients of covariation between time series?

With regard to No. 7, the crux of the question appears to be: how many *independent* observations are there in a given time series? If the original series of observations $y_i \dots$ has considerable alternating error, it may be replaced by a series

$$Y_1, Y_j, Y_{2j}, Y_{3j}, \dots, Y_{nj}$$

where

$$Y_j = y_j + y_{j+1} + \dots + y_{2j-1}.$$

As j (the length of the unitary time interval) is increased, the alternating errors are damped out, and the successive Y 's tend less and less to be repeated observations of the same quantity (if x changes slowly, y_i and y_{i+1} are clearly not independent at all, but are, to a large extent, repeated observations). Is a point finally reached where the Y 's satisfy a test of independence (presumably where functions of the differences of Y take values within assigned ranges)? If so, the number of independent observations in a time series, or in a pair of correlated series, may be found, and the probable error of the coefficient of covariation determined.

APPENDIX III

RELIABILITY OF DATA RELATING TO RESIDENTIAL BUILDING

The following data on residential building apply to St. Louis proper. In the last fifteen years the chief development has taken place outside the city limits, in St. Louis County.

1) *New Building*. This is the number of family accommodations provided for as stated in permits issued by the city of St. Louis. The figures include apartment hotels, apartment houses and flats as well as single-family residences. An apartment building with 24 apartments is counted as 24. Transient hotels are excluded, and of course all buildings of a non-residential character. The figures are those of Roy Wenzlick, real estate analyst, who compiled them from the original records of the city.

2) *Number of Families*. The source is the Bureau of the Census. The definition of family, as used by the Bureau, is rather broad. It includes groups living under one roof but not related by blood, such as institutional families. For this reason, if for no other, the one-to-one relationship of family accommodations to families postulated in this study cannot be expected to hold exactly.

The Census figures were interpolated by overlapping cubics fitted in such a way that the first derivative of the interpolated function is continuous throughout.

3) *Gross Rents*. Accurate rent statistics are difficult to obtain. The most widely used rent indexes, those of the Bureau of Labor Statistics, and the National Industrial Conference Board, are based on small samples of *estimated* rentals. Rents remain one of the most important gaps in statistical information.

The figures used in this study were built up from actual rentals in recent years, and the asking rentals in the classified newspaper advertisements in other periods. For 1924 to 1932, the rental figures are averages of actual annual rentals received on fully occupied four-family flats of a certain type quite common in St. Louis. Rentals for 1897 to 1923 were estimated by linking on an index of asking rentals. The rents at which houses were offered

were collected from the advertising section of the leading St. Louis newspaper, and classified according to type and number of rooms. In the earlier part of the period, up to and through 1912, observations were obtained for January of each year only, unless too few cases were found in that month, in which case additional cases were collected for February, March and so on until a sufficiently large sample had been obtained. In 1913, rents were obtained for four months, January, April, July and September; in 1914 and 1915 for January and July; in 1916 for January, February and July; in 1917 for January, April, July, October and December. They were obtained for every month in 1918, for four months in 1919, five months in 1920 and 1921, three months in 1922 and once a year (January) in 1923 to 1928 inclusive. The 1929 figure was based on rents for January and July and the 1930 figure for January, July, September, October and November. Starting in 1931, asking rentals were collected monthly.

The rents for each class and number of rooms were put on a per room basis and averaged. The per room figures were then charted on a semi-logarithmic paper. It was found that the lines ran very closely parallel with one another, a circumstance which lent color to the statistical stability of the asking rental as a measure of actual rental. An average was then picked out on the graphs by inspection. These averages ranged from a low of about \$2.50 per room in 1899-1900 to a high of somewhat over \$10.00 per room in 1920.

The next step was to convert these scattered observations representing isolated months into estimated averages for the year as a whole. Prior to 1913, this was accomplished simply by averaging two succeeding observations, weighting the first two-thirds and the second one-third. From April 1912 the observations were interpolated graphically by months, and then a weighting function applied. This weighting function was based on the seasonal distribution of moving in St. Louis as determined from the records of moving companies. The yearly averages determined in this way were put on an index base with 1926 = 100.

