

**MATHEMATICS
AND
STATISTICS
FOR
ECONOMISTS**

Rare sec

by

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Mathematics, and Statistics

IOWA STATE COLLEGE

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PREFACE

The present book grew out of a definite need felt during many years of teaching at Iowa State College. There seems to be no text adapted to the American student of economics (undergraduate or graduate) who has had no thorough training in college mathematics, but who is willing to acquire some of the mathematical equipment necessary nowadays, for a serious study of economics.

This book has been written in an attempt to meet such a specific need. It includes some applications of elementary mathematics to economics, as well as topics in calculus, probability, and elementary statistics. The examples are taken from economics. The student is not burdened by the necessity of familiarizing himself with mechanics and other branches of physics, which traditionally supply much of the illustrative materials in elementary calculus texts. It is felt that the economics student who wants to learn mathematics and statistics has not the time to study those topics which are somewhat remote from his field of interest.

This book is addressed specifically to the future econometrician—a student of economics who is willing to use the tools of mathematics and statistics in his economic investigations. Little mathematical preparation is required of the student. It is believed that some knowledge of algebra and elementary trigonometry will be sufficient, although familiarity with elementary economics is required. In this connection, two somewhat advanced books on economic theory are recommended: Kenneth E. Boulding, *Economic Analysis* (New York: Harper & Brothers, 1941) and George J. Stigler, *The Theory of Price* (New York: The Macmillan Company, 1946). The student will gain a greater insight into the theoretical economic problems used as illustrations if he reads the relevant chapters of one or both texts. It is to be hoped that, after mastering this text, the student or reader will possess sufficient knowledge in mathematics and statistics to understand most of the articles published in such journals as *Econometrica*, the *Review of Economic Studies*, and the *Journal of the American Statistical Association*.

It is evident that a book which is planned, not for the future professional mathematician, but for the future econometrician, cannot be entirely rigorous in the proofs of the mathematical theorems involved. Intuitive proofs and demonstrations are frequently substituted for mathematical rigor, where the

more adequate proof is beyond the scope of the book and also beyond the powers of most readers or students. Rigorous treatment is already available in many books on advanced calculus, algebra, and statistics. Some of these books are indicated in the postscript.

Many empirical examples are included in the exercises of this book. They represent the efforts of econometricians to utilize statistical methods to obtain theoretically meaningful economic relationships. The statistical methods used by these econometricians are not always the most modern ones. In spite of this fact, it seemed worth while to include, as illustrations, some of the older results, found with the help of somewhat antiquated statistical methods. It should be emphasized that the empirical relationships given in the examples are to be interpreted with some caution. They represent merely efforts to estimate some kind of average relationship between the variables indicated. It is to be hoped that these examples, indicative of the great theoretical interest and practical potentialities of econometrics, will make the study of mathematics and statistics more interesting to the economist, and will inspire him to future studies in the field.

Some of the examples in the text require the use of mathematical tables. The following set of tables can be recommended: H. D. Larsen, *Rinehart Mathematical Tables, Formulas, and Curves*, Enlarged Edition (New York: Rinehart & Company, Inc., 1953).

I should like to express my gratitude to a number of my colleagues at Iowa State College who have taken a kindly interest in this text and given me assistance in various ways. I am particularly obliged to Professor Edward S. Allen (Department of Mathematics), Professor Dio L. Holl (Head, Department of Mathematics), Professor William G. Murray (Head, Department of Economics), and Professor E. R. Smith (Department of Mathematics). I am also indebted to Professor C. V. Newsom for improvements in the manuscript. I have to thank Mr. F. Jarred (Melbourne, Australia) for helping me with the answers to the problems.

Problems marked by * contain important ideas and theorems which will be required later. Problems and sections marked ** are somewhat more difficult and may be omitted.

Sources of numerical examples

The numerical examples given in the text are taken from the following publications:

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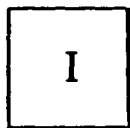
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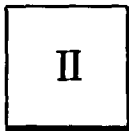
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I

**SOME APPLICATIONS OF
ELEMENTARY MATHEMATICS
TO ECONOMICS**

FUNCTIONS AND GRAPHS

1. Functions

The function concept is one of the most important in both pure and applied mathematics. The variable y is said to be a function of the variable x when y depends upon x in such a way that the fixing of x determines one or more corresponding values of y . The variable x , whose value may be arbitrarily assigned (except for the nonpermissible values) is called the *independent variable*. The variable y , whose numerical value is determined after a permissible value has been given to x , is the *dependent variable*. If we want to indicate that y is a function of x , without fixing the specific form of the function, we write $y = f(x)$. Instead of f , we may use also, as functional symbols, such letters as g , h , F , G , and so on. Also, we write $y = y(x)$.

EXAMPLE 1

Let y depend on x through the medium of the formula $y = 1/x$. Thus the formula denotes a functional relationship. If $x = 3$, we have $y = 1/3$. If $x = -7$, we have $y = -1/7$. If $x = 8$, we have $y = 1/8$. If $x = 1/3$, we have $y = 1/(1/3) = 3$, and so forth. We see that for all permissible values of x (that is, all values except $x = 0$) we can find the corresponding y . Specifically, we say then that y is a function of x .

It is easy to illustrate, from economics, the concept of functional relationship. Consider, for instance, a demand function. Here the quantity demanded (dependent variable y) may be considered as a function of the price (independent variable x) of a certain commodity. In production theory,

NOTE: Problems marked by contain important ideas and theorems which will be required later. Problems and sections marked ** are somewhat more difficult and may be omitted.

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the quantity of the product (dependent variable y) may be considered as a function of the amount of labor involved in production (independent variable x), and so on.

EXAMPLE 2

If $f(x)$ is the function of Example 1, we may write, for instance,

$$y = f(x) = y(x) = \frac{1}{x}.$$

When we substitute specific values for x , we may write

$f(4) = 1/4$, which means that x has been replaced by 4,

$f(-3) = 1/(-3) = -1/3$, which means that -3 has been substituted for x ,

$f(5/3) = 1/(5/3) = 3/5$, and so on.

Instead of a specific number we may substitute into $f(x)$ an algebraic expression for x ; thus,

$$f\left(\frac{1}{a}\right) = \frac{1}{\frac{1}{a}} = a,$$

$$f\left(\frac{x}{y}\right) = \frac{y}{x}.$$

EXERCISES 1

- Let $y = \frac{1}{3}f(x) = 3 - 2x + x^2$. Find $f(0)$, $f(-2)$, $f(5)$, $f(-1)$.
- Let $y = f(x) = 2x/(x^2 - 1)$. (a) Find $f(0)$; $f(-6)$; $f(5)$; $f(2)$; $f(-3/4)$.
(b) Are the values $x = 1$ and $x = -1$ permissible for the given function?
- Let $y = f(x) = (2x^2 - 4x + 6)/2x$. (a) Find $f(1)$; $f(0)$; $f(-1)$; $f(1/5)$; $f(-1/3)$. (b) Is the value $x = -2$ permissible? (c) Is the value $x = 0$ permissible?
- Let $y = f(x) = 2x$. Find $f(1)$; $f(2)$; $f(4)$.
- Let $y = f(x) = (2x - 1)^2$. Find $f(0)$; $f(-1)$; $f(3)$; $f(-1/5)$.
- Let $y = f(x) = [2(x - 1)^2 + 5]/(x + 3)^2$. (a) Find $f(0)$; $f(-1)$; $f(5)$; $f(-1/2)$. (b) Is $x = 3$ a permissible value? (c) Is $x = -3$ a permissible value of x ?
- Let $y = f(x) = x^2$. Find $f(0)$; $f(-2)$; $f(4)$; $f(10)$; $f(-10)$.
- Let $f(x) = 2x^2 - 1$. Find (a) $f(1)$, $-f(0)$; (b) $2f(1)$, $-3f(-1)$; (c) $[f(3)]/[f(-2)]$.
- Let $f(x) = a + bx$, where a and b are arbitrary constants. Find (a) $f(0)$; (b) $f(-2)$; (c) $f(1)$; (d) $f(a)$; (e) $f(-a)$; (f) $f(a/b)$; (g) $f(-a/b)$.
- Let $f(x) = ax^2 + bx + c$, where a , b and c are constants. Find (a) $f(0)$; (b) $f(-1)$; (c) $f(1)$; (d) $f(a - b)$; (e) $f(b - a)$.
- Let $f(x) = 5x$. Find $f(a)$; $f(b)$; $f(a + b)$; $f(a - b)$. Show that $f(a + b) = f(a) + f(b)$. Show that $f(a - b) = f(a) - f(b)$.

2. Rectangular coordinates

The graphical representation of theoretical or statistical relationships between 2 variables is very important in economics. To represent a pair of related numbers x and y , which may be positive or negative, we must first draw a system of 2 axes, 1 horizontal and 1 vertical. The point of intersection of the 2 axes is labeled O and is called the *origin*. The horizontal axis is called the X axis and the vertical axis the Y axis. The positive directions are to the right and upward, respectively.

Convenient units of x and y are chosen. These units need not be the same, since x and y may represent entirely different types of quantities. The units chosen will depend on the ranges of the values of x and y which are to be included.

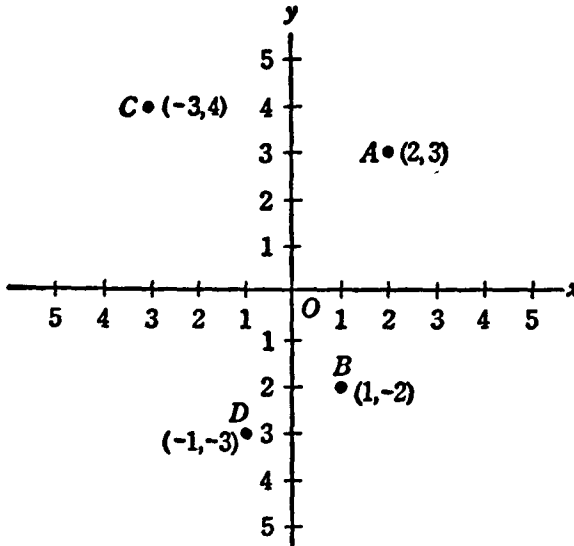


Figure 1

A pair of related numbers x and y is designated by (x,y) . The geometric representation of such a pair upon an axis system is a point. The x number is said to be the x coordinate, or *abscissa*, of the point, and the y number is the y coordinate, or *ordinate*, of the point. To plot a point we proceed as follows:

EXAMPLE 1

To plot the point $(2,3)$ we note that $x = +2$, $y = +3$. We proceed from the origin 2 units to the right (positive direction) and then 3 units upward (positive direction). This gives the required point $(2,3)$. This point is labeled A in Figure 1.

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EXAMPLE 2

To plot the point $(1, -2)$ we note that $x = 1$, $y = -2$. We proceed from the origin 1 unit to the right (positive direction) and then 2 units downward (negative direction). This gives the point $(1, -2)$. The point is designated by B in Figure 1.

EXAMPLE 3

To plot the point $(-3, 4)$ we note that $x = -3$, $y = 4$. We proceed from the origin 3 units to the left (negative direction). Then we go 4 units upward (positive direction). This gives the point $(-3, 4)$ as required. This is point C in Figure 1.

EXAMPLE 4

To plot the point $(-1, -3)$ we note that $x = -1$, $y = -3$. We proceed from the origin 1 unit to the left (negative direction) and then 3 units down (negative direction). This gives the required point $(-1, -3)$. This is point D in Figure 1.

EXERCISES 2

1. Plot the following points in a system of rectangular coordinates: (a) $(1, 6)$; (b) $(1/2, 2)$; (c) $(1, 1/4)$; (d) $(-1, 6)$; (e) $(-3, 4)$; (f) $(3, -7)$; (g) $(2, -5)$; (h) $(-1, -1)$; (i) $(-4, -1/2)$.
2. Plot the points $(0, 0)$; $(1, 1)$; $(2, 2)$; $(3, 3)$. Show that they lie on a straight line.
3. Plot the points $(-3, -5)$; $(-2, -3)$; $(-1, -1)$; $(0, 1)$; $(1, 3)$; $(2, 5)$. Show that they are all located on a straight line.
4. Plot the points $(0, -2)$; $(0, -1)$; $(0, 0)$; $(0, 1)$; $(0, 2)$.
5. Plot the points $(-3, 0)$; $(-2, 0)$; $(-1, 0)$; $(0, 0)$; $(1, 0)$; $(2, 0)$; connect consecutive points by straight-line segments.
6. Measure the distance from the origin to the following points: (a) $(0, 4)$; (b) $(5, 0)$; (c) $(1, 2)$; (d) $(-2, 1)$; (e) $(0, -5)$.
7. Locate the points $(1, 1)$; $(-1, 1)$; $(1, -1)$; $(-1, -1)$. Show that they are symmetrically located with respect to the 2 axes. What is the distance from each axis?
8. Find the distance between the points $(0, 2)$ and $(4, -1)$.
9. Find the distance between $(1, 1)$ and $(5, 5)$.
10. Find the distance between $(-2, 3)$ and $(2, 6)$.

3. Graphs of Equations

Suppose that a function $y = f(x)$ is given by means of a formula. It is possible to learn the simultaneous behavior of x and y by plotting a set of points (x, y) , where each x is one of the permissible values, selected arbitrarily, and the y in each case is the corresponding value of y , determined by the relation $y = f(x)$.

■ EXAMPLE 1

Let $y = f(x) = 5x - 2$. We would like to know something about the behavior of x and y for the range $x = -3$ to $x = +3$. Therefore we construct a table for the choices, $x = -3, -2, -1, 0, 1, 2, 3$, along with the corresponding values of y :

x	$y = f(x) = 5x - 2$
-3	$5(-3) - 2 = -17$
-2	$5(-2) - 2 = -12$
-1	$5(-1) - 2 = -7$
<u>0</u>	$5(0) - 2 = \underline{-2}$
1	$5(1) - 2 = 3$
2	$5(2) - 2 = 8$
3	$5(3) - 2 = 13$

The related values of x and y from this table, namely, $(-3, -17)$, $(-2, -12)$, $(-1, -7)$, and so forth, determine a set of points that are plotted on the same graph and connected. The points appear to be on a straight line (Figure 2).

■ EXAMPLE 2

Assume that $y = f(x) = 2x^2$. Make a graph indicating the behavior of the function from $x = -4$ to $x = +4$.

x	$y = f(x) = 2x^2$
-4	32
-3	18
-2	8
-1	2
<u>0</u>	0
1	2
2	8
3	18
4	32

The graph determined by the resulting collection of points is shown in Figure 3. If more points are determined in the prescribed range, it is found that a smooth curve, known as a *parabola*, is determined.

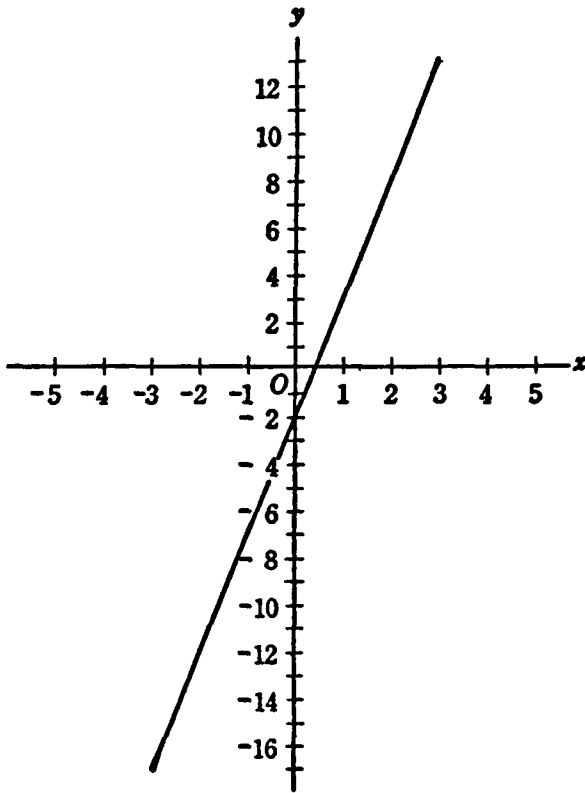


Figure 2

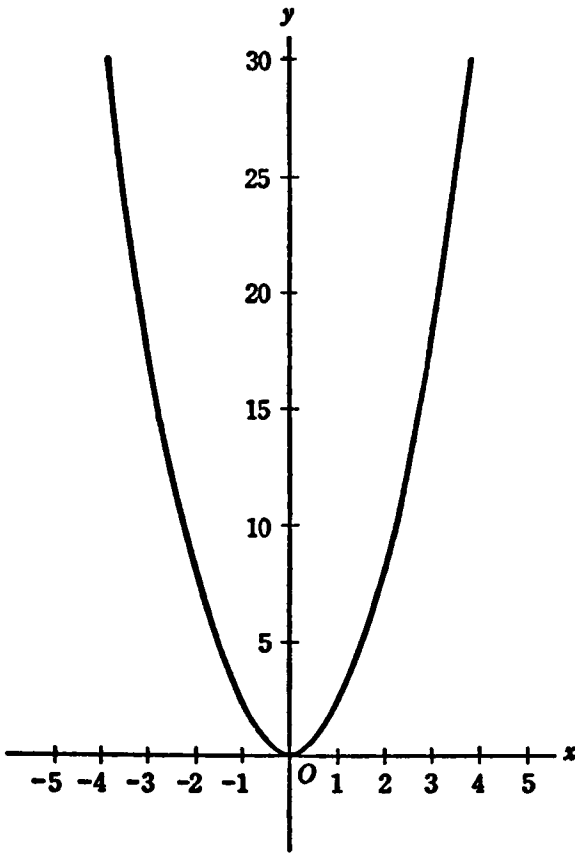


Figure 3

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● EXERCISES 3

1. Graph the function $y = f(x) = 4 - 3x$ for the range $x = -3$ to $x = 5$.
2. Plot the function $y = f(x) = 5x - 3$ for the range $x = -10$ to $x = 10$.
3. Graph the function $y = 20 + 3x$ for the range $x = 0$ to $x = 5$, where x denotes the amount of labor (in weeks) and y the amount of product (in tons).
(a) How many tons can be produced with 2 weeks of labor ($x = 2$)? (b) How many weeks of labor are necessary to produce 23 tons of the product ($y = 23$)?
4. Plot the function $y = 2x^2 - 5x + 1$ for the range $x = -5$ to $x = 5$. Obtain sufficient points to gain an idea of the smooth curve.
5. Plot a smooth curve representing the function $y = 24/x$ for the range $x = 1$ to $x = 10$. Assume that x is the price (in cents) and y the quantity demanded (in pounds). (a) What is the quantity demanded if the price is 6 cents ($x = 6$)? (b) Which price will cause a demand of 4 lb. ($y = 4$)?
6. Plot the function $y = x^4$ from $x = -3$ to $x = 3$.
7. Plot the 2 functions, $y = 2x - 5$ and $y = 10 - x$, on the same coordinate system from $x = -6$ to $x = 6$, and determine graphically their intersection.
8. Plot, on the same coordinate system, the functions $y = x^3$ and $y = 1 - x$ from $x = -4$ to $x = 4$, and find graphically their intersection.
9. Plot, on the same coordinate system, the functions $y = x^4$ and $y = 2x - 1$ from $x = -4$ to $x = 4$, and determine graphically their intersection.
10. The following production function has been estimated by W. H. Nichols for certain operations in the Chicago meat-packing industry: $y = -2.05 + 1.06x - 0.04x^2$. In this equation y is the weekly total live-weight of hogs (in millions of pounds) and x is the weekly total of man-hours (thousands). (a) Plot the function between $x = 10$ and $x = 15$. (b) Estimate from the graph the y which corresponds to $x = 11$. (c) What x corresponds to $y = 4.7$?
11. W. H. Nichols estimated the relation between weekly payrolls (y , in millions of cents) and weekly total live-weight of hogs (x , in millions of pounds) for a Chicago meat-packing house as follows: $y = 0.32 - 0.007x + 0.02x^2$. (a) Plot the function between $x = 1,500$ and $x = 6,000$. (b) What y corresponds to $x = 3,500$? (c) What x corresponds to $y = 500,000$?

LINEAR EQUATIONS IN ONE UNKNOWN

4. Linear Equations

A statement of equality between two quantities is an equation. For instance, $2(a - b) = 2a - 2b$ is an equation. This particular equation is an identity, since the statement is true for all possible values of a and b . We have, for instance, if $a = 1$, $b = 3$, $2(1 - 3) = 2 - 6 = -4$; that is, each side of the equation is equal to -4 . By making $a = 10$, $b = 5$, we have $2(10 - 5) = 20 - 10 = 10$; here each side of the identity is equal to 10. The statement remains true irrespective of the values assigned to a and b .

Consider now the equation

$$2x - 1 = 7.$$

This equation differs in nature from the previous one, for it is true for $x = 4$, and only for $x = 4$. We have $2(4) - 1 = 8 - 1 = 7$. This is an illustration of an equation in one unknown, x . Only the first power of the unknown x appears in the equation, so we call it a *linear equation*.

By solving a linear equation we mean the determination of the particular value of x which satisfies the equation. Certain basic rules pertaining to equalities are employed in the process of solving; for instance, (a) the same quantity may be added to or subtracted from both sides of an equality; (b) both members of the equation may be multiplied or divided by any arbitrary quantity (except 0). These methods are combined frequently to solve linear equations.

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EXAMPLE

Solve the equation

$$5x - 3 = 12.$$

Add 3 to both sides:

$$5x - 3 + 3 = 12 + 3,$$

$$5x = 15.$$

Divide both sides by 5:

$$\frac{5x}{5} = \frac{15}{5},$$

$$x = 3.$$

To check such a solution in the original equation, we write

$$5(3) - 3 = 12,$$

$$15 - 3 = 12.$$

This is true. Evidently the solution $x = 3$ is correct.

EXERCISES 4

Solve the following equations and check:

1. $-x + 6 = 5x + 10.$

2. $3x - 7 = 19x + 1.$

3. $9(-x + 1) + 5(3x - 5) - (x + 1) = 0.$

4. $(2x - 1)/(3 - 5x) = 10.$

5. $1/(2x - 3) = -5.$

6. $10/(1 - 5x) = 5.$

7. $10 - (1/x) = 4.$

8. $2 - 1/(x + 4) = 4.$

9. (Linear programming.) A commodity is produced by using just one factor, for example, labor. Let x_1 be the first activity, which produces \$1. It utilizes one unit of the factor. Let x_2 be the second activity, which produces \$1. It utilizes two units of the factor. The total revenue from both activities is $r = x_1 + x_2$. This has to be maximized under the condition that x_1 and x_2 are not negative (they may be zero). But the total labor supply is limited by the relationship $x_1 + 2x_2 = 10$. Find the solution which actually maximizes the revenue r . (HINT: Assume first $x_2 = 0$, and find the revenue. Then assume $x_1 = 0$ and find the revenue. The particular activity will be selected for which the revenue is a maximum.)

10. (Linear programming.) A commodity is produced with the use of one single factor. There are 3 activities, x_1, x_2, x_3 , which will each yield \$1. Hence the total revenue is $r = x_1 + x_2 + x_3$. Activity x_1 uses one unit, activity x_2 two units, activity x_3 three units of the factor. But the supply of the factor is limited by the relationship $x_1 + 2x_2 + 3x_3 = 6$. Find the activity which yields the maximum revenue. (HINT:

Assume (a) $x_1 = x_2 = 0$, (b) $x_1 = x_2 = 0$, (c) $x_1 = x_2 = 0$, and select the activity which gives maximum revenue.)

5. Graphs of Linear Functions

A function of the form $y = ax + b$ is said to be a *linear function*. The graph corresponding to a linear function is a *straight line*.

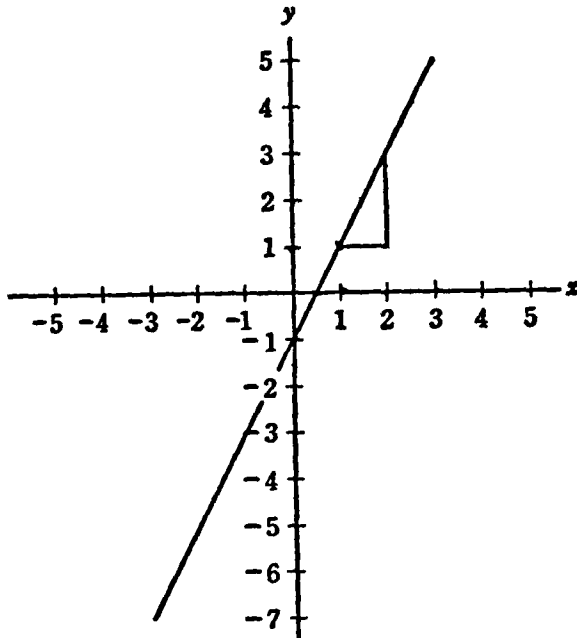


Figure 4

EXAMPLE 1

In Figure 4 is sketched the graph of the function $y = f(x) = -1 + 2x$. To obtain this graph we construct the table:

x	$y = f(x)$
-3	-7
-2	-5
-1	-3
0	-1
1	1
2	3
3	5

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Connecting the points, which are obtained by reference to the table, we get the straight line in Figure 4.

Consider now a line which is supposed to *go through 2 given points*. This condition determines the particular line without ambiguity. The equation of a line, which goes through the 2 fixed points (x_1, y_1) and (x_2, y_2) , is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1).$$

It is confirmed readily that such a line goes through the 2 points (x_1, y_1) and (x_2, y_2) by substituting the pair $x = x_1, y = y_1$ and also the pair $x = x_2, y = y_2$ into the equation.

■ EXAMPLE 2

Consider, for instance, the straight line which goes through the points $(1,1)$ and $(3,5)$. We have

$$x_1 = 1, y_1 = 1, x_2 = 3, y_2 = 5.$$

The above equation becomes

$$y - 1 = \frac{5 - 1}{3 - 1} (x - 1).$$

Simplifying this, we get

$$y = 2x - 1.$$

This is the equation of the line shown in Figure 4. It will be observed that this line passes through the points $(1,1)$ and $(3,5)$. The expression

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

is known as the *slope of the line*.

The geometric significance of the slope may be shown as follows: From any point on the line, proceed horizontally by 1 unit to the right. Then from the new point, proceed vertically upward by m units, since the slope is m . The point determined in this way is also on the line. If the slope is negative, proceed downward. The construction, as just described, is shown in Figure 4. From the point $(1,1)$ we moved 1 unit horizontally to the right to the point $(2,1)$. Since the slope $m = 2$ is positive, we go up 2 units from $(2,1)$ to the point $(2,3)$. This point is again on the line. The equation

$$y - y_1 = m(x - x_1),$$

gives a straight line, which passes through the point (x_1, y_1) , with slope m . This fact follows immediately from the definition of slope and the equation of a straight line just considered.

EXAMPLE 3

Consider, for instance, the construction of a line which goes through the point $(-4,2)$ and has the slope $m = -2$. To construct this line, we proceed from the point $(-4,2)$ 1 unit horizontally to the right and reach the point $(-3,2)$. Since the slope is negative we move 2 units vertically downward to the point $(-3,0)$. We then draw a straight line through the 2 points $(-4,2)$ and $(-3,0)$. This is the required line. The construction is shown in Figure 5.

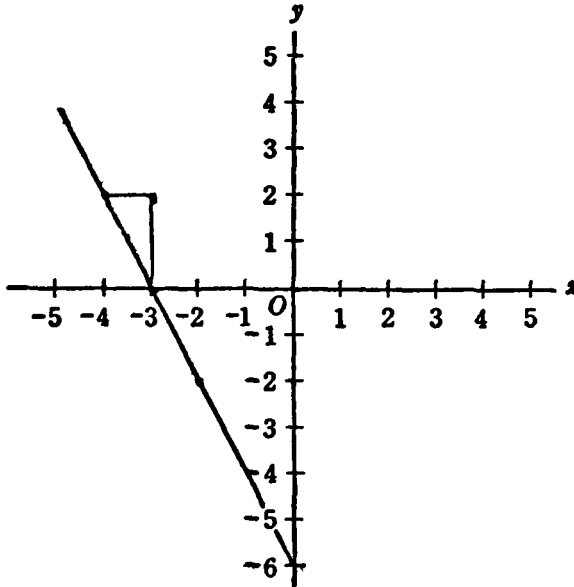


Figure 5

The equation of the line shown in Figure 5 is

$$y - 2 = -2(x + 4).$$

This equation can be simplified to

$$y = -2x - 6.$$

It is verified easily that this is the function corresponding to the graph in Figure 5.

It is verified readily that the coefficient a in the function $y = ax + b$ is the slope m . So, let us write the equation of a straight line in the form

$$y = f(x) = mx + b.$$

By taking $x = 0$, we find that

$$f(0) = b.$$

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Hence b is the value of y which corresponds to $x = 0$, that is, it is the ordinate of a point on the y axis. It is called the *y intercept*. The above equation of a straight line is called the *slope-intercept equation*.

EXAMPLE 4

Let us, for instance, construct the graph of the equation $y = 3x - 5$. We know that the y intercept is -5 . Hence we find $y = -5$ on the y axis. Then to construct the slope $m = 3$ we proceed 1 unit horizontally to the right and from there 3 units vertically upward, since the slope is positive. This determines the point $(1, -2)$. So, the straight line passes through the points $(0, -5)$ and $(1, -2)$, as shown in Figure 6

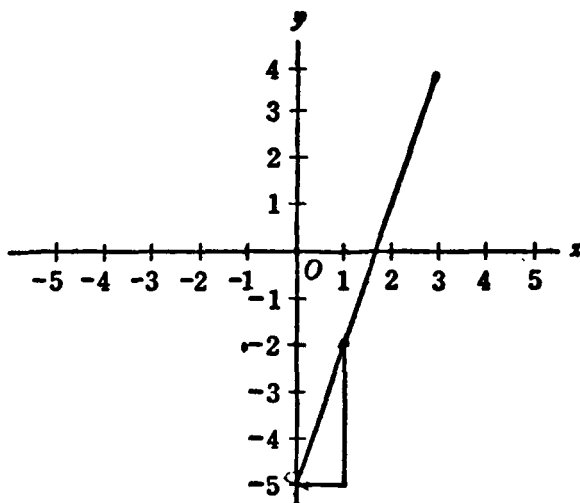


Figure 6

EXERCISES 5

1. Find the equations and graphs of the straight lines which pass through the following pairs of points: (a) $(-1, 4)$, $(1, 0)$; (b) $(-1, -2)$, $(-5, -2)$; (c) $(0, 0)$, $(1, 5)$; (d) $(1, 4)$, $(2, 3)$.

2. Make a graph and find the equation of the straight line with slope -4 which passes through the point (a) $(0, 1)$; (b) $(-3, 2)$; (c) $(-3, 7)$; (d) $(-5, -2)$.

3. Find the equation and the graph of the straight line which passes through the point $(-1, 3)$ and has the slope (a) -1 ; (b) 2 ; (c) 5 ; (d) 0 .

4. Plot each of the following linear equations, using the slope-intercept method: (a) $y = 4x - 6$; (b) $y = 2 - 5x$; (c) $y = -5 - 7x$; (d) $y = 3x$; (e) $y = 8$; (f) $y = 2x + 6$.

5. What is the slope and intercept of each of the following equations: (a) $y = -x$; (b) $y = -4x - 3$; (c) $y = 3x - 7$; (d) $y = 4x + 6$. Plot these equations.

6. Linear-Demand Functions

We will deal in this book mainly with *market* or *collective-demand functions*. Each of these demand functions shows the relationship between the quantity demanded D and the price p charged on a given market. In general, the higher the price the lower the demand, though there may be exceptions to this rule.

Collective or market-demand functions can be constructed by adding the *individual-demand functions* for all individuals in a market. The quantities demanded at a given price by all individuals are added.

■ EXAMPLE 1

The comments on demand functions may be illustrated in the case of a market consisting of 5 individuals, designated by $A, B, C, E,$ and F :

Price p	Quantity Demanded by					Market Demand D
	A	B	C	E	F	
\$1	6	10	20	5	12	53
\$2	5	6	15	4	11	41
\$3	4	2	10	3	10	29

The demand functions of the individuals $A, B, C, E,$ and F can be derived from their utility functions or indifference maps. This will be demonstrated later. See especially Section 70 and Problems 6 and 7 in Exercises 70.

The above table tells us that at a price of \$1, the individual A will demand 6 units of the commodity. At the same price, the demand of B will be 10 units, the demand of C 20 units, the demand of E 5 units, and the demand of F 12 units. There are, by assumption, only these 5 individuals in the market. Adding up all the quantities demanded at the price \$1, we obtain the collective or market-demand D of 53 units. The process is repeated for a price of \$2, and for a price of \$3.

Now the assumption is made for the sake of simplicity that the market or collective-demand function is linear. In other words, if plotted, the graph will show a straight line. Probably this is never strictly true with empirical market-demand curves. Linear-demand functions must be considered as approximations to the true demand functions, which may be much more complicated.

We have represented graphically in Figure 7 the individual-demand curves of $A, B, C, E,$ and F , as well as the market or collective-demand curve D . It should be noted that, by convention, we always put the quantities D on the horizontal axis and the prices p on the vertical axis.

Linear-demand functions can be derived, by statistical methods, from

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actual market data, that is, by using the prices and quantities recorded at various times on the market. These methods will be discussed in Section 100.

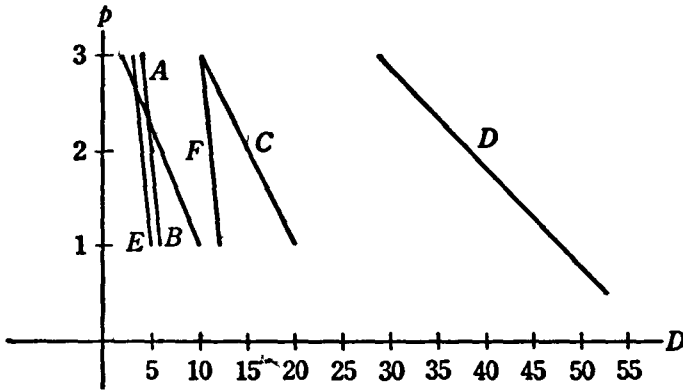


Figure 7

EXAMPLE 2

Assume a collective-demand curve $D = f(p) = 25 - 5p$. (a) What is the quantity demanded if the price is \$3? We have $D = f(3) = 25 - (5)(3) = 10$. (b) Assume the demand is 18 units. What is the corresponding price? We have $18 = 25 - 5p$, and, by solving this equation, it is found that $p = \$1.40$. (c) What would be the demand if the commodity in question were a *free good*, that is if $p = 0$? We have $D = 25 - (5)(0) = 25$. (d) What is the highest price anybody will pay for the commodity? Put $D = 0 = 25 - 5p$. Then $p = 5$. Actually $D = f(5) = 0$. The price must be *less* than 5 in order that any of the commodity be sold on the market.

EXERCISES 6

1. A collective-demand curve is $D = 100 - 5p$. Find the price if the quantity demanded is (a) 50; (b) 16; (c) 5 units. Find the quantity demanded if the price is (d) 19; (e) 10; (f) 1. (g) Find the demand if the commodity were a free good. (h) What is the highest price anybody would pay for this commodity? (i) Plot the curve.

2. Given the collective-demand curve $D = 36 - 3p$. Find the price if the quantity demanded is (a) 18, (b) 2. Find the quantity demanded if the price is (c) 11, (d) 5. (e) What is the highest price anybody would pay for this commodity? (f) How much of the commodity would be demanded if it were a free good? (g) Plot the curve.

3. Given the collective-demand curve $D = 10 - \frac{p}{4}$. Find the quantity demanded if the price is (a) 4; (b) 16; (c) 25. What is the price if the quantity demanded is (d) 9; (e) 7; (f) 2? (g) What is the highest price anybody would pay for this commodity? (h) How much would be demanded if the commodity were a free good? (i) Plot the curve.

4. Given the collective-demand curve $D = 1 - p$, find the price if the quantity

demanded is (a) 0.9; (b) 0.7; (c) 0.2. Find the quantity demanded which corresponds to a price of (d) 0.75; (e) 0.125; (f) 0.0025. (g) What would be the demand if the commodity were a free good? (h) What is the highest price anybody would pay for this commodity? (i) Plot the curve.

✓ 5. The collective-demand curve for a commodity is $D = 12.5 - 2.25p$. Find the price if the quantity demanded is (a) 11.75; (b) 10; (c) 1.125. What is the quantity demanded if the price is (d) 3.5; (e) 1; (f) 0.5? (g) Find the highest price anybody would pay for this commodity? (h) How much would be demanded if it were a free good? (i) Plot the curve.

6. The demand curve for sugar in the United States for 1915–1929 was approximately $D = 135 - 8p$, where D and p are suitably defined (Henry Schultz). Find the quantity demanded if the price is (a) 3; (b) 10; (c) 5. What is the price corresponding to a demand of (d) 80? (e) 95? (f) 35? (g) What is the highest price anybody will pay for sugar? (h) How much sugar will be consumed if it is a free good? (i) Plot the demand curve.

7. The demand for cotton in the United States for 1915–1929 was estimated as $10D = 64 - 3p$ (Henry Schultz). Find the quantity demanded if the price is (a) 10; (b) 12.25; (c) 5. What is the price corresponding to a demand of (d) 5? (e) 2.44? (f) 3.36? (g) What would be the demand for cotton if it were a free good? (h) What is the highest price anybody would pay for cotton? (i) Plot the demand curve.

8. The demand curve for barley in the United States for the period 1915–1929 was estimated as $100D = 207 - p$ (Henry Schultz). Find the demand for a price of (a) 55, (b) 160, (c) 85. What is the price if the demand is (d) 1.50; (e) 0.90; (f) 0.85? (g) What is the highest price anybody will pay for barley? (h) How much barley would be consumed if it were a free good? (i) Plot the demand curve.

9. The demand curve for rye in the United States, 1915–1929, was estimated as $100D = 88 - p$ (Henry Schultz). Find the demand if the price is (a) 20, (b) 35, (c) 42.5. What is the price if the demand is (d) 0.40, (e) 0.50, (f) 0.60? (g) What would be the consumption of rye if it were a free good? (h) What is the maximum price anybody will pay for rye? (i) Plot the demand curve.

10. The demand for a commodity is $D = A - Bp$, where A and B are positive constants. (a) Find the price if the demand is $A/3$. (b) Find the demand if the price is $A/2B$. (c) Find the demand if the commodity is a free good. (d) What is the highest price anybody will pay for it?

11. Given the relationship $2D + 3p = 5$. (a) Find the highest price anybody will pay for this commodity. (b) Find the demand if the commodity is a free good. (c) Express D in terms of p in the form of an ordinary linear-demand curve. (d) Plot the demand curve.

7. Linear-Supply Functions

The individual-supply function of a firm, or of a private individual, shows the amount of a commodity that will be offered on the market at a given price. The collective or market-supply function is the sum of the amounts supplied by various individuals or firms at a given price.

■ EXAMPLE 1

As an example let us consider a market where a commodity is supplied by three firms (or individuals) A , B , and C :

Price p	Quantity Supplied			Market Supply S
	A	B	C	
\$1	0	1	3	4
\$2	1	3	6	10
\$3	2	5	9	16

At a price of \$1 the firm A does not supply any of the commodity. The firm B supplies 1 unit and C offers 3 units of the commodity. Adding all these quantities, we have 4 units, as the market or collective-supply S of the commodity at a price of \$1.

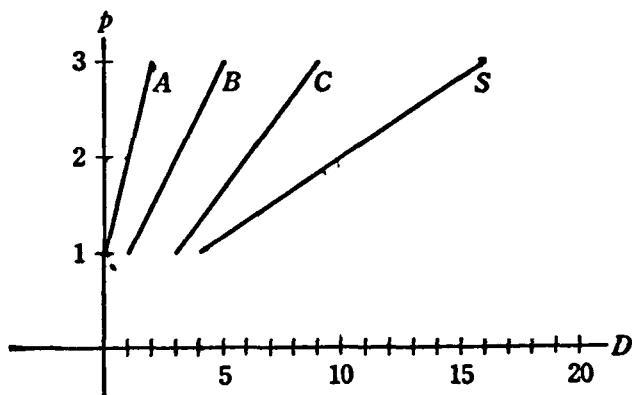


Figure 8

If the market price is \$2, the firm A supplies 1 unit, the firm B 3 units, and the firm C 6 units. Adding these quantities, we have a market supply S of 10 units if the price is \$2. In a similar way we add up all the quantities supplied by firms A , B , and C at a price of \$3, and have a market supply of 16 units.

We show in Figure 8 the individual-supply functions of the individuals (or firms) A , B and C , as well as the market-supply curve S .

We assume that the market-supply function is linear; that is, if plotted in a graph, it appears as a straight line. Such a function must be considered as an approximation to the actual situation.

Supply functions are derived from production theory, which will be dealt with in Section 71. Statistically they are obtained from market data,

that is, from the record of prices and quantities sold on a market at various dates. The problem of deriving statistical-supply curves from market data will be discussed in Section 100.

■ EXAMPLE 2

Let $S = f(p) = 3p - 2$ be the supply curve for some commodity. (a) Given $p = 5$, what is the supply? We have $S = f(5) = (3)(5) - 2 = 13$. (b) What price will cause a supply of 10? The equation $10 = 3p - 2$ has the solution $p = 4$. (c) What is the lowest price at which the commodity will be supplied? We put $S = 0 = 3p - 2$, which gives $p = 2/3$. Actually, $S = f(2/3) = 0$. At a price of $2/3$ nothing will be supplied. Hence the price must be greater than $2/3$ in order that any of the commodity be supplied.

● EXERCISES 7

1. Let the supply curve for some commodity be $S = 2p - 1$. (a) Find the supply if the price is (a) 2; (b) 10; (c) 100. Determine the price if the supply is (d) 10; (e) 25; (f) 100. (g) What is the lowest price at which any of the commodity will be supplied? (h) Plot the curve.

2. The supply of a commodity is $S = 3p$. Find the supply if the price is (a) 5; (b) 10; (c) 25. Assume that the supply is (c) 90; (e) 25; (f) 2; determine the price. (g) What is the lowest price at which the commodity will be supplied? (h) Plot the curve.

3. The supply of a commodity is $S = 5p - 10$. Find the price if the supply is (a) 10; (b) 20; (c) 100; (d) 150. Find the supply if the price is (e) 3; (f) 5.50; (g) 10. (h) What is the lowest price at which any of the commodity will be supplied? (i) Plot the curve.

4. Let the supply curve of a commodity be $S = 3p - 10$. Determine the price if the supply is (a) 160; (b) 13; (c) 40. (d) What is the lowest price at which the commodity will be supplied? (e) Plot the curve.

5. Henry Schultz estimates the supply curve for sugar (imported) for the United States in 1903-1913 as $S = 1.1p - 0.1$, where supply and price are measured in convenient units. Find the price if the supply is (a) 1.00; (b) 0.80; (c) 0.50. Determine the supply if the price is (d) 8; (e) 6; (f) 4.10. (g) What is the lowest price that will cause any sugar to be supplied? (h) Plot the supply curve.

6. The supply of a commodity is $S = ap - b$, where a and b are positive constants. Find the price if the supply is (a) $5a - b$; (b) $a + 2b$. Determine the supply if the price is (c) $3b/a$; (d) $5b/a$. (e) What is the lowest price which will cause any supply of the commodity?

7. Let the supply curve of a commodity be $S = mp$, where m is a positive constant. Find the supply if the price is (a) 1; (b) 10; (c) m ; (d) $1/3m$. Determine the price if the supply is (e) 8; (f) 6; (g) m ; (h) $5/m$. (i) What is the lowest price at which any supply will be supplied on the market?

8. Let the supply curve of a commodity be $S = 8p - 10$. Find the quantity

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supplied if the price is (a) 11; (b) 10; (c) 39. Find the price if the supply is (d) 1; (e) 6; (f) 100. (g) What is the lowest price at which this commodity will be supplied on the market?

9. The supply function of a commodity is $S = 15p - 75$. (a) Find the supply if the price is 10; (b) if it is 6. Determine the price if the supply is (c) 7; (d) if it is 90. (e) What is the lowest price at which any of the commodity will be supplied on the market? (f) Plot the curve.

8. Market Equilibrium

Under free competition no individual or firm can by itself influence the market price. There is free movement in and out of various industries.

Market equilibrium exists under free competition if the quantity of a commodity demanded is equal to the quantity supplied. This fact determines the equilibrium price and the quantity exchanged.

It is demonstrated in elementary economics how the equilibrium price is established. If the actual price were higher than the equilibrium price, where demand and supply are equal, then the quantity demanded would be less than the quantity supplied. Some of the sellers would not be able to sell their products at the prevailing price, but would be willing to sell at a lower price rather than not sell at all. Competition among the sellers will reduce the price until it is established at the level where demand is equal to supply.

Assume on the other hand that the actual price is lower than the equilibrium price. Then the quantity demanded will exceed the quantity supplied. This means that some buyers will not be able to satisfy their demand at the prevailing price. Hence they will be willing to pay higher prices rather than forego the satisfaction of their demand. Competition among the buyers will drive the price up until it is established at the equilibrium level where demand is equal to supply.

These propositions are demonstrated in Problem 8, Exercises 8. Market equilibrium under monopoly will be discussed in Section 56.

It should be noted that we deal here with partial equilibrium (Marshall). We analyze the market of one commodity in isolation and assume that the prices of all other commodities are constant. Problems of general equilibrium will be discussed in Section 11.

■ EXAMPLE

Let the demand for a commodity be $D = 2 - p$ and the supply be $S = p - 1$ (Figure 9). We must have demand equal to supply, that is, $S = D$, or $p - 1 = 2 - p$. Solving this equation we get for the equilibrium price, $p = 3/2$. Inserting this value of p into the demand equation, we have $D = 2 - 3/2 = 1/2$. Substituting the same value into the supply equation, we have $S = 3/2 - 1 = 1/2$. Hence the equilibrium price is $3/2$ and the quantity exchanged is $1/2$.

Let us assume for the given D and S that the actual price is lower than the equilibrium price, for example, that $p = 1$. Then we would have $S = 0$ and $D = 1$. Hence demand is larger than supply, and bidding among the buyers will force the price up.

Assume on the contrary that the actual price is higher than the equilibrium price, for example, that $p = 2$. Then we have $S = 1$, $D = 0$. Now the supply is larger than the demand. Competition among the sellers will

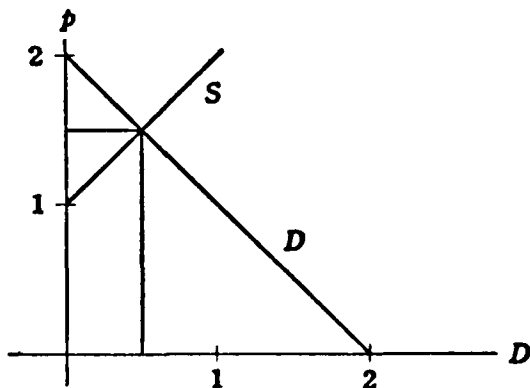


Figure 9

force the price down until the equilibrium level is reached. Such is the case only if $p = 3/2$, which is the very point of intersection between the demand and the supply function in our example.

In Figure 9 we show a graphic representation of the demand and the supply curves used in this example. The intersection of the demand and the supply curves determines the price established on the market (\$1.50) and the quantity sold (0.5).

