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**THE MATHEMATICAL PART  
OF ELEMENTARY STATISTICS**

**A TEXTBOOK FOR COLLEGE STUDENTS**

**BY**

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**1934**

**D. C. HEATH AND COMPANY**

**BOSTON**

**NEW YORK**

**CHICAGO**

**ATLANTA**

**SAN FRANCISCO**

**DALLAS**

**LONDON**

B28  
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PRINTED IN THE UNITED STATES OF AMERICA

## PREFACE

This is an elementary textbook, dealing with the mathematical part of statistics. Statistical methods are used very generally by investigators in widely different fields, by economists, biologists, psychologists, physicists, and astronomers. The emphasis in one field is different from that in another, but there is a well-defined body of material, mathematical in nature, which is common to all. These mathematical rudiments are set forth in this text.

The course for which this book is intended will usually be offered to sophomores, and may, like the book, be divided into two parts, the first of which may be taken without the second. The book is, in fact, a reduction to printed form of two half-year lecture courses, which the author has been giving for the past ten years, and it has been quite the custom for certain students to take the first part only, substituting for the second part an applied course in economic or educational statistics. This arrangement has been advantageous to all the departments concerned. The mathematics has been taught in the department of mathematics, and the applications in the departments in which they belonged.

A brief course in analytic geometry is presupposed, but no calculus — although an occasional notation is introduced which is borrowed from calculus, such as the integral sign to indicate the area under a curve. Analytics is inherently necessary to the proper understanding of statistical methods, and it seems to the author better to suppose that it has been taught already by the use of one of the excellent texts in analytics than to attempt to teach it along with statistics.

Part I is distinctly easier than Part II, and it is intended

to provide a sufficient mathematical introduction to most of the applied courses. It will probably be found about as difficult as a semester course in analytics. Part II begins with the theory of probability. This subject involves intrinsically difficult notions, comparable in difficulty with those of the calculus. It must be presented because it lies at the basis of the theory of sampling. However, students who would be characteristically unable to think through for themselves problems in probability such as are given here should not be encouraged to study the second part. It is not desirable to try to teach them sampling theory, for they would not really understand it, and it is better that they should not acquire a superficial facility in using the formulae. The subject of finite differences, briefly considered in the closing chapter, may be taught with Part I if desired, being logically detached from the rest of the theory. Part III comprises the more necessary statistical tables and a separate introduction, including a more complete discussion of the point binomial than might be desired in an elementary text. Many of the simplest problems in probability require for their solution the summation of a number of consecutive terms of a point binomial. There is no single short method of obtaining even a fair approximation to this sum which is available for all cases. One set of formulae works best in one case, another in another. In this introduction certain selected formulae are listed, with instructions as to when each is to be used. By the aid of the tables, the summation can be made quickly so as to obtain the required probability, correct to two or three places. Part III is also published separately. It is supposed throughout, but especially in the instructions as to the use of these tables, that a computing machine is at the disposal of the student; nevertheless most of the problems in this text can be handled satisfactorily with only a slide rule and a four-place table of logarithms.

## PREFACE

In writing this text, the author has held in mind two principles. First, every idea presented must be illustrated with a numerical example, and this must be followed by short "exercises," which can be done — at the board if desired — without any mechanical aids, and in five or ten minutes each. These are artificial problems involving simple numbers, and have been invented solely for the purpose of teaching the methods. Longer numerical problems, having applications in various statistical fields, and problems in theory, are listed at the close of each chapter. The second principle is that tacit hypotheses underlying the various methods shall be exposed. This is the more necessary because in practice these hypotheses often fail to be realized, and it is important that, at the very outset of his training, the statistical worker should appreciate the tentative character of results in such cases. Moreover, it makes possible the maintenance of an attitude of rigorousness of treatment which is very desirable for the student's proper mathematical development.

Problem material and statistical data have been selected freely from several books and journals. The author's name is usually indicated where this has been done. I am deeply indebted to many of my students for their assistance in preparing the manuscript, particularly perhaps to H. G. Neebe, whose thorough work on the tables was very valuable to me, and to H. A. Lewis, who read with keenly critical mind the whole manuscript and made further special investigations at several points.

B. H. C.

MIDDLETOWN, CONNECTICUT,  
SEPTEMBER, 1930.