Finally, the actual rentals for the period 1924-1932 were extended backwards by use of this asking rental index. For this purpose \$2,186 was taken equal to 100. This will give a correct result provided that asking rentals are subject to a constant percentage error. This can be assumed to be true in periods where the asking rental index remains relatively constant. Where, however, it is not

constant, but changes rapidly, it cannot be assumed that asking rentals are always a constant fraction or multiple of actual rentals. A comparison of the index of asking rentals with the rent index of the Bureau of Labor Statistics showed wide discrepancies during the period 1919-1923 when rents were rising so rapidly. There was pretty fair agreement for other periods. The reason that the index of asking rentals does not represent true conditions during periods of rapid change of rent levels is that asking rentals are rentals on a few parcels of properties that happen to be in the market at that moment. It is a time when occupancy is high and there are few houses on the market. They do not represent the rentals actually paid on the houses. Rents change slowly because they are long-term contracts for the most part. Also, during a time of boom, landlords are carried away by the general good feeling and ask much higher prices than they expect to get.

At any rate, the index of asking rentals seems definitely at fault in the early post-war period and it was arbitrarily changed for the years 1918 to 1923, inclusive, to conform more closely to the movements of the Bureau of Labor Statistics rental index.

It is assumed that the same phenomenon occurred during the boom of 1903 and 1904, and the index of asking rentals for those years was arbitrarily lowered.

In summary, the rents given in Column 6, Table 1, are based on asking rentals for 1897-1917, Bureau of Labor Statistics rent index, adjusted, for 1918-1923, and actual rents during 1924-32-33.

The estimated rents for the earlier years must be regarded with some suspicion. It is certainly difficult to believe that the fast drop in rentals shown to have occurred from 1897 to 1899 was in accord with the fact. These were years of emergence from the deep depression of the nineties. Commodity prices, national incomes and volume of production were rising and it is exceedingly doubtful whether rentals had a sharp trend in the opposite direction.

4). *Taxes.* Taxes are averages of actual figures for the period 1924 on. They were extrapolated backwards by means of an index of the average real estate tax collections per family in St. Louis. On the assumption that in the long run the number of dwellings equals the number of families, this index gives a fairly accurate picture of the average tax burden per individual dwelling. The chief objection to the figure as it stands is that it includes taxes on non-residential property. It was not possible to eliminate these.

The purpose of putting taxes on a per family basis is that it automatically corrects for the change in the size of the average family, and the consequent change in the size of unit dwelling.

5). *Construction Costs.* Construction costs are for a typical St. Louis residential building. The particular building selected was a four-family flat. This type was probably built in greater numbers with very slight variations than any other general type during the period studied. Cost figures for this building are very much more than a simple aggregative or price index of a few building materials. They represent a very detailed and careful calculation of costs covering all items entering into building, down to the corner bead on the walls. These specifications were changed several times during the period to give recognition to changes in available materials and differences in building practice. For example, during the past year conduit and B X replaced the knob and tube wiring formerly allowed. In 1907 the kitchen drainboards were wooden, the toilets had high wooden tanks, the bath tubs were on legs, the bath floors were not tiled and the other floors were not hardwood. Labor costs were computed on what was actually paid, rather than on some scale which, at least in periods of depression, is only nominal. Further details are exhibited in the accompanying table.

Cost of lot is included in figures given in Column 11, Table 1.

6). *Change in Units of Measurement.*

One question which must be faced in regard to the foregoing data is whether the units of measurement retain their significance unchanged throughout the period. We have already alluded to the change in the size of a typical family since the turn of the century. This is to some extent taken care of by the *b*-function developed in the text, but not wholly. Some families obtained accommodations in the latter part of the period by remodeling large single-family houses built during the earlier part of the period, two or more families entering what had been when built a large single-family residence.

The rental figure represents rents on a uniform size of house throughout the period. On a per family basis, it should be relatively too low in the early years because there were more rooms per family at that time. It should be relatively high in the later years because the average family now needs fewer rooms. Fortunately

the cost of land and building is also for a building of fixed size, and since rents and costs are respectively in the numerator and denominator of the incentive fraction, the change in the fundamental unit of measurement has little effect on the result. The errors, if any, compensate partially or wholly. However, taxes are not on a constant space base unit basis, but on a per family basis and therefore do not agree in regard to unit of measurement with rents and costs. Consequently, some systematic error in measurement is apparently introduced into the measurement of incentive.