● EXERCISES 8

1. Let the demand curve for a commodity be $D = 10 - 3p$ and the supply curve be $S = 2p - 1$. (a) Find the equilibrium price. (b) Find the quantity exchanged on the market. (c) Make a graph of the demand and supply curves.

2. The demand curve for a commodity is $D = 25 - 2p$. The supply curve is $S = p - 2$. (a) Find the equilibrium price; (b) the equilibrium quantity. (c) Make a graph of the demand and supply curves.

3. The demand for a commodity is $D = 35 - 7p$. The supply curve is $S = 2p - 5$. (a) Find the equilibrium price. (b) Find the quantity exchanged on the market. (c) Graph the demand curve and the supply curve.

4. The demand curve for a commodity is $D = 15 - 2p$. The supply curve is $S = 3p$. (a) Find the equilibrium price, and (b) find the equilibrium quantity. (c) Make a graph of the demand and supply curves.

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5. The demand curve for a commodity is $D = 35 - 3p$. The supply curve is $S = 2p$. (a) Find the equilibrium price, and (b) find the equilibrium quantity. (c) Graph the demand and supply curves.

6. Let the demand for a commodity be $D = 10 - 2p$, and the supply curve be $S = 4p$. (a) Find the equilibrium price. (b) Find the equilibrium quantity exchanged. (c) Plot the demand and supply curves.

7. The demand curve for a commodity is $D = 10 - 3p$, and the supply curve is $S = 5$ (perishable commodity). (a) Determine the equilibrium price. (b) Find the quantity exchanged. (c) Plot the demand and supply functions.

8. Let the demand curve for a commodity be $D = 1 - p$, and the supply curve be $S = 2p$. (a) Determine the equilibrium price. (b) What is the quantity exchanged in equilibrium? (c) Assume that the government fixes the price at $1/2$; show that the supply is greater than the demand. (d) Assume that the government fixes the price at $1/5$; show that the supply is smaller than the demand. (c) Plot the demand and supply curves.

9. Let the demand curve for a commodity be $D = A - Bp$, and the supply curve be $S = Mp - N$, where A, B, M, N are positive constants. (a) Find the equilibrium price on the market. (b) Determine the quantity exchanged.

10. Henry Schultz estimated the demand for buckwheat for the United States during 1915–1929 as $100D = 234 - 2p$. Assume that the supply of buckwheat is $S = 1.85$. (a) Find the equilibrium price. (b) Find the equilibrium quantity exchanged. (c) Plot the demand and supply curves.

11. The demand and the supply of sugar for the United States during 1890–1915 is estimated according to Henry Schultz as $D = 1.6 - 0.5p$ and $S = 0.7p + 0.4$, respectively. (a) Determine the equilibrium price. (b) Find the equilibrium quantity exchanged on the market. (c) Make a graph of the demand and supply curves.

12. For agricultural products in the United States during 1920–1943 the estimated demand was $D = 224.125 - 0.097p$, and the supply was $S = -49.375 + 1.721p$ (G. Tintner). (a) Find the equilibrium price and quantity. (b) Make a graph of the demand and supply functions.

9. Taxation

Suppose that a specific tax is imposed on sales. That is, for each unit of a commodity sold, a certain fixed amount of money has to be paid to the government. This is equivalent to an upward shift of the supply function by the amount of the tax.

EXAMPLE 1

Consider the following supply function:

Price	Quantity Supplied	Price after Tax	Price after Subsidy
1	3	3	0
2	5	4	1
3	7	5	2

For instance, before the imposition of the tax, 3 units were supplied at a price of \$1. If the tax is \$2 per unit, the consumer has to pay \$3 if \$1 is to be left for the supplier. Hence, with the tax, it takes a price of \$3 to bring forth a supply of 3 units on the market. Similarly, before imposition of the tax, 5 units were supplied if the price of the commodity was \$2. Now a tax of \$2 per unit is imposed. Hence it takes an actual price of \$4 (of which \$2 go to the government and \$2 to the supplier) to bring forth a supply of 5 units. In the same fashion we can argue that, after imposition of the tax of \$2 per unit, it takes a price of \$5 (instead of \$3 as before the imposition of the tax) to bring about a supply of 7 units on the market. Hence, we can say that the supply curve has shifted upward by the amount of the tax (in this case, \$2).

Next, consider a subsidy. The government pays the sellers of a commodity a certain fixed amount per unit. We assume a subsidy of \$1 per unit. Consider the above table again. If the original price without subsidy is \$1, then 3 units are supplied on the market. But since the government pays a subsidy of \$1, the price to be charged to the public to bring about a supply of 3 units is now \$0. Similarly, before the institution of the subsidy it took a price of \$2 to obtain a supply of 5 units; of this amount the subsidy \$1 is paid by the government. Hence the consumers have only to pay \$1 to obtain a supply of 5 units on the market, and so forth. It appears that the supply curve has shifted downward by the amount of the subsidy (in this case \$1).

EXAMPLE 2

Let the demand curve of a commodity be $D = 10 - p$, and let the supply curve be $S = -2 + 2p$, before taxation. By setting supply equal to demand we get the following equilibrium values: $p = 4$ and $D = S = 6$.

We represent this situation in Figure 10. S_1 is the supply curve before imposition of a tax or subsidy.

Assume now that a tax of \$2 per unit is imposed. The new supply curve becomes $S = -2 + 2(p - 2) = -6 + 2p$. This is the curve S_2 in Figure 10. Again putting demand equal to supply we find that $p = 16/3 = 5\frac{1}{3}$, $D = S = 14/3 = 4\frac{2}{3}$. The total amount of the tax is $2(14/3) = 28/3 = 9\frac{1}{3}$. This is the revenue of the government from the tax.

The price realized by the entrepreneur is now only $16/3 - 2 = 10/3 = 3\frac{1}{3}$.

Now let us assume a subsidy of \$1 per unit. The new supply curve becomes $S = -2 + 2(p + 1) = 2p$. This is now the curve S_3 in Figure 10. Equilibrium is established if demand is equal to supply. This gives $p = 10/3 = 3\frac{1}{3}$; $D = S = 20/3 = 6\frac{2}{3}$. The total amount of the subsidy is $(20/3)(1) = 6\frac{2}{3}$. This is the amount which the government has to pay. The price realized by the sellers is now $10/3 + 1 = 13/3 = 4\frac{1}{3}$.

EXERCISES 9

1. Given the demand curve $D = 20 - 2p$, and the supply curve $S = -4 + 3p$.
 (a) Find the equilibrium amount sold and the price before the imposition of a tax

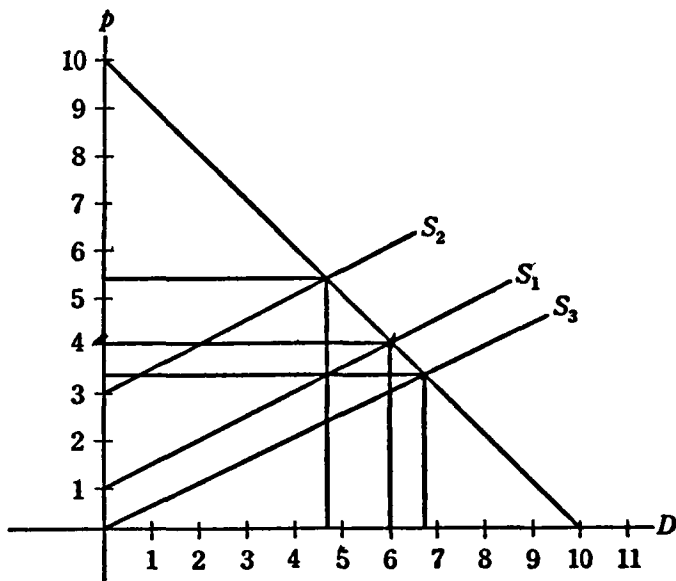


Figure 10

or subsidy. Find the price, the quantity sold, and the total amount of tax or subsidy after the imposition of a tax of (b) \$1; (c) \$2; (d) \$0.50; a subsidy (e) of \$1; (f) of \$0.25.

2. Find the equilibrium price, quantity sold, total amount of tax or subsidy for the demand and supply curve in Problem 1, Exercises 8, after the imposition of (a) a tax of \$1; (b) a tax of \$0.50; (c) a tax of \$0.10; (d) a subsidy of \$1; (e) a subsidy of \$0.60. (f) Plot the original demand and supply curve and the supply curves after the imposition of the taxes and subsidies.

3. Find the equilibrium price, quantity sold, total amount of tax or subsidy, for the demand and supply curve in Problem 2, Exercises 8, after the imposition of (a) a tax of \$0.05; (b) \$0.10; (c) \$0.50; (d) a subsidy of \$0.02; (e) a subsidy of \$0.25.

4. Consider the data in Problem 5, Exercises 8. What amount of tax or subsidy would have to be imposed (a) to double the equilibrium price without the tax; (b) to halve the equilibrium price; (c) to make the commodity a free good (price \$0)?

5. Consider the data in Problem 6, Exercises 8. What amount of tax or subsidy must be imposed in order to bring about (a) an increase by $\frac{1}{8}$ of the equilibrium amount sold before imposition of the tax; (b) decrease by half the amount sold; (c) no sale effected at all?

6. Problem 11, Exercises 8, gives demand and supply curves for sugar in the United States. Find the equilibrium price, quantity sold, total amount of tax or subsidy after the imposition of (a) a tax of \$0.01; (b) a tax of \$0.05; (c) a subsidy of \$0.03; (d) a subsidy of \$0.05. Plot the original demand and supply curves and also the supply curves after the imposition of taxes and subsidies.

7. The demand curve is $D = a - bp$ and the supply curve is $S = -m + np$, where a, b, m, n , are positive constants. (a) Find the equilibrium price and the amount sold on the market before the imposition of a tax or subsidy. (b) Repeat (a) if we assume the imposition of a specific tax or a subsidy t , (if t is negative); also establish (c) the new equilibrium quantity and price; (d) the total amount of tax or subsidy.

8. Consider the data in Problem 12, Exercises 8. What is the effect on price and quantity sold if there is (a) a specific tax of 10; (b) a subsidy of 10 for agricultural products.

9. Consider the demand and supply curves in Problem 4, Exercises 8. Consider the effect (a) of the imposition of a specific tax of 1.5 and (b) of a subsidy of 2 on equilibrium price and quantity. Analyze the situation graphically.

10. Let the demand curve be $D = 1 - p$ and the supply curve be $S = p$. Consider the equilibrium quantity and price (a) before and (b) after the imposition of a specific tax of t per unit.

SYSTEMS OF LINEAR EQUATIONS

10. Linear Equations in More Than One Unknown

Assume that we have a system of n linear equations in n unknowns. Except for the unusual situation when one equation is essentially a restatement of one or more of the other equations, such a system has a solution; that is, a value for each unknown so that each equation is satisfied. In such a case, the given system may be reduced to a system of $n - 1$ equations in $n - 1$ unknowns. This can be accomplished by eliminating one of the n original variables. We can carry out this elimination by combining the n equations two by two, by adding or subtracting, after first multiplying them by suitable constants. This leaves a system of $n - 1$ linear equations in $n - 1$ unknowns.

This new system can be treated in the same fashion. We combine the $n - 1$ linear equations two by two after multiplying by appropriate constants. Thus we can eliminate another unknown. This leaves a system of $n - 2$ linear equations with $n - 2$ unknowns.

This process may be continued, until we have 1 linear equation in 1 unknown. This equation can be solved by methods discussed in Section 4, thereby providing the value of 1 unknown. Using 1 equation of the system of 2 equations with 2 unknowns, we may derive the necessary value for another unknown. The third is found from an equation of the system of 3 equations with 3 unknowns, and so forth.

The complete solution should be checked by substituting into the equations of the original system.

■ EXAMPLE

Assume that we have the following system of 2 equations in 2 unknowns:

$$2x + 6y = 9,$$

$$3x + y = 1.$$

Multiply the members of the first equation by 3 and those of the second by 2, in order to eliminate x :

$$6x + 18y = 27.$$

$$6x + 2y = 2.$$

Subtract the second of these equations from the first, thereby obtaining

$$16y = 25.$$

Dividing both sides of this equation by 16, we have immediately

$$y = \frac{25}{16}.$$

We obtain from the first of the original equations:

$$2x + 6\left(\frac{25}{16}\right) = 9,$$

$$2x + \frac{150}{16} = 9,$$

$$2x = 9 - \frac{150}{16},$$

$$2x = \frac{144 - 150}{16},$$

$$2x = -\frac{6}{16} = -\frac{3}{8},$$

$$x = -\frac{3}{16}.$$

The solution of our original system of equations is therefore $x = -3/16$ and $y = 25/16$. This solution is checked by substitution into the original equations as shown below:

$$2\left(-\frac{3}{16}\right) + 6\left(\frac{25}{16}\right) = \frac{144}{16} = 9,$$

$$3\left(-\frac{3}{16}\right) + \frac{25}{16} = \frac{16}{16} = 1.$$

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● EXERCISES 10

Solve the following systems of equations and check:

1. $2x + 5y = -1$ and $x + 2y = 9$.

2. $-x + 5y = 10$ and $3x + y = 0$.

3. $4x + 19y = 10$ and $x - y = 1$.

4. $9x - y = 10$ and $5x + 4y = 1$.

5. $-3x - 5y = 0$ and $x + y = 0$.

6. (Linear programming.) A commodity can be produced in three ways, using two factors. If it is produced by the first method, an activity which yields \$1's worth of the commodity (x_1) will use 1 unit of the first factor and 1 unit of the second factor. If it is being produced in the second way, the activity which produces \$1's worth of the commodity (x_2) will use 1 unit of the first factor and 2 units of the second factor. Finally, if the third method of production is adopted, we must use 2 units of the first factor and 1 unit of the second factor in order to produce \$1's worth of the final commodity (x_3). Note that x_1 , x_2 and x_3 cannot be negative. The total profit is $r = x_1 + x_2 + x_3$. This has to be maximized under the condition that the total supply of the first factor is 10 units: $x_1 + x_2 + 2x_3 = 10$; and that the total supply of the second factor is 15 units: $x_1 + 2x_2 + x_3 = 15$. Find the combination of activities which maximizes the total receipts r . (HINT: Assume (a) $x_3 = 0$, (b) $x_2 = 0$, (c) $x_1 = 0$, and compute the total receipts r in each case. Discard any solution where the x 's are negative and select the solution where r is the maximum.)

Solve each of the four systems of equations that follow:

7. $2x - y = 0$,

$$y + z = 0,$$

$$z = 1.$$

8. $2x + 3y + 5z = -9$,

$$x + 10y + 6z = -13,$$

$$-5x + y + 10z = 14.$$

9. $5x + 4y + 3z = 7$,

$$x + y + 2z = 4,$$

$$3x + 5y + z = 0.$$

10. $5x + 6y + z = 1$,

$$2x + 7y - z = -1,$$

$$3x + 10y + 2z = 2.$$

11. Market Equilibrium for Several Commodities

The demand and the supply of a commodity depend frequently, not only on the price of the particular commodity, but also upon the prices of other related commodities. Again we can determine the equilibrium quantities exchanged and the prices of all commodities concerned. The condition for equilibrium is that for all commodities: the supply is equal to the demand.

If we had information about the demand and supply functions of *all*

commodities in the economy, we could determine the *Walrasian general equilibrium*, that is, *all prices and all quantities exchanged in the economy*.

■ EXAMPLE

Denote by p_A the price of commodity A and by p_B the price of commodity B . Let D_A and D_B be the demand, respectively, for A and B , and S_A and S_B the supply of A and B . Let the demand function for A be $D_A = 5 - p_A + p_B$, and the demand function for B be $D_B = 7 - p_A - p_B$. The respective supply functions are $S_A = -5 + p_A + 5p_B$ and $S_B = -1 + 3p_A + p_B$. By putting $D_A = S_A$ and $D_B = S_B$ we get the 2 linear equations

$$10 = 2p_A + 4p_B \quad \text{and} \quad 8 = 4p_A + 2p_B.$$

By solving this system of equations, we compute the equilibrium prices $p_A = 1$ and $p_B = 2$. Substituting back into the original equations, we have for the equilibrium quantities $D_A = S_A = 6$ and $D_B = S_B = 4$.

● EXERCISES 11

1. $D_A = 10 - p_A - 2p_B$; $D_B = 6 - p_A - p_B$; $S_A = -3 + p_A + p_B$; $S_B = -2 + p_B$. (a) Find the equilibrium prices. (b) Determine the equilibrium quantities exchanged on the market.

2. $D_A = 10 - 2p_A + p_B - p_C$; $D_B = 12 + p_A - 4p_B + p_C$; $D_C = 20 - p_A - p_B - 5p_C$; $S_A = 4$; $S_B = 13$; $S_C = 2$. (a) Find the equilibrium prices. (b) Determine the equilibrium quantities exchanged on the market.

3. $D_A = 10 - 3p_A + p_B$; $D_B = 20 + 4p_A - 5p_B$; $S_A = 9$; $S_B = 14$. (a) Find the equilibrium prices. (b) Find the equilibrium quantities.

4. $D_A = 100 - 2p_A - p_B - 2p_C$; $D_B = 200 - 10p_A - 2p_B - 3p_C$; $D_C = 150 - 2p_A - 3p_B - 5p_C$; $S_A = 40$; $S_B = 70$; $S_C = 10$. Find the equilibrium prices and quantities.

5. Let A represent beef, B pork, and C mutton. Henry Schultz estimated the demand functions for beef, pork, and mutton in the United States in 1922–1933 to be, respectively, $D_A = 63.3 - 1.9p_A + 0.2p_B + 0.5p_C$, $D_B = 71.0 + 0.4p_A - 1.2p_B - 0.1p_C$, $D_C = 10.3 + 0.1p_A + 0.1p_B - 0.3p_C$. Assume that $S_A = 60$, $S_B = 70$, $S_C = 7$. (a) Determine the equilibrium prices. (b) Find the quantities exchanged on the market.

6. Let A, B, C, D , designate barley, corn, hay and oats. Henry Schultz has estimated the demand functions for the United States in 1896–1914 as follows: $D_A = 2.24 - 0.01p_A - 0.01p_B + 0.01p_D$; $D_B = 49.07 - 0.02p_A - 0.36p_B - 0.03p_C + 0.03p_D$; $D_C = 1.30 - 0.05p_C + 0.01p_D$; $D_D = 24.16 + 0.03p_A + 0.07p_B - 0.61p_C - 0.30p_D$. Determine the equilibrium prices if $S_A = 2.24$, $S_B = 48.92$, $S_C = 1.29$, $S_D = 23.92$.

7. Let $D_A = a + bp_A + cp_B$; $D_B = e + fp_A + gp_B$; $S_A = h + kp_A$; and $S_B = m + np_B$. Find the equilibrium prices on the market.

8. Let the demand functions for 2 commodities A and B be $D_A = 10 - 2p_A + p_B$, $D_B = 20 + p_A - 5p_B$. The supply functions of A and B are $S_A = 4p_A$ and

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$S_B = -1 + 6p_B$. Find the equilibrium prices and quantities exchanged on the markets.

9. The demand functions of 3 commodities A , B , and C are $D_A = 20 - 3p_A + p_B + p_C$; $D_B = 30 + p_A - 5p_B$; $D_C = 15 + p_A - 3p_C$. The corresponding supply functions are $S_A = 9p_A$, $S_B = 30p_B - 3$, $S_C = 3p_C - 1$. Find the equilibrium prices and quantities exchanged on the markets.

10. If y denotes disposable income, c consumers' expenditure, r gross business savings, and x gross investments, the following simple model of general equilibrium has been established by T. Haavelmo for the United States for the years 1929-1941:

$$c = 0.712y + 95.05,$$

$$r = 0.158(c + x) - 34.30,$$

$$y = c + x - r,$$

$$x = 93.53.$$

(a) Solve for the unknowns y , c , r , x , and check. (b) Assume $x = 200$ rather than $x = 93.53$, as in the last equation in the above system. Solve for the unknowns and check. (c) What are the estimated disposable income, consumer expenditure, and gross business savings if the gross investment is 300? (HINT: Substitute $x = 300$ for the last equation in the above system, solve for the unknowns, and check.)

12. Imputation

The *theory of economic imputation* deals with this question: *In what way is the value (price) of a final product divided among the various factors of production (that is, land, labor, and capital) which have collaborated in the production?*

We assume static conditions; that is, the basic conditions of production and demand remain the same. We also assume free competition. Under free competition, the action of no one individual in the economy is important enough to influence, by itself, the workings of the market. There are a great number of independent firms. No firm can make profits.

Under free competition we have free movement of the factors of production. If the compensations for the use of the factors is not the same in the production of all commodities, the factors will tend to move away from the poorly paid occupations into the better paid ones, until the compensation is the same in all lines.

We make also the unrealistic assumption that there are fixed coefficients of production. This would be the case if the various factors of production had to be combined in fixed proportions in order to produce a given product. For instance, a certain product A can only be produced by combining 1 unit (acre) of land with 2 units (weeks) of labor. If, on the other hand, 2 units of land are combined with 3 units of labor we get another product B ; and so on.

Under these very simplified and artificial conditions, the problem of imputation reduces itself to the solution of a system of linear equations.

■ EXAMPLE

Assume that 1 unit of commodity A is produced by using 1 unit of land and 2 units of labor; and 1 unit of commodity B is produced by using 2 units of land and 3 units of labor. Let the price of commodity A be $p_A = 25$ and of commodity B be $p_B = 40$. Designate the wage rate by W and the rent per acre by R . The sum of the shares of the factors must be added to give the price, since there is no profit. The 2 equations are

$$R + 2W = 25 \quad \text{and} \quad 2R + 3W = 40.$$

We solve these two equations by elimination and obtain for the wage rate and the rent the values, $W = 10$ and $R = 5$.

● EXERCISES 12

1. Let a unit of commodity A be produced by combining 5 units of land and 6 units of labor. A unit of B is produced by using 10 units of land and 1 unit of labor. Let $p_A = 23$ and $p_B = 13$. Find the wage W and the rent R .

2. Let 3 units of land, 2 units of labor, and 10 units of capital be combined to produce 1 unit of commodity A . If 1 unit of land, 3 units of labor, and 5 units of capital are used, 1 unit of commodity B is produced. By using 2 units of land, 2 units of labor, and 3 units of capital we get a unit of commodity C . The prices are $p_A = 62$, $p_B = 36$, $p_C = 25$. Determine the rent R , wage W , and interest I . Show that the sum of the shares of all factors equals the price of each commodity. Hence there is no profit.

3. One unit of commodity A is produced by combining 1 unit of land, 2 units of labor, and 5 units of capital. One unit of commodity B is produced by 2 units of land, 3 units of labor, and 1 unit of capital. One unit of commodity C results if we use 3 units of land, 1 unit of labor, and 2 units of capital. Assume that the prices are $p_A = 27$, $p_B = 16$, $p_C = 19$. Find the rent R , wage W , and rate of interest I .

4. Commodity A is produced by combining 2 acres of land, 3 weeks of labor, and 5 units of capital. Commodity B is produced by combining 1 acre of land, 1 week of labor, and 1 unit of capital. Commodity C is produced by combining 2 acres of land, 5 weeks of labor, and 10 units of capital. The prices are $p_A = 78$, $p_B = 23$, $p_C = 135$. Determine the rent R , wage W , and interest I .

5. Commodity A is produced by combining 1 acre of land, 2 weeks of labor, and 3 units of capital; B is produced by combining 2 units of land, 1 unit of labor, and 2 units of capital; C results from a combination of 5 units of land, 2 units of labor, and 1 unit of capital. Given the prices, $p_A = 84$, $p_B = 56$, $p_C = 108$, find the rent R , wage W , and interest rate I .

6. Assume that a units of labor and b units of land are combined in order to produce 1 unit of commodity A . By combining c units of land and d units of labor, we get 1 unit of commodity B . Let the prices be $p_A = m$ and $p_B = n$; the quantities a, b, c, d, m, n are constants. Find the wage W and the rent R .

7. A unit of commodity A is produced by combining 1 acre of land and 2 days of labor. A unit of B requires 2 acres of land and 4 days of labor. Let the prices be

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$p_A = 20$ and $p_B = 40$. Show that it is impossible to determine the rent R and the wage W . Why?

8. A product which is produced by combining 1 week of labor, 3 acres of land, and 10 units of capital sells for \$14 per unit. Another product produced by combining 2 weeks of labor, 4 acres of land, and 20 units of capital sells for \$24 for each unit. A third is produced by combining 9 acres of land, 20 weeks of labor, and 30 units of capital, and sells for \$49. Determine, under conditions of perfect competition, the wage, rent, and interest rate.

9. Let us distinguish between 4 factors of production: skilled labor, unskilled labor, land, and capital. By combining 1 week of skilled labor, 10 weeks of unskilled labor, 2 acres of land, and 100 units of capital, we produce a unit of commodity A , which sells for \$420. Using 3 weeks of skilled labor, 20 weeks of unskilled labor, 1 acre of land, and 200 units of capital, we produce a unit of commodity B , which sells for \$835. Using 5 weeks of skilled labor, 20 weeks of unskilled labor, 1 acre of land, and 100 units of capital, we produce a unit of C which sells for \$555. If we use 1 week of skilled labor, 100 weeks of unskilled labor, 3 acres of land, and 500 units of capital we produce a unit of D which sells for \$2,525. Assuming free competition, determine the compensation of the 4 factors of production, namely, the wage of skilled labor W_1 , the wage of unskilled labor W_2 , the rent for land R , and the interest for capital I . Show that the shares of the factors in the products add up to the prices of the commodities. Hence there is no profit.

10. Using data given by W. Leontief for the United States in 1939, we assume constant coefficients of production. By using 0.242 units of labor and 5.461 units of capital we get a unit of agricultural products, which sells for \$1. By using 0.282 units of labor and 5.191 units of capital, one unit of industrial products is produced, which sells for \$1. Assume free competition and compute the compensations for labor and capital.

QUADRATIC EQUATIONS IN ONE UNKNOWN

13. Quadratic Demand and Supply Curves

Assume a demand curve $D = 2 - p^2$ and a supply curve $S = p$. Putting demand equal to supply to obtain equilibrium, we get an equation in which the highest power of the variable is 2, namely, $p^2 + p - 2 = 0$. Such an equation is said to be of *second degree*, and is described frequently as a *quadratic equation*.

The solutions of the general quadratic equation $ax^2 + bx + c = 0$ are given by the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the case of the equation $p^2 + p - 2 = 0$, where the unknown is p , the coefficient a is 1, $b = +1$, $c = -2$; thus

$$p = \frac{-1 \pm \sqrt{1 + 8}}{2} = \frac{-1 \pm 3}{2} = -2 \text{ or } +1.$$

We discard the first solution since price cannot be negative. The equilibrium quantity is then found to be $D = S = 1$. This checks our solution.

● EXERCISES 13

1. The demand curve of a commodity is $D = 20 - 3p - p^2$ and the supply curve is $S = 5p - 1$. Find the equilibrium price and the quantity exchanged. Plot both curves.

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In the following exercises, given the demand curve and supply curve, find the equilibrium price and quantity, and plot both curves:

2. Demand curve: $D = 100 - p^2$. Supply curve: $S = p + 5p^2$.
3. Demand curve: $D = 20 - p$; supply curve: $S = 4p + p^2$.
4. Demand curve: $D = 25 - p - p^2$; supply curve: $S = 10$.
5. Demand curve: $D = 250 - 10p - 2p^2$; supply curve: $S = 5p + 6p^2$.
6. Demand curve: $D = 10 - 2p$; supply curve: $S = 5p^2$.
7. Demand curve: $D = 250 - 3p^2$; supply curve: $S = p^2 + 2p^4$. (HINT: Consider p^2 as the unknown.)
8. Demand curve: $D = 10,000 - p^3$; supply curve: $S = 2p^6$. (HINT: Consider p^3 as the unknown.)
9. Demand: $D = 100/p$; supply: $S = 4p$.
10. Demand curve: $D = 24/p$; supply curve: $S = 5p - 1$.
11. Demand curve: $D = 100/(3p^2)$; supply curve: $S = 4p^2/3$.

LOGARITHMS

14. Pareto Distribution of Income

The Italian economist Vilfredo Pareto indicated the following empirical law for the distribution of incomes: If N denotes the number of people who have an income x or higher, then $N = A/x^\alpha$, where A and α are constants and α is approximately 1.5. This function fits the data rather well for relatively high incomes.

The statistical fitting of Pareto distributions by the method of least squares will be considered later. (See Section 97, especially Problems 22–25, Exercises 97.)

EXAMPLE

Let the Pareto distribution of incomes in a community be given by $N = 2,000,000,000/x^{1.5}$. (a) How many people have an income of \$100,000 or higher? (b) What is the lowest income of the 100 richest people in the community?

The solution of the two parts of this problem is facilitated by the use of logarithms. The following definitions and principles pertaining to the use of logarithms are recalled at this point. In the various statements that follow, the symbol \log is the abbreviation of logarithm.

1. $\log_b N = x$ is equivalent to $b^x = N$. If $b = 10$, the logarithms are said to be common logarithms. When the subscript b is omitted, it is understood in this book that $b = 10$.

2. $\log 10 = 1$.
3. $\log N^t = t \log N$.
4. $\log MN = \log M + \log N$.
5. $\log M/N = \log M - \log N$.

6. By reference to Table 1 in the Appendix, the common logarithm of any number between 1 and 10 may be found. In this table a decimal point is to be understood after the first digit of each number in the N column and a decimal point is to be understood before each 4-digit array within the body of the table. Thus, to obtain $\log 4.67$, the first 2 digits, namely 4.6, are located in the N column; the third digit 7 is a column heading; so the desired value, that is, $\log 4.67 = 0.6693$, is the number opposite the 46 and appearing in the column headed by 7. Similarly, $\log 2.72 = 0.4346$, and $\log 2.00 = 0.3010$.

7. It is possible to obtain the common logarithm of any number greater than 10 or less than 1 by the use of Table 1 and Principles 2, 3, 4, and 5. Thus

$$\log 4.67 = 0.6693,$$

$$\begin{aligned}\log 46.7 &= \log (4.67)(10) = \log 4.67 + \log 10 \\ &= 0.6693 + 1, \text{ or } 1.6693.\end{aligned}$$

$$\begin{aligned}\log 467 &= \log (4.67)(10^2) = \log 4.67 + \log 10^2 \\ &= \log 4.67 + 2 \log 10 = 0.6693 + 2, \text{ or } 2.6693.\end{aligned}$$

$$\begin{aligned}\log 0.467 &= \log (4.67)(10^{-1}) = \log (4.67) + \log 10^{-1} \\ &= \log 4.67 - 1 \log 10 = 0.6693 - 1.\end{aligned}$$

In summary,

$$\log 0.00467 = 0.6693 - 3.$$

$$\log 0.0467 = 0.6693 - 2.$$

$$\log 0.467 = 0.6693 - 1.$$

$$\log 4.67 = 0.6693.$$

$$\log 46.7 = 1.6693.$$

$$\log 467 = 2.6693.$$

$$\log 4670 = 3.6693.$$

Let us return to a consideration of question (a) in the example, which involves a study of $N = 2,000,000,000/x^{1.5}$, where $x = 100,000$. We may take the common logarithm of each member, thereby obtaining the following sequence of equations:

$$\log N = \log \frac{2,000,000,000}{100,000^{1.5}},$$

$$\log N = \log 2,000,000,000 - \log (100,000)^{1.5},$$

$$\log N = \log 2,000,000,000 - 1.5 \log 100,000,$$

$$\log N = 9.3010 - 1.5(5) = 9.3010 - 7.5 = 1.8010.$$

It is apparent that N is about 63.2, or roughly 63, since $\log 63.2 = 1.8007$.

Question (b) involves a study of the Pareto equation when $N = 100$. Thus

$$100 = \frac{2,000,000,000}{x^{1.5}}$$

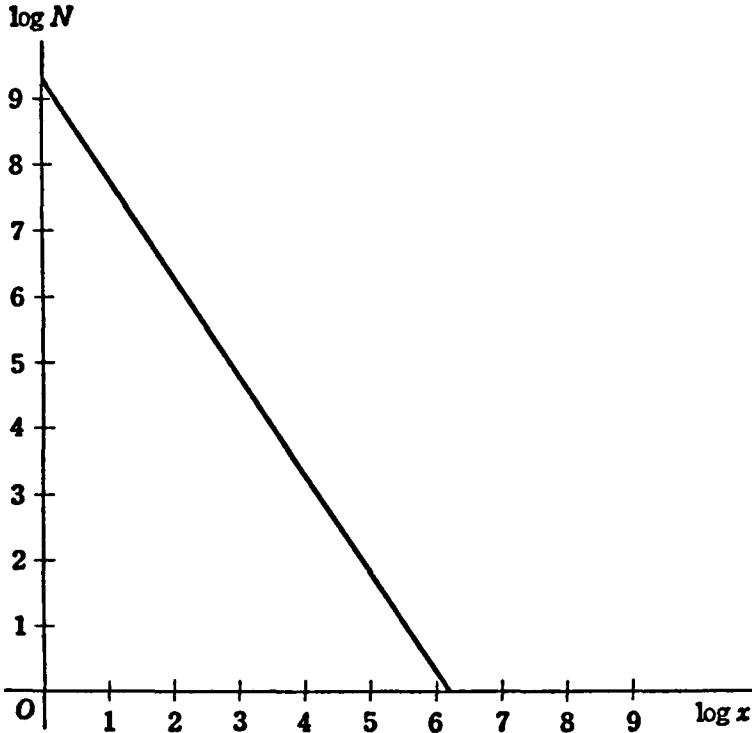


Figure 11

The solution of this equation for x leads to the following sequence of equations:

$$100 x^{1.5} = 2,000,000,000,$$

$$x^{1.5} = 20,000,000.$$

$$\log x^{1.5} = \log 20,000,000,$$

$$1.5 \log x = 7.3010,$$

$$\log x = 4.8673.$$

So x is about 73,700, since $\log 73,700 = 4.8675$.

A graph of the above equation is shown in Figure 11. The graph, as shown, is the result of plotting the logarithm of N (number of people having

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a given income x or larger) against the logarithm of the income x . It is apparent that the result is a straight line.

● EXERCISES 14

1. Assume that in a city the Pareto distribution is $N = 50,000,000,000/x^{1.2}$. Find the number of people who have an income of (a) over \$500,000; (b) over \$2,500,000; (c) over \$3,000; (d) over \$125,400. What is the lowest income of (e) the 10 richest people? (f) the 250 richest people? (g) the 10,000 richest people? (h) the 15 richest people? (k) the 2 richest people? (l) Plot the distribution in terms of $\log N$ and $\log x$.

2. The Pareto distribution in a community is $N = 3,000,000,000/x^{4/3}$. Find the number of people who have an income of (a) \$150,000 or larger; (b) \$2,125,500 or larger; (c) \$5,500,000 or larger; (d) \$10,000,000 or larger; (e) \$2,500 or larger; (f) \$5,275 or larger. Find the lowest income of (g) the 10 richest people; (h) the 150 richest people; (i) the 255 richest people; (j) the richest man in the community. (k) Plot the distribution, using $\log x$ and $\log N$ as coordinates.

3. Assume the Pareto distribution of a community to be $N = 1,236,000,000/x^{1.1}$. Find the number of people who have an income of (a) \$125,200 or higher; (b) \$236,567 or higher; (c) \$5,234,125 or higher; (d) \$1,567 or higher; (e) \$12,345,789 or higher. Find the lowest income of (f) the 11 richest people; (g) the 124 richest people; (h) the 10,000 richest people in the community. (i) Plot the distribution, using $\log x$ and $\log N$ as coordinates.

4. The income distribution of the United States for 1918 has been estimated by Von Szeliski to be $N = 813,000,000,000/x^{1.48}$. Estimate the number of people who had an income (a) of \$100,000 or higher; (b) of \$2,000,000 or higher; (c) of \$150,000 or higher; (d) of \$5,000 or higher; (e) of \$2,500 or higher. Find the lowest income of (f) the 10 richest people; (g) the 3 richest people; (h) the 125 richest people; (i) the 40 richest people. (j) How many people had an income between \$12,500 and \$20,000? (k) Plot the distribution using $\log x$ and $\log N$.

5. H. T. Davis estimated the Pareto distribution of income for the United States in 1918 as follows: $N = 1,926,000,000,000/x^{1.7}$. Find the number of people who had an income of (a) \$100,000 or higher; (b) \$5,000 or higher; (c) \$2,125 or higher; (d) \$2,125,000 or higher; (e) \$500,000 or higher. Find the lowest income of (f) the 12 richest men; (g) the 100 richest; (h) the 1,200 richest; (i) the 2 richest. (j) How many people had an income between \$20,000 and \$25,000? (k) between \$100,000 and \$250,000? (l) Plot the distribution in terms of $\log x$ and $\log N$.

6. H. T. Davis estimated the Pareto distribution for salaries paid by General Motors in 1936 to be $N = 67,620,000,000/x^3$. Find the number of employees who had an income of (a) \$100,000 or over? (b) \$50,000 or over? (c) \$12,500 or over? (d) \$4,000 or over? What is the lowest income of (e) the 100 best paid employees? (f) the 5 best paid employees? (g) the 125 best paid employees? (h) How many people had an income between \$7,500 and \$10,000? (i) Plot the distribution in terms of $\log x$ and $\log N$.

7. Assume the Pareto distribution of a community to be $N = a/x^b$, where a

and b are constants. (a) How many people have an income larger than K ? (b) larger than \$100,000? (c) What is the lowest income of the 100 richest people? (d) of the 10 richest people? (e) How many people have incomes between s and t ? (f) between \$500,000 and \$1,500,000?

8. Let the Pareto distribution in a community be $N = 1,000/\sqrt[3]{x^4}$. NOTE: $\sqrt[3]{x^4} = x^{4/3}$. How many people have an income of (a) \$10 or higher? (b) \$100 or higher? (c) What is the lowest income of the 10 richest people? (d) the 100 richest people? (e) How many people have incomes between 15 and 20?

9. Let the Pareto distribution in a community be $N = 100,000/x^2$. How many people have an income of (a) 10 or higher? (b) 15 or higher? What is the lowest income of (c) the 100 richest people; (d) the 5 richest people? (e) How many people have incomes between 100 and 120? (f) between 50 and 75?

10. Assume that the income distribution in a city is $N = 1,000,000,000/x^{1.5}$. How many people have an income of (a) more than \$100,000? (b) more than \$250,000? (c) What is the lowest income of the 1,500 richest people? (d) of the 3,000 richest people? (e) How many people have incomes between \$500,000 and \$750,000?

15. Demand Curves with Constant Elasticity

A demand curve of the general form $D = A/p^B$, where A and B are constants, has constant elasticity, as will be proved later (see Section 49). The statistical fitting of demand curves with constant elasticity is discussed in Problem 1 ff, Exercises 100.

By a demand curve with constant elasticity B we understand the following property: If the price increases by 1 per cent, the demand will decrease approximately by B per cent.

EXAMPLE 1

Assume a demand curve in the form $D = 25/p^3$. (a) What is the demand if the price is 0.5? We have $D = 25/(0.5)^3 = 200$. (b) What price causes a demand of 65 units? Inserting $D = 65$ into the above equation, we get

$$65p^3 = 25,$$

$$p^3 = 5/13.$$

$$3 \log p = \log 5 - \log 13,$$

$$3 \log p = 0.5851 - 1,$$

or $2.5851 - 3.$

$$\log p = 0.8617 - 1,$$

$$p = 0.727.$$

EXAMPLE 2

In the special case of $B = +1$ in the formula $D = A/p^B$, we have demand curves with unit elasticity. An approximate decrease of 1 per cent in

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the quantity sold corresponds to a 1 per cent increase in price. Price times quantity is constant. That is to say, the total outlay of the buyers or the total receipts of the sellers are the same, whatever the price and quantity sold. Consider for example the demand curve

$$D = \frac{24}{p}.$$

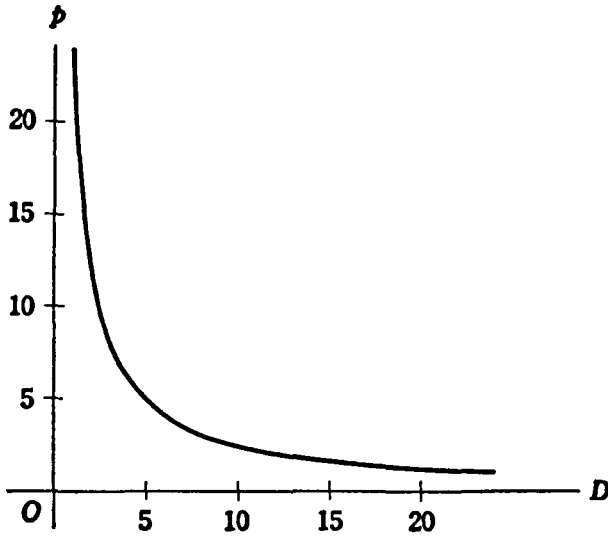


Figure 12

The curve is shown in Figure 12; since D and p are positive, only the positive branch of the curve is drawn. This type of mathematical curve is called a *rectangular hyperbola*. We have, for instance, if $p = 2$, $D = 24/2 = 12$. The total outlay or revenue, $pD = (2)(12) = 24$. On the other hand, if $p = 8$, we have $D = 3$. Hence again $pD = (3)(8) = 24$, as before. This constancy is true for any price.

● EXERCISES 15

1. Assume a demand curve in the form $D = 100/p^5$. Find the demand if the price is (a) 4; (b) 2.5; (c) 0.60; (d) 0.002. Find the price which will cause a demand of (e) 10; (f) 250; (g) 0.5; (h) 0.003 units. (i) Plot the demand curve.

2. Assume a demand curve in the form $D = 1,245/(p)\sqrt[3]{p}$. Find the price that will cause a demand of (a) 20,000; (b) 1,500; (c) 600; (d) 1. Find the demand if the price is (e) 2; (f) 0.85; (g) $1/32$; (h) 1. (i) Plot the demand curve. (NOTE: $\sqrt[3]{p} = p^{1/3}$.)

3. A demand curve is given by $D = 0.657/p^{1.57}$. Find the price which will cause a demand of (a) 10; (b) 3; (c) 0.6; (d) 0.0012. Given the price of (e) 1, (f)

35, (g) 0.86, (h) 0.00065; find the corresponding quantity demanded. (i) Plot the demand curve.

4. The demand curve for sugar in the United States has been estimated by Henry Schultz to be $D = 157.8/p^{0.3}$ for the period 1915–1929. Find the demand for sugar if the price is (a) 5; (b) 8; (c) 11; (d) 12. What price will call forth a demand of (e) 60? (f) 30? (g) 55? (h) 42? (i) Plot the demand curve.

5. The demand curve for corn in the United States, 1915–1929, has been estimated by Henry Schultz to be $D = 172.8/p^{0.49}$. Find the demand if the price is (a) 30; (b) 42; (c) 75; (d) 81. What is the price if the demand is (e) 20; (f) 35; (g) 41; (h) 50. (i) Plot the demand curve.

6. The demand curve for cotton in the United States, 1915–1929, is estimated by Henry Schultz to be $D = 0.59/p^{0.13}$. Find the demand if the price is (a) 30; (b) 50; (c) 75; (d) 51; (e) 32. What is the price if the demand is (f) 0.20; (g) 0.25; (h) 0.30; (i) 0.32. (j) Plot the demand curve.

7. The demand for wheat in the United States, 1915–1929, has been estimated by Henry Schultz as $D = 12.03/p^{0.21}$. Find the price if the demand is (a) 6; (b) 5; (c) 4.5; (d) 2.1; (e) 3.3. Find the demand if the price is (f) 100; (g) 120; (h) 150; (i) 85; (j) 67. (k) Plot the demand curve.

8. The demand for potatoes in the United States for the period 1915–1929 has been estimated by Henry Schultz as $D = 12.05/p^{0.3}$. Find the demand if the price is (a) 30; (b) 45; (c) 25; (d) 33; (e) 37; (f) 40. Find the price if the demand is (g) 3; (h) 4; (i) 4.5; (j) 5; (k) 7. (l) Plot the demand curve.

9. Let the demand curve for a commodity be $D = a/p^b$, where a and b are positive constants. Determine the price if the demand is (a) $5a$; (b) 1. Find the demand if the price is (c) 1; (d) if the price is a .

**10. Assume that the demand curve of a commodity is $D = a/p^b$, where a and b are positive constants. The supply curve is $S = p$. Find the equilibrium price and quantity on the market.

11. Assume a demand function of the form $D = 100/p$. (a) Find the demand if $p = 5$. (b) Compute the total outlay and revenue pD . (c) Compute the demand if $p = 25$. (d) Compute the total outlay or revenue. (e) Compute the quantity demanded if the price is $p = 2$. (f) Compute the total outlay or revenue. (g) Plot the demand curve.

PROGRESSIONS

16. The Arithmetic Progression

Progressions. In algebra we define a *sequence* as a set of numbers ordered in such a manner that there is a first, a second, a third and so on. For instance, the numbers 1, 2, 3, \dots constitute the sequence of positive integers in natural order. Or, the numbers 1, 4, 9, 16, \dots are the squares of the integers ordered according to magnitude. Again, the numbers 1, $1/2$, $1/3$, \dots form what is called a *harmonic sequence in descending order of magnitude*, and so on.

The numbers in a sequence are called *terms*. For instance, in our last sequence, 1 is the first term, $1/2$ is the second term, $1/3$ the third term, and so on.

We are concerned frequently with the *sums* of the terms of sequences. Let u_1 be the first term of a sequence, u_2 the second term, u_3 the third term, and in general u_i the i th term. The sum of the first n terms of the sequence can be written

$$S_n = u_1 + u_2 + u_3 + \dots + u_n = \sum_{i=1}^n u_i.$$

The Greek Sigma Σ is used as the summation sign. The series $\sum_{i=1}^n u_i$ denotes a summation of terms from the lower limit $i = 1$, that is, u_1 , to the upper limit $i = n$, that is, u_n . The limits are sometimes omitted if they are clear from the context. We may write then $\sum u_i$.

■ EXAMPLE 1

Consider the sequence $u_1 = 1$, $u_2 = 1/2$, $u_3 = 1/3$, $u_4 = 1/4$. The

general term is $u_i = 1/i$. The sum of the 4 terms of the sequence is

$$\begin{aligned} S_4 &= u_1 + u_2 + u_3 + u_4 \\ &= \sum_{i=1}^4 1/i \\ &= 1 + 1/2 + 1/3 + 1/4 \\ &= 25/12. \end{aligned}$$

An *arithmetic progression* is formed by adding to each term after the first the same (positive or negative) quantity. The number which is added is called the *common difference*.

EXAMPLE 2

Consider, for instance, the sequence 2, 5, 8, 11. The first term is 2. The second term is $5 = 2 + 3$. The third term is $8 = 5 + 3$. Each term is formed by adding the common difference 3 to the preceding term.

EXAMPLE 3

Another example is the sequence 20, 18, 16, 14. The first term is 20. The second term is $18 = 20 - 2$. The third term is $16 = 18 - 2$. The fourth term is $14 = 16 - 2$. Hence each term is formed by adding to the preceding term the common difference -2 .