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$$\beta = \frac{n \Sigma ty - \Sigma t \Sigma y}{D}, D = n \Sigma t^2 - (\Sigma t)^2.$$

$$y = A + Bx, A = \frac{1}{n} \Sigma y, B = \frac{\Sigma xy}{\Sigma x^2}, \Sigma x^2 = \frac{n(n^2 - 1)}{12}.$$

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$$y = A + Bx + Cx^2, A = \frac{15}{n(n^2 - 4)} \left( \frac{3n^2 - 7}{20} \Sigma y - \Sigma x^2 y \right),$$

$$B = \frac{12}{n(n^2 - 1)} \Sigma xy, C = \frac{15}{n(n^2 - 4)} \left( \frac{12}{n^2 - 1} \Sigma x^2 y - \Sigma y \right).$$

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$$f(X) = \Sigma_Y f(X, Y), f(Y) = \Sigma_X f(X, Y).$$

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$$\Sigma_{X,Y} f(X, Y) = N, \bar{X} = \frac{1}{N} \Sigma_X X f(X), \bar{Y} = \frac{1}{N} \Sigma_Y Y f(Y).$$

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$$\mu_{x'} = \frac{1}{N} \Sigma_{x,y} x f(x, y) = \frac{1}{N} \Sigma_x x f(x),$$

$$\mu_y = \frac{1}{N} \sum_{x,y} yf(x, y) = \frac{1}{N} \sum_y yf(y),$$

$$p_{xy} = \frac{1}{N} \sum_{x,y} xyf(x, y), \quad r = \frac{p_{xy}}{\sigma_x \sigma_y}.$$

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$$u = \frac{X - A}{h}, \quad \bar{u} = \frac{1}{N} \sum u f(u), \quad \sigma_u^2 = \frac{1}{N} \sum u^2 f(u) - \bar{u}^2,$$

$$U = \sum_u u f(u, v), \quad V = \sum_v v f(u, v), \quad \sum_v U = \sum_u V, \quad \sigma_x = h \sigma_u.$$

$$r = \frac{\frac{1}{N} \sum_v U - \bar{u} \bar{v}}{\sigma_u \sigma_v}.$$

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$$r = \frac{1}{\sqrt{\frac{1}{N} \sum u^2 - \bar{u}^2} \sqrt{\frac{1}{N} \sum v^2 - \bar{v}^2}} \left( \frac{1}{N} \sum uv - \bar{u} \bar{v} \right).$$

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$$\frac{Y - \bar{Y}}{\sigma_y} = r \frac{X - \bar{X}}{\sigma_x}, \quad \frac{X - \bar{X}}{\sigma_x} = r \frac{Y - \bar{Y}}{\sigma_y}.$$

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for  $x = ry$ ,  $\frac{1}{N} \sum \delta^2 f = 1 - r^2$ ;

for  $y = \pm x$ ,  $\frac{1}{N} \sum \delta^2 f = 1 - |r|$ .

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NON-MEASURABLE CHARACTERS

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$$z = \frac{N}{2\pi\sqrt{1-r^2}\sigma_x\sigma_y} e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)},$$

$$x^2 - 2rxy + y^2 = k, a = \sqrt{\frac{k}{1-r}}, b = \sqrt{\frac{k}{1+r}},$$

$$r = 1 - \frac{2}{1 + \frac{a^2}{b^2}}.$$

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$$r = \frac{1}{N} \sum \frac{x}{\sigma_x} \frac{y}{\sigma_y} f(x, y).$$

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$$r = \frac{1}{N} \sum \frac{x}{\sigma_x} \frac{v - \bar{v}}{\sigma_v} f(x, v).$$

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$$r = 1 - \frac{6\Sigma(X - Y)^2}{N(N^2 - 1)}.$$

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$${}_n P_r = {}_n C_r \cdot {}_r P_r, {}_n P_r = n(n-1) \cdots (n-r+1),$$

$${}_n P_r = r!, {}_n P_r = \frac{n!}{(n-r)!}$$

$${}_n C_r = \frac{n!}{r!(n-r)!}, \quad {}_n C_r = {}_n C_{n-r},$$

$$(a+b)^n = \sum_{r=0}^n {}_n C_r a^{n-r} b^r.$$

$$n! \cong e^{-n} n^n \sqrt{2\pi n}.$$

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$$(p+q)^n = p^n + \dots + {}_n C_s p^s q^t + \dots + q^n, \quad s = n - t.$$

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$$\frac{1}{{}_m C_n} [{}_p m C_n {}_q m C_0 + \dots + {}_p m C_{n-t} {}_q m C_t + \dots + {}_q m C_n].$$

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$$\bar{t} = nq, \quad \sigma = \sqrt{pqn}, \quad \alpha_3 = \frac{p-q}{\sigma}, \quad \alpha_4 = 3 + \frac{1}{\sigma^2} - \frac{6}{n},$$

| mean - mode | ≤ 1.