There is some doubt, then, as to whether the formulas developed in the text have taken account of all relevant and important factors. It would not be surprising if the figures for actual building were found to deviate systematically from the calculated figures. That is, a trend might be expected to appear in the residuals, showing that not all variables had been taken into account.

In view of these considerations regarding possible change in the unit of measurement since 1900, there is ample justification for including some time trend factor in the equation to take care of it. There being no indications whatsoever as to what functional form this factor should take, it could be assumed to be linear. However, the writers have not introduced such a factor because the observation seemed to fit naturally without the introduction of a time-trend. The residuals do not appear to exhibit any systematic variation of a trend type. As developed in the text, fitting a trend is to be accepted only as a last resort, and after its implications have been carefully thought out. The writers are of the opinion that the compensation mentioned above has actually taken place and suggest that in many other problems in which the fitting of trends seems to be indicated, the suitable selection of functions and factors would render their use unnecessary in many instances.

APPENDIX IV

TRANSFORMATION OF INTEGRAL RANGE

To transform from the infinite range on which W is defined to a finite one, write

$$\begin{aligned}
 W &= \int_{-\infty}^t \frac{b_1}{\sqrt{2\pi} \sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx \\
 &+ \int_{-\infty}^t \frac{a_1 I(x)}{\sqrt{2\pi} \sigma_2} e^{-(t-x-M_1)^2/2\sigma_1^2} dx \\
 &= \int_{-\infty}^t \frac{b_1}{\sqrt{2\pi} \sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx \\
 &+ \int_{-\infty}^{t-t_0} \frac{a_1 I(x)}{\sqrt{2\pi} \sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx \\
 &+ \int_{t-t_0}^t \frac{a_1 I(x)}{\sqrt{2\pi} \sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx .
 \end{aligned}$$

By putting $(x + M_1 - t)/\sqrt{2}\sigma_1 = z$ in the first and second integral, W may be written in the form

$$\begin{aligned}
 W &= \int_{-\infty}^{M_1/\sqrt{2}\sigma_1} \frac{b_1}{\sqrt{\pi}} e^{-z^2} dz \\
 &+ \int_{-\infty}^{M_1-t_0/\sqrt{2}\sigma_1} \frac{a_1}{\sqrt{\pi}} I(\sqrt{2}\sigma_1 z + t - M_1) e^{-z^2} dz \\
 &+ \int_{t-t_0}^t \frac{a_1 I(x)}{\sqrt{2\pi} \sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx .
 \end{aligned}$$

Now, the first integral in W is a constant which may be called A_{01} . In the second integral let I_0 be the average value of I in the interval $-\infty \leq z \leq (M_1 - t_0)/\sqrt{2}\sigma_1$ and let $I_{\Delta}(\sqrt{2}\sigma_1 z + t - M_1)$ be a function such that $I(\sqrt{2}\sigma_1 z + t - M_1) = I_0 + I_{\Delta}(\sqrt{2}\sigma_1 z + t - M_1)$. Then by an application of the law of the mean for integrals,*

$$\begin{aligned} W &= A_{01} + \int_{-\infty}^{(M_1 - t_0)/\sqrt{2}\sigma_1} \frac{a_1}{\sqrt{\pi}} I_0 e^{-z^2} dz \\ &+ \int_{-\infty}^{(M_1 - t_0)/\sqrt{2}\sigma_1} \frac{a_1}{\sqrt{\pi}} I_{\Delta}(\sqrt{2}\sigma_1 z + t - M_1) e^{-z^2} dz \\ &+ \int_{t-t_0}^t \frac{a_1 I(x)}{\sqrt{2\pi}\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx \\ &= A_{01} + A_{02} I_0 + I_{\Delta}(t - M_1 - \lambda_1) \frac{a_1}{\sqrt{\pi}} \int_{-\infty}^{(M_1 - t_0)/\sqrt{2}\sigma_1} e^{-z^2} dz \\ &+ \int_{t-t_0}^t \frac{a_1 I(x)}{\sqrt{2\pi}\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx, \end{aligned}$$

where

$$A_{02} = \int_{-\infty}^{(M_1 - t_0)/\sqrt{2}\sigma_1} \frac{a_1}{\sqrt{\pi}} e^{-z^2} dz$$

and λ_1 is a positive quantity such that $t - M_1 - \lambda_1$ represents a time previous to the present time; thus, $I_{\Delta}(t - M_1 - \lambda_1)$ represents an average value of past fluctuations in incentive from an assumed normal I_0 .