Let a be the first term, d the common difference. Denoting by $x_1, x_2, x_3, \dots, x_n$, respectively, the first, second, third, and n th terms of the arithmetic progression, we have

$$\begin{aligned} x_1 &= a, \\ x_2 &= x_1 + d = a + d, \\ x_3 &= x_2 + d = (a + d) + d = a + 2d, \\ x_4 &= x_3 + d = (a + 2d) + d = a + 3d. \end{aligned}$$

We see that the first term x_1 is a . The second term is formed by adding $1d$ to a . The third term is obtained by adding $2d$ to a . The fourth term results if we add $3d$ to a . We may deduce the general rule: to find the n th term we have to add $(n - 1)d$ to a , that is, $x_n = a + (n - 1)d$.

EXAMPLE 4

In the first arithmetic progression given above, the first term is $a = 2$, the common difference $d = 3$. The third term is

$$x_3 = a + 2d = 2 + 2 \cdot 3 = 8.$$

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EXAMPLE 5

Consider the arithmetic progression 10, 8, 6, \dots . Find the fifth term x_5 . We have $a = 10$, $d = -2$, $n = 5$. Hence

$$x_5 = 10 + 4(-2) = 10 - 8 = 2.$$

EXERCISES 16

1. Consider the arithmetic progression 5, 8, 11, \dots . Find (a) x_4 ; (b) x_7 ; (c) x_{150} .
2. Consider the arithmetic progression $-8, -6, -4, \dots$. Find (a) x_6 ; (b) x_{10} ; (c) x_{80} .
3. Consider the arithmetic progression 100, 90, 80, \dots . Find (a) x_7 ; (b) x_{10} ; (c) $x_{1,000}$.
4. Consider the arithmetic progression 3, 6, 9, \dots . Find (a) x_7 ; (b) x_{12} . (c) For what value of n is $x_n = 81$?
5. An arithmetic progression has $x_2 = 10$ and $x_4 = 30$. (a) Find a and d ; (b) x_{10} ; (c) x_{25} .
6. An arithmetic progression has $a = 50$ and $x_6 = 0$. Find (a) d ; (b) x_{10} ; (c) x_{15} .
7. An arithmetic progression has $a = 10$, $x_6 = 0$. Find (a) d ; (b) x_2 . (c) For what n is $x_n = -25$?
8. An arithmetic progression has the difference $d = (1/2)a$. Which term of the series will reach the value $10a$?
9. An arithmetic progression has the difference $d = -a/3$. Which term will be equal to 0?
10. An arithmetic progression has $a = -5d$. Which term will be equal to 0?

17. Sums of Arithmetic Progressions

Denote by S_n the sum of the first n terms of an arithmetic progression; that is,

$$S_n = x_1 + x_2 + x_3 + \dots + x_n = \sum_{i=1}^n x_i.$$

We can write this sum as follows:

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \dots + x_n.$$

This can also be written in the reverse order

$$S_n = x_n + (x_n - d) + (x_n - 2d) + (x_n - 3d) + \dots + a.$$

Adding the members of these 2 equalities, term by term, all d 's cancel and we have

$$2S_n = (a + x_n) + (a + x_n) + \dots + (a + x_n) = n(a + x_n).$$

The general formula for the sum of terms of an arithmetic progression is

$$S_n = \frac{n(a + x_n)}{2}.$$

■ EXAMPLE

Take the previous example, namely, the progression $10, 8, 6, \dots$. We have $a = 10, d = -2$. Find S_5 , the sum of the first 5 terms.

We have from the previous formula $x_5 = 2$. So the sum is given by $S_5 = 5(10 + 2)/2 = 30$.

● EXERCISES 17

1. Given the arithmetic progression $50, 40, 30, \dots$. Find (a) S_4 ; (b) S_6 ; (c) S_{20} .
2. Given the arithmetic progression $-5, -3, -1, \dots$. Find (a) S_6 ; (b) S_7 ; (c) S_{50} .
3. Given the arithmetic progression $19, 22, 25, \dots$. Find (a) S_4 ; (b) S_{10} ; (c) S_{100} .
4. An arithmetic progression has $a = 100, S_{10} = 550$. Find (a) d ; (b) x_{12} ; (c) S_{16} .
5. Express S in terms of a, d and n , only. (HINT: Use the definition of x_n .)
6. In a certain arithmetic progression $S_6 = 100$ and $S_7 = 140$. Find (a) a and d ; (b) x_6 ; (c) S_{10} .
7. In a particular arithmetic progression $x_2 = 80, S_4 = 240$. Find (a) a and d ; (b) x_{10} ; (c) S_6 .
8. In a certain arithmetic progression $S_3 = 33, S_6 = 50$. Find (a) a and d ; (b) x_8 ; (c) S_7 .
9. Assume in an arithmetic progression that we know that $S_n = A$ and $S_m = B$. Determine a and d .
10. In an arithmetic progression we know that $x_n = C, S_m = D$. Determine a and d .

18. Growth of Enterprise

We assume that an enterprise grows over short periods in an arithmetic progression. This means that the output of the enterprise increases (or decreases) each year by the same number of units. This assumption is not strictly true but frequently is a fair approximation to the truth.

■ EXAMPLE

An enterprise starts by producing 100 units in its 1st year and the production increases by 20 units each year. (a) How much will it produce in the 4th year? We use the formula $x_n = a + (n - 1)d$, where x_n is the production in the n th year, n the number of the year, d the increase or decrease. We know that $a = 100, n = 4, d = 20$. The formula gives $x_4 = 160$. (b) What will be the sum total of the whole production in the first 3 years? The sum of an arithmetic progression is given by $S_n = (a + x_n)(n/2)$. Since $x_3 = 140$, we have $S_3 = 360$.

■ EXERCISES 18

1. An enterprise produces 200 units in the 1st year and increases production by 50 units each year. How much will it produce (a) in the 5th year? (b) in the 10th

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year? (c) What is the sum total of its production in the first 7 years? (d) the first 4 years?

2. An enterprise starts by producing 10,000 units in the 1st year. It decreases its production by 500 units each year. How much will it produce (a) in the 3rd year? (b) the 5th year? (c) What is the total production in the first 4 years? (d) When will it produce 0? (e) What is the sum total of its production until it produces 0.

3. An enterprise starts by producing 700 units in the 1st year. It produced 1,500 units in its 5th year. (a) By how much does production increase each year? (b) When will the enterprise produce 2,100 units? (c) What is the sum total of its production during the first 3 years of its existence?

4. An enterprise produced 600 units in the 3rd year of its existence and 700 units in its 7th year. (a) What was the initial production in the 1st year? (b) What was the production in the 5th year? (c) the total production during the first 4 years?

5. A firm produced 500 units in its 1st year. The sum total of its production in the first 4 years of its existence was 2,300 units. (a) By how much did production increase or decrease each year? (b) How much will it produce in its 5th year? (c) in its 10th year?

6. A firm produced 0 in its 6th year. The sum total of its production during the first 5 years was 3,000 units. (a) What was its original production during the 1st year? (b) By how much did production increase or decrease each year? (c) How much was produced in the 2nd year? (d) What was the sum total of production in the first 4 years?

✓ 7. A firm produced 0 in its 5th year. Its total production during the first 4 years was 2,400 units. (a) Find the initial production in year 1, (b) the yearly increase or decrease. (c) What was the production in year 4? (d) What was the sum total of its production in the first 3 years?

8. Firm *A* starts producing 1,000 units, and decreases production by 100 units yearly. Firm *B* starts by producing 500 units and increases production by 25 units each year. (a) In what year will *A* and *B* produce the same amount? (b) When will firm *A* produce 0? (c) What is the production of firm *B* in the same year?

9. An index of industrial production in the United States was 91 in 1930 and 125 in 1940. Assuming an arithmetic progression, predict the index for 1950.

10. An index of agricultural production in the United States was 98 in 1930 and 126 in 1950. Assuming an arithmetic progression, predict agricultural production for 1960.

19. Geometric Progression

In a geometric progression each term after the first one is formed by multiplying the preceding term by a constant. The ratio between consecutive terms is called the *common ratio*.

Consider, for instance, the sequence 3, 6, 12, 24. The first term is 3; the second term is $3 \cdot 2 = 6$; the third term is $6 \cdot 2 = 12$; the fourth term is $12 \cdot 2 = 24$. The common ratio of two consecutive terms is 2; that is, each term in the sequence is formed by multiplying the preceding one by 2.

As another example, consider the sequence 27, 9, 3, 1, 1/3, 1/9. The first term is 27; the second term is $27 \cdot (1/3) = 9$; the third term is $9 \cdot (1/3) = 3$; the fourth term is $3 \cdot (1/3) = 1$; the fifth term is $1 \cdot (1/3) = 1/3$; the sixth term is $(1/3)(1/3) = 1/9$. The common ratio is 1/3. Each term is formed by multiplying the preceding one by 1/3.

Let y_1 denote the first term of a geometric progression by a and the common ratio by r . Let $y_1, y_2, y_3, y_4, \dots, y_n$ be the first, second, third, fourth, and n th terms, respectively, of a geometric progression. Then

$$\begin{aligned} y_1 &= a, \\ y_2 &= y_1 \cdot r = ar, \\ y_3 &= y_2 \cdot r = (ar)r = ar^2, \\ y_4 &= y_3 \cdot r = (ar^2) \cdot r = ar^3, \text{ and so forth.} \end{aligned}$$

We see that the first term is a . The second term is formed by multiplying a by r raised to the first power. The third term results if a is multiplied by the second power of r . The fourth term is computed by multiplying a by the third power of r . In general, the n th term of a geometric progression is given by multiplying a (the first term) by the $(n - 1)$ power of r (the common ratio). Thus

$$y_n = ar^{n-1}.$$

EXAMPLE

Consider the geometric progression 2, 6, 18, \dots . We have $a = 2, r = 3$. Find the 5th term.

$$y_5 = 2(3)^4 = 2 \cdot 81 = 162.$$

EXERCISES 19

1. Consider the geometric progression 10, 20, 40, \dots . Find (a) y_4 ; (b) y_{10} ; (c) y_{16} .
2. Take the geometric progression 6, 2, 2/3, \dots . Find (a) y_6 ; (b) y_8 ; (c) y_{10} .
3. Take the geometric progression 1, 5, 25, \dots . Find (a) y_4 ; (b) y_6 ; (c) y_{10} .
4. Consider the geometric progression 12, 6, 3, \dots . Find (a) y_{10} ; (b) y_{100} . (HINT: Use logarithms.)
5. A geometric progression has $a = 10$ and $y_3 = 1,000$. (a) Find r ; (b) Find y_6 ; (c) Find y_{10} .
6. The second term of a geometric progression is $y_2 = 8$, and the fourth term is $y_4 = 4$. (a) Find a and r . (b) What is y_3 ? (c) Find y_6 .
7. Find a general expression for a and r if $y_n = A, y_m = B$.
8. Take the geometric progression 1, $(a - b), (a - b)^2, \dots$. (a) Find the general expression for y_n . (b) Find y_6 .
9. Find the geometric progression whose second term is 8 and whose fifth term is 8. Find a and r .
10. The second term of a geometric progression is 100 and the fourth term is 1. Find a and r .

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20. Sums of Geometric Progressions

Let S_n be the sum of the first n terms of a geometric progression; that is,

$$S_n = y_1 + y_2 + y_3 + \cdots + y_n = \sum_{i=1}^n y_i.$$

This may be rewritten in the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}.$$

After multiplying the 2 members by r , there results

$$rS_n = ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n.$$

By subtracting the 2 members just obtained from those of the previous expression for S_n , we obtain

$$S_n - rS_n = a - ar^n,$$

and, hence,

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{or} \quad \frac{a(r^n - 1)}{r - 1}.$$

EXAMPLE

Given the geometric progression 2, 6, 18, \cdots , find S_4 . We have $a = 2$, $r = 3$, $n = 4$. Hence

$$S_n = \frac{2(3^4 - 1)}{3 - 1} = \frac{2(81 - 1)}{2} = 80.$$

EXERCISES 20

1. Given the geometric progression 1, 4, 16, \cdots . Find (a) S_3 ; (b) S_5 ; (c) S_8 .
2. Given the geometric progression 24, 12, 6, \cdots . Find (a) S_3 ; (b) S_5 ; (c) S_{10} .
3. Given the geometric progression 5, 10, 20, \cdots . Find (a) S_3 ; (b) S_5 ; (c) S_7 .
4. A geometric progression has the first term $a = 1$, and the sum of the first two terms is $S_2 = 6$. (a) Find r . (b) Find y_7 . (c) Find S_3 .
5. A geometric progression has the ratio $r = 1/2$. Assume that $S_3 = 10$. Find (a) a ; (b) y_6 ; (c) S_6 .
6. Assume that a geometric progression has $a = 1$. Let $S_3 = A$, where A is a positive constant. Find r .
7. Find a general expression for the difference of the sum of a geometric series $(n + 1)$ terms and the sum of n terms, that is, $S_{n+1} - S_n$. Interpret.
8. As in Problem 7, find $S_{n+m} - S_n$, where n and m are positive integers.
9. Assume that in a geometric progression we have $S_m = A$, $S_{2m} = B$, where A and B are positive constants. (a) Find r . (b) Find a . (c) Check.
10. Use the formula for the sum of a geometric progression to investigate the sum of the progression 1, -1 , 1, -1 , \cdots . Investigate the sum for various values of n .

21. Population

It was the contention of Malthus that population developed like a geometrical progression. The assumption is that each year the population of a given area increases by the same per cent. This is not strictly true since it is probable that population follows a more complicated law, namely, the logistic. But a geometric series can be used for purposes of interpolation or extrapolation over relatively short periods.

■ EXAMPLE

A population is 100,000 in the 1st year and increases by 3 per cent each year. What will it be in the 5th year? The formula for $y_n = ar^{n-1}$, where y_n is the value of the term in the n th year, n is the number of the year, and r is the common ratio.

We have $a = 100,000$, $n = 5$, and $r = 1.03$. Thus, $y_n = 100,000 (1.03)^4$. Taking logarithms, we get $\log y_5 = \log 100,000 + 4 \log 1.03$. This gives $\log y_5 = 5.0512$, so y_5 is about 112,500.

● EXERCISES 21

1. The population of a city was 1,000 in year 1. It increases by 6 per cent each year. How much will it be (a) in year 2? (b) in year 3? (c) in year 10? (d) When will it reach 1,500? (e) 1,100? (f) 2,000?

2. A population is 185,000 in year 1. It decreases by 5 per cent each year. How much will the population be (a) in year 3? (b) in year 10? (c) in year 13? When will it decline to (d) 100,000? (e) 184,500?

3. A population of a city is 250,000 in year 3 and 300,000 in year 7. (a) How much was it in year 1? (b) What is the yearly per cent of increase or decrease? (c) What will be the population in year 12? (d) in year 20? (e) When will it reach 350,000? (f) 500,000? (g) When was it 275,000?

4. The population of a country was 6,000,000 in year 5, and 7,150,000 in year 15. (a) How much was it in year 1? (b) What is the yearly per cent of increase or decrease? (c) When will the population reach 8,000,000? (d) 10,000,000? When did it amount to (e) 6,500,000? (f) 7,000,000? (g) What will it be in the year 20? (h) in the year 35?

5. The population of the United States was 123,000,000 in 1930 and 132,000,000 in 1940. (a) Find the yearly per cent of increase. Determine approximately the population for (b) 1932; (c) 1936; (d) 1939. Determine the approximate population for (e) 1950; (f) 1965.

6. The population of the United States was 4,000,000 in 1790 and 63,000,000 in 1890. (a) Estimate the yearly per cent of increase. (b) Obtain an approximate value for the population in 1900; (c) 1930; (d) 1940.

7. The population of the United States was 76,000,000 in 1900. Estimate the population in 1940 under the assumption of an increase each year of (a) 10 per cent; (b) 5 per cent; (c) 2.5 per cent; (d) 1 per cent; (e) 0.5 per cent.

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8. The population of the United States was 131,670,000 in 1940. Predict the population in 1960, assuming (a) an increase of 6 per cent yearly; (b) 3 per cent yearly; (c) 1 per cent yearly; (d) a decrease of 2 per cent each year.

9. The population of New York City was 79,000 in 1800 and 3,437,000 in 1900. (a) Find the annual per cent of increase. Determine a value for the population in (b) 1955; (c) 1960.

10. The population of New York City was 6,930,000 in 1930 and 7,835,000 in 1950. (a) Find the annual per cent of increase or decrease. According to the law of Malthus, determine the population for (b) 1960; (c) 2000.

11. The population of the United States has been estimated as 131,940,000 in 1940 and as 133,060,000 in 1941. From these data, and assuming growth that follows a geometric progression, predict the population for 1960.

22. Compound Interest

The principle of compound interest is well known. Let i be the interest rate, and let P be the principal. As a convenience, let $r = 1 + i$. Then after 1 year we have the amount

$$A_1 = P + Pi = Pr;$$

after two years

$$A_2 = A_1r = Pr^2, \text{ and so forth.}$$

After n years

$$A_n = Pr^n$$

The amounts for $n = 1$, $n = 2$, and so forth, are the terms of a geometric progression.

■ EXAMPLE

Assume \$100 is invested at 5 per cent interest. What is the total amount after 5 years? We have $P = 100$, $i = 0.05$, $r = 1 + 0.05 = 1.05$. Hence $A_5 = (100)(1.05)^5$. Solving by logarithms, $\log A_5 = \log 100 + 5 \log 1.05 = 2.00000 + 5(0.0212) = 2.1060$. Hence A_5 is about \$127.60.

● EXERCISES 22

1. Find the total amount if \$10 is invested at compound interest at 4 per cent for (a) 2 years; (b) 10 years; (c) 50 years.

2. Find the total amount if \$1,000 is invested at compound interest at 3 per cent for (a) 3 years; (b) 10 years; (c) 100 years.

3. A sum of \$500 is invested at compound interest. Find the total amount after 5 years at (a) 1 per cent; (b) 5 per cent; (c) 10 per cent; (d) 0 per cent.

4. How long will it take \$100 at 3 per cent, compound interest, to amount to (a) \$105? (b) \$150? (c) \$1,000?

5. What rate of interest will cause \$50 invested at compound interest to amount to \$75 after (a) 10 years? (b) 25 years? (c) 100 years?
6. A sum of \$25,000 has been invested at compound interest. At what rate of interest will this sum accumulate to \$30,000 after (a) 2 years? (b) 10 years? (c) 50 years?
7. A sum of \$1,000,000 is invested at compound interest. Find the total amount after 10 years at (a) 1 per cent; (b) 2.5 per cent; (c) 5 per cent.
8. A certain amount of money was invested at compound interest at 4 per cent. After 5 years the accumulated amount is \$2,356,089. What was the original investment?
9. At a 2 per cent rate of interest, how long will it take an original amount of \$100 to accumulate to (a) \$200? (b) \$500? (c) \$1,000?
10. What is the rate of compound interest that will triple an amount of money in 5 years?
11. The total public debt of the United States was \$257,400,000,000 in 1950. How much will this debt be in 1960, assuming a compound interest rate of (a) 1 per cent? (b) 2 per cent? (c) 5 per cent?

**DETERMINANTS

**23. Determinants of the Second Order

Let us consider the solution of the following general system of 2 linear equations in 2 unknowns:

$$\begin{aligned} ax + by &= h, \\ cx + dy &= k, \end{aligned}$$

where $a, b, c, d, h,$ and k are constants.

By the usual process of elimination, $x = (hd - kb)/(ad - cb)$ and $y = (ka - hc)/(ad - cb)$.

If we define the second-order determinant

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix}$$

as having the value $ps - rq$, it follows that the solution of our system of equations can be written

$$x = \frac{\begin{vmatrix} h & b \\ k & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & h \\ c & k \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Note that the elements of the determinant in the denominators comprise the array of coefficients of x and y in the 2 equations. We get the determinant, which appears in the numerator for x , by substituting for the coefficients of x (a and c) the constants (h and k) on the right side of the equations. For y , we substitute the constants h and k for the coefficients of y (b and d) in the determinant of the coefficients.

EXAMPLE

Let us take the example considered previously (Section 10, Example 1):

$$\begin{aligned} 2x + 6y &= 9, \\ 3x + y &= 1. \end{aligned}$$

The solutions are

$$x = \frac{\begin{vmatrix} 9 & 6 \\ 1 & 1 \\ 2 & 6 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 6 \\ 3 & 1 \end{vmatrix}} = \frac{9 \cdot 1 - 1 \cdot 6}{2 \cdot 1 - 3 \cdot 6} = \frac{9 - 6}{2 - 18} = \frac{3}{-16} = -\frac{3}{16}.$$

$$y = \frac{\begin{vmatrix} 2 & 9 \\ 3 & 1 \\ 2 & 6 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 6 \\ 3 & 1 \end{vmatrix}} = \frac{2 \cdot 1 - 3 \cdot 9}{2 \cdot 1 - 3 \cdot 6} = \frac{-25}{-16} = \frac{25}{16}.$$

EXERCISES 23

1. Solve the equations in Problem 1, Exercises 10, by determinants.
2. Solve the equations in Problem 2, Exercises 10, by determinants.
3. Solve the equations in Problem 3, Exercises 10, by determinants.
4. Solve the equations in Problem 4, Exercises 10, by determinants.
5. Solve the equations in Problem 5, Exercises 10, by determinants.
6. Consider the solution of the system $x + y = 10$ and $2x + 2y = +5$ by determinants. Why is no solution possible?
7. Consider the conditions under which a second-order determinant becomes 0.

HINT: Expand $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

- **8. State generally the conditions under which a system of 2 linear equations has no solution. (HINT: use the results of Problem 7.)
9. State the conditions under which the solutions of a system of 2 linear equations in 2 unknowns are 0.
10. Get the general solution of the following system by determinants.

$$\begin{aligned} a_1x + b_1y &= m_1, \\ a_2x + b_2y &= m_2. \end{aligned}$$

- **11. Consider the system

$$\begin{aligned} 2x + 4y &= 0, \\ x + 2y &= 0. \end{aligned}$$

- (a) Show that the determinant of the coefficients is 0. (b) Assume that $x = k$ (an arbitrary number). Solve for y , and show that there is an infinite number of solutions to the given system.

****12.** Consider the system

$$\begin{aligned}(1 - \lambda)x + 2y &= 0, \\ 3x + y &= 0.\end{aligned}$$

(a) Determine λ from the condition that the determinant of the coefficients should be 0. (b) Proceed as in Problem 11.

****13.** Consider the system

$$\begin{aligned}(2 - \lambda)x + y &= 0, \\ 2x + (3 - \lambda)y &= 0.\end{aligned}$$

Proceed as in Problem 12.

****24. Development of Determinants by Minors**

A determinant of the third order has 3 rows (horizontal) and 3 columns (vertical); its appearance is as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Such a third-order determinant can be evaluated by a process known as *expansion* in terms of the elements (that is, numbers) of a row or column. The *minor* of any element in a third-order determinant is the second-order determinant obtained by striking out the row and column containing the particular element.

Let us expand the given determinant in terms of the elements in the first column, that is, a_1, b_1, c_1 .

The minor of a_1 is the second-order determinant obtained by striking out the first row and column:

$$\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}.$$

The minor of b_1 is obtained by striking out the second row and first column:

$$\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}.$$

The minor of the element c_1 is the second-order determinant obtained by striking out the third row and first column:

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

The value of the determinant is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

Note the alternating signs in the development. The complete expansion is $a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$.

■ EXAMPLE

Find the value of

$$\begin{vmatrix} 1 & -4 & 3 \\ 2 & 0 & 1 \\ 0 & 5 & -4 \end{vmatrix}.$$

We expand in terms of the elements of the first column. This gives

$$1 \begin{vmatrix} 0 & 1 \\ 5 & -4 \end{vmatrix} - 2 \begin{vmatrix} -4 & 3 \\ 5 & -4 \end{vmatrix} + 0 \begin{vmatrix} -4 & 3 \\ 0 & 1 \end{vmatrix}.$$

By developing the second-order determinants we get

$$(1)(-5) - (2)(1) + (0)(-4) = -5 - 2 + 0 = -7.$$

● EXERCISES 24

1. Expand the following determinants in terms of the elements in the first column, and find their values:

$$(a) \begin{vmatrix} 1 & 3 & 1 \\ 4 & 6 & 0 \\ 0 & -1 & 7 \end{vmatrix}; \quad (b) \begin{vmatrix} 0 & 1 & -6 \\ 1 & 0 & 8 \\ 4 & -2 & 1 \end{vmatrix}; \quad (c) \begin{vmatrix} 1 & 4 & 8 \\ 3 & 1 & -5 \\ 1 & 0 & 1 \end{vmatrix}.$$

2. Find the value of the following determinant:

$$\begin{vmatrix} a & -1 & 0 \\ 0 & 0 & a \\ 1 & a & 0 \end{vmatrix}.$$

3. Expand the following determinant:

$$\begin{vmatrix} x & 2 & -1 \\ 1 & 0 & 6 \\ 5 & -1 & 8 \end{vmatrix}.$$

4. Equate the determinant in Problem 3 to 0. Find x . Check by substituting the value of x back into the determinant.

5. Find the value of the determinant:

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}.$$

6. Find the value of the determinant:

$$\begin{vmatrix} 5 & -1 & 4 \\ 0 & -2 & 1 \\ 4 & 1 & 0 \end{vmatrix}$$

by developing the minors (a) in terms of the first column; (b) in terms of the second

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column (HINT: When expanding according to the elements of the second column, the alternating signs start with a negative); (c) according to the third column.

7. Compute the value of each of the following:

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 4 \\ 6 & 1 & 0 \end{vmatrix}; \quad \begin{vmatrix} 1 & 2 & 6 \\ 3 & 5 & 1 \\ 1 & 4 & 0 \end{vmatrix}.$$

Note that the rows of the first determinant are the same, respectively, as the columns of the second determinant.

****25. Solutions of Systems of Linear Equations in Three Unknowns by Determinants**

To solve a system of 3 linear equations in 3 unknowns we proceed in a manner analogous to the procedure for 2 linear equations in 2 unknowns.

We form first the determinant of the coefficients of the unknowns of the equations; this gives the denominator determinant. Then, to obtain the value for x , for example, we form the numerator determinant by replacing the coefficients of x in the denominator determinant by the constants on the right side of the equations.

To find the value for y we replace the coefficients of y in the same denominator determinant by the column of the constants. This determinant divided by the determinant of the coefficients gives the value of y . In a similar fashion we get the value for z . Usually all third-order determinants are developed by minors.

EXAMPLE

Solve the following system of equations by determinants:

$$\begin{aligned} x + 2y - z &= 0, \\ 2x + 5y + 2z &= 14, \\ y - 3z &= -7. \end{aligned}$$

We form first the determinant of the coefficients of the unknowns in the system; it is

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 5 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

To find the value of x we replace the column of the coefficients of x (first column) in the above determinant by the column of constants on the right side of the equations. This is divided by the determinant of the co-

efficients as follows:

$$x = \frac{\begin{vmatrix} 0 & 2 & -1 \\ 14 & 5 & 2 \\ -7 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 2 & 5 & 2 \\ 0 & 1 & -3 \end{vmatrix}}.$$

Both determinants are developed in terms of minors, and there results the value $x = -1$. The value of y is given by

$$y = \frac{\begin{vmatrix} 1 & 0 & -1 \\ 2 & 14 & 2 \\ 0 & -7 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 2 & 5 & 2 \\ 0 & 1 & -3 \end{vmatrix}}.$$

Again we develop these determinants in terms of minors. We get $y = 2$. Similarly,

$$z = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 14 \\ 0 & 1 & -7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 2 & 5 & 2 \\ 0 & 1 & -3 \end{vmatrix}}.$$

We develop this again by the method of minors and get the value $z = 3$.

The 3 values, constituting the desired solution of the system, ought to be checked by substituting into the system of the original linear equations.

● EXERCISES 25

1. Given the system of linear equations:

$$x - 2y = 6,$$

$$y - z = 5,$$

$$x + y - z = 0.$$

(a) Solve by the use of determinants. (b) Check by substituting back into the equations.

2. Solve the system of equations in Problem 8, Exercises 10, by determinants. Check.

3. Solve the system of equations in Problem 9, Exercises 10, by determinants. Check.

4. Given the system of equations:

$$x + y = 2,$$

$$y + z = 2,$$

$$x + z = 2.$$

Solve by determinants. Check.

5. Solve the following system of equations by determinants:

$$-x + 2y + z = 2,$$

$$x - y + 3z = 16,$$

$$2x + 2y = -11.$$

Check.

6. Solve by determinants the general system of 3 equations in 3 unknowns, namely,

$$a_1x + a_2y + a_3z = k_1,$$

$$b_1x + b_2y + b_3z = k_2,$$

$$c_1x + c_2y + c_3z = k_3.$$

7. (Input-output analysis.) The following data refer to the American economy, 1939 (Leontief). The total output of agriculture was 17.0 billion dollars, of which 2.5 billion went to industry, 14.5 billion to households. The total output of industry was 133.0 billion, of which 13.7 billion went to agriculture and 119.3 billion to households. The total output of households was 68.8 billion, of which 4.2 billion went to agriculture and 64.6 billion to industry. To determine the prices: P_1 (agricultural prices), P_2 (industrial prices), P_3 (wages), we have the following system of equations:

$$-17.0 P_1 + 2.5 P_2 + 14.5 P_3 = 0,$$

$$13.7 P_1 - 133.0 P_2 + 119.3 P_3 = 0,$$

$$4.2 P_1 + 64.6 P_2 - 68.8 P_3 = 0.$$

(a) Show that the determinant of this system of equations is zero; (b) assume that $P_3 = 1$; solve the first two equations, and show that the third equation is also satisfied.

8. Show that the value of a second-order determinant is not altered if rows are exchanged, respectively, for columns. Prove for the general second-order determinant.

9. Show that the value of a second-order determinant is 0 if 2 rows or 2 columns are equal. Prove for the general second-order determinant.

LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

26. First-Order Homogeneous Difference Equations

If we subtract the value of $y = y(x)$, at the point x , from the value of the same function at the point $x + 1$, we obtain the *first difference*:

$$\Delta y = y(x + 1) - y(x).$$

Equations which involve values of $y(x)$ and $y(x + k)$, where k is an integer, are called difference equations. As illustrations, we might observe $ay(x + 1) + by(x) = 0$, or $y(x + 1) = (-b/a)y(x)$.

If a and b are constant, this equation is said to be a linear homogeneous difference equation of the first-order with constant coefficients. To find its solution we need to know the value of the function at one point, say at $x = 0$. Let us say that $y(0) = K$. Then we can construct step by step the general solution $y(x)$ when x is an integer:

$$y(1) = \left(-\frac{b}{a}\right)y(0) = \left(-\frac{b}{a}\right)K,$$

$$y(2) = \left(-\frac{b}{a}\right)y(1) = \left(-\frac{b}{a}\right)^2 K,$$

$$y(3) = \left(-\frac{b}{a}\right)y(2) = \left(-\frac{b}{a}\right)^3 K,$$

.....

$$y(x) = \left(-\frac{b}{a}\right)^x K.$$

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To check this result we substitute it into the original equation and obtain

$$a\left(-\frac{b}{a}\right)^{x+1} + b\left(-\frac{b}{a}\right)^x = \left(-\frac{b}{a}\right)^x \left[a\left(-\frac{b}{a}\right) + b \right] = 0.$$

Hence our solution satisfies the original difference equation.

EXAMPLE

Let the difference equation be

$$(x + 1) - 2y(x) = 0.$$

In terms of the general equation just considered, we have $a = 1$, $b = -2$, $(-b/a) = +2$. Let us assume that the initial value $y(0) = K = 5$. Hence we have as a general solution: $y(x) = (5)2^x$. So, for $x = 3$, we have

$$y(3) = (5)(2)^3 = (5)(8) = 40.$$

EXERCISES 26

1. Given the difference equation $y(x + 1) - 5y(x) = 0$ and the condition that $y(0) = 10$. Find (a) the general solution; (b) $y(4)$; (c) $y(6)$.

2. Given the difference equation $y(x + 1) - (1/2)y(x) = 0$ and the initial condition $y(0) = 16$. Find (a) the general solution; (b) $y(5)$; (c) $y(10)$.

3. Given the difference equation $y(x + 1) - y(x)/10 = 0$ and the condition $y(0) = 10,000$. Find (a) the general solution of the equation; (b) $y(2)$, (c) $y(5)$.

4. Using the notation given in Section 22 for compound interest, show that the amount $A(x)$ paid in year x satisfies the difference equation $A(x + 1) - rA(x) = 0$, where $r = 1 + i$ and i is the rate of interest. Solve the difference equation with $A(0) = P$ (principal), and show that it is identical with the result given in the section quoted.

5. Establish the difference equation for the example in Section 22.

6. Find the difference equation corresponding to the population law of Section 21.

*7. Under what conditions will a linear homogeneous difference equation have fluctuating solutions, that is, solutions which change sign from term to term?

*8. Solve the difference equation $y(x + 1) + 2y(x) = 0$, where $y(0) = 1$, and show that it has a fluctuating solution, as described in Problem 7.

*9. What is the condition under which the solution of a linear homogeneous equation diminishes from term to term?

**10. Consider the difference equation $ay(x + 1) + by(x) = 0$. Assume the solution in the form $y(x) = km^x$, where k and m are constants. (a) Show that m can be derived from the equation $am + b = 0$. (b) Show also that the constant k can be determined from the initial condition $y(0) = K$. (c) Exemplify with the data given in the above example.

27. First-Order Nonhomogeneous Difference Equations

A nonhomogeneous linear first-order difference equation is an equation of the form $ay(x + 1) + by(x) = c$, where a , b , and c may or may not be constant; in this treatment they will be taken as constant. The equation as given may be written in the form $y(x + 1) = (-b/a)y(x) + (c/a)$. Again let the initial value $y(0) = K$. Then we have by successive application of the difference equation itself:

$$\begin{aligned}
 y(1) &= K\left(-\frac{b}{a}\right) + \frac{c}{a} \\
 y(2) &= K\left(-\frac{b}{a}\right)^2 + \frac{c}{a}\left[\left(-\frac{b}{a}\right) + 1\right] = K\left(-\frac{b}{a}\right)^2 + \frac{c}{a}\left[\frac{(-b/a)^2 - 1}{(-b/a) - 1}\right] \\
 y(3) &= K\left(-\frac{b}{a}\right)^3 + \frac{c}{a}\left[\left(-\frac{b}{a}\right)^2 + \left(-\frac{b}{a}\right) + 1\right] \\
 &= K\left(-\frac{b}{a}\right)^3 + \frac{c}{a}\left[\frac{(-b/a)^3 - 1}{(-b/a) - 1}\right] \\
 &\quad \dots\dots\dots \\
 y(x) &= K\left(-\frac{b}{a}\right)^x + \frac{c}{a}\left[\frac{(-b/a)^x - 1}{(-b/a) - 1}\right]
 \end{aligned}$$

This last result is premised on the assumption that x is an integer.

■ **EXAMPLE**

Let the difference equation be

$$y(x + 1) - 3y(x) = 10.$$

We have $a = 1$, $b = -3$, $c = 10$. Assume that $y(0) = 12$. The general solution is

$$y(x) = (12)(3)^x + 10\left(\frac{3^x - 1}{3 - 1}\right) = (17)(3^x) - 5.$$

For instance, for $x = 2$, we have $y(2) = 148$.

● **EXERCISES 27**

1. Solve the difference equation $y(x + 1) - (1/2)y(x) = 4$, if $y(0) = 8$. Find (a) the general solution; (b) $y(3)$; (c) $y(5)$. (d) Check by substituting back into the difference equation.
2. Solve the difference equation $y(x + 1) - 10y(x) = 1$, if $y(0) = 1$. (a) Find the general solution. (b) Find $y(2)$; $y(4)$. (c) Check by substituting back into the difference equation.

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3. Solve the difference equation $y(x+1) + 2y(x) = 3$, if $y(0) = 16$. (a) Find the general solution. Find (b) $y(2)$; $y(3)$.

*4. Use the notation in Section 22 to show the following: If i is the rate of interest, $r = 1 + i$, P is the principal, and $A(x)$ is the amount in year x , and if B is a bonus paid every year, we obtain the difference equation $A(x+1) - rA(x) = B$, and the amount is given by $A(x) = Pr^x - (B/i)(1 - r^x)$.

5. A principal $P = \$10,000$ is invested at 5 per cent interest, and the firm pays a yearly bonus of \$100. What is the total amount after (a) 5 yrs., (b) 10 yrs.?

6. A principal sum of $P = \$100,000$ is invested at 2 per cent. The firm pays a yearly bonus of \$5,000. What is the total amount after (a) 3 yrs., (b) 12 yrs.?

**7. Consider the difference equation $ay(x+1) + by(x) = c$. Assume the solution in the form $y(x) = km^x + n$, where k, m, n are constants. (a) Show that, in order to make the equation homogeneous, we must have $n = c/(a+b)$; (b) show that, if $n = c/(a+b)$, m can be determined from the equation $am + b = 0$; (c) show that the constant k can be determined from the initial condition $y(0) = K$; (d) demonstrate these propositions with the help of the above example.

*8. Assume an economy in which all income is consumed. Let the extra consumption at the point in time $x+1$ be a constant multiple a of the income (consumption) at time x . Let the initial consumption (income) be A , where A is constant. Then we have the difference equation $Y(x+1) = aY(x) + A$, $Y(0) = A$, where Y is income. (a) Show that the solution of the difference equation is

$$Y(x) = \frac{A}{1-a} - \frac{aAa^x}{1-a}.$$

(b) Show that if $a < 1$, then as x increases Y approaches nearer and nearer to $A/(1-a)$.

9. Use the notation of Problem 8. Assume $a = 1/2$, $A = 100$. Find $Y(2)$; $Y(10)$.

10. In the notation of Problem 8, let $a = 0.712$ (estimate of T. Haavelmo for the United States, 1930-1944). Find $Y(x)$ (a) if $A = 1,000,000,000$; (b) if $A = 5,000,000,000$. (c) Find $Y(3)$, $Y(10)$ under assumptions (a) and (b).

11. The quantity a of Problem 8 was estimated as 0.56 for the United States in 1921-1931 (P. A. Samuelson). Assume $A = 500,000,000$, and find (a) $Y(5)$; (b) $Y(10)$.

*12. Denote by $p(x)$ the price and by $D(x)$ the quantity of a commodity at time x . Assume that the demand function is $p(x) = A + BD(x)$ and the supply function is $D(x) = C + E p(x-1)$; A, B, C, E are constants. The supply depends upon the price a period before, because of a constant period of production. (a) By eliminating $D(x)$, show that the price $p(x)$ satisfies the difference equation $p(x+1) - BE p(x) = A + BC$. (b) Solve the difference equation under the condition that $D(0) = H$, where H is a constant. (c) Find $D(x)$ and check the 2 original equations.

13. Given the demand function $p(x) = 70 - (1/2) D(x)$, the supply function $D(x) = (1/3) p(x-1)$, and the condition that $D(0) = 10$. (a) Find the values of

price and quantity for all points in time x . (b) Verify the solution by substituting into the equations. (c) Plot price and quantity against time and show that the fluctuations diminish as time increases. What is the ultimate limit? (d) Plot the demand and supply functions in one diagram. Connect successive points of equilibrium, and show that the diagram has the appearance of a cobweb (see Figure 51, Section 100).

14. Given the demand function $p(x) = 6 - D(x)$, the supply function $D(x) = p(x - 1)$, and the initial condition $D(0) = 2$. (a) Find the values of price and quantity for all points in time x . (b) Check the solution by substituting into the equations. (c) Plot prices and quantities against time, and show that there are constant fluctuations. (d) Plot the cobweb diagram (Problem 13d).

15. Given the demand function $p(x) = 12 - D(x)$, the supply function $D(x) = 2p(x - 1)$, and the condition $D(0) = 6$. (a) Find the values of price and quantity at all points in time. (b) Check the solutions by substituting into the equations. (c) Plot prices and quantities against time, and show that there are increasing fluctuations. (d) Plot the cobweb diagram (Problem 13d).

16. The demand for sugar in the United States has been estimated as $p(x) = 2.34 - 1.34 D(x)$, and the supply of sugar as $D(x) = 0.5 + 0.6 p(x - 1)$ (H. Schultz). (a) Find the course of price and quantity at all points in time, assuming $D(0) = 1.25$. (b) Verify the results with the help of the previous equations. (c) Plot price and quantity against time. (d) Make a cobweb diagram (Problem 13d).



CALCULUS

FUNCTIONS, LIMITS, AND DERIVATIVES

28. Functions and Variables

As already indicated, a function $y = f(x)$ is sometimes written $y = y(x)$. This emphasizes that y is the dependent variable and x the independent variable; that is, y is determined when x is given. If the relationship is inverted, as is possible in many cases, we have x as the dependent variable and y as the independent variable, and we write $x = g(y)$, or $x = x(y)$. This is the function which is the inverse of $y = f(x)$. For example, if $y = x^2$, then $x = \pm\sqrt{y}$; or if $y = \log x$, then $x = 10^y$. A function that is written by means of an equation which is explicitly solved for the dependent variable is said to be an explicit function of the independent variable; for example, $y = x^2 + 3x + 1$ or $y = \log(x - 1)$.

If a formula involves both x and y and is equal to 0, the situation being typified by $h(x, y) = 0$, we say that we have an implicit function of x and y . For example, $\sqrt{x^2 - y^2} = 0$ is such a function. It may be difficult, but frequently it is possible, to change from an implicit formulation to an explicit, and vice-versa.

■ EXAMPLE 1

Assume the implicit relationship

$$h(x, y) = 4x - 2y - 6 = 0.$$

Obtain y as an explicit function of x . We have, from our equation

$$4x - 2y = 6,$$

$$-2y = 6 - 4x;$$

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so

$$y = y(x) = f(x) = -3 + 2x.$$

Thus $f(0) = -3$, or $y(0) = -3$; $f(4) = 5$ or $y(4) = 5$; $f(-2) = -7$, or $y(-2) = -7$; and so forth. The numbers in parentheses are understood to be values of the independent variable x .

To find the inverse of the function just derived, the dependent variable is x and the independent variable is y . Starting with the original equation, we have

$$4x - 2y = 6,$$

$$4x = 6 + 2y.$$

so

$$x = x(y) = g(y) = 3/2 + y/2.$$

Thus $g(0) = 3/2$, or $x(0) = 3/2$; $g(-3) = 0$, or $x(-3) = 0$; $g(5) = 4$, or $x(5) = 4$; and so forth. The numbers in parentheses are understood to be values of the independent variable y .

It is seen that in this case the 2 functions, $y = f(x)$ and $x = g(x)$, are equivalent. We have, for instance, $f(4) = 5$. This is to say, $y = 5$ corresponds to the value $x = 4$. But on the other hand, $g(5) = 4$. This is to say, the value $x = 4$ corresponds to the value 5 of the independent variable y in the inverse function.

EXAMPLE 2

Assume the implicit relationship

$$y^2 - x = 0.$$

It follows immediately that

$$y = y(x) = f(x) = \pm\sqrt{x}.$$

To each value of the independent variable x corresponds not one but two values of y , except for $x = 0$. For instance, if $x = 4$, then $y(4) = f(4) = \pm\sqrt{4} = \pm 2$. Hence this function is *not single-valued*.

We can also make x the dependent and y the independent variable. Then we have

$$x = x(y) = g(y) = y^2.$$

This is now a single-valued function. For instance, for $y = 3$, we have $x(3) = g(3) = 3^2 = 9$.

EXERCISES 28

1. Assume the implicit relationship, $2x - 5y + 10 = 0$. (a) Make it explicit in the form $y = f(x)$. (b) Is this a single-valued function? (c) Find $y(10) = f(10)$.

(d) Find the inverse function $x = g(y)$. (e) Is it single-valued? (f) Find $x(-2) = g(-2)$. (g) Plot the function.

2. An implicit function is $2y^2 - 5x - 5 = 0$. (a) Make it explicit in the form $y = f(x)$. (b) Is this function single-valued? (c) Find $y(32/25) = f(32/25)$; (d) $y(0) = f(0)$; (e) $y(-1) = f(-1)$. (f) Make the function explicit in the form $x = g(y)$. (g) Is this function single-valued? (h) Find $x(0) = g(0)$; (i) $x(-7) = g(-7)$; (j) $x(1) = g(1)$.

3. An implicit function is $3xy - 24 = 0$. (a) Make it explicit in the form $y = f(x)$. (b) Is this function single-valued? (c) Find $f(-1)$; (d) $f(6)$. (e) Make the function explicit in the form $x = g(y)$. (f) Is this a single-valued function? (g) Find $g(1)$; (h) $g(-6)$. (i) Plot the function.

4. An implicit function is $x^y - 100 = 0$. (a) Make it explicit in the form $y = f(x)$. (HINT: Use logarithms). (b) Is this a single-valued function? (c) Find $f(1)$; (d) $f(-3)$; (e) $f(1/2)$. (f) Make the function explicit in the form $x = g(y)$. (g) Is this a single-valued function? (h) Find $g(10)$; (i) $g(100)$; (j) $g(2)$. (k) Make a graph of the function.

5. An implicit functional relationship is $x^2y - 120 = 0$. (a) Make it explicit in the form $y = f(x)$. (b) Is this a single-valued function? (c) Find $f(1)$; (d) $f(2)$; (e) $f(-3)$. (f) Make the function explicit in the inverse form $x = g(y)$. (g) Is this a single-valued function? (h) Find $g(1)$; (i) $g(-1)$; (j) $g(2)$.

6. An implicit functional relationship is $(x + y)/(x - y) = 2$. (a) Make it explicit in the form $y = f(x)$. (b) Plot the function. (c) Is this function single-valued? (d) Make it explicit in the form $x = g(y)$. (e) Find $g(-a)$.

*7. A function $y = f(x)$ is said to be even if we have $f(x) = f(-x)$. It is odd if $f(x) = -f(-x)$. Investigate for this property the following functions: (a) $y = x$; (b) $y = x^2$; (c) $y = x^3$. (d) Can you draw any conclusion about the function $y = x^n$, where n is an integer.

8. Consider the relationship $x^2 + y^2 = 9$. (a) Make it explicit in the form $y = f(x)$. (b) Plot. (c) Find $f(0)$; $f(1)$; $f(3)$. (d) Make the function explicit in the form $x = g(y)$. (e) Obtain $g(0)$; $g(-1)$; $g(-3)$.

9. Consider the relationship $x^2/2 + y^2/3 = 1$. (a) Make it explicit in the form $y = f(x)$. (b) Plot. (c) Find the inverse function $x = g(y)$.

10. Consider the implicit relationship $x^2/5 - y^2/3 = 1$. (a) Make it explicit as $y = f(x)$. (b) Plot. (c) Find the inverse function $x = g(y)$.

11. Consider the function $y = f(x) = 5x$. (a) Show that $f(1) + f(3) = f(4)$. (b) Show that $f(3) + f(5) = f(8)$. (c) Show that, in general, for any 2 numbers u and v , we have $f(u) + f(v) = f(u + v)$.

12. Consider the function $y = f(x) = 3x$. (a) Show that $f(5) - f(2) = f(3)$; (b) that $f(10) - f(4) = f(6)$. (c) In general, for any 2 numbers u and v , show that $f(u) - f(v) = f(u - v)$.

13. Let a function be defined as $y = f(x) = 2^x$. (a) Show that $f(2) \cdot f(3) = f(5)$. (b) Show that $f(1) \cdot f(4) = f(5)$. (c) Show that, in general, for any 2 numbers u and v , $f(u) \cdot f(v) = f(u + v)$.