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$$y = \frac{\phi(x)}{\sigma}, \quad x = \frac{t - \bar{t}}{\sigma}.$$

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The sum of terms in which  $a \leq t \leq b$  is

$$\int_{x_1}^{x_2} \phi(x) dx, \quad x_1 = \frac{a - \frac{1}{2} - qn}{\sigma}, \quad x_2 = \frac{b + \frac{1}{2} - qn}{\sigma};$$

the sum of those terms in which  $t$  differs from  $qn$  by  $k$  or less is

$$2 \int_0^x \phi(x) dx, \quad x = \frac{k + \frac{1}{2}}{\sigma}.$$

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The sum of those terms in which  $a \leq t \leq b$  is

$$\int_{x_1}^{x_2} \phi(x) dx + \left[ \frac{q-p}{6\sigma} \phi^{(3)}(x) + \frac{1}{24} \left( \frac{1}{\sigma^2} - \frac{6}{n} \right) \phi^{(5)}(x) \right]_{x_1}^{x_2}, \quad x_1, x_2$$

as in § 3.

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*The mean of the finite hypergeometric series is  $\bar{l} = nq$ .*

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$$F(x) = \phi(x) - \frac{\alpha_3}{6} \phi^{(3)}(x) + \frac{\alpha_4 - 3}{24} \phi^{(4)}(x), \quad x = \frac{u - \bar{u}}{\sigma_u}$$

*Frequency over (a, b) =  $N \int_a^b F(x) dx$ ;  $y = \frac{N}{\sigma_1} F(x)$ .*

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$$\int_a^b \phi^{(3)}(x) dx = \phi^{(2)}(x) \Big|_a^b, \quad \int_a^b \phi^{(4)}(x) dx = \phi^{(3)}(x) \Big|_a^b;$$

$$\int_{-\infty}^{\infty} \phi(x) dx = 1, \quad \int_{-\infty}^{\infty} x^2 \phi(x) dx = 1, \quad \int_{-\infty}^{\infty} x^4 \phi(x) dx = 1 \cdot 3,$$

$$\int_{-\infty}^{\infty} x^6 \phi(x) dx = 1 \cdot 3 \cdot 5, \text{ etc.}$$

$$\int_a^b F(x) dx = \int_a^b \phi(x) dx - \frac{\alpha_3}{6} [\phi^{(2)}(b) - \phi^{(2)}(a)] + \frac{\alpha_4 - 3}{6} [\phi^{(3)}(b) - \phi^{(3)}(a)].$$

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*Mean of  $\bar{P}_s = \bar{l}$ .*

$$\bar{\sigma}^2 = \frac{1}{N} \bar{\sigma}^2, \quad \bar{\alpha}_3 = \frac{1}{\sqrt{N}} \bar{\alpha}_3, \quad \bar{\alpha}_4 - 3 = \frac{\bar{\alpha}_4 - 3}{N}.$$

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*The probability that  $\bar{t}$  will lie within  $\delta$  of  $\bar{t}$  is*

$$P_\delta = 2 \int_0^\delta \phi dx + \frac{\bar{\alpha}_4 - 3}{12N} \phi^{(3)}(\delta), \delta \text{ in } \bar{\sigma} \text{ unit.}$$

*$P_\delta \geq 0.999$  almost always if  $\delta \geq 3.5$ , and  $N \geq 25$ .*

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$$\sigma_\sigma = \frac{\bar{\sigma}}{\sqrt{2N}}, \sigma_{\alpha_3} = \sqrt{\frac{6}{N}}, \sigma_{\alpha_4} = \sqrt{\frac{24}{N}}, \sigma_M = \sqrt{\frac{\pi}{2N}} \bar{\sigma}.$$

$$1 - P_\delta \leq \frac{1}{\delta^2}, \quad 1 - P_\delta < \frac{\alpha_{2r}}{(\delta + 2r\delta)^{2r}}, \quad 1 - P_\delta < \frac{1}{2.25\delta^2}.$$