*J. A. Hobson, *Theory of Functions of a Real Variable* Volume 1, page 617. The theorem referred to here does not apply to an integral with an infinite range, but the transformation $1/s = y$ reduces the infinite range to a finite one, the function I is bounded for all values of s and $(s^{-1}/y^2)/y^2$ is also bounded on the transformed finite range. The law of the mean referred to here can thus be applied to the transformed integral and after this application a transformation be made back to s .

It remains to find a t_0 such that

$$I_{\Delta}(t - M_1 - \lambda_1) \frac{a_1}{\sqrt{\pi}} \int_{-\infty}^{(M - t_0)/\sqrt{2}\sigma_1} e^{-z^2} dz$$

is negligible, i.e., so that, for statistical purposes, the error in taking

$$W = A + \int_{t-t_0}^t \frac{a_1 I(x)}{\sqrt{2\pi}\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx,$$

where A is a constant, instead of in the form (7.4), Chapter VI, is negligible. Obviously, since I_{Δ} is bounded above and below and

$$E = \int_{-\infty}^{(M-t_0)/\sqrt{2}\sigma_1} e^{-z^2} dz$$

can be made as small as desired by taking t_0 sufficiently large, it follows that $I_{\Delta} a_1 E / \sqrt{\pi}$ can be made as small as desired.

The problem for two or more integrals differs little from the one just discussed. In fact,

$$\begin{aligned} B(t) = & A_{01} + \int_{-\infty}^t \frac{a_1 I(x)}{\sqrt{\pi}\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx \\ & + A_{11} + \int_{-\infty}^t \frac{a_2 F(x)}{\sqrt{\pi}\sigma_2} e^{-(t-x-M_2)^2/2\sigma_2^2} dx \end{aligned}$$

can be written in the form

$$\begin{aligned} B(t) = & A_{01} + A_{02} I_0 + I_{\Delta}(t - M_1 - \lambda_1) \frac{a_1}{\sqrt{\pi}} \int_{-\infty}^{(M_1 - t_0)/\sqrt{2}\sigma_1} e^{-z^2} dz \\ & + \int_{t-t_0}^t \frac{a_1 I(x)}{\sqrt{2\pi}\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx \\ & + A_{11} + A_{12} F_0 + F_{\Delta}(t - M_2 - \lambda_2) \frac{a_2}{\sqrt{\pi}} \int_{-\infty}^{(M_2 - t_0)/\sqrt{2}\sigma_2} e^{-z^2} dz \end{aligned}$$

$$+ \int_{t-t_0}^t \frac{a_2 F(x)}{\sqrt{2\pi} \sigma_2} e^{-(t-x-M_2)^2/2\sigma_2^2} dx$$

where $F(x) = 1 - A/f(x)$, where $f(x)$ is the foreclosure rate, $A = .108$, that is, the constant determined in Part I, and F_0 and F_{Δ} are defined in a manner analogous to the manner in which I_0 and I_{Δ} were defined. Now, what is desired is a t_0 large enough so that

$$H = I_{\Delta} (t-M_1-\lambda_1) \frac{a_1}{\sqrt{\pi}} \int_{-\infty}^{(M_1-t_0)/\sqrt{2}\sigma_1} e^{-x^2} dz$$

$$+ F_{\Delta} (t-M_2-\lambda_2) \frac{a_2}{\sqrt{\pi}} \int_{-\infty}^{(M_2-t_0)/\sqrt{2}\sigma_2} e^{-x^2} dz$$

will be negligible, so that it will be possible to write

$$B(t) = A_{03} + \frac{A_1}{\sqrt{\pi} \sigma_1} \int_{t-t_0}^t I(x) e^{-(t-x-M_1)^2/2\sigma_1^2} dx$$

$$+ \frac{A_2}{\sqrt{\pi} \sigma_2} \int_{t-t_0}^t F(x) e^{-(t-x-M_2)^2/2\sigma_2^2} dx$$

where A_{03} , A_1 , A_2 , σ_1 , σ_2 , M_1 and M_2 are constants. Such a t_0 can obviously be found.