14. Let a function be $y = f(x) = 10^x$. (a) Show that $f(4)/f(3) = f(1)$. (b) Show

that $f(5)/f(2) = f(3)$. (c) Show that, in general, for any 2 numbers u and v , $f(u)/f(v) = f(u - v)$.

29. Demand Functions and Total-Revenue Functions

A general market-demand function is given in the form, $p = p(D) = f(D)$, where p is the price of the commodity and D is the quantity demanded. This is a generalization of the linear-demand functions, quadratic-demand functions, and demand functions with constant elasticity, which have been introduced earlier.

The inverse function is $D = D(p) = g(D)$. Implicitly, the relation can be written $h(p, D) = 0$. All these expressions are equivalent, and describe the same economic relationship between the quantity demanded D and the price per unit p .

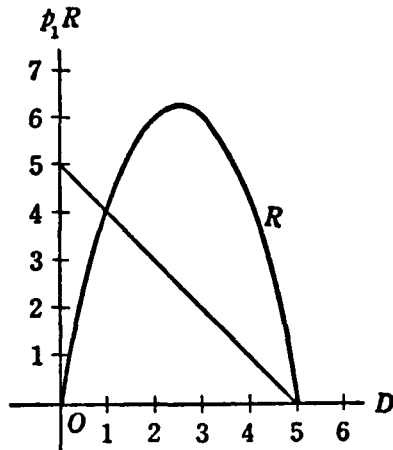


Figure 13

If we multiply price of a commodity by the quantity sold we get the total receipts of the seller or the total outlay of the consumers. This is called the *total-revenue function*; then the revenue function R may be written in functional language as

$$R(D) = pD = f(D) \cdot D = h(D).$$

The total-revenue function is considered as a function of the *quantity* sold, rather than of the price. The total-revenue function gives the total receipts of the sellers, or the total money outlay of the buyers, if a certain quantity D of the commodity is sold.

We show in Figure 13 the graph of a linear-demand function and of the corresponding total-revenue function R . Also the demand function is called sometimes the *average-revenue curve*.

EXAMPLE

Assume the implicit function, $p + D = 1$. (a) Make it explicit for p ; it is $p = 1 - D$. This is said to be the *demand curve*. (b) Find the total-revenue curve; it is $R = pD = D - D^2$. (c) Find the price if $1/2$ unit is sold. The result is $p(1/2) = 1 - 1/2 = 1/2$. (d) Find the total revenue if $1/3$ unit is sold. We obtain $R(1/3) = 1/3 - 1/9 = 2/9$.

EXERCISES 29

1. Given the relationship $2p + 3D = 10$. (a) Find the demand curve; (b) the total-revenue curve; (c) Find $p(1)$; (d) $p(2.5)$; (e) $p(0.755)$; (f) $R(0.75)$; (g) $R(1.125)$; (h) $R(3)$; (i) $R(0)$. (j) Plot the demand curve and total-revenue curve.

2. Given the relationship $3p + D = 60$. (a) Find the demand curve; (b) the total-revenue curve. (c) Find $p(0)$; (d) $p(1)$; (e) $p(6)$; (f) $R(7)$; (g) $R(1.5)$; (h) $R(5.25)$. (i) Plot the demand curve and total-revenue curve.

3. Given the relationship $p^2 + 5D = 100$. (a) Find the demand curve. (b) Find the total-revenue curve. (c) Find $p(10)$; (d) $p(2)$; (e) $p(17)$; (f) $R(2.50)$; (g) $R(3.25)$; (h) $R(0.125)$. (i) Plot the demand curve and total-revenue curve.

4. Given the relationship $p^D = 100$. (a) Find the demand curve; (b) the total-revenue curve. (c) Find $p(1)$; (d) $p(5)$; (e) $R(3)$; (f) $R(10)$; (g) $R(5)$; (h) $R(7.125)$. (i) Plot the demand curve and total-revenue curve.

5. Given the relationship, $p \log D = 24$. (a) Find the demand curve. (b) Find the total-revenue curve. (c) Find $p(10)$; (d) $p(2)$; (e) $p(18.5)$; (f) $R(10)$; (g) $R(5)$; (h) $R(7.125)$. (i) Plot the demand curve and total-revenue curve.

6. Let $10pD^2 = 1,000$. (a) Find the demand curve; (b) the total-revenue curve. (c) Find $p(2)$; (d) $R(1)$; (e) $R(25)$. (f) Plot the demand curve and total-revenue curve.

7. Given the relationship $p^2 - 2p + D = 10$. (a) Find the demand curve; (b) the total-revenue curve. (c) Find $p(0.5)$; (d) $p(0.1)$; (e) $p(0.001)$; (f) $R(0.98)$; (g) $R(6.33)$; (h) $R(0.25)$. (HINT: Take the radical positive.)

8. The implicit demand curve for sugar in the United States (1915–1929), according to Henry Schultz, is estimated by $p + 0.1D = 12$. (a) Find the demand curve; (b) the total-revenue curve. (c) Find $p(60)$; (d) $p(75)$; (e) $p(85)$; (f) $R(50)$; (g) $R(45)$; (h) $R(62)$. (i) Plot the demand curve and total-revenue curve.

9. The implicit demand curve for cotton, 1914–1929, for the United States according to Henry Schultz, is estimated by $pD^{1.4} = 0.11$. (a) Find the demand curve; (b) the total-revenue curve. (c) Find $p(15)$; (d) $p(20)$; (e) $p(12)$; (f) $R(10)$; (g) $R(12)$; (h) $R(15)$. (i) Plot the demand curve and total-revenue curve.

10. The implicit form of the demand function for potatoes in the United States, 1915–1929, according to Henry Schultz, is estimated by $100D + 2p = 440$. (a) Find the demand curve; (b) the total-revenue curve. (c) Find $p(2)$; (d) $p(3)$; (e) $p(4)$; (f) $R(2.5)$; (g) $R(3)$; (h) $R(3.5)$. (i) Plot the demand curve and total-revenue curve.

11. The demand for barley in the United States, 1915–1929, according to Henry Schultz, is estimated by $D = 6.39/p^{0.39}$. (a) Find the demand curve; (b) the total-revenue curve. (c) Find $p(1)$; (d) $p(2)$; (e) $p(3)$; (f) $R(4)$; (g) $R(3)$. (h) Plot the demand curve and total-revenue curve.

12. The demand for hay in the United States for the period 1915–1929, according to Henry Schultz, is estimated by $10D = 53 - 2p$. (a) Find the demand curve $p(D)$; (b) the total-revenue curve. (c) Find $p(0.2)$; (d) $p(0.35)$; (e) $p(0.4)$; (f) $R(0.5)$; (g) $R(0.3)$; (h) $R(0.55)$. (i) Plot the demand curve and total-revenue curve.

13. The demand for butter in Stockholm (1925–1937) has been estimated as: $D = 38/p^{1.2}$ (H. Wold). Find (a) the demand curve; (b) the total-revenue curve; (c) $p(2)$; (d) $R(2)$.

30. Total and Average-Cost Functions

We understand by total cost the sum total of *all* monetary expenditures incurred in order to produce a given amount of the commodity. We deal here only with short-term cost functions. These functions are constructed under the assumption that there is a certain fixed capital equipment.

Denote the amount produced by D and the total cost by C . We have a functional relationship between total cost and amount produced; that is,

$$C = f(D).$$

Average cost is cost per unit of the commodity produced. In order to compute average cost we divide the total cost by the number of units produced. We denote the average cost by A ; thus, in the language of functional relationships,

$$A = \frac{C}{D} = \frac{f(D)}{D} = g(D).$$

On the other hand, if the average cost is known, we get the total cost by multiplying average cost by the number of units produced; that is,

$$C = AD.$$

We exhibit in Figure 14 a total-cost curve C and an average-cost curve A , which have the typical¹ shapes generally assumed in economic theory for short-term cost curves. Note that average costs are high for small amounts produced, then decrease to a minimum, and later increase again as large quantities of the commodity are produced.

■ EXAMPLE

Given the implicit relationship, $3D^3 - C = 0$. (a) Find the total-cost curve; it is $C = 3D^3$. (b) Find the average-cost curve; it is $A = C/D = 3D$. (c) Find the total cost if 10 units are produced. There results $C(10) = 3(10)^3 = 300$. (d) Find the average cost if 3 units are produced. We have $A(3) = (3)(3) = 9$.

● EXERCISES 30

1. Given the relationship $C - 2D - D^2 = 0$. (a) Find the total-cost curve;

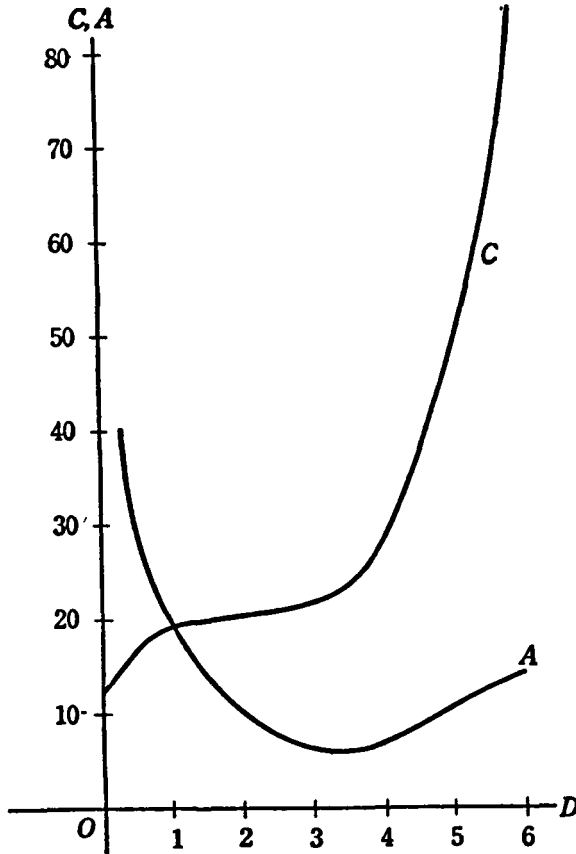


Figure 14

(b) the average-cost curve. (c) Find $C(12)$; (d) $C(4)$; (e) $C(3)$; (f) $A(6)$; (g) $A(11)$; (h) $A(5)$. (i) Plot the total-cost curve and average-cost curve.

2. Given the relationship $D = C^3$. (a) Determine the total-cost curve; (b) the average-cost curve. (c) Find $C(1)$; (d) $C(0)$; (e) $C(5)$; (f) $C(4)$; (g) $A(1)$; (h) $A(6)$; (i) $A(11)$; (j) $A(9)$. (k) Plot the total-cost curve and average-cost curve.

3. Given the relationship $5C - 3D = 100$. (a) Find the total-cost curve; (b) the average-cost curve. (c) Find $C(0)$; (d) $C(3)$; (e) $C(6)$; (f) $A(1)$; (g) $A(10)$. (h) Plot the total-cost curve and average-cost curve.

4. Given the relationship $D = -80 + C^2$. (a) Find the total-cost curve; (b) the average-cost curve. (c) Find $C(1)$; (d) $C(10)$; (e) $C(3)$; (f) $C(12)$; (g) $A(10)$; (h) $A(5)$; (i) $A(6)$; (j) $A(7)$. (k) Plot the total-cost curve and average-cost curve.

5. Given the relationship $C^2 - D - 5D^2 = 45$. (a) Find the total-cost curve; (b) the average-cost curve. (c) Find $C(0)$; (d) $C(10)$; (e) $C(100)$; (f) $A(1)$; (g) $A(5)$; (h) $A(15)$; (i) $A(3)$. (j) Plot the total-cost curve and average-cost curve.

6. Given the relationship $D = \log C - 5$. (a) Establish the total-cost curve; (b) the average-cost curve. (c) Find $C(0)$; (d) $C(10)$; (e) $C(150)$; (f) $A(10)$; (g) $A(100)$; (h) $A(50)$. (i) Plot the total-cost curve and average-cost curve.

7. Given the relationship $C^2/D^8 = 8$. (a) Find the total-cost curve; (b) the average-cost curve. (c) Find $C(4)$; (d) $C(80)$; (e) $C(100)$; (f) $A(1)$; (g) $A(25)$.

8. Given the relationship $5 \log C - 4 \log D = 2$. (a) Find the total-cost curve; (b) the average-cost curve. (c) Find $C(1)$; (d) $C(10)$; (e) $C(32)$; (f) $C(100)$; (g) $A(2)$; (h) $A(10)$; (i) $A(25)$; (j) $A(5)$.

9. Given the relationship $C/\log D = 24$. (a) Find the total-cost curve; (b) the average-cost curve. (c) Find $C(1)$; (d) $C(2)$; (e) $C(100)$; (f) $C(10,000)$; (g) $C(500)$; (h) $A(100)$; (i) $A(25)$; (j) $A(100,000)$.

10. T. O. Yntema has estimated a relationship between the total cost C and the quantity produced D of the United States Steel Corporation, 1928–1938, to be $C - 56D = 182$. (a) Find the total-cost curve; (b) the average-cost curve. (c) Find $C(5)$; (d) $C(10)$; (e) $C(15)$; (f) $C(9.25)$; (g) $C(6.5)$; (h) $A(10)$; (i) $A(12.5)$; (j) $A(19.5)$; (k) $A(6.75)$. (l) Plot the total-cost curve and average-cost curve.

11. The total-cost function of a hosiery mill was estimated by J. Dean as follows: $C = -10,485 + 6.75D - 0.0003D^2$. (a) Find the average-cost curve; (b) $C(4,400)$; (c) $C(6,000)$; (d) $C(8,550)$; (e) $C(6,895)$; (f) $A(5,000)$; (g) $A(8,200)$; (h) $A(4,575)$. (i) Plot the total-cost curve and the average-cost curve.

12. J. Dean estimated the total-cost function of a leather belt shop as follows: $C = 3,000 + 0.8D$. (a) Find the average-cost curve; (b) $C(50,000)$; (c) $C(120,000)$; (d) $C(95,000)$; (e) $A(50,000)$; (f) $A(65,000)$; (g) $A(25,560)$. (h) Plot the total cost curve and the average-cost curve.

31. Difference Quotients

If $y = f(x)$, the difference quotient is defined as

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

where Δx is an increment of x , and Δy is the corresponding increment of y . The ratio $\Delta y/\Delta x$ shows the average *rate of change* in y with respect to x . To x corresponds a y given by $y = f(x)$. To $x + \Delta x$ corresponds $y + \Delta y = f(x + \Delta x)$. This latter value is computed by substituting $x + \Delta x$ in place of x in the equation $y = f(x)$. We have, therefore, the relations

$$y + \Delta y = f(x + \Delta x), \quad y = f(x),$$

from which it follows immediately that

$$\Delta y = f(x + \Delta x) - f(x).$$

To obtain the difference quotient, this has to be divided by the increment in the independent variable Δx ; hence

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

In the study of the trigonometry of the right triangle, the so-called trigonometric functions of an angle such as θ , shown in Figure 15, are the sine of θ ($\sin \theta$), the cosine of θ ($\cos \theta$), the tangent of θ ($\tan \theta$), the cosecant

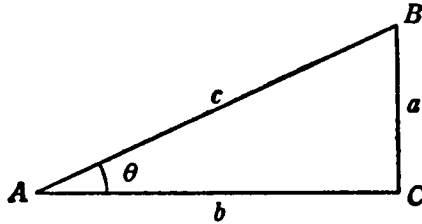


Figure 15

of θ ($\csc \theta$), the secant of θ ($\sec \theta$), and the cotangent of θ ($\cot \theta$). These functions are defined as follows:

$$\begin{aligned} \sin \theta &= \frac{a}{c}, & \csc \theta &= \frac{c}{a}, \\ \cos \theta &= \frac{b}{c}, & \sec \theta &= \frac{c}{b}, \\ \tan \theta &= \frac{a}{b}, & \cot \theta &= \frac{b}{a}. \end{aligned}$$

Of course each definition fails when the denominator is 0.

The values of four of these functions for angles from 0° to 90° are listed in Table 2 in the Appendix. By suitable modifications in the definitions, as given, the concept of the trigonometric functions may be extended to angles greater than 90° . In particular, $\tan (180^\circ - \alpha) = -\tan \alpha$; that is, for instance, $\tan 120^\circ = \tan (180^\circ - 60^\circ) = -\tan 60^\circ = -1.7321$, by reference to Table 2.

The tangent function is of special significance in many studies, for the tangent of the angle of inclination that a straight line makes with the horizontal is known as *the slope of the line*. The increase in slope as an angle increases in size may be observed by an examination of the tangent column in Table 2.

It is a matter of considerable interest to note that the difference quotient as defined above is the slope of a straight line through the 2 points (x, y) and $(x + \Delta x, y + \Delta y)$ on the curve corresponding to the function $y = f(x)$. This fact will be made clear in the examples that follow.

■ EXAMPLE 1

We show in Figure 16 the function $y = f(x) = x^2$. Let us select $x = 2$ and $\Delta x = 1$; of course, $y = f(2) = 4$. As already indicated, $\Delta y = f(x + \Delta x)$

$-f(x) = f(3) - f(2) = 9 - 4 = 5$. So the difference quotient at the point $x = 2, y = 4$ is

$$\frac{\Delta y}{\Delta x} = \frac{5}{1} = 5.$$

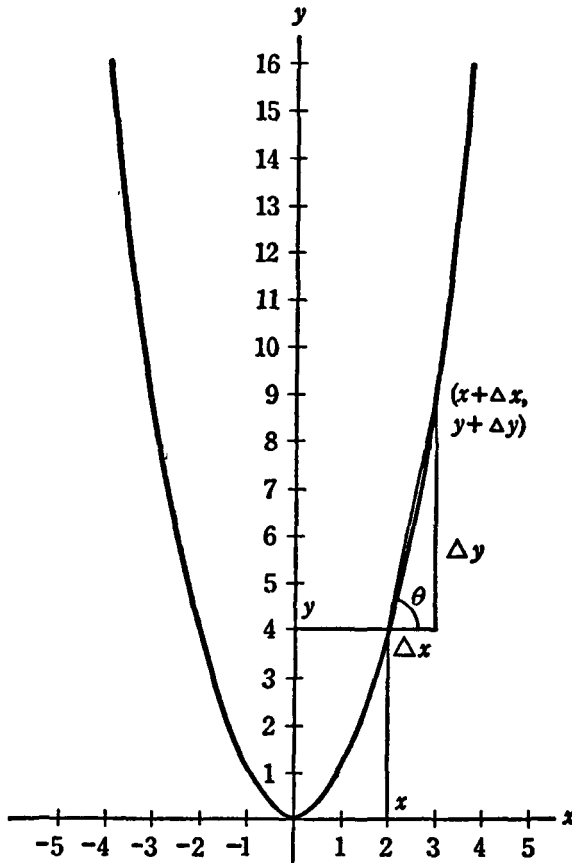


Figure 16

This difference quotient is $\tan \theta$ in Figure 16, that is, the slope of the straight line connecting the points $(2, 4)$ and $(3, 9)$. By reference to Table 2, it is found that $\theta = 78.7^\circ$, approximately.

■ EXAMPLE 2

Let $y = f(x) = 5x^3 + 3x - 1$. Determine the difference quotient. By substituting $x + \Delta x$ for x into the above formula we get

$$y + \Delta y = 5(x + \Delta x)^3 + 3(x + \Delta x) - 1.$$

This can be expanded as follows:

$$y + \Delta y = 5[x^2 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3] + 3(x + \Delta x) \quad - 1,$$

$$y + \Delta y = 5x^2 + 15x^2(\Delta x) + 15x(\Delta x)^2 + 5(\Delta x)^3 + 3x + 3(\Delta x) \quad - 1.$$

But

$$y = 5x^2 \qquad \qquad \qquad + 3x \qquad - 1.$$

By subtracting the members in the second line from those in the first, we get

$$\Delta y = 15x^2(\Delta x) + 15x(\Delta x)^2 + 5(\Delta x)^3 + 3(\Delta x).$$

To obtain the difference quotient $\Delta y/\Delta x$, we divide the last expression term by term by Δx , and obtain

$$\frac{\Delta y}{\Delta x} = 15x^2 + 15x(\Delta x) + 5(\Delta x)^2 + 3.$$

This expression may be evaluated for any given x and Δx to obtain the desired difference quotient. For instance, for $x = 1$ and $\Delta x = 0.1$ we have

$$\frac{\Delta y}{\Delta x} = 15(1)^2 + 15(1)(0.1) + 5(0.1)^2 + 3 = 19.55.$$

This means that for an increment $\Delta x = 0.1$ the average rate of change of y relative to x , when $x = 1$, is 19.55. This value is also the slope of the line connecting the points (1,7) and (1.1,8.955) on the curve. Since $\tan \theta = 19.55$, we find from our tables that $\theta = 87.1^\circ$, approximately.

EXAMPLE 3

Let $y = f(x) = 2x^2 - x$. What is the difference quotient?

$$y + \Delta y = 2(x + \Delta x)^2 - (x + \Delta x).$$

$$y + \Delta y = 2x^2 + 4x(\Delta x) + 2(\Delta x)^2 - x - \Delta x.$$

$$y = 2x^2.$$

$$\Delta y = 4x(\Delta x) + 2(\Delta x)^2 - \Delta x.$$

$$\frac{\Delta y}{\Delta x} = 4x + 2(\Delta x) - 1.$$

For instance, let $x = 10$ and $\Delta x = 2$. Then the difference quotient $\Delta y/\Delta x$ is $(4)(10) + (2)(2) - 1 = 43$. So the slope of the straight line through the 2 points on the curve is 43. Hence, $\tan \theta = 43$, and $\theta = 88.7^\circ$, approximately.

EXERCISES 31

1. Given $y = f(x) = x^3 + 2x^2 - 5$. (a) Find the general formula for the differ-

ence quotient. (b) Compute the difference quotient for $x = 4$, $\Delta x = 2$. (c) What is the angle θ ? (d) Make a graph and indicate the quantities given.

2. Given the function, $y = 3x - 5$. (a) Compute the general expression for the difference quotient. (b) Find the difference quotient for $x = 5$, $\Delta x = 1/2$. (c) What is the angle θ ? (d) Make a graph and indicate the various quantities.

3. Given the function, $y = x^2 + 3x - 6$. (a) Find the general expression for the difference quotient. (b) Assume that $\Delta x = 3$, find the x for which the angle θ is 45° .

4. Let the function be $y = f(x) = 24/x$. (a) Find the general expression for the difference quotient. (b) Compute the difference quotient for $x = 6$, $\Delta x = 1$. (c) Make a graph, and indicate Δx and Δy .

5. Let $y = f(x) = x^3$. (a) Find the general expression for the difference quotient. (b) Assume that $\Delta x = 1$, and find the x for which the angle $\theta = 30^\circ$.

6. Consider the function, $y = 10$. (a) Graph. (b) Find the general expression for the difference quotient.

7. Consider the function, $y = f(x) = x^3$. (a) Plot. (b) Find the general expression for the difference quotient. (c) Find the difference quotient and angle θ if $x = 2$; if $\Delta x = 1$; (d) if $\Delta x = 0.1$; (e) if $\Delta x = 0.01$; (f) if $\Delta x = 0.0001$.

**8. Is there a function whose difference quotient $(\Delta y)/(\Delta x)$ is k , where k is constant? (HINT: Consider the general linear function $y = a + bx$.) What is the interpretation of k ?

**9. Consider the function $y = f(x) = x^2$. Let $\Delta x = 1$. Insert numerical values in the following table of differences Δy :

x	$y = f(x)$	$\Delta y = f(x + 1) - f(x)$
0	$f(0)$	
1	$f(1)$	$f(1) - f(0)$
2	$f(2)$	$f(2) - f(1)$
3	$f(3)$	$f(3) - f(2)$

**10. Use the scheme indicated in Problem 9 to compute the differences Δy for the function $y = 2x^3 - x + 1$ for $\Delta x = 1$; use integral values of x in the range $x = 0$ to $x = 5$.

**11. Use the method indicated in Problem 9 to find the differences Δy of the function $y = 2x - 1$ for $\Delta x = 1/2$; employ values of x in the range $x = -2$ to $x = 2$.

**12. Use the method indicated in Problem 9 to find the differences Δy of the function $y = x^4 - 2x^2$ with $\Delta x = 2$, for integral values of x in the range $x = 0$ to $x = 6$.

32. Limits

An exact mathematical definition of limit is beyond the scope of this book. We will give an intuitive explanation.

By the expression

$$\lim_{x \rightarrow a} f(x) = L,$$

we mean the following: If x is a number close to but different from a , the value of the function $f(x)$ is close to L . Moreover, $f(x)$ can be made as close to L as may be desired by taking x sufficiently close to a .

■ EXAMPLE 1

Consider, for instance,

$$\lim_{x \rightarrow 2} 1 - 2x^2.$$

The desired limit is -7 , the same as $f(2)$. That this is true can be seen from the following table, noting especially the behavior of the function close to $x = 2$:

x	$f(x) = 1 - 2x^2$
1	$1 - 2(1)^2 = -1$
1.5	$1 - 2(1.5)^2 = -3.5$
1.9	$1 - 2(1.9)^2 = -6.22$
1.95	$1 - 2(1.95)^2 = -6.605$
1.99	$1 - 2(1.99)^2 = -6.9202$
2.01	$1 - 2(2.01)^2 = -7.0802$
2.05	$1 - 2(2.05)^2 = -7.405$
2.1	$1 - 2(2.1)^2 = -7.82$
2.5	$1 - 2(2.5)^2 = -11.5$
3	$1 - 2(3)^2 = -17$

We may surmise from a study of the table that the absolute value of the difference between $f(2) = -7$ and the computed $f(x)$ can be made as small as desired, by taking a value of x near enough to $x = 2$. For instance, it appears from our table that we can make the absolute value of the deviation of $f(x)$ from -7 as small as 0.1 by choosing x as any value between 1.99 and 2.01 .

The statement $\lim_{x \rightarrow a} f(x) = L$ is also written $f(x) \rightarrow L$ as $x \rightarrow a$. Sometimes one reads the statement: The function $f(x)$ tends toward the limit L as x tends toward a .

The symbol ∞ stands for infinity; infinity is not a number, and the concept of infinity is used only in connection with limits. By $\lim_{x \rightarrow \infty} f(x) = L$,

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we mean that there is a limit L for the function $f(x)$ which is approached as x assumes greater and greater values, that is, as x becomes successively such a sequence of numbers as 10; 100; 1,000; and so on.

■ EXAMPLE 2

Consider, for instance, the limit

$$\lim_{x \rightarrow \infty} 1/x^2 = 0.$$

It is intuitively evident that x^2 becomes larger and larger as x increases. For instance, the square of 2 is 4, the square of 3 is 9, and so on. Dividing these increasingly larger numbers into 1 produces smaller and smaller numbers. By making x large enough we can get as near as we like to the limit 0, which is L in this case.

This behavior is illustrated by the following table:

x	$f(x) = \frac{1}{x^2}$
5	$\frac{1}{25}$
10	$\frac{1}{100}$
100	$\frac{1}{10,000}$

This table shows that $f(x)$ approaches the limit 0 as x increases indefinitely. For instance, to make the absolute value of the deviation of $f(x)$ from the limiting value 0 as small as $1/100$, we must choose x as large or larger than 10. To make the absolute value of the deviation as small as $1/10,000$, we must choose x as large or larger than 100, and so forth.

We give in Figure 17 a graph of the function $y = f(x) = 1/x^2$. From this it can be seen that smaller and smaller values of y are associated with larger and larger values of x . This graphical study confirms our conclusion

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0.$$

This can also be written

$$f(x) = \frac{1}{x^2} \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty.$$

■ EXAMPLE 3

Consider the function

$$f(x) = \frac{1 + 3x}{2 - x}.$$

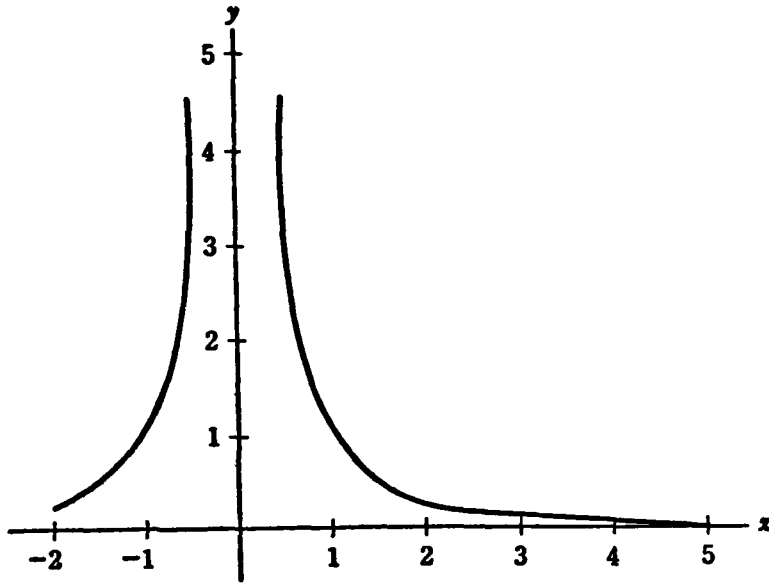


Figure 17

We will argue that $\lim_{x \rightarrow 0} f(x) = 1/2$. It is intuitively evident that the smaller the value of x becomes, the smaller will be the expressions $3x$ in the numerator and $-x$ in the denominator. In fact, as the process continues, $3x$ and $-x$ become entirely negligible; this leaves only 1 in the numerator and 2 in the denominator. These numbers are independent of x . Hence the limit of $f(x)$ is $1/2$ as $x \rightarrow 0$. The behavior of the function as $x \rightarrow 0$ is examined in greater detail in the table that follows:

x	$f(x)$
1	4
0.5	$\frac{5}{3} = 1.667$
0.1	$\frac{13}{19} = 0.684$
0.01	$\frac{103}{199} = 0.518$
0.001	$\frac{1,003}{1,999} = 0.502$

It is apparent that the nearer x approaches 0, the less is the deviation of $f(x)$ from its limiting value 0.5.

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● EXERCISES 32

1. Assume that $y = f(x) = 5x^2 + 10$. Show that $\lim_{x \rightarrow 2} f(x) = 30$ by considering: $f(2.5); f(2.1); f(2.05); f(2.01); f(1.5); f(1.9); f(1.95); f(1.99)$. Show that the absolute value, that is, the value ignoring the sign, of the deviation $f(x) - 30$ becomes smaller, the nearer x is to 2.

2. Consider $y = f(x) = (2x + 5)/4x$. Show that $\lim_{x \rightarrow 5} f(x) = 3/4$ by computing: $f(4), f(4.5), f(4.9), f(4.95), f(4.99), f(4.995), f(4.999), f(5.5), f(5.1), f(5.05), f(5.01), f(5.005), f(5.001)$. Show that the absolute value of the difference $f(x) - 3/4$ becomes smaller the nearer the independent variable x is to 5. Make a graph of the function $f(x)$.

3. Let $f(x) = 100/x$. Show that $\lim_{x \rightarrow \infty} f(x) = 0$. Consider: $f(1), f(2), f(5), f(10), f(20), f(100), f(1,000), f(1,000,000)$. Show that the absolute value of the deviation $f(x) - 0$ becomes smaller the greater the value of the independent variable x .

*4. Recall that the value of the n th term of a geometric progression is $y_n = ar^{n-1}$, where a is the first term and r is the ratio of 2 consecutive terms. Assume that $a = 100$, $r = 1/2$. Show that $\lim_{n \rightarrow \infty} y_n = 0$. (HINT: Compute y_5, y_{10}, y_{100}). Show that the absolute value of y_n becomes smaller the larger the value of n .

5. Using the formula in Problem 4, also show that $\lim_{n \rightarrow \infty} y_n = 0$ if $a = 50$ and $r = 1/5$. Use a method similar to the one suggested in the previous example.

*6. Show on the basis of the 2 previous examples that for any geometric series $\lim_{n \rightarrow \infty} y_n = 0$, irrespective of the value of a , if the absolute value of the common ratio r is less than 1.

*7. Consider the sum of a geometric progression

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

Assume that $a = 100$, $r = 1/2$. Show that $\lim_{n \rightarrow \infty} S_n = 200$. Compute $S_5; S_{10}; S_{100}$.

Show that the deviation of S_n from the limiting value 200 becomes less in absolute value the larger the number n becomes.

8. Using the same method as in Problem 7, show that $\lim_{n \rightarrow \infty} S_n = 5$ for $a = 4$, $r = 1/5$.

*9. Show that the sum S_n of an infinite geometric progression

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \lim_{n \rightarrow \infty} \frac{a(r^n - 1)}{r - 1} = \frac{a}{1 - r},$$

if $r < 1$.

10. Use the results in Problem 9 to find the limit of the sum in the case of each of the following geometric progressions: (a) $4 + 2 + 1 + 1/2 + 1/4 + \cdots$; (b) $100 + 10 + 1 + 0.1 + 0.01 + \cdots$; (c) $81 + 27 + 9 + 3 + 1 + 1/3 + 1/9 + \cdots$; (d) $16 + 4 + 1 + 1/4 + 1/16 + \cdots$; (e) $1 + 9/10 + 81/100 + 729/1,000 + \cdots$.

11. Assume that \$1 is invested. Let a be a proper positive fraction, that is, $0 < a < 1$. Assume that the fraction a of this dollar is again invested in the second period. This procedure is continued; that is, a times the investment of the previous period is reinvested during each period. (a) Assume that investment of this kind goes on indefinitely and compute the limit of the sum that results. (b) Compute a if the sum total of investment from a single dollar over an infinite period of time is \$5. (c) Compute a if the sum total of investment resulting from a single dollar is \$1.50. (d) Compute a if the sum total of all investments in an infinite period is \$10.

12. The quantity $1/(1 - a)$ (Problem 11) is called the investment multiplier. The constant a was estimated by T. Haavelmo as 0.712 (data from United States, 1930–1941). Compute with these data the total investment resulting from the initial investment of \$1,000,000 for an infinite period.

13. The investment multiplier (Problem 12) was estimated as 2.23 from data for the United States, 1921–1939 (Paul A. Samuelson). (a) Find the quantity a (see Problem 11). (b) If a sum of \$10,000,000 is invested indefinitely according to the principle enunciated, what will be the final total over an infinite period? (c) What will be the total investment after 10 years?

33. Derivatives

The difference quotient $\Delta y/\Delta x$ has been introduced in a previous section.

The limit, if it exists, of this difference quotient as $\Delta x \rightarrow 0$ is known as the derivative. The derivative of $y = f(x)$ is denoted by such symbols as y' , $f'(x)$, $D_x y$, $df(x)/dx$, or dy/dx . Thus

$$y' = f'(x) = D_x y = D_x f(x) \\ = \frac{df(x)}{dx} = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

As Δx becomes smaller and smaller, Δy will also in general decrease, as shown in the graph on page 86 (Figure 18). The smaller Δx becomes the closer together will be the two points having the abscissas x and $x + \Delta x$ and the ordinates y and $y + \Delta y$, respectively. The straight line joining the points (x, y) and $(x + \Delta x, y + \Delta y)$, which cuts the curve in at least 2 points, approaches the position of the tangent to the curve $y = f(x)$ at (x, y) as $\Delta x \rightarrow 0$. This latter conclusion follows from the fact that, as $\Delta x \rightarrow 0$, the point $(x + \Delta x, y + \Delta y)$ approaches the point (x, y) , so that the line joining them approaches the tangent to the curve $y = f(x)$ at the point (x, y) .

The limit of the difference quotient is by definition the derivative $f'(x)$. Hence $f'(x)$ is the slope of the straight line which is tangent to the curve $y = f(x)$ at the point (x, y) ; by the slope of the tangent line is meant the tangent of the angle α between the tangent line and the x axis. In symbols,

$$y' = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \tan \alpha.$$

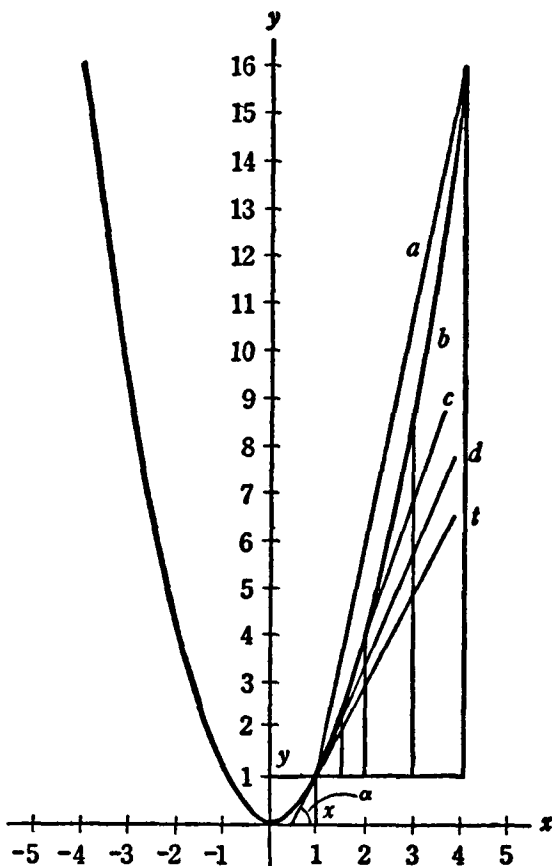


Figure 18

■ EXAMPLE 1

We show in Figure 18 the behavior of $\Delta y/\Delta x$ as $\Delta x \rightarrow 0$, if $y = f(x) = x^2$, and if the point (x,y) is chosen as $(1,1)$. The lines labeled a , b , c , and d are the lines of slope $\Delta y/\Delta x$, drawn for the increments $\Delta x = 3, 2, 1, 1/2$. They converge more and more towards the line t which is tangent to the curve $y = x^2$ at the point $(1,1)$. The angle α made by the tangent line t with x axis is shown.

■ EXAMPLE 2

Find the derivative at the point (x,y) if

$$y = f(x) = 3x^4 - 2x^3 + 1.$$

First we derive the difference quotient by methods described previously:

$$y + \Delta y = 3(x + \Delta x)^4 - 2(x + \Delta x)^3 + 1.$$

This is evaluated by the binomial theorem as follows:

$$y + \Delta y = 3[x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4] - 2[x^3 + 2x(\Delta x) + (\Delta x)^2] + 1.$$

$$y + \Delta y = 3x^4 + 12x^3(\Delta x) + 18x^2(\Delta x)^2 + 12x(\Delta x)^3 + 3(\Delta x)^4 - 2x^3 - 4x(\Delta x) - 2(\Delta x)^2 + 1.$$

$$y = 3x^4 - 2x^3 - 4x(\Delta x) - 2(\Delta x)^2 + 1.$$

$$\Delta y = 12x^3(\Delta x) + 18x^2(\Delta x)^2 + 12x(\Delta x)^3 + 3(\Delta x)^4 - 4x(\Delta x) - 2(\Delta x)^2.$$

Next we divide term by term by Δx , thereby obtaining

$$\frac{\Delta y}{\Delta x} = 12x^3 + 18x^2(\Delta x) + 12x(\Delta x)^2 + 3(\Delta x)^3 - 4x - 2(\Delta x).$$

By definition, the derivative $f'(x)$ is $\lim_{\Delta x \rightarrow 0} \Delta y / \Delta x$. As $\Delta x \rightarrow 0$, $(\Delta x)^2 \rightarrow 0$ and $(\Delta x)^3 \rightarrow 0$. The only terms not containing Δx or powers of (Δx) in the difference quotient are $12x^3 - 4x$. All the other terms become 0. Hence

$$f'(x) = 12x^3 - 4x.$$

This is the desired derivative, which is the slope of the tangent line at some point (x, y) on the curve. For instance, let $x = 1/2$; then

$$f'(1/2) = (12)(1/2)^3 - 4(1/2) = -1/2.$$

Hence, at the point on the curve where $x = 1/2$, the slope of the tangent line is $-1/2$. In other words, this is the angle α between the tangent line to the curve at the point in question and the x axis; thus

$$\tan \alpha = -1/2.$$

We see from Table 2 that the angle α is 153.4° .

● EXERCISES 33

All derivatives in the following exercises are to be computed by obtaining $\lim_{\Delta x \rightarrow 0} \Delta y / \Delta x$.

1. Let $y = f(x) = 3x^3 - 5x + 10$. (a) Find the difference quotient. (b) Find the derivative. (c) Find $f'(5)$. (d) What is the angle the tangent line to the curve at the point $x = 5$ makes with the x axis? (e) Make a graph of the curve and the tangent.

2. Let $y = f(x) = 3x^2$. (a) Find the difference quotient. (b) Find the derivative. (c) Find $f'(3)$; (d) the angle α . (e) Compute the difference quotient and the angle θ for $x = 3$; $\Delta x = 1$. (f) For $x = 3$; $\Delta x = 1/2$. (g) For $x = 3$; $\Delta x = 0.1$. (h) For $x = 3$; $\Delta x = 0.002$. Note that the difference between the difference quotient, the derivative, and the angles θ and α becomes smaller and smaller as Δx decreases. (i) Make a graph of the curve and the tangent.

3. Let $y = f(x) = x^3$. (a) Find the difference quotient. (b) Find the derivative. (c) Find $f'(2)$; (d) the angle α . (e) For what value of x does the angle α have the measure 60° ?

4. Let $y = f(x) = x^4 - 2x^2 + 6$. (a) Find the difference quotient. (b) Find the derivative. (c) Find $f'(0)$ and the corresponding angle α . (d) Find $f'(-2)$ and the corresponding angle α . (e) Plot the function.

5. Let $y = f(x) = 3x^2 - 6x + 8$. (a) Find the difference quotient; (b) the derivative; (c) $f'(0)$ and the corresponding angle α ; (d) $f'(-5)$ and the corresponding angle α ; (e) $f'(2)$ and the corresponding angle α . (f) Where is the angle $\alpha = 15^\circ$? (g) For what x is $\alpha = -30^\circ$? (h) Make a graph of the curve and the tangent lines.

6. Let $y = f(x) = 1 + 2x + 3x^2 - x^3$. (a) Find the difference quotient; (b) the derivative; (c) $f'(0)$ and the angle α ; (d) $f'(-5)$ and the angle α ; (e) the values of x and y for which the angle α is 45° .

7. Let $y = f(x) = -x^3$. (a) Find the difference quotient; (b) the derivative; (c) $f'(1)$ and the angle α ; (d) the x and y where the angle α is -60° .

8. Consider the function $y = f(x) = 3 - 2x + 5x^2 - x^3 + x^4$. (a) Find the difference quotient. (b) Determine the derivative. (c) Find the value of $f'(-4)$ and the corresponding angle α .

9. Consider the function $y = f(x) = 6x^2$. Find the values of x and y where the angle α is (a) 45° ; (b) -45° . (c) Plot the function and check.

**10. Is there a function whose derivative is $y' = k$, where k is a constant? (HINT: Consider the derivative of the general linear function $y = a + bx$, where a and b are constants.) What is the interpretation of k ?

34. Marginal Cost

Marginal cost is defined as *the quotient of the increment of total cost, resulting from an increment in the amount produced, divided by the latter increment*. More precisely, it should be defined as the *limit of the ratio*

$$\frac{\text{increment in total cost}}{\text{increment in output}},$$

as the increment in output tends to 0.

If $C = C(D)$ is the total cost of producing D units, by definition,

$$\text{Marginal cost } C' = \lim_{\Delta D \rightarrow 0} \frac{\Delta C}{\Delta D} = \frac{dC}{dD}.$$

EXAMPLE 1

We show in Figure 19 the graph of the total-cost curve

$$C = f(D) = 10 + 15D - 6D^2 + D^3.$$

From this we derive the marginal-cost curve

$$C' = f'(D) = 15 - 12D + 3D^2,$$

by the process of differentiation, outlined above. It should be noted that the typical short-term marginal-cost curve, like the one shown in the graph labeled C' , first decreases, reaches a minimum, and finally increases, as more and more units of the commodity are produced.

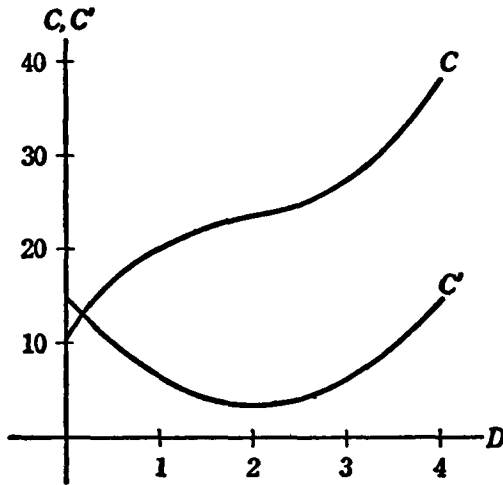


Figure 19

EXAMPLE 2

Let the total-cost curve be $C = D^2$. Then

$$\begin{aligned} \Delta C &= (D + \Delta D)^2 - D^2 = D^2 + 2D(\Delta D) + (\Delta D)^2 - D^2 \\ &= 2D(\Delta D) + (\Delta D)^2. \end{aligned}$$

Thus

$$\frac{\Delta C}{\Delta D} = 2D + \Delta D.$$

Evidently

$$C' = \lim_{\Delta D \rightarrow 0} \frac{\Delta C}{\Delta D} = \underline{2D}.$$

This is the *marginal-cost curve*. For instance, for 3 units produced, $C'(3) = 6$.

EXERCISES 34

1. Given the total-cost curve $C = 2 + D$. (a) Find the marginal-cost curve. (b) Find $C'(0)$; (c) $C'(5)$; (d) $C'(10)$. (e) Make a graph of the total-cost curve and the marginal-cost curve.
2. Given the total-cost curve $C = 1 + 2D + 3D^2$. (a) Find the marginal-cost curve. (b) Find $C'(1)$; (c) $C'(6)$; (d) $C'(10)$; (e) $C'(20)$. (f) Make a graph of the total-cost curve and marginal-cost curve.

3. Given the total-cost curve $C = D^2 + 2D$. (a) Find the marginal-cost curve. (b) Find $C'(10)$; (c) $C'(0)$; (d) $C'(2)$; (e) $C'(15)$; (f) $C'(3)$. (g) Plot the total-cost curve and marginal-cost curve.

4. Given the total-cost curve $C = 10 + 2D + 3D^2 + 4D^3$. (a) Find the marginal-cost curve. (b) Find $C'(0)$; (c) $C'(1)$; (d) $C'(5)$; (e) $C'(10)$; (f) $C'(15)$. (g) Plot the total-cost curve and marginal-cost curve.

5. Find the marginal-cost curve for the total-cost curve in Problem 1, Exercises 30. Also find (a) $C'(1)$; (b) $C'(5)$; (c) $C'(10)$. (d) Plot the total-cost curve and marginal-cost curve.

6. Find the marginal-cost curve belonging to the total curve in Problem 3, Exercises 30. Also calculate (a) $C'(1)$; (b) $C'(10)$; (c) $C'(5)$; (d) $C'(2.35)$. (e) Plot the total and marginal-cost curves.

7. Given the total-cost curve from Problem 10, Exercises 30. Find the marginal cost of steel. Also find (a) $C'(1)$; (b) $C'(100)$. (c) Plot the total-cost curve and marginal-cost curve.

8. Given the total-cost curve $C = m + nD$. (a) Find the marginal-cost curve. (b) Find $C'(0)$; (c) $C'(10)$; (d) $C'(5)$.