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$$\sigma_r = \frac{(1 - \bar{r}^2)}{\sqrt{N}}.$$

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$$\chi^2 = \sum_{i=1}^m \frac{(f_i - Np_i)^2}{Np_i}.$$

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$$\mu_2(F) = \mu_2(f') + \mu_2(f''),$$

$$\mu_4(F) = \mu_4(f') + 6\mu_2(f')\mu_2(f'') + \mu_4(f''), \text{ etc.}$$

$$\sigma_F = \bar{\sigma} \sqrt{\frac{1}{N'} + \frac{1}{N''}}.$$

*The probability that the difference between two means will exceed  $\delta_\sigma$ , numerically, is  $1 - P_\delta$ , where*

$$P_\delta = 2 \int_0^\delta \phi dx.$$

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*The probability that the difference will exceed  $p' - p''$ , numerically, is  $1 - P_\delta$  where*

$$P_\delta = 2 \int_0^\delta \phi dx, \delta = \frac{|p' - p''|}{\sigma}, \sigma = \sqrt{pq \left( \frac{1}{N'} + \frac{1}{N''} \right)}.$$



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If  $\bar{x} = c_1x_1 + \dots + c_Nx_N$ , or if  $\bar{\delta} = c_1\delta_1 + \dots + c_N\delta_N$ ,  
 then  $\bar{\sigma}^2 = c_1^2\sigma_1^2 + \dots + c_N^2\sigma_N^2$ , and  $\bar{s}^2 = c_1^2s_1^2 + \dots + c_N^2s_N^2$ .

If  $\bar{x} = cx^N$ , then  $\frac{\bar{\delta}}{\bar{x}} = N\frac{\delta}{x}$ , and  $\frac{\bar{\sigma}}{\bar{x}} = |N|\frac{\sigma}{x}$ .

If  $\bar{x} = cx_1 \dots x_N$ , then  $\frac{\bar{\delta}}{\bar{x}} = \frac{\delta_1}{x_1} + \dots + \frac{\delta_N}{x_N}$ ,

and

$$\left(\frac{\bar{\sigma}}{\bar{x}}\right)^2 = \left(\frac{\sigma_1}{x_1}\right)^2 + \dots + \left(\frac{\sigma_N}{x_N}\right)^2.$$

If  $\bar{x} = \frac{x_1}{x_2}$ ,  $\left(\frac{\bar{\sigma}}{\bar{x}}\right)^2 = \left(\frac{\sigma_1}{x_1}\right)^2 + \left(\frac{\sigma_2}{x_2}\right)^2$ .

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$$\frac{1}{N} \sum \delta^2 f(x, y) = \sigma_y^2(1 - r^2),$$

$$\hat{y}(x) = \frac{1}{f(x)} \sum_y y f(x, y).$$

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$$\text{Standard error} = \sigma_y \sqrt{1 - r^2}.$$

$$\text{Correlation ratio error} = \sigma_y \sqrt{1 - \eta_v^2}.$$

$$\eta_v^2 = \frac{1}{N\sigma_x^2} \sum \bar{y}^2(x) f(x), \quad 0 \leq \eta_v^2 \leq 1.$$

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$$U = \sum_u v f(u, v), \quad V = \sum_v u f(u, v),$$

$$\eta_u^2 = \frac{1}{\sigma_v^2} \left[ \frac{1}{N} \sum_u \frac{V^2}{f(u)} - \bar{v}^2 \right], \quad \eta_v^2 = \frac{1}{\sigma_u^2} \left[ \frac{1}{N} \sum_v \frac{U^2}{f(v)} - \bar{u}^2 \right].$$

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$$r = \frac{m}{n}.$$

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$$r = \sqrt{\frac{n}{2n-1}}$$

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$$\int_{-\infty}^{y_i} \phi dx = \frac{b_i}{f_i}, i = 1, 2, \dots; m = \frac{y'' - y'}{x'' - x'}$$

$$r = \frac{m}{\sqrt{1+m^2}} = \sin \tan^{-1} m.$$

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$$\int_{-\infty}^x \phi dx = F_1, \int_{-\infty}^{y_1} \phi dx = A_1, \int_{-\infty}^{y_2} \phi dx = B_2,$$

$$m = F_1 F_2 \frac{y_1 + y_2}{\phi(x)}, r = \frac{m}{\sqrt{1+m^2}} = \sin \tan^{-1} m;$$

$$r = \frac{m}{\sqrt{1+\theta m^2}}, \theta \approx 0.6.$$

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CHAPTER VI—MULTIPLE CORRELATION

