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THE
CALCULUS OF OBSERVATIONS

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THE CALCULUS OF OBSERVATIONS

A Treatise on Numerical Mathematics

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PREFACE

THE mathematical problems which arise in dealing with numerical data are attractive and important. Some knowledge regarding them is required by workers in many different fields—astronomers, meteorologists, physicists, engineers, naval architects, actuaries, biometricians, and statisticians, as well as pure mathematicians; but until recently there has been very little instruction on the subject in the mathematical departments of the British universities. Of late, however, there has been a great awakening of interest in it, and it is now included in the syllabus for the Open Competitive Examination for appointments in the Home and Indian Civil Services, the Colonial Service, etc.

The present volume represents courses of lectures given at different times during the years 1913–1923 by Professor Whittaker to undergraduate and graduate students in the Mathematical Laboratory of the University of Edinburgh, and may be regarded as a manual of the teaching and practice of the Laboratory, complete save for the subject of Descriptive Geometry, for which a separate text-book is desirable. The manuscript of the lectures has been prepared for publication by Mr. Robinson, who has performed the whole of the work of numerical verification and has contributed additional examples.

The exposition has been designed to make each chapter, as far as possible, intelligible to those who have not mastered the preceding chapters; so that any one who is interested in some special problem may, by the help of the Table of Contents,

find and understand what is said about it without being obliged to read from the beginning of the book.

One feature which perhaps calls for a word of explanation is the prominence given to arithmetical, as distinguished from graphical, methods. When the Edinburgh Laboratory was established in 1913, a trial was made, as far as possible, of every method which had been proposed for the solution of the problems under consideration, and many of these methods were graphical. During the ten years which have elapsed since then, the graphical methods have almost all been abandoned, as their inferiority has become evident, and at the present time the work of the Laboratory is almost exclusively arithmetical. A rough sketch on squared paper is often useful, but (except in Descriptive Geometry) graphical work performed carefully with instruments on a drawing-board is generally less rapid and less accurate than the arithmetical solution of the same problem.

The material equipment essential for a student's mathematical laboratory is very simple. Each student should have a copy of Barlow's tables of squares, etc., a copy of Crelle's "Calculating Tables," and a seven-place table of logarithms. Further, it is necessary to provide a stock of computing paper (*i.e.* paper ruled into squares by rulings a quarter of an inch apart; each square is intended to hold two digits; the rulings should be very faint, so as not to catch the eye more than is necessary to guide the alignment of the calculation), and lastly, a stock of computing forms for practical Fourier analysis (those used in Chapter X. of this book may be purchased). With this modest apparatus nearly all the computations hereafter described may be performed, although time and labour may often be saved by the use of multiplying and adding machines when these are available.

Attention may be drawn to the opportunities which the subject presents to the research worker in Mathematics. There is an evident need for new and improved methods of dealing with many of the problems discussed in the later chapters.

PREFACE

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The authors wish to acknowledge their indebtedness to Mr. G. J. Lidstone, F.I.A., F.R.S.E., whose unique knowledge of actuarial literature has been of constant value to them. To him mainly it is due that the book is so rich in theorems and methods derived from actuarial sources, with which mathematicians are not usually well acquainted. Mr. Lidstone and Dr. John Dougall, F.R.S.E., have read the proof-sheets; by their suggestions many references have been added, many demonstrations simplified, and many obscurities removed.

E. T. W.

G. R.

PREFACE TO THE SECOND EDITION

Many correspondents have kindly sent notes of errata, which have been corrected in the present impression.

I gratefully acknowledge the help of Dr. A. C. Aitken, Lecturer in Mathematics and Statistics in the University of Edinburgh, whose recent contributions to the theory of Graduation have made it necessary to rewrite the latter part of Chapter XI.

E. T. W.

• OCTOBER, 1925.

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