9. Given the total cost curve $C = mD + nD^2$. (a) Find the marginal-cost curve. (b) Find $C'(1)$; (c) $C'(0)$; (d) $C'(5)$; (e) $C'(m)$; (f) $C'(n)$.

10. Take the total-cost curve from Problem 11, Exercises 30, and find the marginal cost of hosiery. Also find (a) $C'(5,000)$; (b) $C'(3,500)$; (c) $C'(6,568)$; (d) $C'(7,500)$. (e) Plot the total-cost curve and marginal-cost curve.

11. Take the total-cost curve from Problem 12, Exercises 30. Find the marginal-cost curve for leather belts. Also find (a) $C'(60,000)$; (b) $C'(100,000)$; (c) $C'(75,000)$; (d) $C'(45,500)$; (e) $C'(20,000)$. (f) Plot the total-cost curve and marginal-cost curve.

12. The total-cost function of a light plant has been estimated as $C = 16.68 + 0.125D + 0.00439D^2$ (J. A. Nordin). C is the total fuel cost for an 8-hr. period in dollars; D is per cent of capacity. Find the marginal-cost curve. (a) Find $C'(50)$; (b) $C'(30)$; (c) $C'(90)$. (d) Plot the total-cost curve and marginal-cost curve.

35. Marginal Revenue

Total revenue R has been defined (Section 29) as the *product of price and the quantity sold*. Thus R is the total receipts of the sellers or the total outlay of the buyers.

Marginal revenue is the ratio of the increment in total revenue and the increment in the quantity sold. More precisely, it is the *limit of the ratio*

$$\frac{\text{increment in total revenue}}{\text{increment in sales}}$$

as the increment in sales tends to 0. Thus

$$R = pD,$$

and

$$R'(D) = \lim_{\Delta D \rightarrow 0} \frac{\Delta R}{\Delta D}.$$

EXAMPLE

Let the demand function for a commodity be given by

$$p = f(D) = 10 - 2D.$$

The total-revenue function is $R = g(D) = pD = (10 - 2D)D = 10D - 2D^2$.

We get $R + \Delta R$ by replacing D in the last formula by $D + \Delta D$; that is,

$$\begin{aligned} R + \Delta R &= 10(D + \Delta D) - 2(D + \Delta D)^2, \\ &= 10(D + \Delta D) - 2[D^2 + 2D(\Delta D) + (\Delta D)^2], \\ &= 10D + 10(\Delta D) - 2D^2 - 4D(\Delta D) - 2(\Delta D)^2. \end{aligned}$$

We subtract $R = 10D - 2D^2$ to obtain ΔR ; thus

$$\Delta R = 10(\Delta D) - 4D(\Delta D) - 2(\Delta D)^2.$$

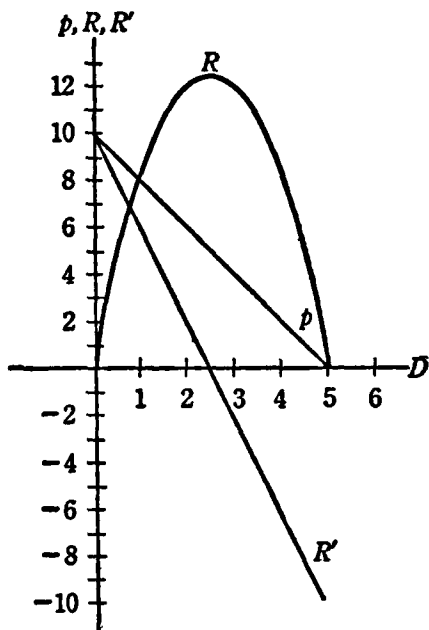


Figure 20

To compute the difference quotient we divide by ΔD , thereby obtaining

$$\frac{\Delta R}{\Delta D} = 10 - 4D - 2(\Delta D).$$

As $\Delta D \rightarrow 0$, the last term approaches 0, so

$$R'(D) = g'(D) = \lim_{\Delta D \rightarrow 0} \frac{\Delta R}{\Delta D} = 10 - 4D.$$

This is the marginal-revenue function. We have, for instance, when $D = 2$,

$$R'(2) = 10 - (4)(2) = 2, \quad \text{and} \quad R'(4) = 10 - (4)(4) = -6.$$

In Figure 20 the demand function is labeled p where $p = 10 - 2D$. The total-revenue function $R = p \cdot D$ is denoted by R , where $R = 10D - 2D^2$. Finally the marginal-revenue function is labeled R' , where $R' = 10 - 4D$.

It should be noted that the marginal-revenue function can become negative. This is obvious, since the increment of the total revenue can become negative.

It is apparent from the graph that the demand function p is a straight line. Note that the marginal-revenue curve, which is also a straight line, cuts the D axis at half the distance from the origin to the point of meeting of the demand curve p and the D axis.

● EXERCISES 35

1. Given the demand curve $p = 10 - 3D$. (a) Find the total-revenue curve; (b) the marginal-revenue curve; (c) $R'(1)$; (d) $R'(0)$; (e) $R'(3)$; (f) $R'(10/3)$. (g) Make a graph of the demand curve; total-revenue curve; and marginal-revenue curve.

2. Given the relationship $2D + 3p = 15$. (a) Find the demand curve; (b) the total-revenue curve; (c) the marginal-revenue curve; (d) $R'(0)$; (e) $R'(1)$; (f) $R'(4)$; (g) $R'(2.5)$.

3. Given the demand curve $p = 16 - D^2/2$. (a) Find the total-revenue curve; (b) the marginal-revenue curve; (c) $R'(1)$; (d) $R'(5)$; (e) $R'(1.125)$. (f) Make a graph of the demand curve; the total-revenue curve; and marginal-revenue curve.

4. Given the relationship in Problem 1, Exercises 29. (a) Determine the demand curve; (b) the total-revenue curve; (c) the marginal-revenue curve; (d) $R'(0)$; (e) $R'(0.5)$; (f) $R'(2.35)$.

5. Given the relationship in Problem 8, Exercises 29. (a) Find the demand curve for sugar; (b) the total-revenue curve; (c) the marginal-revenue curve; (d) $R'(55)$; (e) $R'(40)$; (f) $R'(70)$. (c) Make a graph of the demand curve; the total-revenue curve; and the marginal-revenue curve.

6. Given the relationship in Problem 10, Exercises 29. (a) Find the demand curve for potatoes; (b) the total-revenue curve; (c) the marginal-revenue curve; (d) $R'(1)$; (e) $R'(2)$; (f) $R'(1.5)$; (g) $R'(3)$. (h) Make a graph of the demand curve; the total-revenue curve; and the marginal-revenue curve.

7. Given the relationship in Problem 12, Exercises 29. (a) Find the demand curve for hay; (b) the total-revenue curve; (c) the marginal-revenue curve; (d) $R'(3)$; (e) $R'(4)$; (f) $R'(3.5)$; (g) $R'(2.1)$.

8. Take the demand curve for rye from Problem 9, Exercises 6. (a) Find the total-revenue curve; (b) the marginal-revenue curve; (c) $R'(0.1)$; (d) $R'(0.2)$; (e) $R'(0.3)$; (f) $R'(0.4)$; (g) $R'(0.5)$.

9. Given the relationship $aD + bp = k$, where a , b , k are constants. (a) Find

the total-revenue curve; (b) the marginal-revenue curve; (c) $R'(0)$; (d) $R'(1)$; (e) $R'(a)$; (f) $R'(k)$.

****10.** Assume a demand function $p = a - bD$, where a and b are positive constants. (a) Find R . (b) Find R' . (c) Find the intercept of p with the D axis by solving the equation $p = 0$ for D . (d) Find the intercept of R' with the D axis by solving the equation $R' = 0$ for D . (e) Establish a relationship between the 2 intercepts.

RULES OF DIFFERENTIATION

36. Derivative of a Power

Let us consider the derivative of $y = f(x) = x^n$, where n is a positive integer. An extension of the result obtained to any arbitrary, rational, positive or negative exponent will be given later (Problems 5, 10 in Exercises 42).

By definition of the derivative, we have

$$y' = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

From the formula $y = f(x) = x^n$, we obtain $y + \Delta y$ by substituting $x + \Delta x$ for x . Thus

$$y + \Delta y = (x + \Delta x)^n.$$

This can be developed by the binomial theorem as follows:

$$y + \Delta y = x^n + \frac{n}{1} x^{n-1}(\Delta x) + \frac{n(n-1)}{1 \cdot 2} x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n.$$

From this expression we subtract $y = x^n$ to obtain the increment Δy , which gives

$$\Delta y = \frac{n}{1} x^{n-1}(\Delta x) + \frac{n(n-1)}{1 \cdot 2} x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n.$$

To compute the difference quotient $\Delta y/\Delta x$, we divide term by term by Δx , thereby obtaining

$$\frac{\Delta y}{\Delta x} = \frac{n}{1} x^{n-1} + \frac{n(n-1)}{1 \cdot 2} x^{n-2}(\Delta x) + \cdots + (\Delta x)^{n-1}.$$

The first term does not involve the increment Δx . All other terms involve Δx or powers of (Δx) .

Passing to the limit as $\Delta x \rightarrow 0$, we have

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = nx^{n-1}.$$

So we have the following rule: If $y = f(x) = x^n$, where n is a positive integer, the derivative is

$$y' = f'(x) = nx^{n-1}.$$

EXAMPLE

For instance, the derivative of $y = f(x) = x^5$ is $f'(x) = 5x^4$. If $x = 2$, the derivative is $f'(2) = (5)(2^4) = 80$.

Let us consider an important special case if the function $y = f(x) = x$, where $n = 1$. We have $f'(x) = 1 \cdot x^0 = 1$, since any number raised to the power 0 is 1. It follows that the derivative of x itself with respect to x is 1.

EXERCISES 36

1. (a) Find the derivative of $y = f(x) = x^5$. (b) Find $f'(-1)$.
2. (a) Find the derivative of $y = f(x) = x^3$; (b) find $f'(1)$; (c) $f'(-5)$. (d) Find the x and y for which the angle of inclination of the tangent to the curve is 60°
3. Find the derivative of $y = f(x) = x^7$.
4. Find the derivative of $y = f(x) = x^2$.
5. (a) Find the derivative of $y = f(x) = x^6$. (b) Find $f'(0)$. (c) Where is the slope of the tangent line equal to -1 ?
6. Find the derivative of $y = f(x) = x^8$.
7. Find the derivative of $y = f(x) = x^{20}$.
8. Find the derivative of $y = f(x) = x^{18}$.
9. (a) Find the derivative of $y = f(x) = x^4$. (b) Find $f'(-2)$ and the angle of inclination when $x = -2$.
10. (a) Find the derivative of $y = f(x) = x^{10}$. (b) Find $f'(0)$ and the angle of inclination.

37. Derivative of a Constant Times a Function

If the function $f(x)$ approaches a certain limit L , as x approaches a , then the function $cf(x)$, where c is a constant, approaches cL , as x approaches a . The strict proof of this theorem is beyond the scope of this book, but it is quite fundamental in the analysis that follows. We will give an example.

EXAMPLE 1

Consider the function $f(x) = (5 + 2x)/(3 - x)$. We have $\lim_{x \rightarrow 0} f(x) = 5/3$. This is evident, since the terms $2x$ in the numerator and $-x$ in the denominator will contribute less and less to $f(x)$ the smaller x becomes.

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Now consider the function $g(x) = 3f(x) = (15 + 6x)/(3 - x)$. Evidently $\lim_{x \rightarrow 0} g(x) = 3$ times the previous limit, or $(3)(5/3) = 5$.

We can apply this result about limits to the problem of finding the derivative of a constant times a function. Let $u = f(x)$ be a function of x . Now introduce a new function

$$y = g(x) = cu = cf(x),$$

where c is a constant.

We have, from the definition of a derivative,

$$u' = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}.$$

It follows from the theorem about limits at the beginning of this section that

$$y' = g'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = c \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = cu'.$$

In words: *The derivative of a function of x multiplied by a constant is the constant times the derivative of the function.*

EXAMPLE 2

Suppose we have $y = f(x) = 3x^4$. The derivative of the function $u = x^4$ (from the rule for the derivative of a power) is $u' = 4x^3$. Since x^4 is multiplied by 3, $4x^3$ must be multiplied by 3 to give the derivative of y ; thus

$$y' = f'(x) = (3)(4x^3) = 12x^3.$$

For instance, at the point $x = 2$, we have

$$f'(2) = (12)(2^3) = 96.$$

EXAMPLE 3

Let $y = 2x^3$. The derivative of x^3 is $3x^2$. Hence $y' = 2 \cdot 3x^2 = 6x^2$.

EXERCISES 37

- (a) Differentiate $y = f(x) = 4x^6$. (b) Find $f'(-1)$; (c) $f'(0)$.
- (a) Differentiate $y = f(x) = 10x^3$. (b) Find $f'(1)$; (c) $f'(-2)$.
- (a) Differentiate $y = f(x) = 4x^4$. (b) Find $f'(0)$; (c) Find the x and y where the slope is -1 .
- (a) Find the derivative of $y = f(x) = -5x^2$. (b) Find $f'(0)$; (c) $f'(1)$; (d) $f'(-2)$.
- Find the derivative of $y = f(x) = -3x^4$.
- Find the derivative for $y = f(x) = 6x^2$.
- Find the derivative of $y = f(x) = -5x^6$.
- Consider the function in Problem 7. Find the values of x and y where the angle of inclination is (a) -15° ; (b) -45° .

9. Find the derivative of the function $y = -4x^2$ for $y = -3$; find the angle of inclination.

10. (a) Find the derivative of the function $y = f(x) = -5x^2$. Find the point on the curve where the slope of the tangent line is (b) -6 ; (c) 4 . (d) Plot the curve and the tangent lines.

38. Derivatives of Sums and Differences of Function

We will first state a theorem about limits of sums of functions.

Assume we have a function $f(x)$, for which $\lim_{x \rightarrow a} f(x) = L$; and another function $g(x)$, for which $\lim_{x \rightarrow a} g(x) = M$. Now we form a new function $h(x) = f(x) + g(x)$.

It is intuitively clear that

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = L + M.$$

In words: *The limit of a sum of functions is equal to the sum of their limits. By analogy, the limit of a difference of functions is equal to the difference of the limits.* An actual proof of this theorem is beyond the scope of this book, but let us study the following example:

■ **EXAMPLE 1**

Let $f(x) = 5 + 3x$. Then, evidently, $\lim_{x \rightarrow 0} f(x) = 5$. This is obvious, since the term $3x$ will become smaller and smaller as $x \rightarrow 0$. If $g(x) = 3 + 4x$, it is apparent also that $\lim_{x \rightarrow 0} g(x) = 3$.

Define a new function

$$h(x) = f(x) + g(x) = 8 + 7x.$$

Evidently $\lim_{x \rightarrow 0} h(x) = 8$. Hence we have a confirmation for a particular case of the theorem

$$\lim_{x \rightarrow 0} h(x) = \lim_{x \rightarrow 0} [f(x) + g(x)] = \lim_{x \rightarrow 0} f(x) + \lim_{x \rightarrow 0} g(x),$$

for the limit of the sum is the sum of the limits; that is, $8 = 5 + 3$.

Now consider the difference of the functions. Let

$$H(x) = f(x) - g(x) = 2 - x.$$

We have, evidently, $\lim_{x \rightarrow 0} H(x) = 2$. Hence

$$\lim_{x \rightarrow 0} H(x) = \lim_{x \rightarrow 0} [f(x) - g(x)] = \lim_{x \rightarrow 0} f(x) - \lim_{x \rightarrow 0} g(x),$$

as was expected.

Now let us consider the derivative of the sum of 2 functions. Let $u = f(x)$

and $v = g(x)$ be functions of the same independent variable x . Their derivatives are

$$u' = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x},$$

$$v' = g'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x}.$$

Then introduce a new function $y = h(x) = u + v$. We have for its derivative

$$y' = h'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u + \Delta v}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} = u' + v'.$$

This follows from the fact, discussed above, that the limit of the sum of 2 functions is the sum of the limits. An analogous theorem holds for the difference.

We can summarize as follows: *The derivative of a sum of functions is the sum of the derivatives. The derivative of a difference of functions is the difference of the derivatives.*

EXAMPLE 2

Assume that $y = f(x) = 2x^2 - 5x$. The derivative of $2x^2$ is $4x$, the derivative of $5x$ is 5 . Hence, $y' = f'(x) = 4x - 5$.

EXERCISES 38

1. (a) Find the derivative of $y = f(x) = 4x - 2x^2$. (b) Find $f'(0)$.
2. (a) Find the derivative of $y = f(x) = x - x^4$. (b) Find $f'(3)$.
3. (a) Find the derivative of $y = f(x) = 2x^2 - 5x^3 + x^4$. (b) Find $f'(0)$; (c) $f'(-2)$.
4. (a) Find the derivative of $y = f(x) = 2x - x^2$. (b) Find $f'(-1)$ and the corresponding angle of inclination of the tangent line. (c) Find the x and y where the angle of inclination is -30° .
5. (a) Find the derivative of $y = f(x) = x^4 - 3x^2$. (b) Find $f'(-2)$ and the corresponding angle of inclination.
6. Given the demand curve $p = 10 - 3D$. (a) Find the total-revenue curve R ; (b) the marginal-revenue curve R' ; (c) $R'(2.5)$; (d) $R'(1)$.
7. Given the demand curve $p = 1,000 - 2D - 3D^2$. (a) Find R ; (b) R' ; (c) $R'(1)$; (d) $R'(12)$; (e) $R'(6)$; (f) $R'(5.8)$.
8. Given the demand for sugar in the United States, 1915–1929, (Henry Schultz): $p = 14.7 - 0.1D$. (a) Find R ; (b) R' ; (c) $R'(80)$; (d) $R'(100)$; (e) $R'(65)$.
9. The demand for cotton in the United States, 1915–1929, is given by Henry Schultz: $p = 8 - D$. (a) Find R ; (b) R' ; (c) $R'(5)$; (d) $R'(1)$; (e) $R'(4)$; (f) $R'(6.5)$.

10. The demand for barley in the United States, 1915–1929, is given by Henry Schultz: $p = 86 - 25D$. (a) Find R ; (b) R' ; (c) $R'(1.5)$; (d) $R'(0.75)$; (e) $R'(0.85)$; (f) $R'(0.55)$; (g) $R'(0.725)$. (h) Plot the demand curve and the total-revenue curve.

11. Given the demand for rye in the United States, 1915–1929 (Henry Schultz): $p = 76 - 73D$. (a) Find R ; (b) R' ; (c) $R'(0.40)$; (d) $R'(0.55)$; (e) $R'(0.365)$; (f) $R'(0.255)$. (g) Plot the demand curve and the total-revenue curve.

12. Given the demand for buckwheat in the United States, 1915–1929 (Henry Schultz): $p = 101 - 294D$. (a) Find R ; (b) R' ; (c) $R'(0.2)$; (d) $R'(0.1)$; (e) $R'(0.25)$; (f) $R'(0.05)$. (g) Plot the demand curve and the total-revenue curve.

13. Given the demand for hay in the United States, 1915–1929 (Henry Schultz): $p = 15 - 2D$. (a) Find R ; (b) R' ; (c) $R'(0.5)$; (d) $R'(1)$; (e) $R'(0.2)$; (f) $R'(0.75)$; (g) $R'(0.95)$. (h) Plot the demand curve and the total-revenue curve.

39. Derivative of a Constant

Let $y = f(x) = c$, where c is a constant. Since c is independent of x we have

$$y + \Delta y = c,$$

$$y = c,$$

$$\Delta y = 0,$$

$$\frac{\Delta y}{\Delta x} = 0.$$

Hence

$$y' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = 0.$$

In general, the derivative of a constant is 0.

■ EXAMPLE

Let $y = 5x + 1$. We shall use the formula for the derivative of a sum. The derivative of $5x$ is 5. The derivative of 1 (which is a constant) is 0. Hence $y' = 5$.

● EXERCISES 39

1. Find the derivative of the function $y = -10 - 2x + 5x^2$.
2. (a) Find the derivative of the function $y = f(x) = x^3 - 10x^4 + 2x^3 - x + 10$.
(b) Find $f'(0)$; (c) $f'(-2)$; (d) $f'(5)$.
3. (a) Find the derivative of $y = f(x) = 1 + 2x + x^2$. (b) Draw its graph.
(c) Find $f'(1)$; (d) $f'(-4)$. (e) Find the x and y for which the angle of inclination of the tangent becomes 20° ; (f) -60° .

4. Let c be the consumption expenditure in dollars, y the deflated disposable income. From data in the United States, 1922–1941, we have the estimated relationship $c = 0.672y + 113.1$ (T. Haavelmo). (a) Find the derivative dc/dy , which is the marginal propensity to consume. (b) Compute $1/[1 - (dc/dy)]$. This is the so-called multiplier. (See Problem 11, Exercises 32.) NOTE: c is here the dependent variable and not a constant.

5. Take the demand curve $p = 10/D$. (a) Find R ; (b) R' ; (c) $R'(0)$; (d) $R'(1)$; (e) $R'(2)$; (f) $R'(3)$.

6. Let the total-cost curve of a commodity be $C = 2 + 5D + 3D^4$. (a) Find the marginal-cost curve; (b) $C'(0)$; (c) $C'(3)$.

7. Let the total-cost curve of a commodity be $C = 10 + 3D$. (a) Find the marginal-cost curve; (b) $C'(4)$; (c) $C'(100)$. (d) Plot the total-cost curve and the marginal-cost curve.

8. Consider the total-cost curve of steel (Yntema) as $C = 182 + 56D$. (a) Find the marginal-cost curve; (b) $C'(10)$; (c) $C'(15)$; (d) $C'(20)$. (e) Plot the total and marginal-cost curves.

9. Take as the total cost of hosiery (Dean) $C = -10,485 + 6.75D - 0.0003D^3$. (a) Find the marginal-cost curve; (b) $C'(5,000)$; (c) $C'(7,000)$; (d) $C'(6,500)$. (e) Plot the total and marginal-cost curves.

10. Given the total cost of leather belts (Dean) $C = 3,000 + 0.8D$. (a) Find the marginal-cost curve; (b) $C'(100,000)$; (c) $C'(50,000)$; (d) $C'(75,000)$. (e) Plot the total and marginal-cost curves.

11. Take as the total sales cost in a department store $C = 16.8 + 1.052D - 0.002D^2$ (Dean). (a) Find the marginal-cost curve; (b) $C'(30)$; (c) $C'(150)$; (d) $C'(100)$; (e) $C'(75)$. (f) Plot the total and marginal-cost curves.

40. Derivative of a Product

We state first a theorem about limits as follows: Let $f(x)$ be a function such that $\lim_{x \rightarrow a} f(x) = L$, and $g(x)$ another function such that $\lim_{x \rightarrow a} g(x) = M$. Let us define a new function $h(x) = f(x) \cdot g(x)$, which is the product of the 2 previous functions. It is intuitively evident that we have

$$\lim_{x \rightarrow a} h(x) = \lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M.$$

In words: *The limit of the product of 2 functions is the products of their limits.* The proof of this theorem is beyond the scope of this book. We will give an example of its application.

EXAMPLE 1

Define $f(x) = 2 + 5x$. Then $\lim_{x \rightarrow 0} f(x) = 2$. If $g(x) = 3 - 4x$, then $\lim_{x \rightarrow 0} g(x) = 3$. Define a new function $h(x) = f(x) \cdot g(x) = (2 + 5x)(3 - 4x)$

$= 6 + 7x - 20x^2$. Evidently $\lim_{x \rightarrow 0} h(x) = 6$. This result is the product of the limits of $f(x)$ and $g(x)$, as was expected.

To find the derivative of a product of 2 functions we proceed as follows: Assume that we have 2 functions of the independent variable x , namely, $u = g(x)$ and $v = h(x)$. Their derivatives are defined as

$$u' = g'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x},$$

$$v' = h'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x}.$$

We form a new function y which is the product of the 2 functions u and v ; that is, $y = f(x) = uv$.

The derivative of y , by definition, is $y' = f'(x) = \lim_{\Delta x \rightarrow 0} \Delta y / \Delta x$.

To compute the derivative of a product of 2 functions we note first that u , v , and y are by definition all functions of the same independent variable x . Hence, as x increases from x to $x + \Delta x$, at the same time u changes to $u + \Delta u$, v to $v + \Delta v$, and y to $y + \Delta y$. None of the dependent variables, u , v , and y , are assumed to be constant. As $\Delta x \rightarrow 0$, also $\Delta y \rightarrow 0$, $\Delta u \rightarrow 0$, and $\Delta v \rightarrow 0$. This statement is made on the assumption that all the functions are well behaved in the intervals under consideration; it is possible to create functions for which the assertion is not true.

We have

$$\begin{aligned} y + \Delta y &= (u + \Delta u)(v + \Delta v) = uv + (\Delta u)v + (\Delta v)u + (\Delta u)(\Delta v), \\ y &= uv. \end{aligned}$$

Subtracting the members of the second equation from those of the first,

$$\Delta y = (\Delta u)v + (\Delta v)u + (\Delta u)(\Delta v).$$

To find the difference quotient $\Delta y / \Delta x$, we divide the last expression, term by term, by Δx , thereby obtaining

$$\frac{\Delta y}{\Delta x} = \frac{(\Delta u)v}{\Delta x} + \frac{(\Delta v)u}{\Delta x} + \frac{(\Delta u)(\Delta v)}{\Delta x}.$$

To compute the derivative, we let $\Delta x \rightarrow 0$. Moreover, we must use the rules already discussed for computing limits: in particular, the limit of a sum is the sum of the limits, and the limit of a product is the product of the limits. Consequently,

$$y' = f'(x) = \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \right) v + \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} \right) u + \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \right) \left[\lim_{\Delta x \rightarrow 0} (\Delta v) \right]$$

We note that $\lim_{\Delta x \rightarrow 0} (\Delta v) = 0$. Hence the last term in the above expression becomes 0, since one factor is 0 and the other is assumed to exist.

The limits appearing in the first 2 terms are derivatives, so we have as a result:

$$y' = f'(x) = u'v + v'u.$$

In words: *The derivative of the product of 2 functions is the derivative of the first factor times the second factor plus the derivative of the second factor times the first factor.*

EXAMPLE 2

To illustrate, consider the expression $y = f(x) = (x^2 + 5)(3x - 1)$. Here $u = x^2 + 5$ and $v = 3x - 1$. The derivatives are $u' = 2x$, $v' = 3$. Hence $y' = f'(x) = u'v + v'u = (2x)(3x - 1) + (3)(x^2 + 5) = 9x^2 - 2x + 15$.

EXERCISES 40

- Find the derivative of $y = (x - 6)(2x^2 + 2x + 1)$.
- (a) Find the derivative of the function $y = f(x) = (x^2 + 1)(2x^3 - 3x + 1)$.
(b) Find $f'(0)$; (c) $f'(-2)$; (d) $f'(4)$.
- Find the derivative of the function $y = f(x)$ that is the product of $(1 - 2x + x^2 + 4x^3)$ and $(2 - x - x^2)$.
- (a) Find the derivative of $y = f(x) = (x - 2)(3 - 4x)$. (b) Find $f'(0)$; (c) $f'(-4)$. (d) Find the x and y where the angle of inclination of the tangent is 45° ; (e) -60° ; (f) 0° .
- Find the derivative of $y = (x^3 - 4x^2 + 5x - 10)(x^4 - 7x^2 + 6)$.
- Use the formula of the derivative of a product to find the derivative of $y = x^2$. (HINT: $x^2 = x \cdot x$.)
- (a) Find the derivative of $y = f(x) = (x^3 - 2x^2 + 4x - 1)(-x^4 + 6x^3 - x^2 + 5x - 7)$. (b) Find $y'(1) = f'(1)$, and the corresponding angle of inclination. (c) Find $y'(-2) = f'(-2)$, and the corresponding angle of inclination.
- (a) Use the formula for the derivative of a product to find the derivative of $y = f(x) = 3x(1 - x)$. Is there a point on the curve where the slope of the tangent line is (b) -1 ; (c) 5 ; (d) 0 ? (e) Plot the curve and the tangent lines.
- Given the demand curve $p = (10 - 2D)(20 - D^2)$. (a) Find R ; (b) R' ; (c) $R'(4)$; (d) $R'(3)$; (e) $R'(1)$; (f) $R'(2.5)$.
- Given the demand curve $p = (1 - D)(2 - 3D)$. (a) Find R ; (b) R' ; (c) $R'(0.5)$; (d) $R'(0.3)$; (e) $R'(0.25)$. (f) Plot the demand curve and the marginal-revenue curve.

41. Derivative of a Quotient of Functions

First, we need an additional theorem about limits. Let $f(x)$ be such that $\lim_{x \rightarrow a} f(x) = L$. Also, take another function $g(x)$ such that $\lim_{x \rightarrow a} g(x) = M$. We form a new function which is their quotient; that is, $h(x) = f(x)/g(x)$

It is intuitively evident that

$$\lim_{x \rightarrow a} h(x) = \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}.$$

In words: *The limit of the quotient of two functions, if the denominator function is not zero, is the quotient of their limits.* The proof of this proposition goes beyond the scope of this book. We will simply illustrate it by an example.

■ EXAMPLE 1

Let $f(x) = 10 + 3x^2$; then $\lim_{x \rightarrow 0} f(x) = 10$. Also, take $g(x) = 5 - 3x$; evidently $\lim_{x \rightarrow 0} g(x) = 5$.

Now we form a new function which is the quotient of the two, namely,

$$h(x) = \frac{f(x)}{g(x)} = \frac{10 + 3x^2}{5 - 3x}.$$

It is readily observed that $\lim_{x \rightarrow 0} h(x) = 10/5 = 2$. This follows from the observation that, as $x \rightarrow 0$, the terms $3x^2$ in the numerator and $-3x$ in the denominator approach 0. Hence, the limit of the ratio $h(x)$ is the ratio of the limits, as we expected.

To find the derivative of the quotient of 2 functions we proceed as follows: Let $u = g(x)$ and $v = h(x)$ be functions of the independent variable x . A new function $f(x)$ is defined as their quotient; that is,

$$y = f(x) = \frac{u}{v}.$$

The derivatives of the 3 dependent variables u , v , and y with respect to the independent variable x are defined as follows:

$$u' = g'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x},$$

$$v' = h'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x},$$

$$y' = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

Note again that as x increases to $x + \Delta x$, the 3 dependent variables change also: u to $u + \Delta u$; v to $v + \Delta v$; y to $y + \Delta y$.

Hence we have

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}.$$

From this expression we subtract y in order to compute Δy ; there results

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}.$$

The common denominator is $v(v + \Delta v)$, so

$$\Delta y = \frac{(u + \Delta u)v - (v + \Delta v)u}{(v + \Delta v)v} = \frac{(\Delta u)v - (\Delta v)u}{(v + \Delta v)v}.$$

To find the difference quotient, we divide the latter expression in the numerator, term by term, by Δx ; that is,

$$\frac{\Delta y}{\Delta x} = \frac{(\Delta u/\Delta x)v - (\Delta v/\Delta x)u}{(v + \Delta v)v}.$$

To find the derivative, we let $\Delta x \rightarrow 0$. We recall the rules for limits, such as the limit of a difference is the difference of the limits, and the limit of a quotient is the quotient of the limits. We obtain

$$y' = f'(x) = \frac{\left(\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}\right)v - \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x}\right)u}{\lim_{\Delta x \rightarrow 0} (v + \Delta v)v}$$

We have noted previously that as $\Delta x \rightarrow 0$, also $\Delta v \rightarrow 0$. Hence

$$\lim_{\Delta x \rightarrow 0} (v + \Delta v) = v.$$

It follows that the derivative is

$$y' = f'(x) = \frac{u'v - v'u}{v^2}.$$

In words: *The derivative of a quotient of functions is the derivative of the numerator times the denominator minus the derivative of the denominator times the numerator, divided by the square of the denominator.*

EXAMPLE 2

This rule may be exemplified by the following problem:

$$y = f(x) = \frac{3x^2}{4x - 1}.$$

The numerator is $u = 3x^2$, and the denominator is $v = 4x - 1$. The derivatives are $u' = 6x$ and $v' = 4$. Hence the derivative of y is

$$y' = f'(x) = \frac{(6x)(4x - 1) - (4)(3x^2)}{(4x - 1)^2} = \frac{12x^2 - 6x}{(4x - 1)^2}.$$

● EXERCISES 41

✓1. Find the derivative of $y = \frac{6x^3}{2x + 1}$.

2. (a) Find the derivative of $y = f(x) = (2x^2 - 3x + 1)/(x^2 + 5x + 6)$.
 (b) Find $f'(0)$; (c) $f'(1)$; (d) $f'(-5)$.

3. Find the derivative of $y = (1 - x^2)/(x^2 + 2x)$.

4. (a) Find the derivative of $y = f(x) = (x + 1)/(3x - 2)$. (b) Find $f'(1)$;
 (c) $f'(4)$ and the corresponding angle of inclination of the tangent.

5. Consider the function $y = (5x + 1)/(2x + 1)$. Find the value of x and y where the angle of inclination is (a) 45° ; (b) 0° ; (c) 26° .

✓6. Find the derivative of $y = (ax + b)/(cx + d)$, where a, b, c, d are constants.

7. Take the demand curve $p = 10/(1 + 5D)$. (a) Find R ; (b) R' ; (c) $R'(1)$;
 (d) $R'(10)$. (e) Plot the demand curve and the marginal-revenue curve.

8. Consider the demand curve $p = 6/(2 + 3D)$. (a) Find R ; (b) R' ; (c) $R'(1)$;
 (d) $R'(4)$; (e) $R'(10)$. (f) Plot the demand curve and the marginal-revenue curve.

9. Consider the total-cost function as $C = 2D/(20 - 3D)$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(5)$; (d) Plot the total-cost curve and marginal-cost curve.

10. Let the total-cost curve for a commodity be $C = 5D/(25 - 5D^2)$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(2)$.

42. Derivative of a Function of a Function

A very useful formula gives the derivative of a function of a function. For instance, suppose we want to find the derivative of

$$y = f(x) = (x^3 - 3x^2 + 5)^4.$$

We can regard the expression in brackets, $x^3 - 3x^2 + 5$, as a new function of x , which will be designated by u . So

$$u = g(x) = x^3 - 3x + 5.$$

Then y can be written as

$$y = u^4.$$

The function y is said to be a function of a function. It is a function of u ; but u itself is a function of x .

The derivative of u with respect to x is

$$\frac{du}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}.$$

The derivative of y with respect to u is

$$\frac{dy}{du} = \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u}.$$

The derivative of y with respect to x is

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

It has been noted previously that as $\Delta x \rightarrow 0$, also $\Delta u \rightarrow 0$ and $\Delta y \rightarrow 0$. This follows from the fact that both y and u are functions of x .

Consider now the difference quotient $\Delta y/\Delta x$. This is a fraction. The numerator and denominator can be multiplied by the same quantity Δu without changing the value of the fraction. It is also permitted to rearrange the terms in the following order:

$$\frac{\Delta y}{\Delta x} = \left(\frac{\Delta y}{\Delta u}\right)\left(\frac{\Delta u}{\Delta x}\right).$$

The derivative of y with respect to x is computed by letting $\Delta x \rightarrow 0$; hence

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta u}\right)\left(\frac{\Delta u}{\Delta x}\right) = \left(\lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u}\right) \cdot \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}\right).$$

The justification of the expression on the right is that the limit of a product is the product of the limits. Since $\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$, we can replace Δx by Δu in the first limit.

The 2 limits in the last expression are, by definition, dy/du and du/dx . Hence

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

In words: *To find the derivative of y with respect to x when y is a function of u , which is itself a function of x , proceed as follows: Multiply the derivative of y with respect to u by the derivative of u with respect to x .*

■ EXAMPLE 1

Let us continue our consideration of

$$y = (x^3 - 3x + 5)^4.$$

We put

$$u = x^3 - 3x + 5,$$

and y becomes a function of u , namely,

$$y = u^4.$$

The derivative of y with respect to u is

$$\frac{dy}{du} = 4u^3.$$

The derivative of u with respect to x is

$$\frac{du}{dx} = 3x^2 - 3.$$

Hence the derivative of y with respect to x is the product

$$\frac{dy}{dx} = (4u^3)(3x^2 - 3) = 4(x^3 - 3x + 5)^3(3x^2 - 3).$$

The final result is achieved by substituting for u its expression in terms of x .

■ **EXAMPLE 2**

Let $y = (x^2 - 2x + 5)^3$. We have $u = x^2 - 2x + 5$ and $du/dx = 2x - 2$. In terms of u we have $y = u^3$ and $dy/du = 3u^2$. Hence, $dy/dx = (dy/du)(du/dx) = 3u^2(2x - 2) = 3(x^2 - 2x + 5)^2(2x - 2)$.

● **EXERCISES 42**

1. Find the derivative of $y = (3x - 2)^5$.
2. Find the derivative of $y = (2 - 3x + 4x^3)^5$.
3. Take as the total cost $C = (1 + 5D^2)^5$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(2)$; (d) $C'(3)$.
4. Take as the total cost $C = (1 + 3D)^3$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(25)$; (d) $C'(16)$.
5. Prove that the formula for the derivative of a power holds for fractional n . Let $u = x^{p/q}$, where p and q are positive integers. Hence, $n = p/q$ is a rational number. Let $y = u^q = x^p$. Differentiate this expression with respect to x , using the formula for the derivative of a function of a function. Simplify and prove that $du/dx = (p/q)x^{(p/q)-1} = nx^{n-1}$.
6. Use the result of Problem 5 to find the derivatives of (a) \sqrt{x} ; (b) $\sqrt[3]{x}$; (c) $\sqrt[4]{x^3}$; (d) $\sqrt[5]{x^2}$. (Note that $\sqrt{x} = x^{1/2}$, $\sqrt[3]{x} = x^{1/3}$, $\sqrt[4]{x^3} = x^{3/4}$, and so forth.)
7. Let the total cost curve be $C = \sqrt{1 + 2D}$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(1.5)$; (d) $C'(4)$.
8. Let the total-cost curve of a commodity be $C = 100 + 5\sqrt[3]{D}$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(8)$; (d) $C'(27)$.
9. Take as the demand curve $p = \sqrt{10 - 2D}$. (a) Find the total revenue R ; (b) R' ; (c) $R'(3)$; (d) $R'(4.5)$.
10. Consider the derivative of $y = x^{-n}$, where n is a positive constant. Prove that the general rule for the derivative of a power holds also for negative exponents. (HINT: Let $u = x^n$, so that $y = 1/u$. Differentiate, remembering that y is a function of x . Simplify and demonstrate that $dy/dx = (-n)x^{-n-1}$.)
11. Use the result of Problem 10 to find the derivative of each of the following functions: (a) $y = 1/x$; (b) $y = 1/x^7$; (c) $y = 10/(2x + 1)$; (d) $y = 24/\sqrt{x}$.
12. Given the demand for sugar in the United States 1915–1929 (Henry Schultz):

$p = 6,570,000/D^2$. (a) Find R and R' ; (b) $R'(60)$; (c) $R'(50)$; (d) $R'(5)$; (e) $R'(35)$; (f) $R'(30)$.

13. Given the demand for corn in the United States 1915–1929 (Henry Schultz): $p = 12,800/D^{1.2}$. (a) Find R and R' ; (b) $R'(20)$; (c) $R'(35)$; (d) $R'(45)$; (e) $R'(50)$; (f) $R'(30)$.

14. Given the demand for cotton in the United States 1915–1929 (Henry Schultz): $p = 0.756/D^{0.8}$. (a) Find R and R' ; (b) $R'(20)$; (c) $R'(15)$; (d) $R'(35)$; (e) $R'(50)$; (f) $R'(35)$.

15. Given the demand for wheat in the United States 1922–1934 (Henry Schultz): $p = 2,180/D^2$. (a) Find R and R' ; (b) $R'(6)$; (c) $R'(3.3)$; (d) $R'(6.5)$; (e) $R'(2.125)$; (f) $R'(3.33)$.

16. Given the demand for potatoes in the United States 1915–1929 (Henry Schultz): $p = 2,630/D^2$. (a) Find R and R' ; (b) $R'(3)$; (c) $R'(4.5)$; (d) $R'(3.25)$; (e) $R'(1.25)$; (f) $R'(4)$.

17. Consider the two functions $y = f(x) = x^2$ and $x = g(y) = \sqrt{y}$. (a) Show that g is the inverse of f . (b) Compute $dy/dx = f'(x)$ and $dx/dy = g'(y)$. (c) Show that $f'(4)g'(16) = 1$; (d) $f'(3)g'(9) = 1$; (e) $f'(10)g'(100) = 1$. (f) Show in general that $f'(x)g'(y) = 1$, if $y = f(x)$.

**18. Let $y = f(x)$ and $x = g(y)$ be inverse functions. Show that $f'(x) = 1/g'(y)$, by differentiating $g(y) = x$ on both sides with respect to x , and remembering that y is a function of x .

**19. Use the result of the previous example to find the derivative of $y = \sqrt{x}$ by differentiating $x = y^2$ with respect to y and expressing the reciprocal of this derivative in terms of x . Check by finding the derivative of $y = x^{1/2}$.

**20. Find the derivative of $y = \sqrt[3]{x}$ in the same way as in the previous example. Check by finding the derivative of $y = x^{1/3}$.

**21. Find the derivative of $y = \sqrt[n]{x}$ by using the method of Example 19. Check by finding the derivative of $y = x^{1/n}$.

DERIVATIVES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

43. The Number e .

Consider the following sequence:

$$u_1 = (1 + 1)^1 = 2,$$

$$u_2 = \left(1 + \frac{1}{2}\right)^2 = 1 + 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = 2 + \frac{1}{4},$$

$$u_3 = \left(1 + \frac{1}{3}\right)^3 = 1 + 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 = 2 + \frac{10}{27},$$

$$u_4 = \left(1 + \frac{1}{4}\right)^4 = 1 + 4\left(\frac{1}{4}\right) + 6\left(\frac{1}{4}\right)^2 + 4\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 = 2 + \frac{113}{256}.$$

It can be shown that all u_n , where $u_n = (1 + 1/n)^n$, if n is a positive integer, has the property that

$$2 < u_n < 3.$$

Hence we conjecture that the following limit exists:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

This limit is called e ; its value does not demand that n be integral. The limit can be evaluated by expanding u_n by the binomial theorem, and by

letting $n \rightarrow \infty$. This procedure gives

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

The number e obtained thereby is an irrational number; that is, it cannot be expressed as a ratio of any 2 integers. Its value to 5 decimal places is

$$e = 2.71828.$$

If $m = 1/n$, it is evident that as $n \rightarrow \infty$, $m \rightarrow 0$. Hence, since

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e,$$

it follows that

$$\lim_{m \rightarrow 0} (1 + m)^{1/m} = e.$$

The number e is used frequently in higher mathematics. It is taken as the base of the so-called system of natural logarithms, or logarithms to the base e .

● EXERCISES 43

1. Compute: (a) $(1 + 1/5)^5$; (b) $(1 + 1/10)^{10}$.
2. Use the expansion for e , namely,

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots$$

to compute an approximation to e using (a) 2 terms, (b) 4 terms, and (c) 6 terms.

44. Natural Logarithms

The logarithm of the number a taken to the base e is written $\log_e a$. This is also called a *natural logarithm*. We recall that the logarithm of a number b to the base 10 is written $\log b$. It should also be remembered that $\log 100 = 2$, since $100 = 10^2$; $\log 1,000 = 3$, since $1,000 = 10^3$; $\log 0.01 = -2$, since $0.01 = 1/100 = 10^{-2}$; and so on.

Natural logarithms, or logarithms to the base e , may be interpreted in a similar way; that is, $\log_e a$ is the exponent to which the base e has to be raised in order to get the number a .

■ EXAMPLE 1

Since $e^2 = 7.3891$, $\log_e 7.3891 = 2$; since $e^3 = 20.086$, $\log_e 20.086 = 3$; and since $1/e = e^{-1} = 0.367879$, $\log_e 0.367879 = -1$; and so on.

Logarithms to the base 10 can be converted into logarithms to the base e by a simple process. Let a be the natural logarithm and b the logarithm

to the base 10 of some number N ; that is, $\log_e N = a$ and $\log N = b$. Hence $N = e^a$ and $N = 10^b$. After taking the natural logarithm of the 2 members of the second equality we obtain

$$\log_e N = \log_e 10^b$$

or

$$\log_e N = b \log_e 10.$$

At this point it seems appropriate to call attention to Table 3 in the Appendix, namely, a table of 4-place natural logarithms, that is, logarithms to the base e . In this table will be found $\log_e 10 = 2.3026$. It then becomes apparent, if one recalls the definition of b , that

$$\log_e N = 2.3026 \log N.$$

Also, it follows readily that

$$\log N = 0.4343 \log_e N.$$

In conclusion, it has been shown that the natural logarithm of any number may be computed by multiplying the logarithm of this number to the base 10 by the conversion factor 2.3026. Of course, if such a table as Table 3 is available, natural logarithms may be obtained by direct reference to it. Both methods are employed in the examples that follow.

■ **EXAMPLE 2**

From Table 1, $\log 17 = 1.2304$. Hence the natural logarithm of 17 is

$$\log_e 17 = (1.2304)(2.3026) = 2.8331.$$

By reference to Table 3

$$\log_e 17 = \log_e (10)(1.7) = \log_e 10 + \log_e 1.7 = 2.3026 + 0.5306 = 2.8332.$$

■ **EXAMPLE 3**

$$\log 0.5 = 0.6990 - 1, \quad \text{or} \quad -0.3010.$$

Hence

$$\log_e 0.5 = (-0.3010)(2.3026) = -0.6931.$$

By reference to Table 3, noting the value of $\log_e 0.1$ at the bottom of the page,

$$\begin{aligned} \log_e 0.5 &= \log_e (5)(0.1) = \log_e 5 + \log_e 0.1 = 1.6094 + .6974 - 3 \\ &= 2.3068 - 3 = -0.6932. \end{aligned}$$

● **EXERCISES 44**

1. Find the logarithms to the base e of the following: (a) e ; (b) 1; (c) e^a ; (d) $\sqrt[3]{e}$; (e) $\sqrt[3]{e^2}$; (f) $1/e$; (g) $1/e^a$; (h) $1/e^2$; (i) $1/\sqrt[3]{e}$; (j) $1/\sqrt[3]{e^2}$; (k) $1/\sqrt[3]{e^3}$.
2. Using the formula for the conversion of logarithms to the base 10 to log-

arithms to the base e , compute the natural logarithms of the following: (a) 100; (b) 45; (c) 34.198; (d) 0.189; (e) 1; (f) 0.00001; (g) 0.02347; (h) 0.09872.

3. Check the results obtained in Problem 2 by using Table 3.

4. Find the numbers whose *natural* logarithms are (a) 0.6931; (b) 2.3026; (c) 1.7918; (d) 4.0943; (e) 5.0106; (f) 1.9315; (g) 1.2528. (HINT: Convert the natural logarithms into logarithms to the base 10.)