APPENDIX V

TECHNIQUE FOR FITTING FORMULAS INVOLVING INTEGRALS

Formula (11.1) in Chapter VI is itself not readily adaptable to methods of fitting, but a further approximation can be made. For this purpose replace the functions

$$\varphi(f) = \frac{1}{\sqrt{\pi} \sigma_i} e^{-\frac{(t-x-M_i)^2}{2\sigma_i^2}}, \quad i = 1, 2,$$

by lines as indicated in the accompanying figure. To obtain a function $E_1(t)$ with "moving lag" as illustrated, it is merely necessary to obtain the quantities

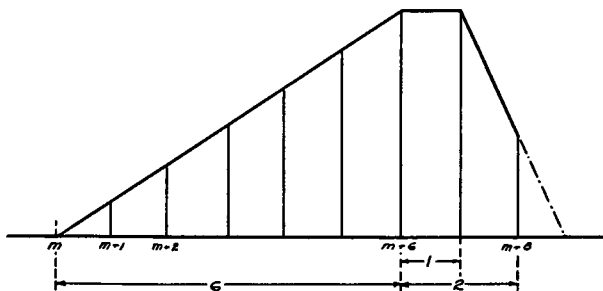


FIGURE 8

$$S_m^0 = E(1) + \dots + E(m)$$

$$S_m^1 = mE(1) + \dots + E(m) =$$

$$\sum_{i=1}^m (m - i + 1)E(i),$$

where 1 is the first year in the series, and $m = t - t_0$, since E_1 is given by the formula*

*See Max Sasuly, *Trend Analysis of Statistics*, published by Brookings Institution, 1934, Chapter X.

$$E_1(m + \mu + \nu) = \frac{2}{\mu \nu (\mu + \nu + 2)} \left\{ \nu [S_{m-1}^1 - S_{m+\mu-1}^1] + \mu [S_{m+\mu+\nu}^1 - S_{m+\mu}^1] \right\}$$

As a first approximation one can take $m = t - 8$, $\mu = 6$, $\nu = 2$, so that

$$E_1(t) = 1/60 \{2[S_{t-9}^1 - S_{t-8}^1] + 6[S_t^1 - S_{t-2}^1]\}.$$

For the foreclosure function $F_1(t)$ with distributed lag, the straight line approximation can be taken to be as shown in Figure 9, and then if $m = t - 8$, $\mu = 8$,

$$F_1(f) = 1/36 (T_{t-8}^1 - T_t^1 + 9T_0^1),$$

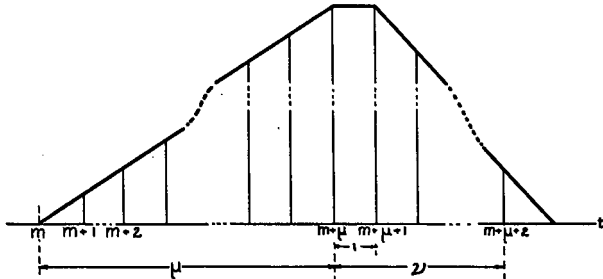


FIGURE 9

where T_0^1 and T_t^1 are defined in ways analogous to the ways in which S_0^1 and S_m^1 were defined, but with the quantities $F(1), \dots, F(m)$ replacing the quantities $E(1), \dots, E(m)$.

Now, if A_0, A_1, A_2 and A_3 are the constants determined previously, using lags (formula 7.1), and if the effect of $E(t)$ is small and if, furthermore, $B(t)$ can be represented by the sum of definite integrals which are approximately proportional to $E_1(t)$ and $F_1(t)$ and a constant term, then it should be possible to write as an approximation

$$B(t) = A_{01} + A_{11}K_{01}E_1(t) + A_{21}K_{02}F_1(t),$$

where K_{01} and K_{02} are effectively constant weight functions, which can be determined by the method of least squares.