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$$f(x, y) = \sum_z f(x, y, z), f(x) = \sum_y f(x, y), N = \sum_x f(x).$$

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$$\bar{u} = \frac{1}{N} \sum_u \sum_v \sum_w u f(u, v, w) = \frac{1}{N} \sum_u u f(u, v) = \frac{1}{N} \sum_u u f(u).$$

$$\mu_{x^2} = \frac{1}{N} \sum_{x,y,z} x^2 f(x, y, z) = \frac{1}{N} \sum_x x^2 f(x).$$

$$p_{x^2 y^2 z^2} = \frac{1}{N} \sum_{x,y,z} x^2 y^2 z^2 f(x, y, z).$$

$$p_{xy} = \frac{1}{N} \sum_{x,y,z} xy f(x, y, z) = \frac{1}{N} \sum_{x,y} xy f(x, y) = r_{xy} \sigma_x \sigma_y.$$

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$$\bar{z}(x, y) = \frac{1}{f(x, y)} \sum z f(x, y, z), \quad z = \bar{z}(x, y).$$

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$$\frac{z}{\sigma_z} (1 - r_{zy}^2) = \frac{x}{\sigma_x} (r_{zx} - r_{zy}r_{yx}) + \frac{y}{\sigma_y} (r_{yz} - r_{zy}r_{yx});$$

$$z = Z - \bar{Z}, \quad x = X - \bar{X}, \quad y = Y - \bar{Y}.$$

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$$\frac{x_1}{\sigma_1} R_{11} + \frac{x_2}{\sigma_2} R_{12} + \cdots + \frac{x_m}{\sigma_m} R_{1m} = 0;$$

$$R = (r_{11}r_{22} \cdots r_{mm}), \quad R_{\lambda\lambda} = \text{cofactor of } r_{\lambda\lambda}.$$

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$$\eta_z^2 = \frac{1}{N\sigma_z^2} \sum \bar{z}^2(x, y) f(x, y).$$

$$\rho_z^2 = \frac{r_{zx}^2 + r_{yz}^2 - 2r_{zy}r_{yx}r_{zx}}{1 - r_{zy}^2}.$$

$$0 \leq \eta_z^2, \rho_z^2 \leq 1.$$

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$$r_{xy \cdot z} = \frac{r_{xy} - r_{xz}r_{yz}}{\sqrt{(1 - r_{zy}^2)(1 - r_{zx}^2)}}.$$

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$$\Delta u(x) = u(x + h) - u(x), \quad \Delta^2 u(x) = \Delta \Delta u(x).$$

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$$\begin{aligned} \Delta^n(f + g) &= \Delta^n f + \Delta^n g; \Delta^n(cf) = c\Delta^n f; \\ \Delta^n(a_0 x^n) &= a_0 h^n n!; \Delta^n(a_0 x^n + \dots + a_n) = a_0 h^n n!; \\ x^{(m)} &= x(x-h) \dots (x-mh+h); \Delta x^{(m)} \\ &= mx^{(m-1)}h; \end{aligned}$$

$$f(x) = f(0) + x^{(1)}\Delta_0 + \frac{x^{(2)}}{2} \Delta_0^2 + \dots + \frac{x^{(n)}}{n!} \Delta_0^n,$$

if  $\Delta x = 1$ ;

$$u(s) = u(0) + s^{(1)}\Delta_0 + \dots + \frac{s^{(n)}}{n!} \Delta_0^n.$$

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At  $x = 1.25$ ,  $u = (.0078125)(-7u_0 + 105u_1 + 35u_2 - 5u_3)$ ;

At  $x = 1.75$ ,  $u = (.0078125)(-5u_0 + 35u_1 + 105u_2 - 7u_3)$ ;

At  $x = 1.50$ ,  $u = (.0625)(-u_0 + 9u_1 + 9u_2 - u_3)$ .

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$$\sum_{x=0}^{n-1} u_x = U_x \Big|_0^n.$$

$$u_0 + u_1 + \dots + u_{n-1} = u_0 \frac{n^{(1)}}{1} + \Delta_0 \frac{n^{(2)}}{2} + \dots + \Delta_0^k \frac{n^{(k+1)}}{k+1}.$$

### PART III

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