5. Check the results obtained in Problem 4 by using Table 3. Give especial attention to Parts (d) and (e).

45. Derivative of the Logarithmic Function

To find the derivative of the logarithmic function, we recall that

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{m \rightarrow 0} (1 + m)^{1/m}.$$

Let us start our analysis by taking

$$y = f(x) = \log_e x.$$

Then

$$y + \Delta y = \log_e (x + \Delta x).$$

Subtracting y from this expression provides a formula for Δy : hence

$$\Delta y = \log_e (x + \Delta x) - \log_e x.$$

We recall that the logarithm of a fraction is the difference of the logarithms. Hence we can write

$$\Delta y = \log_e \frac{x + \Delta x}{x} = \log_e \left(1 + \frac{\Delta x}{x}\right).$$

To find the difference quotient we divide by Δx ; this gives

$$\frac{\Delta y}{\Delta x} = \left(\frac{1}{\Delta x}\right) \log_e \left(1 + \frac{\Delta x}{x}\right).$$

An expression is not changed if it is multiplied and divided by the same quantity. So, multiply and divide the right member of the last equation by x ; this gives

$$\frac{\Delta y}{\Delta x} = \left(\frac{1}{x}\right) \left(\frac{x}{\Delta x}\right) \log_e \left(1 + \frac{\Delta x}{x}\right).$$

We remember that the logarithm of a power of a quantity is computed by multiplying the logarithm of the quantity by the power. Hence we can write

$$\frac{\Delta y}{\Delta x} = \left(\frac{1}{x}\right) \log_e \left(1 + \frac{\Delta x}{x}\right)^{(x/\Delta x)}$$

The derivative is computed by letting $\Delta x \rightarrow 0$; this gives

$$y' = f'(x) = \lim_{\Delta x \rightarrow 0} \left(\frac{1}{x}\right) \log_e \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}$$

Evidently as $\Delta x \rightarrow 0$, $(\Delta x/x) \rightarrow 0$, since x is taken as fixed. Putting $\Delta x/x = m$, we can write

$$y' = f'(x) = \lim_{m \rightarrow 0} \left(\frac{1}{x}\right) \log_e (1 + m)^{1/m}.$$

But it is already known that

$$\lim_{m \rightarrow 0} (1 + m)^{1/m} = e.$$

Consequently,

$$y' = f'(x) = \left(\frac{1}{x}\right) \log_e e = \frac{1}{x},$$

since $\log_e e = 1$.

Hence we have the rule: The derivative of the natural logarithm of x is the reciprocal of x .

■ **EXAMPLE**

Find the derivative of the function

$$y = f(x) = (5x)(\log_e x).$$

We use the formula for the derivative of a product; namely, $y' = u'v + v'u$. In this case, $u = 5x$, $v = \log_e x$. The derivatives are $u' = 5$, $v' = 1/x$. Hence

$$y' = f'(x) = (5)(\log_e x) + \left(\frac{1}{x}\right)(5x) = 5 \log_e x + 5.$$

When $x = 3$, this result yields

$$f'(3) = 5 \log_e 3 + 5 = 5(1.0986) + 5 = 10.4930.$$

● **EXERCISES 45**

1. Find the derivative of the function $y = 10 \log_e x - 5$. Evaluate $f'(1)$; $f'(4)$.
2. Find the derivative of $y = 2x - 3 \log_e x$. Evaluate $f'(2)$; $f'(10)$.
3. Find the derivative of $y = 2x (\log_e x)$. Evaluate $f'(1)$; $f'(e)$; $f'(12)$.
4. Find the derivative of $y = (2 \log_e x)/x^2$. Evaluate $f'(2)$; $f'(8)$.
- ✓ 5. Find the derivative of $y = (\log_e x)^4$. Evaluate $f'(1)$; $f'(e^2)$; $f'(9)$.
6. The demand curve is $p = 100 - 3 \log_e D$. (a) Find R ; (b) R' ; (c) $R'(1)$; (d) $R'(10)$; (e) $R'(2)$.
7. The demand curve is $p = 100/\log_e D$. (a) Find R ; (b) R' ; (c) $R'(5)$; (d) $R'(2)$.

8. The total-cost function is $C = \log_e D$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(2)$; (d) $C'(e^2)$.

9. The total cost function is $C = 10 + 5 \log_e D$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(5)$; (d) $C'(e^2)$.

10. The total cost function is $C = (10 + D) \log_e D$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(e^2)$; (d) $C'(4)$.

11. The total cost function is $C = (\log_e D)/(20 - 3D)$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(e)$; (d) $C'(5)$.

46. General Logarithmic Differentiation

We use the formula for the derivative of a function of a function to derive the general principle of logarithmic differentiation.

Let $y = f(x) = \log_e u$; where $u = g(x)$ is a function of x . The formula for the derivative of a function of a function is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

In our case $y = \log_e u$. Hence

$$\frac{dy}{du} = \frac{1}{u},$$

$$\frac{du}{dx} = u',$$

and, by multiplication,

$$\frac{dy}{dx} = \frac{u'}{u}.$$

This gives the following rule: *The derivative of the natural logarithm of an function of x is the derivative of the function with respect to x divided by the function itself.*

EXAMPLE 1

Assume that $y = \log_e (3 - 2x^3)$. Find the derivative of the function. We have, evidently,

$$u = 3 - 2x^3,$$

$$u' = -6x^2.$$

Hence

$$\frac{dy}{dx} = \frac{-6x^2}{3 - 2x^3}.$$

EXAMPLE 2

Let

$$y = \frac{(2x - 4)(x^2 + 3x + 1)}{(1 + 2x)^4}.$$

Let us take the logarithm to the base e of both sides; this gives

$$\log_e y = \log_e (2x - 4) + \log_e (x^2 + 3x + 1) - 4 \log_e (1 + 2x).$$

We use the above formula for the derivative of the logarithm of a function for all the terms appearing in this equality, and obtain

$$\frac{y'}{y} = \frac{2}{(2x - 4)} + \frac{2x + 3}{(x^2 + 3x + 1)} - \frac{(4)(2)}{(1 + 2x)}.$$

Multiplying the left and right members of this equation by

$$y = \frac{(2x - 4)(x^2 + 3x + 1)}{(1 + 2x)^4}$$

gives

$$y' = \frac{2(x^2 + 3x + 1)}{(1 + 2x)^4} + \frac{(2x + 3)(2x - 4)}{(1 + 2x)^4} - \frac{8(2x - 4)(x^2 + 3x + 1)}{(1 + 2x)^5}.$$

● **EXERCISES 46**

Use logarithmic differentiation to find the derivatives of the following functions:

1. $\log_e (1 - 4x)$.
2. $\log_e (1 + 1/x^2)$.
3. $\log_e (x^2 + 3x - 7)$.
4. $(2 + 4x)(1 + 5x^2)$.
5. $\frac{x^2 - 5x + 7}{5x - 6}$.
6. $(1 - 4x)^2(1 - 2x^2 + x^4)^6$.
7. $\frac{(1 + x)^2(1 - 2x)^3}{(x^2 - 5)^2}$.
8. $\frac{(x^2 - 4x^2 + 3x - 1)^5(x^2 - x)^8}{(x + 5)^5(2x^2 - 4x + 1)^3}$.

9. Total cost function: $C = 6 \log_e (1 + 3D)$. (a) Find the marginal-cost curve; (b) $C'(0)$; (c) $C'(1)$; (d) $C'(6)$.

47. Derivative of the Exponential Function

A special application of logarithmic differentiation pertains to the derivative of the exponential function

$$y = e^x.$$

Taking the logarithm to the base e of both members of the equation yields

$$\log_e y = x.$$

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After differentiating these two members with respect to x , there results

$$\frac{y'}{y} = 1.$$

Hence,

$$y' = y = e^x.$$

This gives the rule for the derivative of the exponential function e^x :
The derivative with respect to x of the exponential function e^x is equal to itself.

■ EXAMPLE 1

Find the derivative of the function

$$y = (1 + 3e^x)^5.$$

We use the principle of the derivative of a function of a function. Here $u = 1 + 3e^x$ and $y = u^5$. Hence

$$\frac{dy}{du} = 5u^4,$$

$$\frac{du}{dx} = 3e^x.$$

By multiplication,

$$\frac{dy}{dx} = (5u^4)(3e^x) = (15)(e^x)(1 + 3e^x)^4.$$

■ EXAMPLE 2

Find the derivative of the function

$$y = \frac{1}{3^x}.$$

This is most easily done by the use of logarithms. We have

$$\log_3 y = -x \log_3 3,$$

$$\frac{y'}{y} = -\log_3 3,$$

since on the right $(-\log_3 3)$ may be regarded as the constant coefficient of x . Thus

$$y' = -y(\log_3 3) = \frac{-\log_3 3}{3^x} = \frac{-1.0986}{3^x}.$$

● EXERCISES 47

Find the derivatives of the following functions:

- | | |
|--------------------------|----------------------------------|
| 1. e^{-3x} . | 5. $\frac{1 + 4e^x}{2 + 3e^x}$. |
| 2. e^{4x} . | 6. 2^x . |
| 3. $\frac{1}{4e^{6x}}$. | 7. 10^x . |
| 4. $(2 + 3e^x)^8$. | |

****8.** What is the derivative of $y = a^x$, where a is any positive number? (HINT: Take the natural logarithm of each member.)

9. Demand curve: $p = 100 - e^D$. (a) Find R ; (b) R' ; (c) $R'(1)$; (d) $R'(3)$.
10. Demand curve: $p = 10e^{-4D}$. (a) Find R ; (b) R' ; (c) $R'(0)$; (d) $R'(1)$; (e) $R'(4)$.
11. Total-cost curve: $C = 10e^{6D}$. (a) Find the marginal-cost curve; (b) $C'(1)$; (c) $C'(10)$; (d) $C'(7)$.

ECONOMIC APPLICATIONS OF THE DERIVATIVE

48. Elasticity

We define the elasticity of a function $y = f(x)$ with respect to x as

$$\frac{Ey}{Ex} = \lim_{\Delta x \rightarrow 0} \frac{(\Delta y/y)}{(\Delta x/x)} = \left(\frac{x}{y}\right) \left(\frac{dy}{dx}\right).$$

It is apparent that the elasticity is the limit of the ratio of the *relative* increment in y to the *relative* increment in x , as the increment of x tends to 0. It is a pure, dimensionless number, independent of the scale employed in the measurement of x and y . The elasticity of y with respect to x is obtained approximately as the per cent of increase or decrease in y that will follow if x is increased by 1 per cent. .

EXAMPLE

Let $y = 3x - 6$. (a) Find the elasticity. We have, by definition,

$$\frac{Ey}{Ex} = \left(\frac{x}{y}\right) \left(\frac{dy}{dx}\right) = \frac{3x}{3x - 6} = \frac{x}{x - 2}.$$

(b) Find the elasticity if $x = 10$. By substituting $x = 10$ into the formula, we have $10/8 = 5/4$. If x is increased by 1 per cent, y will increase by about $5/4$ per cent.

EXERCISES 48

1. $y = 1 + 2x - x^2$. (a) Find the elasticity of y with respect to x ; (b) for $x = 1$; (c) for $x = 0$; (d) for $x = 5$; (e) for $x = 10$.

2. $y = x^2 - 3x + 10$. (a) Find the elasticity Ey/Ex ; (b) if $x = 0$; (c) for $x = 10$; (d) for $x = 1$; (e) for $x = 5$.
3. $y = x^3$. (a) Find Ey/Ex ; (b) for $x = 1$; (c) for $x = 5$.
4. $y = (1 - 2x)/(2 + 3x)$. (a) Find Ey/Ex ; (b) for $x = 0$; (c) for $x = 2$; (d) for $x = 10$; (e) for $x = 3$.
5. $y = e^{5x}$. (a) Find Ey/Ex ; (b) for $x = 1$; (c) for $x = 0$; (d) for $x = 1.2$; (e) for $x = 5$; (f) for $x = 2$.
6. $y = 5 \log_e x$. (a) Find Ey/Ex ; (b) for $x = 10$; (c) for $x = e$; (d) for $x = 3$; (e) for $x = e^4$.
7. Find the elasticity of $y = a + bx$, where a and b are constants.
8. Find the elasticity of $y = ax^m$, where m is a constant.
- **9. Find the elasticity of $y = au$, where a is a constant and u is a function of x .
- **10. Find the elasticity of $y = au + b$, where a and b are constants and u is a function of x .
- **11. Find the elasticity of $y = u + v$, where u and v are functions of x .
- **12. Find the elasticity of $y = u - v$, where u and v are functions of x .
- **13. Find the elasticity of $y = au + bv$, where u and v are functions of x , and a and b are constants.
- **14. Find the elasticity of $y = uv$, where u and v are functions of x .
- **15. Find the elasticity of $y = u/v$, where u and v are functions of x .
- **16. The elasticity of $y = f(x)$ is also written

$$\frac{Ey}{Ex} = \left(\frac{dy}{dx}\right) \left(\frac{x}{y}\right) = \frac{\frac{d}{dx}(\log_e y)}{\frac{d}{dx}(\log_e x)}.$$

Justify this statement.

49. Elasticity of Demand

By definition,

$$\frac{ED}{Ep} = \left(\frac{p}{D}\right) \left(\frac{dD}{dp}\right) = \frac{p}{D} \left(\frac{1}{dp/dD}\right).$$

This follows from the fact that $dD/dp = 1/(dp/dD)$. See Problem 18, Exercises 42.

The elasticity of demand is the limit of the relative decrease in the quantity demanded, resulting from a relative increment in price, if the increment in price tends to 0. It is approximately the per cent of decrease in the quantity demanded resulting from an increase in price of 1 per cent.

EXAMPLE

Let the demand curve be $p = 1 - D$. (a) Find

$$\frac{ED}{Ep} = \frac{p}{D} \left(\frac{1}{dp/dD}\right).$$

Substituting, we have

$$\frac{ED}{Ep} = \frac{(1 - D)}{D} \left(\frac{1}{-1} \right) = \frac{1 - D}{-D}.$$

(b) Find the elasticity for $D = 1/4$. Substituting into the formula, we have $(1 - 1/4)/(-1/4) = -3$. This means that if $1/4$ unit is sold, an increment of 1 per cent in the price will decrease the amount sold by about 3 per cent.

● EXERCISES 49

1. Given the demand curve in Problem 6, Exercises 38. (a) Find the elasticity of demand; (b) for $D = 1$; (c) for $D = 2.5$; (d) for $D = 1.5$; (e) for $D = 0.5$.

2. Given the demand curve in Problem 9, Exercises 40. (a) Find the elasticity of demand; (b) for $D = 1$; (c) for $D = 4$; (d) for $D = 2.5$; (e) for $D = 1.2$.

*3. Demand curve: $Dp^a = b$, where a and b are constants. Prove that this curve has constant elasticity.

*4. Demand curve: $Dp = a$, where a is a constant (rectangular hyperbola). Prove that the elasticity of this curve is -1 .

5. Show that a demand curve that is a horizontal line $p = a$, where a is a constant, has an "infinite" elasticity. This is the demand curve for the product of a firm under perfect competition.

6. Show that a demand curve that is a vertical line $D = a$, where a is a constant, has an elasticity of 0.

7. Why is the elasticity of demand negative for most demand curves? (HINT: Consider the meaning of dp/dD in the definition of the elasticity of demand.)

8. Find the elasticity of the demand for sugar from the demand curve in Problem 8, Exercises 38, (a) for $D = 80$; (b) for $D = 100$; (c) for $D = 50$; (d) for $D = 45$.

9. Find the elasticity of the demand for cotton from Problem 9, Exercises 38, (a) for $D = 1$; (b) for $D = 5$; (c) for $D = 7$; (d) $D = 2$.

10. Find the elasticity of the demand for barley from Problem 10, Exercises 38, (a) for $D = 1$; (b) for $D = 0.80$; (c) for $D = 1.25$; (d) for $D = 1.05$; (e) $D = 2.35$.

11. Find the elasticity of the demand for rye from Problem 11, Exercises 38, (a) for $D = 0.50$; (b) for $D = 0.75$; (c) for $D = 0.4$; (d) for $D = 0.45$; (e) for $D = 0.35$.

12. Find the elasticity of the demand for buckwheat from Problem 12, Exercises 38, (a) for $D = 0.30$; (b) for $D = 0.25$; (c) for $D = 0.20$; (d) for $D = 0.10$.

13. Find the elasticity of the demand for sugar from Problem 12, Exercises 42.

14. Find the elasticity of the demand for cotton from Problem 14, Exercises 42.

15. Find the elasticity of the demand for wheat from Problem 15, Exercises 42.

16. Find the elasticity of the demand for potatoes from Problem 16, Exercises 42.

17. Find the elasticity of the demand for hay from Problem 13, Exercises 38, (a) for $D = 1$; (b) for $D = 0.80$; (c) for $D = 0.65$.

50. Marginal Revenue and Elasticity of Demand

The total revenue is defined as $R = p \cdot D$, where $p = f(D)$ is the demand curve. Hence, we have for the marginal revenue

$$R' = \left(\frac{dp}{dD} \right) D + p.$$

This follows from the formula for the derivative of a product. The elasticity of demand is, by definition,

$$\frac{ED}{Ep} = \frac{p}{D} \left(\frac{1}{dp/dD} \right).$$

From this last equation, we have

$$\frac{dp}{dD} = \frac{p}{D(ED/Ep)}.$$

After substituting this into the formula for marginal revenue, there results

$$R' = \frac{p}{ED/Ep} + p = p \left(1 + \frac{1}{ED/Ep} \right).$$

In words: *The marginal revenue is the price times 1 plus the reciprocal of the elasticity.*

EXAMPLE

Let the demand function be $p = 1 - D$. We derive from this

$$R = p \cdot D = D - D^2,$$

$$R' = 1 - 2D.$$

To show the interrelationship of our formulas, let us take

$$\frac{ED}{Ep} = \left(\frac{p}{D} \right) \left(\frac{1}{dp/dD} \right) = \left(\frac{1-D}{D} \right) \left(\frac{1}{-1} \right) = \frac{1-D}{-D}.$$

Using the formula derived previous to this example,

$$\begin{aligned} R' &= p \left(1 + \frac{1}{ED/Ep} \right) = (1-D) \left(1 - \frac{1}{(1-D)/D} \right) \\ &= (1-D) \left(\frac{1-2D}{1-D} \right) = 1-2D. \end{aligned}$$

This is the same formula for R' as obtained directly.

EXERCISES 50

1. Given the demand curve in Problem 7, Exercises 38, demonstrate the relationship between marginal revenue and elasticity of demand.

2. Given the demand curve in Problem 9, Exercises 40, show the relationship between the marginal revenue and the elasticity of demand.

3. Given the demand curve in Problem 8, Exercises 41, show the relationship between the elasticity of demand and the marginal revenue.

4. Given the demand curve in Problem 9, Exercises 42, show the relationship between elasticity of demand and marginal revenue.

5. Given the demand curve in Problem 6, Exercises 45, show the relationship between the elasticity of demand and the marginal revenue.

6. What is the elasticity of demand if $R' = 0$?

7. What is the marginal revenue for a demand curve which is a horizontal line and has "infinite" elasticity?

8. Show the relationship between the elasticity of demand and the marginal revenue for the demand for sugar in Problem 8, Exercises 38.

9. Show the relationship between the elasticity of demand and the marginal revenue for the demand for rye in Problem 11, Exercises 38.

10. Show the relationship between marginal revenue and elasticity of demand for the demand for corn from Problem 13, Exercises 42.

11. Show the relationship between the marginal revenue and the elasticity of demand for the demand for wheat in Problem 15, Exercises 42.

ADDITIONAL APPLICATIONS OF THE DERIVATIVE

51. Further Geometric Significance of the First Derivative

We recall the definition of the derivative of $y = f(x)$, namely,

$$y' = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

Let us take the increment of x , that is, Δx , to be a positive number. The derivative is positive, then, if Δy is positive. In this case the curve given by the equation $y = f(x)$ must be *rising* at the point (x, y) in question. Moreover, the curve $y = f(x)$ has, at this point, a tangent line with positive slope.

Now consider the case in which Δy is negative. This means that to a positive increment Δx corresponds a negative increment in y . Hence, at the point (x, y) , the derivative $f'(x)$ is negative. So the curve given by the equation $y = f(x)$ is falling, and the slope of the tangent line to the curve at the point (x, y) is negative (illustrated in Figure 21). The case where $\Delta y = 0$ will be dealt with in Section 55.

EXAMPLE 1

We present in Figure 21 the graph of the function $y = f(x) = x^2$. Thus $y' = f'(x) = 2x$. Consider the situation at the point $x = -2$, $y = 4$. It is determined that $f'(-2) = (2)(-2) = -4$. The curve is evidently falling as one moves to the right in the neighborhood of the point $(-2, 4)$. Moreover, the slope of the tangent line is negative.

Next consider the point $(1, 1)$. We have $f'(1) = (2)(1) = 2$. This is positive; hence the curve is rising as one moves to the right in the neighborhood of the point in question, and the slope of the tangent line is positive.

Consider the point where $x = 2$. We have $f'(2) = 4$. This is positive. Hence, at the point $x = 2, y = 4$, the function in question is increasing, and the slope of the tangent line is positive.

Consider the point where $x = -1$. We have $f'(-1) = -2$. Hence the derivative is negative at the point $x = -1, y = 1$. The curve is falling, since the slope of the tangent line at the point $(-1, 1)$ is negative.

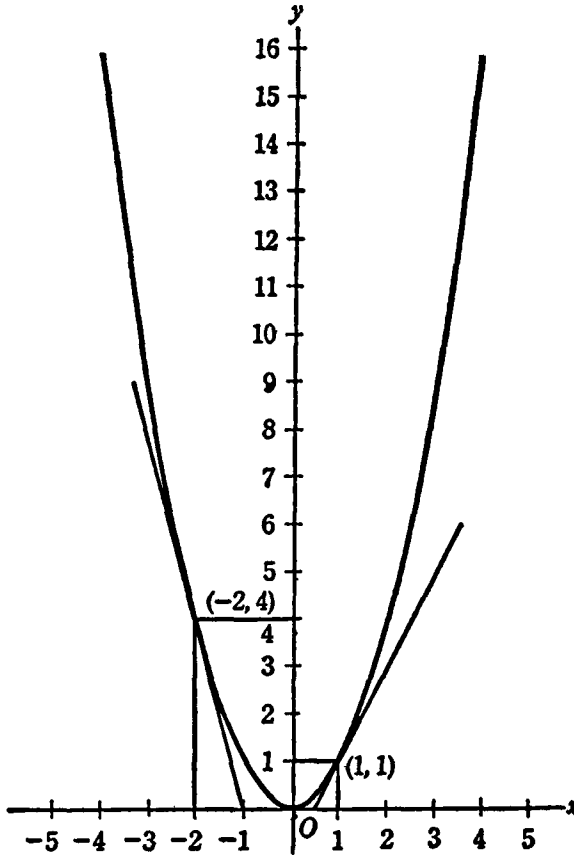


Figure 21

It is apparent that the derivative $y' = 2x$ is positive for $x > 0$ and negative for $x < 0$. Hence the curve in question will increase for all positive x , as one moves to the right, and will decrease for all negative x , as one moves to the right.

■ EXAMPLE 2

Is the function $y = 2x^2 - x + 4$ increasing or decreasing at the point where $x = 2$? We have $y' = f'(x) = 4x - 1$; $f'(2) = 4 \cdot 2 - 1 = 7$. Hence

the curve is increasing at the point $x = 2, y = 10$ as one moves to the right. The slope of the tangent line is positive at this point.

● EXERCISES 51

1. Consider the function $y = f(x) = 2x^3 - x^2$. Is this function increasing or decreasing at the point where (a) $x = 1$; (b) $x = -1$; (c) $x = 5$; (d) $x = -3$? (e) Make a graph of the function for a check.

2. Consider the function $y = f(x) = xe^x$. Can you find the values of x for which this function is always increasing? Where it is always decreasing? (HINT: Find the derivative, then factor, and remember that e^x is always positive.)

3. Where is the function $y = \log_e x$ increasing and where is it decreasing? Remember that $\log_e x$ is only defined for positive x .

4. Consider the behavior of the function $y = f(x) = x^2e^{-2x}$. Where is it increasing and where is it decreasing? (HINT: $e^{-x} = 1/e^x$.)

5. Let $y = a + bx$, where a and b are constant (straight line). Is this function increasing or decreasing?

6. Consider the behavior of the function $y = f(x) = x - x^3$. Where is it increasing and where decreasing?

7. Given the following *demand* functions, find the amounts sold for which the *total revenue* R increases and decreases: (a) $p = 1 - D$; (b) $p = 10 - 3D$; (c) $p = 20 - 5D$; (d) $p = 100 - 2D^2$; (e) $p = 10D - D^2$.

8. Given the demand function for sugar, Problem 8, Exercises 38. For what amounts sold will the total-revenue function be increasing and for what amounts will it be decreasing? Plot the total-revenue curve.

9. Consider the demand function for rye in Problem 11, Exercises 38. For what amounts sold will the total-revenue function be increasing and for what amounts will it be decreasing? Plot the total-revenue curve.

10. Consider the demand function for corn in Problem 13, Exercises 42. Determine for what amounts sold the total revenue is increasing and for what amounts it is decreasing. Check by plotting the total-revenue curve.

HIGHER DERIVATIVES

52. Higher Derivatives

We recall the definition of the first derivative, namely,

$$y' = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

In general, $y' = f'(x)$ will be another function of x which can be differentiated. The derivative of the first derivative gives what is called the second derivative; that is,

$$y'' = f''(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y'}{\Delta x}.$$

To find the third derivative we proceed to differentiate the second derivative, and obtain

$$y''' = f'''(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y''}{\Delta x}.$$

The fourth derivative y^{IV} is computed by finding the derivative of the third derivative. The fifth derivative y^V is computed by finding the derivative of the fourth derivative, and so on.

In general, to find the n th derivative of a function we differentiate the $(n - 1)$ th derivative.

EXAMPLE

Consider the function $y = f(x) = 2 - 3x + 4x^2$. Find the higher derivatives.

The first derivative is

$$y' = f'(x) = -3 + 8x.$$

The second derivative is the derivative of the first derivative

$$y'' = f''(x) = 8.$$

The third derivative is the derivative of the second derivative

$$y''' = f'''(x) = 0.$$

The fourth derivative is the derivative of the third derivative. This is 0, as are all higher derivatives.

To compute the function and its derivatives at the point $x = 2$, we have

$$f(2) = 12,$$

$$f'(2) = 13,$$

$$f''(2) = 8,$$

$$f'''(2) = 0,$$

$$f^{IV}(2) = 0,$$

and so on.

● EXERCISES 52

1. Find the first 3 derivatives of the following functions: (a) $y = 10x - 6$; (b) $y = 2 + 5x - x^2 + x^5$; (c) $y = xe^x$; (d) $y = (\log_e x)/x$.

2. Consider the function $y = e^x$. (a) Find the first 3 derivatives. (b) Try to establish a general rule for the m th derivative, $y^{(m)}$.

3. Consider the function $y = \log_e x$. (a) Find the first 4 derivatives. (b) Try to establish a general rule for the m th derivative, $y^{(m)}$.

4. Consider the function $y = x^a$, where a is a positive integer. (a) Find the first 3 derivatives. (b) Find the derivatives of order a , $y^{(a)}$. (c) Establish formulas for $y^{(a+1)}$ and $y^{(a+2)}$. (d) What conclusion can be drawn about the lowest order for which all future derivatives become 0? (e) Illustrate with the function $y = x^3$.

5. Find the general form of the derivatives of the function $y = 1/x^m$, where m is a positive integer. (HINT: Express the function first with negative exponent.)

6. Try to establish the general form for the n th derivative of $y = (a + bx)^m$, where a and b are constants and m is a positive integer. What are the values for $y^{(m)}$ and higher derivatives?

7. Try to establish the general form of the n th derivative of the function $y = e^{ax}$, where a is a constant.

8. Find all derivatives of the function $y = f(x) = x^4 - 3x^2 + 1$ which are not 0.

9. Find all derivatives of the function $y = f(x) = x^3 - 2x^7 + x^3 - 6$ which are not 0.

10. Find all derivatives of the function $y = f(x) = 5x^3 - 4x + 1$ which are not 0.

53. Geometric Interpretation of the Second Derivative

A geometric interpretation of the first derivative $y' = f'(x)$ has been given previously; if the first derivative is positive at a certain point, it means that the curve corresponding to the function in question is rising at that point. If it is negative, it means that the curve is falling. It should also be recalled that the first derivative $y' = f'(x)$ gives the slope of the line tangent to the curve at the point (x, y) .

We recall that the second derivative is the derivative of the first derivative. Hence a positive second derivative indicates that the first derivative is increasing. This means geometrically that at the point (x, y) the curve must be *concave upward* (see graph).

Consider next the case of a negative second derivative. This condition means that the first derivative, the slope of the line tangent to the curve at the point (x, y) , decreases. So the curve is *concave downward*.

These results may be summarized as follows:

If $y'' = f''(x) > 0$, then the curve is concave upward.

If $y'' = f''(x) < 0$, the curve is concave downward.

EXAMPLE 1

We show in Figure 22 a graph of the function $y = f(x) = x^3$. The first derivative, which gives the slope of the tangent, is $y' = f'(x) = 3x^2$. The second derivative, which indicates the rate of change of the slope of the tangent, becomes $y'' = f''(x) = 6x$.

This is positive for positive x and negative for negative x . Hence our curve is concave upward for positive x and concave downward for negative x . The change in the slope of the tangent lines is illustrated by the value of the slope at the point $x = -2, -1, 1$ and 2 :

$$f'(-2) = 12,$$

$$f'(-1) = 3,$$

$$f'(1) = 3,$$

$$f'(2) = 12.$$

It is seen that the slope of the tangent line decreases for negative x and increases for positive x .

EXAMPLE 2

Let $y = f(x) = 3x^4 - 2x^3$. The first derivative is $y' = f'(x) = 12x^3 - 4x$. The second derivative is the derivative of the first derivative; namely, $y'' = f''(x) = 36x^2 - 4$.

Is the curve in question concave upward or downward at the point $x = 0$? We have $f''(0) = -4 < 0$. This is negative. Hence the curve is concave downward at the point $(0, 0)$.

Is the curve concave upward or downward if $x = 1$? We have $f''(1) = 32 > 0$. This is positive. Hence the curve is concave upward at the point $(1, 8)$. For a general analysis of the situation we note that

$$f''(x) = 36x^2 - 4 = 4(9x^2 - 1).$$

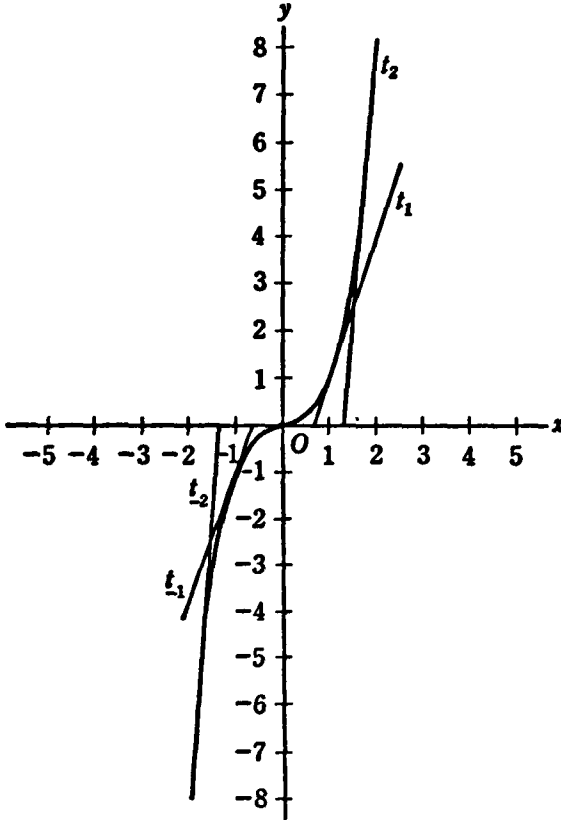


Figure 22

This will be positive for

$$9x^2 - 1 > 0,$$

$$9x^2 > 1,$$

$$x^2 > 1/9.$$

Hence the second derivative will be positive if $x^2 > 1/9$, that is, $x > 1/3$ or $x < -1/3$. Within the interval $-1/3 < x < 1/3$ the second derivative will be negative. So we conclude that the curve is concave downward in the interval $-1/3 < x < 1/3$; elsewhere it is concave upward.

● EXERCISES 53

1. Investigate the concavity of the graph of the function $y = f(x) = x^3 - 2x$ at the following points: (a) (1, -1); (b) (2, 4); (c) (-1, 1); (d) (-3, -21). (e) Can you make a general statement about the concavity of the curve? (f) Make a graph for a check.

2. Investigate the concavity of the graph of the function $y = 2x^2 + 3$ at the following points: (a) (0, 3); (b) (2, 11); (c) (-1, 5); (d) (-5, 53). (e) Can you make a general statement about the concavity of the curve? (f) Make a graph for a check.

3. Investigate the concavity of the curve corresponding to the equation $y = f(x) = 5 - 2x - x^2$ at the following points: (a) (0, 5); (b) (1, 2); (c) (4, 19); (d) (-1, 6); (e) (-3, 2). (f) Can you make a general statement about the concavity of the curve? (g) Graph the curve in order to check.

4. Investigate the concavity of the curve associated with the function $y = f(x) = x^4$ at the points: (a) (1, 1); (b) (2, 16); (c) (3, 81); (d) (5, 625); (e) (-1, 1); (f) (-2, 16); (g) (-3, 81). (h) Establish a general rule for the concavity of the graph in question. (i) Graph the curve.

5. Investigate the concavity of the curves corresponding to the following functions: (a) $y = e^x$; (b) $y = 1/e^x$; (c) $y = \log_e x$; (d) $y = x \cdot e^x$.

6. Investigate the concavity of the curve corresponding to the function $y = x^3 - 6x^2 + 5$. Check by graphing the curve.

7. Investigate the concavity of the curve corresponding to the function $y = x^3 - 2x^2 + 10$. Check by graphing the curve.

8. Determine the concavity of the curve corresponding to the function $y = 2x^4 - x^3 + 4x$. Check by making a graph.

54. Increasing and Decreasing Marginal Costs

We assume the existence of a total-cost function

$$C = f(D),$$

which possesses first and second derivatives. The marginal cost is, by definition, the first derivative

$$C' = f'(D).$$

The sign of the derivative of this marginal-cost curve gives information upon the question of whether there is increasing, decreasing, or constant marginal cost under varying conditions of production. If

$$C'' = f''(D) > 0,$$

we have increasing marginal cost. If

$$C'' = f''(D) < 0,$$

we have decreasing marginal cost. Finally, if

$$C'' = f''(D) = 0,$$

we have constant marginal cost.

■ EXAMPLE

To illustrate this, again consider Figure 19. The total-cost function is

$$C = f(D) = 10 + 15D - 6D^2 + D^3.$$

The marginal-cost curve is the derivative of C ,

$$C' = f'(D) = 15 - 12D + 3D^2.$$

The second derivative, which gives the criterion for increasing or decreasing marginal cost, is

$$C'' = f''(D) = -12 + 6D.$$

This function may be positive or negative. To find the point of constant marginal cost, we put it equal to 0, and obtain

$$f''(D) = -12 + 6D = 0.$$

Hence we have constant marginal cost if $D = 2$.

It is obvious that C'' is negative for D smaller than 2, and positive for D greater than 2. This means that there is decreasing marginal cost up to a production of 2 units and increasing marginal cost if more than 2 units are produced. At a production of exactly 2 units of the commodity we have constant marginal cost. Thus, the total-cost curve C is concave downward for quantities smaller than 2 and concave upward for quantities larger than 2.

● EXERCISES 54

1. Total cost: $C = 50,000D - 300D^2 + D^3$. (a) Find C' ; (b) C'' . (c) Investigate the nature of the marginal cost. (d) Make a graph of the total-cost curve and the marginal-cost curve.

2. Total cost: $C = 100 + 50D - 2D^2 + D^3$. (a) Find C' ; (b) C'' . (c) Investigate the nature of the marginal-cost curve. (d) Make a graph of the total-cost curve and marginal-cost curve.

3. Total cost: $C = 50 + 40D - 12D^2 + D^3$. (a) Find C' ; (b) C'' . (c) Investigate the nature of the marginal-cost curve.

4. Total cost: $C = 500D - 48D^2 + 2D^3$. (a) Find C' ; (b) C'' . (c) Investigate the nature of the marginal-cost curve.

5. Total cost: $C = 200 + 1,000D - 24D^2 + 4D^3 + D^4$. (a) Find C' ; (b) C'' . (c) Investigate the nature of the marginal-cost curve.

6. Investigate the nature of the marginal-cost curve corresponding to the total-cost curve in Problem 10, Exercises 45.

7. Investigate the nature of the marginal-cost curve of steel in Problem 8, Exercises 39. Graph the total-cost curve and marginal-cost curve.

8. Investigate the nature of the marginal-cost curve of hosiery in Problem 9, Exercises 39. Make a graph of the total-cost curve and marginal-cost curve.

9. Investigate the nature of the marginal-cost curve of leather belts in Problem 10, Exercises 39. Make a graph of the total-cost curve and marginal-cost curve.

10. Investigate the nature of the marginal-cost curve of department stores (Problem 11, Exercises 39). Make a graph of the total-cost curve and marginal-cost curve.

MAXIMA AND MINIMA IN ONE VARIABLE. INFLECTION POINTS

55. Maxima and Minima of Functions of One Variable

We shall give a purely geometric argument for the use of the derivative in the study of maximum and minimum points on a curve. Roughly speaking, a maximum point on a curve is a point that is higher than its neighbors; a minimum point is lower than its neighbors. In our consideration we will arbitrarily eliminate those maximum or minimum points that result from cusps; that is, we will consider only those cases of maximum or minimum points where the tangent line is horizontal (note Figure 23). But a horizontal tangent need not necessarily imply a maximum or minimum (for instance, look back at Figure 22). Hence the existence of a horizontal tangent line is a necessary, but not a sufficient, condition for a maximum or minimum if cusps are eliminated from the consideration.

We see from the graph that at a maximum point the curve is concave downward. Also, from the graph, it appears that at a minimum point the curve is concave upward.

A horizontal tangent line implies that the slope of the line tangent to the curve at the point in question must be 0. The slope is given by the first derivative of the curve whose equation is $y = f(x)$. So the necessary condition for a maximum or minimum is $y' = f'(x) = 0$.

For a maximum we must have concavity downward. The concavity of a curve is indicated by the sign of its second derivative; the curve will be concave downward if $y'' = f''(x) < 0$.

For a minimum we must have concavity upward. This implies that $y'' = f''(x) > 0$.

The rules for finding maxima and minima can be summarized in the following way:

For a maximum,

$$f'(x) = 0, \quad f''(x) < 0.$$

For a minimum,

$$f'(x) = 0, \quad f''(x) > 0.$$

There are very special functions such that $f''(x) = 0$ when $f'(x) = 0$; further analysis must be employed under such a circumstance to determine whether the graph might have a maximum or a minimum point.

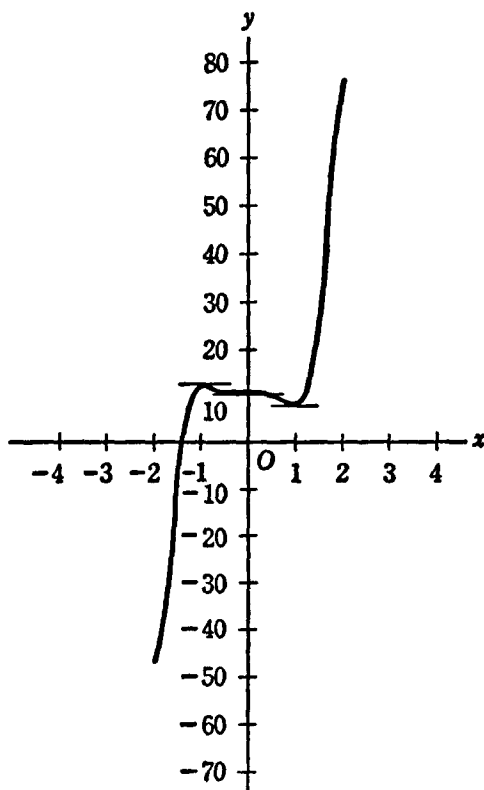


Figure 23

■ EXAMPLE 1

We show in Figure 23 a graph of the function

$$y = f(x) = 3x^5 - 5x^3 + 10.$$

The derivative of this function is

$$y' = f'(x) = 15x^4 - 15x^2.$$

The necessary condition for a maximum or minimum demands that $y' = 0$. We can factor the derivative, and write

$$y' = f'(x) = 15x^2(x - 1)(x + 1) = 0.$$

This has 3 solutions: $x = -1$, $x = 0$, $x = 1$. The graph shows that at all 3 points there are horizontal tangent lines, as one would expect. To ascertain which one of these points is a maximum or a minimum depends upon the sign of the second derivative

$$y'' = f''(x) = 60x^3 - 30x.$$

We have at $x = -1$, $f''(-1) = -30$. This is negative. Hence there is a maximum at the point $x = -1$. The value of the function at $x = -1$ is $f(-1) = 12$.

Next, $f''(0) = 0$. Thus the criteria already established are not valid. However, an examination of the curve indicates that we have neither a maximum nor a minimum. We will see later that we have an inflection point.

Finally, $f''(1) = 30$. This is positive. Hence we have a minimum at $x = 1$. The value of the function is $f(1) = 8$.

EXAMPLE 2

Let us consider the function $y = f(x) = 3x^5 - 10x^3$. The derivative is $y' = f'(x) = 15x^4 - 30x^2$. The second derivative is $y'' = f''(x) = 60x^3 - 60x$. To find the maxima and minima, we put $y' = f'(x) = 0$.

$$\begin{aligned} f'(x) &= 15x^4 - 30x^2 \\ &= (15x^2)(x^2 - 2) = (15x^2)(x + \sqrt{2})(x - \sqrt{2}) = 0. \end{aligned}$$

Hence we have 3 "critical values" for x , namely, $x = 0$, $\sqrt{2} = 1.414$, and $-\sqrt{2} = -1.414$. To find which are maxima and which minima, we substitute these values into the formula for the second derivative, and obtain

$$\begin{aligned} f''(0) &= 0, \\ f''(\sqrt{2}) &= f''(1.414) = 84.84 > 0, \\ f''(-\sqrt{2}) &= f''(-1.414) = -84.84 < 0. \end{aligned}$$

No conclusion can be drawn when $x = 0$; however, it happens to be neither a maximum nor a minimum point. We have a minimum at the point $x = \sqrt{2}$, and a maximum at the point $x = -\sqrt{2}$.

■ EXAMPLE 3

Consider the curve $y = f(x) = x^3 - 3x + 1$. We have $y' = f'(x) = 3x^2 - 3$ and $y'' = f''(x) = 6x$. For a maximum or a minimum we must have $y' = 3x^2 - 3 = 0$. Hence the so-called critical points are $x = 1$ and $x = -1$. When $x = 1$, $f''(1) = 6 > 0$. Thus at the point $x = 1$, $y = -1$ we have a minimum. For $x = -1$, $f''(-1) = -6 < 0$. Therefore at the point $x = -1$, $y = 3$ we have a maximum.

● EXERCISES 55

- Investigate the function $y = 2x^3 + 3x^2 - 2x + 1$ for maxima and minima. Make a graph.
- Investigate the function $y = 10x^3 - 15x^2 + 10$ for maxima and minima. Plot the function.
- Investigate the function $y = 1 + 2x - x^3$ for maxima and minima. Make a graph.
- Investigate the function $y = x^2 - 3x + 1$ for maxima and minima. Make a graph.
- Investigate the function $y = 3x^4 + 16x^3 + 18x^2 + 20$ for maxima and minima. Make a graph.
- Investigate the function $y = xe^{-x}$ for maxima and minima. Make a graph.
- Investigate the function $y = x \log_e x$ for maxima and minima.
- Investigate the function $y = e^{-x^2}$ for maxima and minima.
- Investigate the function $y = (x^5/5) - (13x^3/3) + 36x - 9$ for maxima and minima. Make a graph.
- Investigate the function $y = x^2e^x$ for maxima and minima.

56. Monopoly

The profit of the monopolist is $\pi = R - C$, where the total revenue $R = pD$. It is recalled that p is the demand function and C is the total-cost function. The monopolist tries to maximize the profit by producing the amount and charging the price that will make his profit as large as possible.

The necessary condition for a maximum is $\pi' = R' - C' = 0$, or $R' = C'$; that is, marginal revenue equals marginal cost. For maximum profit, $\pi'' < 0$, or $R'' - C'' < 0$, from which we obtain $R'' < C''$. The last condition assures the *stability* of the situation. There is no incentive to the monopolist to produce more or less or charge a different price. We have a maximum rather than a minimum.

■ EXAMPLE

We show in Figure 24 the demand function

$$p = 10 - 3D$$

and the average-cost curve

$$A = D.$$

From this we derive the total-revenue function

$$R = pD = 10D - 3D^2$$

and the marginal-revenue function R' shown in the graph

$$R' = 10 - 6D.$$

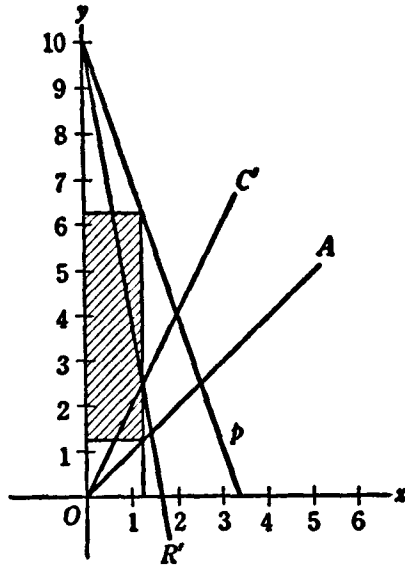


Figure 24

The total-cost curve is

$$C = AD = D^2$$

and from this we obtain the marginal-cost curve

$$C' = 2D,$$

which is shown in the graph.

Total profit is given by

$$\pi = R - C = (10D - 3D^2) - D^2 = 10D - 4D^2.$$

For a maximum, the first derivative π' must be 0 and the second derivative π'' must be negative. So

$$\pi' = 10 - 8D = 0.$$

This gives the solution $D = 10/8 = 5/4$. The second derivative is

$$\pi'' = -8 < 0.$$

This is negative. Hence we have maximum profit when $D = 5/4$ units are produced.

The price is given by inserting $D = 5/4$ in the demand function; that is, $p = 10 - (3)(5/4) = 25/4$.

The average cost is

$$A = \frac{5}{4}.$$

Total revenue is

$$R = pD = \left(\frac{25}{4}\right)\left(\frac{5}{4}\right) = \frac{125}{16}.$$

Total cost is

$$C = AD = \left(\frac{5}{4}\right)\left(\frac{5}{4}\right) = \frac{25}{16}.$$

Hence the maximum profit itself is

$$\pi = R - C = \frac{125}{16} - \frac{25}{16} = \frac{100}{16} = \frac{25}{4}.$$

This is indicated by the shaded area in the graph.

Alternatively, we could have found maximum profit by making marginal revenue equal to marginal cost; thus

$$10 - 6D = 2D.$$

The solution of this equation is again $D = 5/4$. It is seen from the graph that the intersection of the marginal-revenue curve and the marginal-cost curve determines *the quantity* produced.

The second derivatives of the total-revenue function and total-cost function are

$$R'' = -6; \quad C'' = 2.$$

It is evident that $-6 < 2$. Hence the equilibrium is *stable* and we have a true maximum of profits.