When K_{σ_1} and K_{σ_2} are known, the maximum heights of the weighting functions indicated in Figure 8 and Figure 9 are known. It is then possible to replace the broken line weights functions by normal probability functions, thus determining approximate values for M_1 , σ_1 and M_2 , σ_2 in (11.1); i.e., neglecting the effect of $A_1 E(t)$.

Once approximate values of σ_i , M_i , $i = 1, 2$, are obtained, the coefficients A_0 , A_1 , A_2 and A_3 can be determined by the method of least squares, and better values of M_i and σ_i determined by means of a Taylor expansion. When M_1 , M_2 , σ_1 and σ_2 were corrected once* it was found that

$$A_0 = .443; \quad A_1 = 126.002; \quad A_2 = -11.909; \quad M_1 = 2.25;$$

$$M_2 = 1.75; \quad \sigma_1 = 1.00; \quad \text{and } \sigma_2 = 1.25 .$$

*Only a qualitative analysis seemed necessary for the σ_i . One knowing the mechanics of a formula like (11.1) can tell almost by inspection what quantities if recalculated might lead to better "fits". Empirical estimates involve smaller errors than those involved in approximating to the σ_i with the first term of a Taylor expansion.

APPENDIX VI

INADEQUACY OF LAG FORMULAS FOR MAXIMIZING INCOME OVER PERIODS OF TIME

The formula

$$(7.4) \quad B(t) = [16.63 I_{-2.5}^{**} W_{-2.5} - 1.022 F + 0.20] b ,$$

where $W_{-2.5}$ is the chance of not losing the income through foreclosure, etc., and b is the number of family accommodations needed each year, gives an excellent representation of new building as a function of the variables R = rent, T = taxes, C = replacement cost, P = per cent of occupancy, and f = number of foreclosures per year per 100,000 families.

In order to obtain a C that will maximize income — that is, maximize $v = B(t)C(t)$ — it is obviously incorrect to multiply (7.4) by $C(t)$ and differentiate, since C in the bracket above is $C(t-2.5)$ and not $C(t)$. It is natural to ask here if the difficulty can be bridged by means of the calculus of variations,* that is, by maximizing

$$V = \int_0^{\tau} C(t)B(t)dt ,$$

where τ represents some future time, 1, 2, 3, 4, etc., years in advance of present time $t = 0$. This is not possible. The introduction of leads of 2.5 and 2 years in the formula for $B(t)$ makes the calculus of variations problem one which in general does not have a solution. Only a short mathematical analysis is needed to demonstrate this.

For simplicity, write

$$B(t) = G_0(t-2.5) \{C(t-2.5)\}^{-.88} + G_1(t-2) ,$$

where

$$G_0(t-2.5) = A_0[(Rp-T)]W$$

and

$$G_1(t-2) = A_1/f(t-2) + A_2 ,$$

*See for example, W. F. Osgood, *Advanced Calculus*, New York, 1925.

and suppose that $\tau = 5$.

Then, it is desired to maximize (if possible)

$$\begin{aligned} V = V_s &= \int_0^5 B(t)C(t)dt = \int_0^5 C(t)[G_0(t-2.5)(C(t-2.5))^{-.86} \\ &\quad + G_1(t-2)]dt \\ &= \int_{2.5}^5 C(t)[G_0(t-2.5)(C(t-2.5))^{-.86} + G_1(t-2)]dt \\ &\quad + \int_{2.5}^5 C(t)[G_0(t-2.5)(C(t-2.5))^{-.86} + G_1(t-2)]dt . \end{aligned}$$

Write*

$$(1) \quad \begin{array}{ll} y_0(t) \text{ for } C(t) \text{ on} & -2.5 \leq t \leq 0 \\ y_1(t) \text{ for } C(t) \text{ on} & 0 \leq t \leq 2.5 \\ y_2(t) \text{ for } C(t) \text{ on} & 2.5 \leq t \leq 5 \end{array}$$