● EXERCISES 56

1. Let the demand curve of a commodity be $p = 10 - 5D$. The average cost is $A = 3$. (a) Find C ; (b) C' ; (c) R ; (d) R' ; (e) π . (f) Find the necessary condition for a maximum of profits. (g) Find the sufficient condition for a maximum. (h) Make a graph of p , A , R , C . (i) Show how the intersection of R' and C' determines the quantity produced. (j) Find p , R , C , π under conditions of maximum profit and interpret them in the graph.

2. The demand curve of a commodity is $p = 300 - 2D$ and the average-cost curve is $A = D$. (a) Find R ; (b) C ; (c) R' ; (d) C' ; (e) π . (f) Determine the amount produced which will maximize profits. (g) Show that at this point $R' = C'$. (h) Show

the stability of the position. (i) Show that $R'' < C''$. (j) Determine price, total revenue, total cost, and profit of the monopolist.

3. The demand curve for some commodity is $p = 12 - 3D$. The average cost curve is $A = 2D^2$. (a) Find the necessary condition for a maximum of profits; (b) the sufficient condition. (c) Find the price, total revenue, total cost, and profit for the maximum. (d) Make a graph of p , A , R' , and C' . Show how the quantity produced, the price, the total revenue, total cost, and profit can be determined graphically.

4. The demand curve for a commodity is $p = 36 - 2D^2$. The average cost curve is $A = 15D$. (a) Find the necessary conditions for a maximum of profits. (b) Show the stability of equilibrium. (c) Find the quantity produced, the price, total revenue, total cost, and profit for the maximum. (d) Demonstrate that $R' = C'$ and $R'' < C''$ at the point of equilibrium.

5. The demand curve for a commodity is $p = 10e^{-2D}$. The average cost curve is $A = 1/D$. (a) Find the necessary conditions for a maximum of profits. (b) Show that the equilibrium is stable.

6. The demand curve for a commodity is $p = -\log_e D$. The average cost is 0. (a) Find the necessary condition for maximum profits. (b) Show that the equilibrium is stable.

7. The demand curve for a commodity is $p = 50e^{-3D}$. The average cost is 0. (a) Find the necessary conditions for a maximum of profits. (b) Determine the sufficient conditions.

8. The demand for a commodity is $p = m - nD$. The average-cost curve is $A = r + sD$, where m , n , r and s are positive constants. (a) Find R , C , π . (b) Determine the necessary condition for maximum profit; (c) the sufficient condition. (d) Find price, total revenue, total cost, and profit for the maximum position.

9. The demand for sugar in the United States is estimated as $p = 2.34 - 1.34D$ (Henry Schultz). Assume that the average-cost curve of sugar is $A = 1/D - 0.83 + 0.85D$. (a) Find R , C , π . (b) Determine the necessary conditions for a maximum of profits, assuming a sugar monopoly. (c) Show that the equilibrium is stable. (d) Find price, total revenue, total cost, and profits under monopoly. (e) Assume the same demand curve and a supply curve on the competitive market: $p = 1.7D - 0.83$. Establish the competitive equilibrium on the sugar market. Determine the quantity produced and the price, and compare with the monopoly solution. (f) Make a graphical comparison by plotting on the same graph the demand curve, the average-cost curve, the marginal-cost curve (same as the supply curve), and marginal-revenue curve. Demonstrate the price formation under monopoly and under free competition.

10. The demand for steel in the United States is estimated to be $p = 250 - 50D$ (Whitmann). The estimated average cost of making steel is $A = 182/D + 56$ (Yntema). (a) Find R , C , C' , R' . (b) Find the necessary condition for a maximum of steel profits assuming a monopoly in steel. (c) Find the sufficient condition of maximum profits. (d) Establish the quantity produced, price, total revenue, total cost, and profits under monopoly. (e) Assume the same demand curve and a competitive-supply curve (marginal-cost curve) for steel, $p = 56$, assuming the same cost curve under free competition and monopoly. Find the quantity produced and the price

established under conditions of free competition. (f) Plot the demand curve, average-cost curve, marginal-cost curve (same as supply curve), marginal-revenue curve. Demonstrate the price formation under monopoly and under free competition.

57. Average and Marginal Cost

We denote by C the total cost, by D the quantity produced, by A the average cost. We have, by definition, $A = C/D$.

Using the formula for differentiation of a quotient we get as a necessary condition for minimum average cost

$$\frac{dA}{dD} = A' = \frac{DC' - C}{D^2} = 0.$$

Hence $DC' - C = 0$ and $C' = C/D = A$. It follows that for minimum average cost, the average cost is equal to marginal cost.

■ EXAMPLE

We start with the total-cost curve

$$C = 15D - 6D^2 + D^3.$$

We derive the equation of the average-cost curve by writing

$$A = \frac{C}{D} = 15 - 6D + D^2.$$

The marginal-cost curve has the equation

$$C' = \frac{dC}{dD} = 15 - 12D + 3D^2.$$

Both the average-cost curve and the marginal-cost curve are shown in Figure 25.

For the minimum, average cost A we must have

$$A' = 0, \quad A'' > 0.$$

Thus

$$A' = -6 + 2D = 0,$$

which yields $D = 3$. Further,

$$A'' = 2 > 0.$$

This is positive; hence there is a minimum rather than a maximum when $D = 3$.

The value of the minimum average cost, that is, A when $D = 3$, is

$$A = 15 - (6)(3) + (3)^2 = 6.$$

The marginal cost for $D = 3$ is

$$C' = 15 - (12)(3) + (3)(3)^2 = 6.$$

Hence marginal and average costs are equal (both are 6), as was expected, at the point of minimum average cost ($D = 3$).

We see from the graph that the marginal-cost curve C' cuts the average-cost curve A at its minimum value. It also appears that the marginal-cost curve cuts the average-cost curve from below.

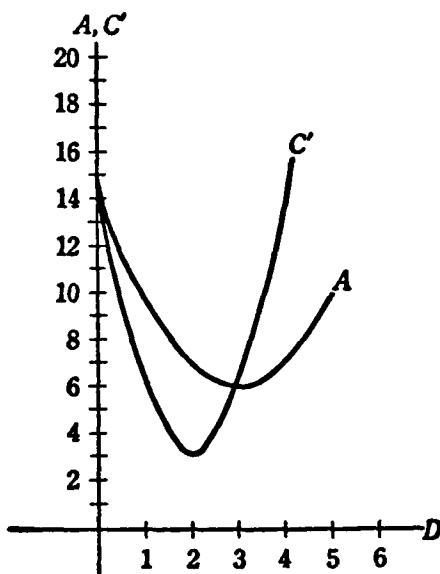


Figure 25

● EXERCISES 57

1. Let $A = 20 - 6D + D^2$. (a) Find the minimum of A . (b) Show that for the minimum $A = C'$. (c) Make a graph of A and C' . Verify that they are equal for the minimum of A .

2. Let $A = 50 - 8D + D^2$. (a) Find the minimum of A . (b) Show that at the minimum $A = C'$. (c) Make a graph of A and C' and verify the previous relationship.

3. Let the average-cost curve be $A = 10 - 4D^3 + 3D^4$. (a) Find the minimum of A . (b) Verify that at the minimum $A = C'$.

4. The average-cost curve of a commodity is $A = 1 + 120D^3 - 6D^2$. (a) Find the minimum of A . (b) Verify that at the minimum $A = C'$.

5. The average-cost curve of a commodity is $A = e^D + e^{-D}$. (a) Find the minimum of A . (b) Verify that at the minimum $A = C'$.

6. The average-cost curve is $A = m - nD + kD^2$, where m , n , and k are positive constants. (a) Find the minimum of A . (b) Verify that at the minimum $A = C'$.

7. Let the average-cost curve be $A = 1 + D \log_e D$. (a) Determine the minimum of A . (b) Show that, at the point of minimum of average cost, average cost equals marginal cost.

8. The total-cost function is $C = 2D - 2D^2 + D^3$. (a) Find the average-cost function A . (b) Establish the minimum of average cost. (c) Find the marginal-cost function C' . (d) Show that, at the minimum of average cost, average cost is equal to marginal cost. (e) Plot the average-cost and marginal-cost functions.

9. The total-cost function is $C = 5D - 3D^2 + 2D^3$. (a) Find the average-cost function A . (b) Establish the minimum of the average cost. (c) Find the marginal-cost function C' . (d) Show that, at the minimum of average cost, average cost is equal to marginal cost. (e) Plot the average-cost and marginal-cost curves.

10. The total-cost function is $C = 4D - D^2 + 2D^3$. (a) Find the average-cost function A . (b) Establish the minimum of the average-cost function. (c) Find the marginal-cost function C' . (d) Show that, at the minimum of average cost, average cost is equal to marginal cost. (e) Plot the average-cost and marginal-cost curves.

58. Points of Inflection

We will present a purely geometrical argument for the criteria employed in locating points of inflection. By definition, a point of inflection exists at a point where the concavity of the curve changes. The curve changes either from concave upward to concave downward or from concave downward to concave upward.

When we were discussing the geometric interpretation of second derivatives, we showed that the concavity of a curve is given by considering the sign of the second derivative of the function corresponding to the curve. If the second derivative $y'' = f''(x)$ is positive, the curve is concave upward. If the second derivative of the function is negative, the curve is concave downward.

It is obvious that, at the point of change of the concavity of the curve, the second derivative has in general to change from positive to negative or from negative to positive. Hence at the point of change the second derivative must, in general, be 0. There are some special functions, rarely encountered in this field, where the argument just given is not appropriate.

The following rule can be derived from these considerations:

In general, if $f''(x) = 0$, there is an inflection point on the curve.

One of the exceptions to this will be discussed below in Problem 5, Exercises 58.

EXAMPLE

We present in Figure 26 the graph of the function

$$y = f(x) = x^3 - 2x^2.$$

We want to examine the function for inflection points.

The first derivative is

$$y' = f'(x) = 3x^2 - 4x.$$

The second derivative must vanish if there is an inflection point; so let us take

$$y'' = f''(x) = 6x - 4 = 0.$$

The solution of this equation is

$$x = 2/3.$$

Hence we expect an inflection point at the point $(2/3, -16/27)$ on the curve.

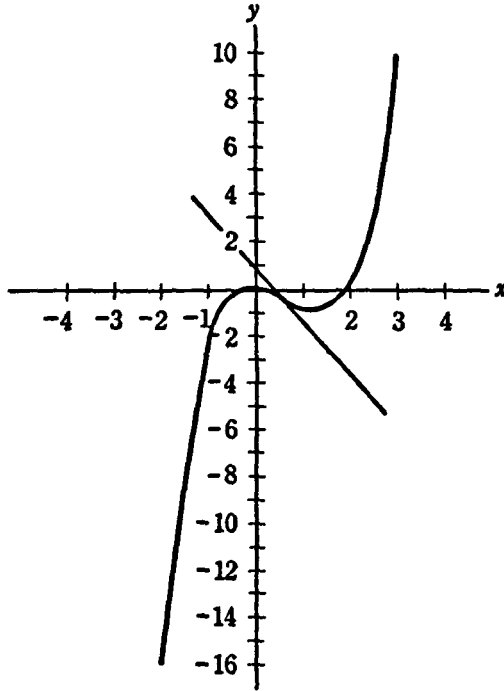


Figure 26

The slope of the tangent line at the inflection or turning point is computed by inserting $x = 2/3$ into the formula for the first derivative; this gives

$$f'(2/3) = (3)(2/3)^2 - (4)(2/3) = -4/3.$$

● EXERCISES 58

1. Consider the function $y = f(x) = 2x^3 - x^2$ for concavity. (b) Is there an inflection point? (c) What is the slope of the tangent line at the point of inflection? (d) Make a graph of the function.
2. Consider the function $y = e^x$ for concavity. Is there an inflection point?
3. Consider the concavity of the function $y = 2 + 3x - x^2$. Has this curve an inflection point? Make a graph.

4. Consider the concavity of the function $y = 3 + 5x + 4x^2$. Has this curve an inflection point?

***5.** Consider the function $y = x^4$. Graph. Compute y' , y'' . Show that there is *no* inflection point at $x = 0$ in spite of the fact that $y'' = 0$ for this value. Compare the behavior of this function with that of $y = x^5$. Make graphs of both functions.

6. Find the total-cost curve for Problem 1, Exercises 57. (a) Find the point of inflection of the total-cost curve C . (b) Show that the inflection point of the total-cost curve is the point of minimum marginal cost. (c) Make a graph of the total-cost and marginal-cost curves and consider the relationship between the concavity of the total-cost curve and the phenomenon of increasing and decreasing marginal cost.

7. Find the total-cost function for the case described in Problem 2, Exercises 57, and proceed as in Problem 6.

8. Consider the total-cost function corresponding to the average-cost function given in Problem 3, Exercises 57. Proceed as in Problem 6.

****9.** Show that, in general, for any total-cost function $C = f(D)$ the point of inflection corresponds to the *minimum of the marginal-cost curve* $C' = f'(D)$.

DERIVATIVES OF FUNCTIONS OF SEVERAL VARIABLES

59. Functions of Several Independent Variables

Up to now we have considered only functions of 1 (independent) variable. But in economics we have frequently a given quantity depending not just upon 1 but on several independent variables, for instance, the demand for beef depends on the price of beef, the price of pork, and the price of mutton.

If z is a function of 2 variables x and y , we write

$$z = f(x,y).$$

This means that for every permissible *pair* of values of the independent variables x and y we can find at least 1 number z which corresponds to this pair.

■ EXAMPLE 1

Let us define a function of two variables by the formula,

$$z = f(x,y) = x^2 - 2y + 10.$$

We get, for instance, for $x = 1, y = 3$,

$$f(1,3) = (1)^2 - 2(3) + 10 = 5.$$

Similarly, by substituting other values for x and y ,

$$f(-4,2) = (-4)^2 - 2(2) + 10 = 22,$$

$$f(2,-5) = (2)^2 - 2(-5) + 10 = 24,$$

$$f(0,3) = (0)^2 - 2(3) + 10 = 4,$$

$$f(3,0) = (3)^2 - 2(0) + 10 = 19.$$

If u is a function of 3 variables x , y , and z , we write

$$u = f(x, y, z).$$

By assigning specific values to the 3 independent variables x , y , and z , we can compute the number u which is associated with them.

■ **EXAMPLE 2**

Define, for instance,

$$u = f(x, y, z) = xy - 4z.$$

Substitute first $x = 1$, $y = 3$, $z = 10$, we have

$$f(1, 3, 10) = (1)(3) - 4(10) = -37.$$

By substituting $x = -5$, $y = 2$, $z = -1$ we have

$$f(-5, 2, -1) = (-5)(2) - 4(-1) = -6,$$

and so on.

● **EXERCISES 59**

1. Given the function $z = f(x, y) = (x/2)(x + y)$. Find (a) $f(0, 1)$; (b) $f(2, 4)$; (c) $f(1, 2)$; (d) $f(-1, -5)$.

2. Given the function $z = f(x, y) = x^2 + 2xy + 5y^2 - x + 3y + 6$. Find (a) $f(0, 0)$; (b) $f(1, 0)$; (c) $f(0, -1)$; (d) $f(-2, -3)$.

3. Given the function $u = f(x, y, z) = 2x + 3y + 5z + 1$. Find (a) $f(0, 1, -2)$; (b) $f(1, 1, 1)$; (c) $f(5, 2, -4)$; (d) $f(a, -a, -a)$.

4. Given the function $z = f(x, y) = xe^y$. Find (a) $f(0, 0)$; (b) $f(2, 0)$; (c) $f(1, 1)$; (d) $f(-1, -1)$; (e) $f(3, -2)$.

5. Given the function $u = f(x, y, z) = xy/z$. Find (a) $f(1, 1, 1)$; (b) $f(0, 1, 2)$; (c) $f(a, a - a)$.

6. Let $w = f(x, y, z, u) = (x + y)/(z + u)$. Find (a) $f(0, 1, 1, 2)$; (b) $f(-1, 0, 0, 1)$; (c) $f(3, 5, -1, 6)$.

7. Given the function $z = f(x, y) = x^2 - xy + y^2 - 4x$. Find (a) $f(0, 0)$; (b) $f(2, 5)$; (c) $f(-a, a)$.

8. Given the function $z = f(x, y) = x/y$. Find (a) $f(a, a)$; (b) $f(a, -a)$; (c) $f(-a, -a)$.

9. Let $u = f(x, y, z) = x^2 - 2xy + xz - z^2$. Find (a) $f(1, 1, 0)$; (b) $f(-3, -6, 1)$; (c) $f(a, a, a)$.

60. Partial Derivatives

Let us consider a function of two independent variables

$$z = f(x, y).$$

The partial derivative of z with respect to x is defined as

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

Other symbols used for the partial derivative are

$$\frac{\partial z}{\partial x} = f_x(x,y) = D_x z.$$

Note that the variable x takes on an increment Δx , but the variable y is kept constant, if the partial derivative is taken with respect to x . Hence, in practice the partial derivative of $z = f(x,y)$ with respect to x is computed by taking the derivative of z with respect to x , treating y as if it were a constant.

In a similar manner, the partial derivative of z with respect to y is defined as

$$\frac{\partial z}{\partial y} = f_y(x,y) = D_y z = \lim_{\Delta y \rightarrow 0} \frac{\Delta z}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{f(x,y + \Delta y) - f(x,y)}{\Delta y}.$$

We note in this case that y takes on an increment Δy , but x is kept fixed. Hence, the partial derivative of $z = f(x,y)$ with respect to y is computed in practice by taking the derivative with respect to y and treating x like a constant.

EXAMPLE 1

Consider the function

$$z = f(x,y) = x^2y + 4x - 2y + 5.$$

To find the partial derivative with respect to x , we treat y as a constant and take the derivative with respect to x .

$$\frac{\partial z}{\partial x} = f_x(x,y) = 2xy + 4.$$

This can be evaluated for various values of x and y . We have, for instance, for the partial derivative of z with respect to x at the point $x = 1, y = -2$

$$f_x(1,-2) = (2)(1)(-2) + 4 = 0.$$

At the point $x = 5, y = 1$ we have for the same partial derivative

$$f_x(5,1) = (2)(5)(1) + 4 = 14.$$

To find the partial derivative with respect to y , we differentiate the function with respect to y , treating x as a constant,

$$\frac{\partial z}{\partial y} = f_y(x,y) = x^2 - 2.$$

This can be evaluated for various values of x and y . For instance, we have for $x = 6, y = 1$

$$f_y(6,1) = (6)^2 - 2 = 34.$$

We proceed similarly with functions of more than 2 independent variables. We compute the partial derivative of a function of several variables

with respect to 1 of the independent variables by taking the derivative with respect to the assigned variable, treating all other independent variables as constant.

■ EXAMPLE 2

Consider, for instance,

$$u = f(x, y, z) = x^2 - 2xy + y^3 + yz.$$

The partial derivative with respect to x is computed by differentiating u with respect to x , treating y and z as constants,

$$\frac{\partial u}{\partial x} = f_x(x, y, z) = 2x - 2y.$$

The partial derivative with respect to y is computed by differentiating u with respect to y , holding x and z constant,

$$\frac{\partial u}{\partial y} = f_y(x, y, z) = -2x + 3y^2 + z.$$

The partial derivative with respect to z results by taking the derivative of u with respect to z , holding x and y constant,

$$\frac{\partial u}{\partial z} = f_z(x, y, z) = y.$$

These partial derivatives can be evaluated for any given values of x , y , and z . We have, for instance,

$$\begin{aligned} f_x(1, 2, 6) &= (2)(1) - (2)(2) = -2, \\ f_y(-1, 4, 2) &= (-2)(-1) + (3)(4)^2 + 2 = 52, \\ f_z(3, 5, -3) &= 5. \end{aligned}$$

● EXERCISES 60

1. Given the function $z = x^2 + xy^2 - 2x + 5y$. (a) Find both partial derivatives. (b) Evaluate them for $x = 1$, $y = 2$.

2. Given the function $z = x^3 + 2x^2 + y^3 - y$. (a) Find both partial derivatives. (b) Evaluate for $x = -1$, $y = 3$.

3. Given the function $u = 2x + 3xy - xz$. (a) Find the partial derivatives of u with respect to x , y , and z . (b) Evaluate them for $x = 0$, $y = 1$, $z = -2$.

4. Given the function $z = x + ye^{-x}$. (a) Find both partial derivatives. (b) Evaluate them for $x = 1$, $y = -1$.

5. Given the function $z = x \log_e(xy)$. (a) Find both partial derivatives. (b) Evaluate them for $x = 1$, $y = 1$.

6. Given the function $u = 2x^2 - xy + z^3 - 4xz + 6z - 8$. (a) Find all partial derivatives. (b) Evaluate them for $x = 1$, $y = -1$, $z = 0$.

7. The consumption of beer in the United Kingdom has been estimated from data covering the period 1920–1938 to be $q = 177.6Q^{-0.023}p^{-1.040}\pi^{0.939}$; where q is the quantity of beer consumed, Q is the aggregate real income, p the average retail price for beer, and π the average retail price of all other commodities (R. Stone). (a) Find the partial derivatives: $\partial q/\partial Q$; $\partial q/\partial p$; $\partial q/\partial \pi$. (b) Evaluate the partial derivatives for $Q = 100$, $p = 100$, $\pi = 100$.

8. The consumption of spirits in the United Kingdom has been estimated for the period 1920–1938 to be $q = 228Q^{0.538}p^{-0.717}\pi^{0.717}$; where q is the quantity of spirits consumed, Q the aggregate real income, p the average retail price of spirits, and π the average retail price of all other commodities (R. Stone). (a) Find the partial derivatives $\partial q/\partial Q$; $\partial q/\partial p$; $\partial q/\partial \pi$. (b) Evaluate the partial derivatives for $Q = 90$, $p = 110$, $\pi = 100$.

9. The consumption of tobacco in the United Kingdom has been estimated for the period 1920–1938 to be $q = 750.2Q^{0.072}p^{-0.513}\pi^{0.513}$ (R. Stone). Here q is the amount of tobacco consumed, Q the aggregate real income, p the retail price of tobacco, and π the average retail price of all other commodities. (a) Find the partial derivatives $\partial q/\partial Q$; $\partial q/\partial p$; $\partial q/\partial \pi$. (b) Evaluate the partial derivatives for $Q = 120$, $p = 100$, $\pi = 150$.

10. The consumption of soap in the United Kingdom has been estimated for the period 1920–1938 to be $q = 28.97Q^{0.317}p^{-0.38}\pi^{0.327}$ (R. Stone). Here q is the consumption of soap, Q the aggregate real income, p the retail price of soap, and π the average retail price of all other commodities. (a) Find the partial derivatives $\partial q/\partial Q$; $\partial q/\partial p$; $\partial q/\partial \pi$. (b) Evaluate the partial derivatives for $Q = 80$, $p = 110$, $\pi = 95$.

61. Marginal Productivity

A production function

$$x = f(a, b)$$

shows the amount x of the product X produced, if a units of production A (for example labor) are used, simultaneously with b units of production B (say, land).

Now let us investigate what happens if there is an increment in the amount of A while the amount of B is held constant. The average rate of increase in x with respect to a is given by

$$\frac{f(a + \Delta a, b) - f(a, b)}{\Delta a}$$

The limit of this ratio as $\Delta a \rightarrow 0$ will be the partial derivative

$$\frac{\partial x}{\partial a} = f_a(a, b).$$

This is the marginal productivity of the factor A .

Similarly, if the amount of A is held constant and the amount of B increases, we would be interested in considering

$$\frac{f(a, b + \Delta b) - f(a, b)}{\Delta b}.$$

The limit of this ratio, as the increment Δb tends to 0, will be the partial derivative

$$\frac{\partial x}{\partial b} = f_b(a, b).$$

This is the marginal productivity of the factor B .

■ EXAMPLE

The production function of a commodity is $x = f(a, b) = 10a + 5b - a^2 - 2b^2 + 3ab$. (a) Find the marginal productivity of A .

By partial differentiation, $\partial x / \partial a = f_a(a, b) = 10 - 2a + 3b$. (b) What is the marginal productivity of A if $a = 1$ and $b = 5$? Substituting, we have $f_a(1, 5) = 23$. Thus, on the assumption that b can be held constant, the limit of the ratio between the increment in the product x and the increment in the factor a tends to 23.

● EXERCISES 61

1. The production function for a commodity is $x = 10a - a^2 + ab$. (a) Find the marginal productivities of A and B . (b) Find the two marginal productivities if $a = 2$ and $b = 6$.

2. The production function of a commodity is $x = 10a + 20b + 8c - a^2 + 2b^2 - c^2abc$. (a) Find the marginal productivities of A , B , and C . (b) Determine these marginal productivities if $a = 1$, $b = 2$, $c = 3$.

3. The production function of a commodity is $x = 100a + 200b + 50c - a^2 - 2b^2 - 3c^2 - 5ab + 3ac - bc$. (a) Find the marginal productivities of A , B , and C . (b) Determine these productivities if $a = 1$, $b = 2$, $c = 5$.

4. The production function for the United States, during 1899–1922, was estimated by Paul Douglas to be $x = 1.01L^{0.76}C^{0.26}$, where x is the total production, L is labor, and C is capital. (a) Find the marginal productivities of L and C . (b) Establish the marginal productivities for $L = 160$, $C = 130$.

5. The production function for Australia, during 1934–1935, was estimated by Douglas and Gunn as $x = L^{0.64}C^{0.36}$, where x is production, L is labor, and C is capital. (a) Find the marginal productivities of L and C . (b) Determine the marginal productivities for $L = 1.5$, $C = 1.1$.

6. The production function for the United States in 1919 was estimated by Douglas and Gunn as $x = L^{0.76}C^{0.26}$, where x is the product, L is labor, and C is capital. (a) Find the marginal productivities of L and C . (b) Estimate the marginal productivities for $L = 1$, $C = 1$.

7. The production function for Canada in 1937 was estimated by Douglas and

Daly as $x = L^{0.43}C^{0.53}$, where x is the product, L the amount of labor, and C of capital. (a) Determine the marginal productivities of C and L . (b) Find the marginal productivities if $C = 0.9$, $L = 0.8$.

8. The production function of Iowa farms was estimated for 1942 as $x = a^{0.287}b^{0.156}c^{0.063}d^{0.211}f^{0.158}$, where x is the product, a the land, b the labor, c the improvements, d the liquid assets, f the cash operating expenses (Tintner). (a) Determine the marginal productivities of a , b , c , d , and f . (b) Estimate the marginal productivities for $a = b = c = d = f = 1$.

9. The production function for hogs in Iowa has been estimated for 1942 as $x = 100a^{0.28}b^{0.23}c^{0.04}d^{0.17}f^{0.18}$, where the symbols a , b , c , d , f have the same meaning as in Problem 8 and x is the production of hogs (Tintner). (a) Find the marginal productivities for $a = 1$, $b = 2$, $c = 1$, $d = 1.5$, $f = 0.5$.

10. The production function of farms in Iowa in 1939 has been estimated as $x = a^{0.34}b^{0.24}c^{0.04}d^{0.17}e^{0.08}f^{0.11}$, where a , b , c , d , and f have the same meaning as in Problem 8, and e is working assets (Tintner and Brownlee). (a) Find the marginal productivities of a , b , c , d , e , and f . (b) Estimate the marginal productivities for $a = 2.37$, $b = 2.4$, $c = 5.134$, $d = 5.171$, $e = 2.685$, and $f = 2.127$.

11. The production function for the United States, during 1921–1941, was estimated to be $x = a^2b^{0.34}$, where x is the total product, a labor, and b fixed capital (Tintner). (a) Find the marginal productivities of labor and capital. (b) Estimate the marginal productivities for $a = 1.7$, $b = 2$.

62. Partial Elasticities of Demand

Let $D_A = f(p_A, p_B)$ be the demand for commodity A which depends upon the prices of commodities A and B , denoted, respectively, by p_A and p_B . We define the partial elasticity of demand to be

$$\frac{ED_A}{Ep_A} = \left(\frac{\partial D_A}{\partial p_A} \right) \left(\frac{p_A}{D_A} \right)$$

and, similarly,

$$\frac{ED_A}{Ep_B} = \left(\frac{\partial D_A}{\partial p_B} \right) \left(\frac{p_B}{D_A} \right)$$

The partial elasticity of the demand for commodity A with respect to the price of A is (approximately) the per cent of increase or decrease in the demand for A if the price of A increases by 1 per cent and the price of B is constant. The partial elasticity of demand for A with respect to the price of B is (approximately) the per cent of increase or decrease in the demand for A , if the price of B increases by 1 per cent while the price of A remains constant.

■ EXAMPLE

Let the demand function for commodity A be $D_A = f(p_A, p_B) = 25 - 2p_A + p_B$. (a) Find the partial elasticities. We have the partial derivatives

$\partial D_A / \partial p_A = -2$ and $\partial D_A / \partial p_B = 1$. Inserting them into the above formulas we have

$$\frac{ED_A}{Ep_A} = \frac{-2p_A}{25 - 2p_A + p_B} \quad \text{and} \quad \frac{ED_A}{Ep_B} = \frac{p_B}{25 - 2p_A + p_B}.$$

(b) Evaluate the elasticities for $p_A = 3$, $p_B = 1$. We get

$$\frac{ED_A}{Ep_A} = \frac{-6}{20} = -0.3.$$

This means that, as the price of *A* increases by 1 per cent, if the price of *B* remains constant, the demand for *A* will decrease by about 0.3 per cent. Further,

$$\frac{ED_A}{Ep_B} = \frac{1}{20} = 0.05.$$

If the price of *A* stays constant and the price of *B* increases by 1 per cent, the demand for *A* will increase by about 0.05 per cent.

● EXERCISES 62

1. The demand for a commodity *A* is $D_A = 100 - 10p_A - 2p_B$. (a) Find the partial elasticities ED_A/Ep_A and ED_A/Ep_B . (b) Evaluate the partial elasticities for $p_A = 2$, $p_B = 5$.

2. The demand for commodity *A* is $D_A = 50 - 3p_A + 5p_B$. (a) Find the partial elasticities ED_A/Ep_A and ED_A/Ep_B . (b) Evaluate the partial elasticities of demand for $p_A = 10$, $p_B = 4$.

3. Let *A* be beef, *B* pork, and *C* mutton. The demand for beef is estimated, according to Henry Schultz, to be $D_A = 63.3 - 1.9p_A + 0.2p_B + 0.5p_C$. (a) Find the partial elasticities of the demand for beef with respect to the price of beef, the price of pork, and the price of mutton. (b) Estimate the partial elasticities for $p_A = 10$, $p_B = 8$, $p_C = 7$.

4. Using the same notation as in Problem 3, we have for the estimated demand for pork: $D_B = 71.0 + 0.4p_A - 1.2p_B - 0.1p_C$ (Henry Schultz). (a) Find the partial elasticities of the demand for pork with respect to the prices of beef, pork, and mutton. (b) Determine the partial elasticities for $p_A = 8$, $p_B = 10$, $p_C = 12$.

5. Using the same notation as in Problem 3, the demand for mutton is estimated as $D_C = 10.3 + 0.1p_A + 0.1p_B - 0.3p_C$ (Henry Schultz). (a) Find the partial elasticities of the demand for mutton with respect to the prices of beef, pork, and mutton. (b) Determine the partial elasticities for $p_A = 8$, $p_B = 9$, $p_C = 7$.

6. Let *A* be barley, *B* corn, *C* hay, *D* oats. Henry Schultz estimated the demand for barley in the United States as follows: $D_A = 2.24 - 0.01p_A - 0.01p_B + 0.01p_D$. (a) Find the partial elasticities of the demand for barley with respect to the prices of barley, corn, hay, and oats. (b) Estimate the partial elasticities for $p_A = 1$, $p_B = 1$, $p_D = 1$.

7. Using the same notation as in Problem 6, we have for the estimated demand for corn $D_B = 49.07 - 0.02p_A - 0.36p_B - 0.03p_C + 0.03p_D$ (Henry Schultz)

(a) Find the partial elasticities of the demand for corn with respect to the prices of corn and hay. (b) Estimate the partial elasticities for $p_A = 1, p_B = 2, p_C = 3, p_D = 1$.

8. Henry Schultz estimated the demand for hay in the United States (notation in Problem 6) as $D_C = 1.30 - 0.05 p_C + 0.01 p_D$. (a) Find the partial elasticities of the demand for hay with respect to the prices of corn, hay, and oats. (b) Estimate the partial elasticities for $p_C = 2, p_D = 1$.

9. The demand for oats in the United States is estimated as $D_D = 24.2 + 0.1p_B - 0.6p_C - 0.3p_D$ (the same notation as in Problem 6). (a) Find the partial elasticity of the demand for oats with respect to the prices of barley, corn, hay and oats. (b) Estimate the partial elasticities for $p_B = 1, p_C = 1, p_D = 1$.

10. Recall the demand function for beer in the United Kingdom, Problem 7, Exercises 60. Define in the notation of this problem the income elasticity of the demand for beer as $E_q/EQ = (\partial q/\partial Q)(Q/q)$, and the price elasticities as $E_q/Ep = (\partial q/\partial p)(\partial p/\partial q)$, $E_q/E\pi = (\partial q/\partial \pi)(\pi/q)$. Find the income elasticity and the two price elasticities of the demand for beer.

11. Consider the demand function for spirits in the United Kingdom, Problem 8, Exercises 60. Using the definitions in Problem 10 above, compute the income elasticity and both price elasticities of the demand for spirits.

12. Consider the demand for tobacco in the United Kingdom, Problem 9, Exercises 60. Using the definitions of Problem 10 above, find the income elasticity and both price elasticities of the demand for tobacco.

13. Consider the demand for soap in the United Kingdom, Problem 10, Exercises 60. Using the definitions of Problem 10 above, derive the income elasticity and both price elasticities of the demand for soap.

14. C. F. Roos and V. von Szeliski give the following formula for the estimated demand for passenger automobiles in the United States: $D = 0.92I^{1.07}p^{-0.74}T^{1.10}$, where D denotes replacement sales of automobiles, I is income, p the average price per car, T the index of the scrapping of cars. Find (a) the income elasticity of demand $ED/EI = (\partial D/\partial I)(I/D)$, (b) the price elasticity of demand $ED/Ep = (\partial D/\partial p)(p/D)$.

15. The demand for bicycles in the Netherlands has been estimated from data covering the period 1922–1933 as $V = 11.2K - 8.6P - 379$ (J. B. D. Derksen and A. Rombouts), where V is the annual total consumption of bicycles, K an index of purchasing power, and P the price of bicycles. (a) Find the income elasticity $EV/EK = (\partial V/\partial K)(K/V)$; (b) the price elasticity $EV/EP = (\partial V/\partial P)(P/V)$. (c) Evaluate the elasticities for $K = 100, P = 45$.

16. The demand for shoes in the United States has been estimated for the period 1919–1934 as $v = 0.053i^{0.25}p^{-0.16}$ (V. von Szeliski and L. J. Paradiso). Here v is the number of pairs of shoes per capita, p a price index of shoes and boots, and i national income per capita. (a) Find the price elasticity $E_v/Ep = (\partial v/\partial p)(p/v)$; (b) the income elasticity $E_v/Ei = (\partial v/\partial i)(i/v)$.

17. Denote oleomargarine by A and butter by B . The demand for oleomargarine has been estimated by W. R. Pabst for the United States, 1904–1933, as $D_A = p_A^{-1.32} p_B^{0.4}$. Find the partial elasticities of demand.

18. Using the notation of the previous problem, the demand for butter is $D_B = p_A^{0.32} p_B^{-0.1}$ (W. R. Pabst). Compute the partial elasticities of demand.

**63. Differentiation of Implicit Functions

In order to find the various derivatives, we have up to now assumed that one variable is the dependent variable and the other variables are independent variables. But in economics functions appear frequently which cannot easily be expressed in this form. Hence it is of some importance to be able to differentiate implicit functions.

Assume, for instance, that we have a function of 2 variables

$$f(x, y) = 0.$$

We want to find the derivative dy/dx without solving first for y , because, for some cases, this may not be possible. Since y is a function of x , it is to be presumed that y will take on an increment Δy as x takes on an increment Δx . Thus, the total increment in the function, as x takes on an increment Δx , is $f(x + \Delta x, y + \Delta y) - f(x, y)$. It follows that

$$\begin{aligned} \frac{f(x + \Delta x, y + \Delta y) - f(x, y)}{\Delta x} &= \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x} \\ &+ \frac{f(x, y + \Delta y) - f(x, y)}{\Delta x} = 0. \end{aligned}$$

As $\Delta x \rightarrow 0$, the first term on the right approaches $f_x(x, y + \Delta y)$, or, of course, $f_x(x, y)$, since $\Delta y \rightarrow 0$ as $\Delta x \rightarrow 0$. The second term may be rewritten in the form

$$\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \left(\frac{\Delta y}{\Delta x} \right).$$

As $\Delta x \rightarrow 0$, this term approaches $f_y(x, y) dy/dx$. Consequently, we have

$$f_x(x, y) + f_y(x, y) \left(\frac{dy}{dx} \right) = 0,$$

or

$$\frac{dy}{dx} = -\frac{f_x}{f_y}.$$

The derivation as just made requires special properties of continuity on the part of the functions; however, in general, such conditions are satisfied by the functions employed in economics.

■ **EXAMPLE 1**

Find the derivative of the implicit function

$$y^5 + 2x^2y^3 + xy - 42 = 0$$

at the point $x = 1, y = 2$. We have

$$f_x = 4xy^3 + y,$$

$$f_y = 5y^4 + 4x^2y + x.$$

Thus

$$\frac{dy}{dx} = -\frac{4xy^3 + y}{5y^4 + 4x^2y + x}.$$

For $x = 1, y = 2$ we have

$$\frac{dy}{dx} = -\frac{18}{89}.$$

Let us now consider an implicit function of 3 (or more) variables; for example,

$$f(x, y, z) = 0,$$

wherein x and y are regarded as independent variables, but z is treated as a function of both x and y . We want to find $\partial z/\partial x$ and $\partial z/\partial y$, the two partial derivatives. By an analysis similar to that given above,

$$f_x + f_z \left(\frac{\partial z}{\partial x} \right) = 0,$$

$$f_y + f_z \left(\frac{\partial z}{\partial y} \right) = 0,$$

Hence

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z},$$

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}.$$

■ **EXAMPLE 2**

Let the implicit function be

$$z^5 - x^2z^3 + xy + 2 = 0.$$

We want to find the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ at the point $x = 3, y = 2, z = 1$.

$$f_x = -2xz^3 + y,$$

$$f_y = x,$$

$$f_z = 5z^4 - 2x^2z.$$

Thus

$$\frac{\partial z}{\partial x} = -\frac{(-2xz^2 + y)}{5z^4 - 2x^2z},$$

$$\frac{\partial z}{\partial y} = -\frac{x}{5z^4 - 2x^2z}.$$

At the point $x = 3$, $y = 2$, $z = 1$, we have $\partial z/\partial x = -4/13$ and $\partial z/\partial y = +3/13$.

Assume now that we have 2 quantities y and z which are both functions of x . We have 2 independent equations defining the relation between the 3 variables

$$f(x, y, z) = 0,$$

$$g(x, y, z) = 0.$$

In order to find the derivatives dy/dx and dz/dx , we differentiate both relations with respect to x , again extending the analysis considered above, and then solve the resulting system of equations:

$$f_x + f_y \left(\frac{dy}{dx}\right) + f_z \left(\frac{dz}{dx}\right) = 0,$$

$$g_x + g_y \left(\frac{dy}{dx}\right) + g_z \left(\frac{dz}{dx}\right) = 0.$$

Hence we have

$$\frac{dy}{dx} = \frac{f_z g_x - f_x g_z}{f_y g_x - f_x g_y},$$

$$\frac{dz}{dx} = \frac{f_x g_y - f_y g_x}{f_y g_x - f_x g_y}.$$

EXAMPLE 3

Let the relations between the 3 variables x , y , z be

$$f(x, y, z) = x^2 + 2xy + z - 4 = 0,$$

$$g(x, y, z) = x^3 - yz + z^3 - 1 = 0.$$

We want to find the derivatives dy/dx and dz/dx at the point $x = y = z = 1$. We have

$$f_x = 2x + 2y,$$

$$f_y = 2x,$$

$$f_z = 1,$$

$$g_x = 3x^2,$$

$$g_y = -z,$$

$$g_z = 3z^2,$$

$$\frac{dy}{dx} = \frac{3x^2 - 6xz^2 - 6yz^2}{6xz^2 + z}$$

$$\frac{dz}{dx} = \frac{-2xz - 2yz - 6x^2}{6xz^2 + z}$$

At the point $x = y = z = 1$ we have $dy/dx = -9/7$ and $dz/dx = -10/7$.

● EXERCISES 63

1. Find the derivative dy/dx of the following implicit functions: (a) $x^4 - 2x^2y^2 + y^2 + 101 = 0$, at the point $x = 2, y = 3$; (b) $x^5y^4 - x^3y^2 + xy - 1 = 0$, if $x = y = 1$; (c) $xe^{xy} - e^y = 0$, if $x = 1, y = 0$; (d) $x^2y^2 + x^3y - 90 = 0$ at the point $x = 2, y = 3$.

2. Assume the function $x^2 + y^2 = 4$. Find the derivatives dy/dx and dx/dy , (a) by differentiating the implicit function; (b) by making the function explicit.

3. A demand function is defined implicitly as $D^2p^3 = 1$. Find the elasticity of demand (a) by differentiating the implicit relationship; (b) by making the relation explicit and differentiating it.

4. Find the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ of the following implicit functions: (a) $z^5 + y^2z^2 - 2xy = 0$, if $x = y = z = 1$; (b) $xe^z + ze^y - 2e = 0$, if $x = 2, y = 0, z = 1$; (c) $x^2 \log_e z + ze^{xy} - y - 1 = 0$, for $x = 2, y = 0, z = 1$; (d) $x^5y^2 - z^2xy = 0$, for $x = y = z = 1$.

5. Let $x^2 + y^2 + z^2 = 16$. Find the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, (a) by differentiating the implicit relationship; (b) by making the relationship explicit for z .

6. The implicit form of a production function is $x^2ab = 1$. Find the marginal productivities (a) by differentiating the implicit relationship; (b) by making the relationship explicit for x .

7. A demand function is given by $D_a^2p_a^3p_b = 10$. Find the partial elasticities of demand (a) by differentiating the implicit relationship; (b) by making the relationship explicit.

8. Find the derivatives dy/dx and dz/dx from the equations

$$xy^2 + 4zx - 5 = 0,$$

$$xz + y^2z^2 - 2 = 0,$$

if $x = y = z = 1$.

9. Find the derivatives dy/dx and dz/dx of the system

$$xy + z - 5 = 0,$$

$$xz + 2y - 7 = 0,$$

at the point $x = 1, y = 2, z = 3$.

10. What is the effect of a tax levied on each unit sold by a monopolist? (HINT: the profit after tax is, in the notation of Section 56, $\pi = R(D) - C(D) - tD$, if t

is the tax. Hence the necessary condition for a maximum is

$$\pi' = R'(D) - C'(D) - t = 0,$$

and the sufficient condition is

$$\pi'' = R''(D) - C''(D) < 0.$$

Differentiate the necessary condition with respect to t and find dD/dt . Determine the sign from the sufficient condition. What can you say about the sign of dp/dt ?

11. Equilibrium on the market of a commodity is defined by the 2 equations

$$f(D, a) - p = 0,$$

$$g(D) - p = 0,$$

where D is the quantity, p is the price, f is the demand function, g is the supply function, and a is a shift parameter (for example, signifying change in taste). Let $f_a > 0$, $f_D < 0$, $g_D > 0$ and find the signs of the change in quantity dD/da and change in price dp/da .

HOMOGENEITY

64. Homogeneous Functions

A special type of function which is of importance in many fields of economics is the so-called homogeneous function. Consider the function of 2 variables

$$z = f(x,y).$$

This function is said to be a homogeneous function of degree k if the following relationship holds:

$$f(tx,ty) = t^k f(x,y).$$

In words: *a function is said to be homogeneous of degree k if when each of the independent variables is multiplied by a positive constant t , the new function is t^k times the original function.*

■ EXAMPLE 1

Let $z = f(x,y) = x^2 + 4xy + 3y^2$. If we multiply x and y by a positive constant t , we get

$$\begin{aligned} f(tx,ty) &= (tx)^2 + 4(tx)(ty) + 3(ty)^2 \\ &= t^2x^2 + 4t^2xy + 3t^2y^2 \\ &= t^2(x^2 + 4xy + 3y^2) \\ &= t^2f(x,y). \end{aligned}$$

We conclude that $f(x,y) = x^2 + 4xy + y^2$ is a homogeneous function of degree 2 ($k = 2$).

Let us, for instance, double the independent variables x and y . We have immediately

$$f(2x, 2y) = 2^0 f(x, y).$$

■ EXAMPLE 2

Consider the function $z = f(x, y, z) = (3x/z) - (2y/z)$. Multiplying the 3 independent variables x , y and z by an arbitrary positive constant t , we have

$$\begin{aligned} f(tx, ty, tz) &= \frac{3(tx)}{(tz)} - \frac{2(ty)}{(tz)} \\ &= \frac{3x}{z} - \frac{2y}{z} \\ &= t^0 f(x, y, z). \end{aligned}$$

Since $t^0 = 1$, the function $f(x, y, z)$ is said to be a homogeneous function of degree 0. If all independent variables x , y , and z are multiplied by any arbitrary positive constant, the value of the function remains unchanged. As an illustration, if x , y , and z are multiplied by 5, we have merely

$$f(5x, 5y, 5z) = f(x, y, z).$$

● EXERCISES 64

1. Establish the degree of homogeneity of the function $y(x, y) = x^3 + 4x^2y - 2xy^2 + y^3$. Check the result by multiplying the variables x, y by 5.

2. Establish the degree of homogeneity of the function $f(x, y, z) = x^2 + 3xy + z^2$. Check by multiplying the independent variables x, y , and z by 4.

3. What is the degree of homogeneity of the function

$$f(x, y, z, u, v) = \sqrt[3]{2x^2 - y^2 + 5z^2 - 3u^2}.$$

Check by multiplying the independent variables by $1/8$.

4. What is the degree of homogeneity of the function $f(x, y, z) = (2x - 3y)/5z$? Check by multiplying the independent variables by $1/3$.

5. Let the price of commodity A be denoted by p_A . The price of another commodity B is denoted by p_B . The demand for A is $D_A = 50p_B/p_A$ and the demand for B is $D_B = 100p_A/p_B$. (a) Show that the 2 demand functions are homogeneous functions of 0 degree. (b) Check by assuming that the prices p_A and p_B are doubled. (c) Check by assuming that the prices p_A and p_B are halved. (d) What conclusions can you draw for the change in the demand for A and B if all prices increase or decrease by the same per cent? (e) Compute the demand for the commodities A and B if $p_A = 5$ and $p_B = 3$. (f) Show that the demand remains the same if the prices given in (e) are tripled.

6. Let p_A , p_B , and p_C be the prices of 3 commodities A , B , and C . Assume the

demand functions for the 3 commodities in the following form:

$$D_A = \frac{2p_C - p_A}{p_C},$$

$$D_B = \frac{5p_C - p_A - 2p_B}{p_C},$$

$$D_C = \frac{10p_A - p_B}{p_C}.$$

(a) Show that the 3 demand functions are homogeneous functions of 0 degree in the 3 prices. (b) Check by assuming that all prices increase by 20 per cent. (c) Check by assuming that all prices decrease by 40 per cent. (d) Compute the quantities demanded of the commodities A , B , and C if $p_A = 2$, $p_B = 1$, and $p_C = 10$. (e) Compute the quantities demanded if the prices given in (d) are doubled.