In the second integral put $t = t_1 + 2.5$. Then $C(t-2.5)$ for $2.5 \leq t \leq 5$ becomes $C(t_1)$ for $0 \leq t_1 \leq 2.5$ and $C(t)$ for $2.5 \leq t \leq 5$ becomes $C(t_1 + 2.5)$ for $0 \leq t_1 \leq 2.5$. By means of (1), the integral V_s becomes

$$V_s = \int_0^{2.5} y_1[G_0 y_0^{-.86} + G_1]dt + \int_0^{2.5} y_2[G_0^{(1)} y_1^{-.86} + G_1^{(1)}]dt ,$$

where $G_0^{(1)} = G_0(t + 2.5)$ and $G_1^{(1)} = G_1(t + 2.5)$; that is,

$$V_s = \int_0^{2.5} \sum_{i=1}^2 y_i(G_0^{(i-1)} y_{i-1}^{-.86} + G_1^{(i-1)}) dt.$$

Here the function y_0 is known, so that the only variables are y_1 and y_2 . If the integrand be denoted by F , the necessary conditions for a maximum are $\partial F / \partial y_1 = \partial F / \partial y_2 = 0$, or,

$$G_0 y_0^{-.86} + G_1 - .86 G_0^{(1)} y_2 y_1^{-1.86} = 0 .$$

and

$$G_0 y_1^{-.86} + G_1^{(1)} = 0 .$$

*The treatment given here was suggested by Professor Arnold Dresden, Swarthmore College.

From these equations, y_1 and y_2 can in general be determined, but there is a joker. If the period of time had been only two and a half years,

$$V = V_{2.5} = \int_0^{2.5} y_1 [G_0 y_0^{-.88} + G_1] dt ,$$

where the only variable is y_1 , the first necessary condition is $G_0 y_0^{-.88} + G_1 = 0$ on $(0, 2.5)$, and this is, in general, not satisfied. Furthermore, if a nine-year period had been used, still different results would have been obtained. From the point of view of economics this mathematical analysis is entirely relevant.

APPENDIX VII

PROBLEM OF LAGRANGE IN THE CALCULUS OF VARIATIONS

To find the values of f_α satisfying

$$\psi(F_1, F_2, \dots, F_m, u, p, t) = 0,$$

where $F_\alpha = f_\alpha u$, so that

$$Q = \int_{t_1}^{t_2} \sum_{\alpha=1}^m f_\alpha(u, p, t) p_\alpha(t) u(t) dt$$

is a minimum, it can be assumed that there is a solution

$$\bar{F}_1(t), \dots, \bar{F}_s(t)$$

and the conditions which this solution must satisfy can then be found.

Let $\mu_1(t), \dots, \mu_s(t)$ be s functions of t which possess continuous derivatives of the first order on the range $t_1 \leq t \leq t_2$ and vanish when $t = t_1$. Form the functions

$$(1) \quad F_i(t) = \bar{F}_i(t) + \varepsilon_i \mu_i(t), \quad i = 1, 2, \dots, s,$$

where $\varepsilon_1, \dots, \varepsilon_s$ are parameters. This family of functions evidently contains $\bar{F}_i(t)$ for the parametric values $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_s = 0$. The parameters $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s$ are not independent since the functions $\bar{F}_1, \bar{F}_2, \dots, \bar{F}_s$ must satisfy (2.8) of Chapter IX; that is,

$$(2) \quad \psi(\bar{F}_1 + \varepsilon_1 \mu_1, \dots, \bar{F}_s + \varepsilon_s \mu_s, u, p, t) = 0.$$

When one substitutes the family (1) in Q , it becomes a function of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s$ as follows:

$$Q(\varepsilon_1, \dots, \varepsilon_s) = \int_{t_1}^{t_2} \sum_{i=1}^s (\bar{F}_i + \varepsilon_i \mu_i) p_i(t) u(t) dt$$

$$+ \int_{t_1}^{t_2} \sum_{j=t_1}^m F_j(t) p_j(t) u(t) dt .$$

This function $Q(\varepsilon_1, \dots, \varepsilon_s)$ is to be a minimum when $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_s = 0$ for $\varepsilon_1, \dots, \varepsilon_s$ satisfying (2). The relation (2) may be taken to define one of the parameters ε_i in terms of the others. Any one of the ε_i may be taken to be this dependent variable. For the sake of definiteness assume that $\varepsilon_s, \dots, \varepsilon_{s-1}$ are the independent variables defining ε_s by (2).

For a minimum of $Q(\varepsilon_1, \dots, \varepsilon_s)$ the partial derivatives with respect to the independent variables must vanish and hence

$$\begin{aligned} \partial Q / \partial \varepsilon_j &= \int_{t_1}^{t_2} \mu_j(t) p_j(t) u(t) dt \\ &+ \int_{t_1}^{t_2} \mu_s(t) (\partial \varepsilon_s / \partial \varepsilon_j) p_s(t) u(t) dt = 0 \end{aligned}$$

for $j = 1, 2, \dots, s-1$ when $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_s = 0$.

Now a differentiation of (2) yields

$$(\partial \psi / \partial \bar{F}_j) \mu_j(t) + (\partial \psi / \partial \bar{F}_s) \mu_s(t) (\partial \varepsilon_s / \partial \varepsilon_j) = 0, \quad j = 1, 2, \dots, s-1,$$

so that

$$\mu_s(t) \partial \varepsilon_s / \partial \varepsilon_j = - [(\partial \psi / \partial \bar{F}_j) / (\partial \psi / \partial \bar{F}_s)] \mu_j(t) .$$

When this latter value is substituted in $\partial Q / \partial \varepsilon_j$, one obtains

$$\int_{t_1}^{t_2} \{p_j(t) - p_s(t) [(\partial \psi / \partial \bar{F}_j) / (\partial \psi / \partial \bar{F}_s)]\} u(t) u_s(t) dt = 0$$

for $j = 1, 2, \dots, s-1$.

The functions $\mu_j(t)$ are arbitrary except that they possess continuous derivatives of the first order and vanish when $t = t_1$. As a result the coefficient of each $\mu_j(t)$ must vanish for every value of t in the interval (t_1, t_2) , that is,

$$(3) \quad p_j(t) - p_s(t) [(\partial \psi / \partial \bar{F}_j) / (\partial \psi / \partial \bar{F}_s)] = 0$$

except at values of t for which $u(t)$ is zero; for, suppose that at a point $t = \tau$ of this interval, the left-hand side were, say, positive.

Since it represents a continuous function (by hypothesis) it will be positive throughout a certain neighborhood of τ . One could then choose $\mu_j(t)$ to be positive throughout this neighborhood and zero everywhere else in the interval (t_1, t_2) . It would follow that the integral could not be zero and a contradiction would result.

A system of $(s-1)$ partial differential equations of the type (3) will in general have unique solutions in terms of $(s-1)$ arbitrary constants. In general, therefore, because of the history of the problem, one may assume that these $(s-1)$ equations plus $(s-1)$ independent initial conditions defining the status of the F 's at $t = t_1$ uniquely determine the s coefficients of production.

By defining a quantity $\lambda(t)$ by the equation

$$\lambda(t) = (\partial \psi / \partial \bar{F}_s) p_s(t)$$

the conditions (3) can be written in the more symmetrical form

$$\partial \psi / \partial \bar{F}_i = \lambda(t) p_i(t), \quad i = 1, 2, \dots, s,$$

where the additional equation given by $i = s$ may be taken to be the equation of definition of $\lambda(t)$.

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DYNAMIC ECONOMICS ERRATA

Page XV, Line 10—Change "Industry" to "Industrial."

Between lines 10 and 11 insert:

June 13, 1933, and signed by President Franklin D. Roosevelt.

Page 4—Interchange lines 1 and 2.

Page 20, Equation (2)—Enclose $p^* + c$ in parentheses.

Page 21, Line 26—Change $>$ to $<$.

Line 27—Change "greater" to "less".

Page 22, Line 29—Change dy/dy to dy/dp .

Page 59—At end of line which precedes (3.1) insert * and at bottom of page add footnote:

*Vito Volterra, Sur la théorie mathématique des phénomènes héréditaires, *Journ. de Math. pures et appliquées*, Vol. VII, 1928, has considered general integral equations of the type (3.1).

Page 62, Lines 21, 22, 23—Change n to n .

Page 86, Equation (6.2)—First bracket turned wrong.

Last line of footnote—delete "not".

Page 106—Bottom of table replace first "1934" by "1933."

Page 155, Line 3—Replace H_o and C_o by H_c and C_c respectively.

Line 13—Replace C_o by C_c .

Page 166, Fourth line from bottom—Replace j by j .