7. Investigate for homogeneity the production function for the United States given in Problem 4, Exercises 61.

8. Investigate for homogeneity the production function for Australia given in Problem 5, Exercises 61.

9. Investigate for homogeneity in the prices p and π the consumption function for spirits in the United Kingdom given in Problem 8, Exercises 60.

10. Investigate for homogeneity in the prices p and π the consumption function of tobacco given in Problem 9, Exercises 60.

65. Euler Theorem

An important relationship for homogeneous functions involving their partial derivatives is as follows: Let $f(x, y)$ be a homogeneous function of the k th degree. Then $xf_x(x, y) + yf_y(x, y) = kf(x, y)$. This identity is called the *Euler theorem*.

**To derive this theorem we differentiate

$$f(tx, ty) = t^k f(x, y)$$

with respect to the parameter t ; we obtain

$$xf_{1x}(tx, ty) + yf_{1y}(tx, ty) = kt^{k-1} f(x, y).$$

Since t is arbitrary we may take $t = 1$. From this follows the Euler theorem:

$$xf_x(x, y) + yf_y(x, y) = kf(x, y).$$

This can be generalized for any number of independent variables.

■ EXAMPLE 1

We have shown that the function

$$f(x, y) = x^2 + 4xy + 3y^2$$

is a homogeneous function of second degree. We compute the partial derivatives

$$f_x(x,y) = 2x + 4y,$$

$$f_y(x,y) = 4x + 6y.$$

Then

$$\begin{aligned} xf_x(x,y) + yf_y(x,y) &= x(2x + 4y) + y(4x + 6y) \\ &= 2x^2 + 8xy + 6y^2 \\ &= 2(x^2 + 4xy + 3y^2) \\ &= 2f(x,y). \end{aligned}$$

Since $f(x,y)$ is a homogeneous function of second degree, the sum of the partial derivatives, each times its respective independent variable, is twice the original function $f(x,y)$, as was expected.

EXAMPLE 2

The function

$$f(x,y,z,u) = 3x - 2y + 4z - u$$

is a homogeneous function of first degree, also called a linear homogeneous function. We find the partial derivatives

$$f_x(x,y,z,u) = 3,$$

$$f_y(x,y,z,u) = -2,$$

$$f_z(x,y,z,u) = 4,$$

$$f_u(x,y,z,u) = -1.$$

Thus

$$\begin{aligned} xf_x(x,y,z,u) + yf_y(x,y,z,u) + zf_z(x,y,z,u) + uf_u(x,y,z,u) \\ = 3x - 2y + 4z - u \\ = f(x,y,z,u). \end{aligned}$$

Hence we see that in this linear homogeneous function the sum of all the partial derivatives, each times its respective independent variable, is equal to the original function $f(x,y,z,u)$.

EXERCISES 65

1. Given the function $u = 2x^4 - 4y^4$. (a) Determine whether it is a homogeneous function. (b) Apply the Euler theorem.

2. Given the function $u = x + 2y - 5z$. (a) Determine whether it is a homogeneous function. (b) Apply the Euler theorem.

3. Given the function $u = x^2 + y^2$. (a) Determine whether it is a homogeneous function. (b) Apply the Euler theorem.

****4.** Let $u = ax^b y^c$; a , b , and c are constants. (a) Find the condition under which this is a linear homogeneous function. (b) Apply the Euler theorem if these conditions hold true.

5. Is the production function for the United States in the period 1899–1922, given in Problem 4, Exercises 61, linear and homogeneous? Apply the Euler theorem. What would be the resulting production if amounts of labor and capital were doubled?

6. Is the production function for Australia in 1934–1935 given in Problem 5, Exercises 61, linear and homogeneous? Apply the Euler theorem.

7. Is the production function for Canada in 1937 given in Problem 7, Exercises 61, linear and homogeneous? Apply the Euler theorem.

8. Is the production function for Iowa farms in 1942 given in Problem 8, Exercises 61, linear and homogeneous? Apply the Euler theorem.

9. Investigate the homogeneity of the production function for hogs in Iowa, Problem 9, Exercises 61. Apply the Euler theorem.

10. Investigate the production function of farms in Iowa, Problem 10, Exercises 61, for homogeneity. Apply the Euler theorem.

11. Is the production function for the United States, 1921–1941, given in Problem 11, Exercises 61, linear and homogeneous?

HIGHER PARTIAL DERIVATIVES AND APPLICATIONS

66. Higher Partial Derivatives

Assume that we have a function of 2 independent variables $z = f(x, y)$. The 2 first derivatives are

$$\frac{\partial z}{\partial x} = f_x(x, y) = \frac{\partial f(x, y)}{\partial x},$$

$$\frac{\partial z}{\partial y} = f_y(x, y) = \frac{\partial f(x, y)}{\partial y}.$$

As already indicated, the partial derivative $f_x(x, y)$ is the derivative of $f(x, y)$ with respect to x , treating y as a constant. The partial derivative $f_y(x, y)$ is the derivative of $f(x, y)$ with respect to y , treating x as a constant.

We can also define higher derivatives in an analogous manner. For instance,

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = \frac{\partial f_x(x, y)}{\partial x}.$$

This means that the second partial derivative with respect to x is computed by differentiating the first partial derivative $f_x(x, y)$ partially with respect to x . That is, we take the derivative of the function $f_x(x, y)$ with respect to x , holding y constant.

Similarly,

$$\frac{\partial^2 z}{\partial y^2} = f_{vv}(x, y) = \frac{\partial f_v(x, y)}{\partial y}.$$

That is, the second partial derivative with respect to y is computed by differentiating partially with respect to y the first partial derivative with respect to y .

Apart from these two second-order derivatives there are also the mixed partial derivatives

$$\frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y) = \frac{\partial f_y(x, y)}{\partial x},$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y) = \frac{\partial f_x(x, y)}{\partial y}.$$

Under proper conditions of continuity, that are found in virtually all functions, $f_{xy}(x, y) = f_{yx}(x, y)$. Hence, in practice, the mixed partial derivative $f_{xy}(x, y)$ can be computed in two ways which yield the same result: Either we differentiate the first partial derivative with respect to x , partially with respect to y , or we differentiate the first partial derivative with respect to y , partially with respect to x . The generalization of these methods for functions of more than 2 independent variables is obvious.

EXAMPLE

Assume a function of 2 variables

$$z = f(x, y) = 2x^2 - 3xy + 5y^2.$$

The first partial derivatives are

$$\frac{\partial z}{\partial x} = f_x(x, y) = 4x - 3y,$$

$$\frac{\partial z}{\partial y} = f_y(x, y) = -3x + 10y.$$

The second partial derivative with respect to x is derived by taking the partial derivative of $f_x(x, y) = 4x - 3y$ with respect to x ; that is,

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = 4.$$

The second partial derivative with respect to y is computed by finding the partial derivative of $f_y(x, y) = -3x + 10y$ with respect to y . Thus

$$\frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = 10.$$

The mixed second-order partial derivative $f_{xy}(x, y)$ may be computed in two ways. The first is to find the partial derivative of the function $f_x(x, y) =$

$4x - 3y$ with respect to y ; this gives

$$\frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y) = -3.$$

The same result is achieved by finding the partial derivative of the function $f_y(x, y) = -3x + 10y$ with respect to x ; that is,

$$\frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y) = -3.$$

● EXERCISES 66

1. Given $z = x^4 - 5xy^3 + 6xy - x + 4y + 10$. Find all the second-order partial derivatives.

2. Given $z = t = ye^{-z}$. Find all the second-order partial derivatives.

3. Given $z = x \log_e y$. Find all the second-order partial derivatives.

4. Given $z = e^x \log_e xy$. Find all the second-order partial derivatives.

5. Given $u = x^3 - xy + 4xz - z^2$. Find all second-order partial derivatives.

6. Given $z = x^3 - 6x^2y + xy - 7$. (a) Find all second-order partial derivatives. (b) Evaluate for $x = 1, y = -1$.

7. Given the function $z = x/y$. (a) Find all second-order partial derivatives. (b) Evaluate for $x = 2, y = -6$.

8. Given the function $u = x^4yz - z^6$. (a) Find all the second-order partial derivatives. (b) Evaluate for $x = y = z = 1$.

9. Given the function $w = x^3 - xyz + u^4z$. (a) Find all second-order partial derivatives. (b) Evaluate for $x = 1, y = -1, z = 0, u = 2$.

10. Given the function $w = xy/zu$. (a) Determine all second-order partial derivatives. (b) Evaluate for $x = 1, y = -1, z = 2, u = 4$.

67. Maxima and Minima in Several Variables,

We remember that the necessary condition for a maximum or minimum for a function $y = f(x)$ of 1 independent variable was

$$f'(x) = 0.$$

Assume now that we have a function of 2 variables,

$$z = f(x, y).$$

What are the necessary conditions for a maximum or minimum of this function?

First, assume that we keep the independent variable y fixed. Let us assign to it the value $y = b$, where b is a constant. Then the function

$$z = f(x, b)$$

is a function of the single variable x , since by assumption b is a constant. The necessary condition for a maximum of this function is, as before, that its

derivative (with respect to the single variable x) vanish; that is,

$$f_x(x, b) = 0.$$

Next, in the function $f(x, y)$ let us assign a constant value to x . For instance, let $x = a$, where a is a constant. The function becomes now

$$z = f(a, y).$$

This is a function of the single independent variable y . The necessary condition for a maximum is that its derivative (with respect to the single variable y) vanish; that is,

$$f_y(a, y) = 0.$$

From this demonstration, although it is not completely rigorous, we may conclude that the necessary condition for the existence of a maximum or a minimum of a function of 2 variables is that both first-order partial derivatives become 0:

$$f_x(x, y) = 0,$$

$$f_y(x, y) = 0.$$

By analogy, we can extend this theorem to any number of independent variables and we have the general rule: *a necessary condition for the existence of a maximum or minimum of a function of several variables is that all the first-order partial derivatives be 0.*

The sufficient conditions for a maximum or minimum of a function of several variables are too difficult to derive here. We will simply state them for the case of 2 independent variables.

A maximum or minimum exists, in general, if the following condition involving a determinant is fulfilled:

$$\begin{vmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{xy}(x, y) & f_{yy}(x, y) \end{vmatrix} = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 > 0.$$

In other words, the determinant given above must be positive in order to have a maximum or minimum. Moreover,

$$f_{xx}(x, y) > 0, \quad f_{yy}(x, y) > 0 \quad \text{for a minimum,}$$

$$f_{xx}(x, y) < 0, \quad f_{yy}(x, y) < 0 \quad \text{for a maximum.}$$

We need actually consider the sign of only 1 of the derivatives f_{xx} and f_{yy} , since the other has necessarily the same sign if the determinant is positive.

■ EXAMPLE

Consider the function

$$z = f(x, y) = 2x + 8y - x^2 - 2y^2.$$

The first-order partial derivatives are

$$f_x(x,y) = 2 - 2x,$$

$$f_y(x,y) = 8 - 4y.$$

The necessary conditions for a maximum or a minimum demand that the 2 first-order derivatives are 0; that is,

$$f_x(x,y) = 2 - 2x = 0,$$

$$f_y(x,y) = 8 - 4y = 0.$$

The solution of this system of equations is $x = 1, y = 2$.

In order to investigate whether there is actually a maximum or minimum at the point $x = 1, y = 2$, we compute the second-order derivatives

$$f_{xx}(x,y) = -2,$$

$$f_{xy}(x,y) = 0,$$

$$f_{yy}(x,y) = -4,$$

$$\begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix} = (-2)(-4) - (0)^2 = 8 > 0.$$

The determinant is positive; hence we have either a maximum or a minimum.

We note that both second-order derivatives are negative; that is,

$$f_{xx}(x,y) = -2 < 0,$$

$$f_{yy}(x,y) = -4 < 0.$$

Hence we conclude that we have a maximum of the function $z = f(x,y)$ at the point $x = 1, y = 2$. The maximum value of the function is computed by substituting the values of $x = 1$ and $y = 2$ into the function $f(x,y)$:

$$f(1,2) = (2)(1) + (8)(2) - (1)^2 - (2)(2)^2 = 9.$$

This is the maximum value of z .

● EXERCISES 67

1. Consider the function $z = x^2 + 2x + y + y^2$ for maxima and minima. Find the corresponding value of z .

2. Consider the function $z = 10 - x^2 - 5y^2 + 3xy - x + 2y$ for maxima and minima. Find the corresponding value of z .

3. Consider the function $z = x^2 - 12x + y^2 - 27y$ for maxima and minima. Find the corresponding value of z .

4. Consider the function $z = xy - x^2$ for maxima and minima.

5. Consider the function $z = xy - x + y$ for maxima or minima.

6. Find the maxima and minima of the function $z = 12 - x^2 + 2y - y^2$.

- 7. Consider the function $z = xy$ for maxima and minima.
- 8. Consider the maxima and minima of the function $z = x^3 + y^2 - 3x - 12y + 10$.
- **9. Consider the function $z = ax^3 + 2bxy + cy^2$, where $a, b,$ and c are constants. What are the values of the parameters $a, b,$ and c for (a) a maximum; (b) a minimum; (c) neither a maximum nor a minimum of z ?
- 10. Consider the function $z = ax + by + c$ ($a, b,$ and c constants) for maxima and minima.
- 11. Consider the maxima and minima of the function $w = x^2 + y^2 + z^2 + u^2 + v^2$.

68. Joint Production

Assume that a manufacturer produces 2 commodities A and B . Denote by D_A the amount of A produced, and by D_B the amount of B . The demand curves are $p_A = f(D_A)$ and $p_B = g(D_B)$. The joint-cost function is (total cost) $C = h(D_A, D_B)$. The profit is $\pi = p_A D_A + p_B D_B - C$. The entrepreneur endeavors to maximize his profit by producing the appropriate amounts of the commodities and charging the prices which will give largest profits.

The necessary conditions for a maximum of profit is $\partial\pi/\partial D_A = 0$, and $\partial\pi/\partial D_B = 0$. The sufficient conditions for a maximum of profits can also be given.

■ EXAMPLE

Let the demand curves for 2 commodities be $p_A = 1 - D_A$ and $p_B = 1 - D_B$. The total-cost curve for the joint production is $C = D_A D_B$. The profit function is $\pi = D_A - D_A^2 + D_B - D_B^2 - D_A D_B$. The necessary conditions for maximum profits are

$$\frac{\partial\pi}{\partial D_A} = 1 - 2D_A - D_B = 0 \quad \text{and} \quad \frac{\partial\pi}{\partial D_B} = 1 - D_A - 2D_B = 0.$$

The solution of these 2 equations is $D_A = 1/3, D_B = 1/3$. Consequently, the prices are $p_A = 2/3$ and $p_B = 2/3$, and the total cost is $C = 1/9$. The total profit is $\pi = 1/3$.

The sufficient conditions for maximum profits involve the second derivatives; thus

$$\frac{\partial^2\pi}{\partial D_A^2} = -2, \quad \frac{\partial^2\pi}{\partial D_B^2} = -2, \quad \frac{\partial^2\pi}{\partial D_A \partial D_B} = -1.$$

Since

$$\begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = (-2)(-2) - (-1)(-1) = 3 > 0,$$

we have a maximum or minimum.

However, since

$$\frac{\partial^2 \pi}{\partial D_A^2} = -2 \quad \text{and} \quad \frac{\partial^2 \pi}{\partial D_B^2} = -2$$

are both negative, the values $D_A = 1/3$ and $D_B = 1/3$ give maximum profits.

● EXERCISES 68

1. The demand curves for 2 commodities are $p_A = 28 - 3D_A$ and $p_B = 22 - 2D_B$. The joint-cost function is $C = D_A^2 + 3D_B^2 + 4D_A D_B$. (a) Find the necessary and sufficient conditions for maximum profits. (b) Determine the prices, total cost, and profit.

2. The demand curves for 2 commodities are $p_A = 35 - 4D$ and $p_B = 26 - D_B$. The joint-cost function is $C = D_A^2 + D_B^2 + 3D_A D_B$. (a) Find the necessary and sufficient conditions for maximum profit. (b) Determine the prices, total cost, and profit.

3. The demand curves for 2 commodities are $p_A = 7$ and $p_B = 20$. The joint-cost function is $C = D_A^2 + 3D_B^2 + D_A D_B$. (a) Find the necessary and sufficient conditions for maximum profits. (b) Find the prices, total cost, and profits.

4. The demand functions of 2 commodities are $p_A = 80 - 8D_A$ and $p_B = 100 - 2D_B$. The joint-cost function is $C = 20D_A + 2D_B + 2D_A^2 + 2D_B^2 - 2D_A D_B$. (a) Find the necessary and sufficient conditions for maximum profit. (b) Determine the prices, total cost, and profits.

5. The demand curves for 3 commodities are $p_A = 10 - 3D_A$, $p_B = 20 - 5D_B$ and $p_C = 60 - 7D_C$. The joint-cost function is $C = 10 + 5D_A + 2D_B + 6D_C$. (a) Find the necessary conditions for maximum profits. (b) Determine the prices, total cost, and profit.

6. The demand functions for 3 commodities are $p_A = 21 - 5D_A$, $p_B = 77 - 10D_B$ and $p_C = 30 - 2D_C$. The joint-cost function is $C = 2D_A D_B + D_A D_C + 3D_B D_C$. (a) Find the necessary conditions for maximum profit. (b) Determine the prices, total cost, and profit.

7. The demand functions for 4 commodities are $p_A = 220 - 10D_A$, $p_B = 90 - 5D_B$, $p_C = 60 - 6D_C$, and $p_D = 30 - 2D_D$. The joint-cost function is $C = 5D_A + 9D_B + 5D_C + D_D + 2D_A D_B + D_A D_C + D_A D_D + 3D_B D_C + 2D_C D_D$. (a) Find the necessary conditions for maximum profits. (b) Find the prices, total cost, and profit.

8. The demand curves for 2 commodities are $p_A = 5 - D_A$ and $p_B = 5 - D_B$. The joint-cost function is $C = \log_e D_A D_B$. (a) Find the necessary and sufficient conditions for maximum profits. (b) What are the prices, total cost, and profits?

9. The demand curves for 2 commodities are $p_A = a - bD_A$ and $p_B = c - dD_B$. The joint-cost function is $C = mD_A^2 + nD_B^2 + qD_A D_B$, where a, b, c, d, m, n , and q are constants. (a) Find the necessary and sufficient conditions for maximum profits. (b) What are the prices?

69. Constrained Maxima and Minima

In mathematical economics we meet problems frequently in which certain functions have to be maximized and minimized under side conditions. For instance, in utility theory, utility has to be maximized, but the maximization is subject to the budget equation. In production theory, the profit of the firm has to be maximized, while the existence of a production function has to be taken into account, and so on.

Such problems are called *problems of constrained maxima and minima*, or *maxima and minima subject to side conditions*. The exact theory of these maxima and minima is beyond the scope of this book. We will state only without proof a method by means of which one can derive the necessary conditions for the existence of constrained maxima and minima.

We will state the conditions for problems in 2 variables. The results can be extended easily to any number of variables. Assume that we have a function of 2 variables

$$z = f(x, y).$$

This function $f(x, y)$ has to be maximized or minimized under the condition

$$g(x, y) = 0.$$

We form a new function

$$F(x, y) = f(x, y) + \lambda g(x, y),$$

where λ is a constant, the so-called *Lagrange multiplier*.

The necessary conditions for a maximum or minimum involve the partial derivatives of the function $F(x, y)$:

$$F_x(x, y) = f_x(x, y) + \lambda g_x(x, y) = 0,$$

$$F_y(x, y) = f_y(x, y) + \lambda g_y(x, y) = 0.$$

These 2 equations, together with the conditional equation $g(x, y) = 0$, permit us to determine in general the unknowns x, y , and the constant λ , associated with a maximum or minimum of z .

■ EXAMPLE

Assume that the function

$$z = f(x, y) = 10 + x + 3y - \frac{1}{2}x^2 - y^2$$

has to be maximized or minimized, under the condition that

$$g(x, y) = 2x + 3y - 13 = 0.$$

First we construct the function

$$\begin{aligned} F(x, y) &= f(x, y) + \lambda g(x, y) \\ &= 10 + x + 3y - \frac{1}{2}x^2 - y^2 + \lambda(2x + 3y - 13), \end{aligned}$$

where λ is an undetermined constant (Lagrange multiplier).

We find the partial derivatives of $F(x, y)$ as follows:

$$F_x(x, y) = 1 - x + 2\lambda,$$

$$F_y(x, y) = 3 - 2y + 3\lambda.$$

Putting the partial derivatives equal to 0 and using the conditional equation $g(x, y) = 0$, we get 3 linear equations in 3 unknowns; thus

$$-x + 2\lambda = -1,$$

$$-2y + 3\lambda = -3,$$

$$2x + 3y = 13.$$

The solution of this system of equations is $x = 43/17$, $y = 45/17$, $\lambda = 13/17$. Hence the values $x = 43/17$ and $y = 45/17$ give a maximum or minimum.

● EXERCISES 69

1. Find the maximum of $u = xy$ under the condition that $x + 3y = 5$. What is the maximum value of u ?

2. Find the minimum of $u = 2x^2 + 3y^2$ under the condition that $5x + 6y = 10$. Compare it with the unconstrained minimum.

3. Find the minimum of $u = x^2 + 3y^2 + 5z^2$ under the condition that $2x + 3y + 5z = 100$. What is the value of the constrained minimum of u ?

4. Find the maximum of $u = 10x + 20y - x^2 - y^2$ under the condition $2x + 5y = 10$. Compare it with the unrestricted maximum.

5. Find the minimum of $u = 3x + 6y$ under the condition that $x^2 + 5y^2 = 25$. What is the value of the constrained maximum?

6. Find the maximum of $u = e^{2y}$ under the conditions that $2x + 5y = 1$. What is the value of the constrained maximum?

7. Find the maximum of $u = xyz$ under the condition that $2x - 3y + z = 10$. What is the value of the constrained maximum?

8. Find the maximum of $u = \log_e xy$ under the condition that $2x + 3y = 5$. What is the value of the constrained maximum?

9. Let $u = ax^2 + bxy + cy^2$. Find the maximum or minimum of u if $mx + ny = k$, where a, b, c, m, n , and k are constants. What is the value of the constrained maximum or minimum?

10. Find the maximum of $u = kxy$ if $mx + ny = r$, where k, m, n , and r are constants. What is the value of the constrained maximum?

**11. Consider the maximum of $z = f(x, y)$ under the condition that $g(x, y) = 0$. Eliminate the Lagrange multiplier λ , and show that $f_x/f_y = g_x/g_y$. Show by implicit differentiation (Section 63) that the expressions on both sides of this equation are equal to the slope of the tangent $-dy/dx$.

**12. Show that the result of Problem 11 holds for the situation described in Example 1.

70. Utility Theory

Let us consider for the sake of simplicity the case in which an individual spends all his money on just 2 commodities X and Y .

There is a function $U = f(x, y)$, which (in a sense) indicates the satisfaction derived by the individual from varying combinations of the amounts of the commodities X and Y . The amounts of these commodities are x and y . We will see later that U need not be measurable (Problem 13, Exercises 70).

Assume that the prices of X and Y are p_x and p_y . They are established on the market independent of any actions of the individuals in question. Assume that his income I is also given. Then we have the so-called *budget equation*

$$p_x x + p_y y = I,$$

where p_x , p_y , and I are given constants.

The individual will try to maximize U by choosing appropriate amounts of X and Y while taking the condition represented by the budget equation into account.

This is a problem in restricted maxima. We form the new function

$$F(x, y) = f(x, y) + \lambda(p_x x + p_y y - I).$$

The partial derivatives of F are equated to 0; that is,

$$F_x(x, y) = 0,$$

$$F_y(x, y) = 0.$$

The budget equation should, in general, give enough equations to determine x , y , and λ . Then x and y are the quantities of the commodities X and Y demanded by the individual under the conditions stated above.

■ EXAMPLE

Let us consider the following equation:

$$U = 4x + 17y - x^2 - xy - 3y^2.$$

Let $p_x = 1$, $p_y = 2$, and $I = 7$. The budget equation is $x + 2y = 7$; U has to be maximized under this condition. We form the function $F(x, y) = 4x + 17y - x^2 - xy - 3y^2 + \lambda(x + 2y - 7)$. λ is a Lagrange multiplier. After differentiating $F(x, y)$ partially and equating both partial derivatives to 0, we have

$$\frac{\partial F}{\partial x} = 4 - 2x - y + \lambda = 0,$$

$$\frac{\partial F}{\partial y} = 17 - x - 6y + 2\lambda = 0.$$

Combining these 2 equations with the budget equation $x + 2y = 7$ provides $x = 1$ and $y = 3$ for the demand for X and Y .

● EXERCISES 70

1. Let the utility index be $U = xy$. Assume that $p_x = 1$, $p_y = 2$, and $I = 10$. Find the demand for X and Y .

2. Assume the utility index to be $U = 58x + 76y - 5x^2 - 2xy + 10y^2$. Let $p_x = 1$, $p_y = 3$, and $I = 14$. Find the demand for X and Y .

3. A utility function for food (x) and nonfood (y) has been derived for American data, 1935–1941 (J. A. Nordin). The utility function is $U = -0.000890x^2 + 0.008353y^2 + 0.022401xy + 104.572144x + 96.686771y$. Derive from this the demand for food and for nonfood if we have $p_x = 1$, $p_y = 1.1$, $I = 2500$.

4. A utility index is $U = e^{xy}$. Let $p_x = 1$, $p_y = 5$, and $I = 10$. Find the demand for X and Y .

5. A utility index is $U = x^2y^3$. Assume $p_x = 1$, $p_y = 4$, and $I = 10$. Find the demand for X and Y .

6. Assume that $U = xy$. Let $p_x = p$ (a constant), $p_y = 2$, and $I = 10$. (a) Find the demand for X and Y . (b) Find the elasticity of demand for X , that is $E_x/E_p = (dx/dp)(p/x)$. (c) Find the elasticity for $x = 5$.

7. Assume $U = xy$. Let $p_x = 1$, $p_y = q$ (a constant), and $I = 10$. (a) Find the demand for X and Y . (b) Determine the elasticity of the demand for Y , that is, $E_y/E_q = (dy/dq)(q/y)$. (c) Find the elasticity for $y = 10$.

8. Assume $U = xy$. Let $p_x = p$ (a constant), $p_y = q$ (a constant), and $I = 10$. (a) Find the demand for X and Y . (b) Determine the partial elasticities of demand $E_x/E_p = (\partial x/\partial p)(p/x)$, $E_x/E_q = (\partial x/\partial q)(q/x)$, $E_y/E_p = (\partial y/\partial p)(p/y)$, $E_y/E_q = (\partial y/\partial q)(q/y)$. (c) Evaluate the partial elasticities for $x = 5$, $y = 10$.

9. Assume that $U = xy$. Let $p_x = 1$, $p_y = 2$, and $I = J$ (a constant). (a) Find the demand for X and Y . (b) Find the income elasticities of demand $E_x/E_J = (dx/dJ)(J/x)$ and $E_y/E_J = (dy/dJ)(J/Y)$. (c) Evaluate the income elasticities for $J = 10$.

**10. An indifference curve is derived by letting $U = \text{constant}$. Take the utility index of Problem 1, and draw indifference curves for $U = 12, 12.5, 13$. Plot the budget equation and show that the point of maximum utility is determined at the point of tangency of the budget equation with an indifference curve.

**11. Marginal utility of X is $\partial U/\partial x$; marginal utility of Y is $\partial U/\partial y$. Show with the help of the data in Problem 1 that $(\partial U/\partial x)/p_x = (\partial U/\partial y)/p_y$, if utility is maximized. These are the weighted marginal utilities. They represent the marginal utility of \$1 worth of each commodity.

12. The marginal rate of substitution of X for Y is the amount of X which is necessary to compensate the individual for the loss of a small unit of Y . It is defined as $R = (\partial U/\partial x)/(\partial U/\partial y)$. (a) Show by the elimination of λ from the necessary conditions for maximum utility that we have always $R = p_x/p_y$. (b) Verify for the utility index in Problem 1.

**13. Utility theory assumes only the ordering and not the measurability of

Show that the new utility indexes $V = U^2$, $W = U^5$, and $Z = \log_e U$ give the same result for the demand for X and Y as the utility index in Problem 1 under the same conditions for prices and incomes.

****14.** Assume a utility index $V = g(U)$, where $U = f(x,y)$. Show with the help of the theorem for the differentiation of a function of a function that maximization of U and V leads to the same results for the demand for X and Y .

****15.** It is important to note that $-\lambda$ is the marginal utility of money. Prove this by using the results of Problem 11 and remember that the price of money is 1.

****16.** Take an indifference curve defined by $U(x,y) = c$, where c is constant.
(a) Find the slope of the indifference curve by partial differentiation. (b) Verify analytically the result of the graphical analysis in Problem 10. (HINT: Use the method indicated in Section 63.)

71. Production under Free Competition

Under free competition the individual firm cannot exert an influence upon the price of the product or upon the prices of the factors of production. They are determined independently of the actions of the firm on the markets. On each market there are so many firms that the action of any single one can be neglected.

Consider, for example, a firm which produces a product X and uses 2 factors A and B in the process of production. The production function is

$$x = f(a,b)$$

Here x is the amount of the product, and a and b the amounts of the factors. Denote by p_x the price of the product X , and by p_a and p_b the prices of the 2 factors of production. The profit of the firm is

$$\pi = xp_x - ap_a - bp_b.$$

It should be noted that $xp_x = R$, the total revenue, and that $ap_a + bp_b = C$, total cost.

The profit has to be maximized subject to the existence of the production function. We form the function

$$F(x,a,b) = xp_x - ap_a - bp_b + \lambda[x - f(a,b)],$$

where λ is a Lagrange multiplier.

The following equations are, in general, enough for the solution of the problem

$$\begin{aligned} F_x(x,a,b) &= p_x + \lambda = 0, \\ F_a(x,a,b) &= -p_a - \lambda f_a = 0, \\ F_b(x,a,b) &= -p_b - \lambda f_b = 0, \\ x &= f(a,b). \end{aligned}$$

An alternative method without the use of the Lagrange multiplier λ is

as follows: Substitute $x = f(a, b)$ into the expression for π . Then we have

$$\pi = p_x f(a, b) - ap_a - bp_b.$$

The necessary conditions for maximum profit are

$$\pi_a = p_x f_a(a, b) - p_a = 0,$$

$$\pi_b = p_x f_b(a, b) - p_b = 0.$$

■ **EXAMPLE**

Assume a production function of the form

$$x = f(a, b) = 10 - a^{-1} - b^{-1}.$$

The prices are $p_x = 9$, $p_a = 1$, $p_b = 4$. The profit function is $\pi = 9x - a - 4b$. This function has to be maximized under the condition that the relation between x , a , and b described by the production function holds.

We introduce the new function

$$F(x, a, b) = 9x - a - 4b + \lambda(x - 10 + a^{-1} + b^{-1}),$$

where the constant λ is a Lagrange multiplier. To deduce the necessary conditions for a maximum of profits, we find the partial derivatives and put them equal to 0:

$$F_x(x, a, b) = 9 + \lambda = 0,$$

$$F_a(x, a, b) = -1 - \lambda a^{-2} = 0,$$

$$F_b(x, a, b) = -4 - \lambda b^{-2} = 0.$$

We deduce from the first equation that $\lambda = -9$. We substitute this value into the two remaining equations and obtain

$$-1 + 9a^{-2} = 0,$$

$$-4 + 9b^{-2} = 0.$$

Hence, we have for the amounts of the factors, $a = 3$ and $b = 3/2$. From the production function we get the amount of the product, $x = 9$. Finally the profit equation gives $\pi = 72$.

As an alternate procedure, we can substitute the equation for x from the production function into the profit equation. This gives

$$\pi = 90 - 9a^{-1} - 9b^{-1} - a - 4b.$$

We have now to find the unrestricted maximum of the function with respect to a and b . We form the partial derivatives and put them equal to 0 in order to find the necessary conditions for a maximum, thus

$$\pi_a = 9a^{-2} - 1 = 0,$$

$$\pi_b = 9b^{-2} - 4 = 0.$$

These are exactly the same equations as before, and we get the same solutions for x , a , b and π .

● EXERCISES 71

1. The production function of a firm is $x = f(a,b) = 5 - a^{-1/2} - b^{-1/2}$. The prices are $p_x = 2$, $p_a = 1$, $p_b = 8$. Find the amounts of the factors demanded, the amounts of the product produced, and the amount of profits made by the firm.

2. The production function for a firm is $x = f(a,b) = 10 - a^{-2} - b^{-2}$. The prices are $p_x = 4$, $p_a = 27$, $p_b = 1$. Find the amounts of the factors demanded, the amount of the product supplied, and the profit made by the firm.

3. The production function of a firm is

$$x = \frac{12ab - a - b}{ab}.$$

The prices are $p_x = 9$, $p_a = 1$, $p_b = 4$. Find the amounts of the factors demanded, the amount of the product supplied, and the amount of profit made by the firm.

*4. Given the production function $x = f(a,b)$, the profit function becomes $\pi = p_x f(a,b) - p_a a - p_b b$. Find the sufficient conditions for a maximum of profit (see Section 67).

5. Show that the sufficient conditions for profit maximization are fulfilled for the production function given in the example above and for the functions in Problems 1 to 3.

6. Assume a production function for a commodity in the form $x = f(a,b)$. Maximize profits. Show that in equilibrium the marginal productivities are $f_a(a,b) = p_a/p_x$, $f_b(a,b) = p_b/p_x$, and also that

$$\frac{f_a(a,b)}{f_b(a,b)} = \frac{p_a}{p_b};$$

that is, the ratio of the marginal productivities, when in equilibrium, is equal to the price ratio. (HINT: Eliminate the Lagrange multiplier λ .) Apply to the data in Problem 1.

7. An implicit function of the variables x , a , and b , denoted by $g(x,a,b) = 0$, is called a transformation function. Derive the equilibrium conditions of production under the assumption that profit is being maximized while the transformation function $g(x,a,b) = 0$ is satisfied.

8. The production function of a firm is $x = f(a,b,c) = 20 - a^{-1} - b^{-1} - c^{-1}$. The prices are $p_x = 4$, $p_a = 1$, $p_b = 9$, and $p_c = 16$. Find the amounts of the factors demanded, the amount of the product produced, and the total profit made by the firm.

9. The production function of a firm is $x = f(a,b) = 20 - a^{-1} - b^{-2}$. The prices are $p_x = 5$, $p_a = 20$, and $p_b = 40$. (a) Find the necessary and sufficient conditions for a maximum of profits. (b) Determine the amounts of the factors demanded, the amount of the product produced, and the amount of profit made by the firm.

**10. Use the results of Problem 6 to give a geometric interpretation for the first-order conditions of profit maximization. (HINT: Consider da/db for fixed x , the constant product curve.)

ELEMENTS OF INTEGRATION

72. Indefinite Integrals

Indefinite integration is the inverse of the process of differentiation; that is, it consists in finding a function whose derivative is given. We write

$$\int f(x) dx = F(x) + c,$$

where c is an arbitrary constant, if $F'(x) = f(x)$. The constant c is added for the purpose of generality, since the derivative of a constant is 0; that is, the result of differentiation is not affected if an arbitrary constant is added to the function $F(x)$. The c is called the *constant of integration*, and $f(x)$ is known as *the integrand*.

■ EXAMPLE 1

Evaluate the indefinite integral $\int (2x) dx$. The problem is to find the function whose derivative is $2x$. We know that the derivative of x^2 is $2x$. Hence the desired value is $x^2 + c$, where c is an arbitrary constant. To check, we compute the derivative

$$\frac{d}{dx}(x^2 + c) = 2x.$$

So we write

$$\int (2x) dx = x^2 + c.$$

There are rules for integration that are similar to those for differentiation; for instance, *the integral of a sum is the sum of the integrals; the integral of a*

difference is the difference of the integrals; and the integral of a constant times a function is the constant times the integral of the function. The demonstration of these rules depends in each instance on the fact that integration is the operation which is inverse to differentiation.

To illustrate these rules we compute some more integrals.

■ **EXAMPLE 2**

$$\int (3x^2) dx = x^3 + c.$$

For a check we have

$$\frac{d}{dx} (x^3 + c) = 3x^2.$$

This is the given integrand.

■ **EXAMPLE 3**

Show that

$$\int (3x^2 + 2x) dx = \int (3x^2) dx + \int (2x) dx = x^3 + x^2 + c,$$

where c may be regarded as the arbitrary constant that is the sum of the constants provided by the 2 integrations.

To check this result we note that

$$\frac{d}{dx} (x^3 + x^2 + c) = 3x^2 + 2x.$$

This is the given integrand; It should be observed that in this problem we have employed the fact that the integral of a sum is the sum of the integrals.

■ **EXAMPLE 4**

Show that

$$\int (3x^2 - 2x) dx = x^3 - x^2 + c.$$

To confirm this result, we differentiate

$$\frac{d}{dx} (x^3 - x^2 + c) = 3x^2 - 2x.$$

This result is the given integrand; so the confirmation is complete. We note that

$$\int (3x^2 - 2x) dx = \int (3x^2) dx - \int (2x) dx;$$

that is, the integral of the difference is the difference of the integrals.

■ EXAMPLE 5

Show that

$$\int (4x) dx = 2 \int (2x) dx = 2(x^2 + k) = 2x^2 + c,$$

where $2k = c$. This is the correct result for the given integral, since

$$\frac{d}{dx}(2x^2 + c) = 4x.$$

Moreover, $\int (2x) dx = x^2 + c$, as can readily be confirmed by differentiation. So we observe the truth of the statement that the integral of a constant times a function is the constant times the integral of the function.

To facilitate computation it is customary to learn the integrals of common functions; we give below a short table of indefinite integrals of various function.

1. $\int dx = x + c.$
2. $\int x^n dx = x^{n+1}/(n+1) + c, n \neq -1.$
3. $\int e^x dx = e^x + c.$
4. $\int e^{ax} dx = \frac{e^{ax}}{a} + c.$
5. $\int \frac{dx}{x} = \log_e x + c.$

All these formulas can be verified by differentiation.

■ EXAMPLE 6

Show that

$$\int (x^2 - 2x + 1) dx = \frac{x^3}{3} - x^2 + x + c.$$

$$\int (x^2 - 2x + 1) dx = \int x^2 dx - 2 \int x dx + \int dx.$$

But

$$\int x^2 dx = \frac{x^3}{3} + c, \text{ by Formula 2;}$$

$$\int x dx = \frac{x^2}{2} + c, \text{ by Formula 2;}$$

and

$$\int dx = x + c, \text{ by Formula 1.}$$

So we obtain the desired confirmation,

$$\int x^2 dx - 2 \int x dx + \int dx = \frac{x^3}{3} - x^2 + x + c,$$

where the last c is a composite of all the arbitrary constants provided by the 3 formulas.

● **EXERCISES 72**

1. Evaluate $\int (1 - 2x + x^2) dx$. Check.
2. Evaluate $\int (x - 5x^2 + x^4) dx$. Check.
3. Evaluate $\int (10/x) dx$. Check.
4. Evaluate $\int 15 \sqrt{x} dx$. Check.
5. Evaluate $\int 17 dx/\sqrt[3]{x}$. Check.
6. Evaluate $\int 3 e^x dx$. Check.
7. Evaluate $\int (1 + 2x^2 + 8x^5) dx$. Check.
8. Evaluate $\int 20 dx/\sqrt[4]{x}$. Check.
9. Evaluate $\int (1 - 2/x^6) dx$. Check.
10. Evaluate $\int (a + b\tilde{x}^m)$, (a, b, m constants.) Check.

73. Marginal Cost, Total Cost, Average Cost

Since marginal cost $C' = dC/dD$, we have total cost $C = \int C' dD$ and average cost $A = C/D$. This follows from the definition of marginal cost as the derivative of total cost.

Integration introduces an arbitrary constant. But if, besides the marginal-cost function, the value of the total cost at any arbitrary value is also given, we can determine the function completely.

■ EXAMPLE

The marginal-cost curve for some product is $C' = 1 + 2D + 6D^2$.
 (a) Find the total-cost curve if $C(0) = 100$. By integration,

$$C = \int (1 + 2D + 6D^2) dD = D + D^2 + 2D^3 + c.$$

From the additional condition that $C(0) = 100$, $0 + 0^2 + 2(0^3) + c = 100$. Hence $c = 100$, and $C = 100 + D + D^2 + 2D^3$. (b) Find the average-cost curve $A = C/D$. Obviously $A = C/D = 100/D + 1 + D + 2D^2$.

● EXERCISES 73

1. The marginal-cost function is $C' = 1 + 100D - 5D^2$. (a) Find C if $C(1) = 200$. (b) Find A .

2. $C' = 10/D + 5$. (a) Find C if $C(1) = 20$. (b) Find A . (c) Evaluate C , C' , and A for $D = 10$.

3. $C' = 2 + 5e^D$. (a) Find C if $C(0) = 100$. (b) Find A . (c) Evaluate C , C' , and A for $D = 60$.

4. $C' = 100 - 2D + 5D^2 + D^4$. (a) Find C if $C(0) = 100$. (b) Find A . (c) Evaluate C , C' , and A for $D = 10$.

5. $C' = 100\sqrt{D}$. (a) Find C if $C(9) = 20$. (b) Find A . (c) Evaluate C , C' , and A for $D = 16$.

6. $C' = 10\sqrt[3]{D}$. (a) Find C if $C(8) = 100$. (b) Find A . (c) Evaluate C , C' , and A for $D = 27$.

7. $C' = 100/\sqrt{D}$. (a) Find C if $C(16) = 100$. (b) Find A . (c) Evaluate C , C' , and A for $D = 100$.

8. The marginal-cost curve of steel is $C' = 56$. Find the total-cost curve and average-cost curve if $C(0) = 182$.

9. The marginal-cost curve of hosiery is $C' = 6.75 - 0.0006D$. Find the total-cost curve and the average-cost curve if $C(0) = -10,485$.

10. The marginal-cost curve of leather belts is $C' = 0.8$. Find the total-cost curve and the average-cost curve if $C(10) = 3,008$.

11. The marginal-cost curve for sales in a department store is $C' = 1.052 - 0.004D$. Find the total-cost curve and average-cost curve if $C(0) = 16.8$.

74. Definite Integrals

Suppose we know the equation of a curve,

$$y = f(x).$$

We will show how to find the area under the curve and above the X axis,

between the limits $x = a$ and $x = b$. This is illustrated in Figure 28.

One way to proceed is to find the approximate area. Divide the interval from a to b into n parts of equal width,

$$\Delta x = \frac{b - a}{n}.$$

The points of subdivision, including $x = a$ and $x = b$, are

$$\begin{aligned} x_0 &= a, \\ x_1 &= a + \Delta x, \\ x_2 &= a + 2(\Delta x), \\ x_3 &= a + 3(\Delta x), \\ &\dots\dots\dots \\ x_n &= a + n(\Delta x) = b. \end{aligned}$$

The area to be evaluated can be approximated by the sum of the areas of n rectangles, namely,

$$f(x_1)(\Delta x) + f(x_2)(\Delta x) + f(x_3)(\Delta x) + \dots + f(x_n)(\Delta x).$$

This sum can be written in a more concise way, that is,

$$\sum_{i=1}^n f(x_i)(\Delta x).$$

We will get a better approximation to the area if we have more and more rectangles. In the limit, as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)(\Delta x) = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(x_i)(\Delta x) = \int_a^b f(x) dx.$$

The last expression in this formula is the definite integral, and it is defined as

$$[F(x)]_a^b = F(b) - F(a), \text{ where } \int f(x) dx = F(x) + c.$$

The fact that the definite integral gives the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is one of the most important results in mathematics and is known as the *fundamental theorem of the integral calculus*. A proof of this theorem is found in most works on the calculus.

We show, in Figure 27, an approximation to the area under the curve

$$y = f(x) = x^2$$

between $x = 1$ and $x = 3$.

We have divided the interval into $n = 2$ equal parts. Hence

$$\Delta x = 1.$$

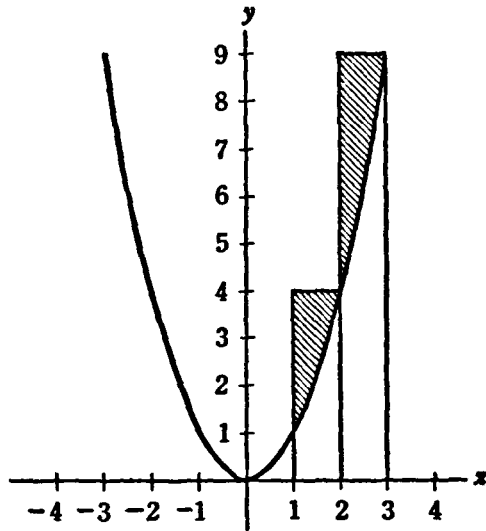


Figure 27

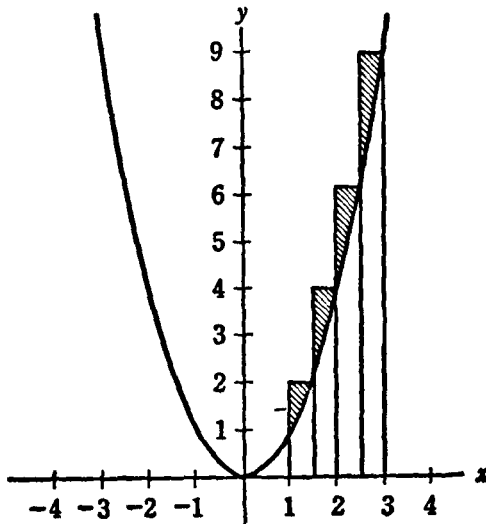


Figure 28

The 2 rectangles have the base $\Delta x = 1$ and the heights $f(1) = 1$ and $f(2) = 4$, respectively. Hence the sum of the areas of the 2 rectangles in our first approximation is

$$(1)(1) + (4)(1) = 5.$$

The shaded area represents the error committed in this approximation.

Next we divide the same interval from a to b , that is, from 1 to 3, into 4 equal parts. This means that $n = 4$. Hence $\Delta x = (b - a)/n = 1/2$. Now the 4 rectangles have the base $\Delta x = 1/2$ and the heights $f(1.5) = 2.25$, $f(2) = 4$, $f(2.5) = 6.25$, $f(3) = 9$, respectively.

In this second approximation to the area under the curve, the sum of the areas of the rectangles is

$$(2.25)(0.5) + (4)(0.5) + (6.25)(0.5) + (9)(0.5) = 10.75.$$

The error committed in our second approximation is represented again by the shaded area in Figure 28. It is apparent that the total error is smaller than in the previous case (Figure 27), which represented $n = 2$ or $\Delta x = 1$.

It seems reasonable to expect that our approximations will get better and better, and that the total error will become smaller and smaller as we continue this process of taking n larger and larger, or making Δx smaller and smaller.

Finally as $n \rightarrow \infty$ and $\Delta x \rightarrow 0$, the limit of the sum of the rectangular areas gives the true area under the curve. By the fundamental theorem of the integral calculus, this area can be computed accurately. It is

$$\int_1^3 x^2 dx = \left[\frac{x^3}{3} \right]_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = 9 - \frac{1}{3} = \frac{26}{3} = 8.667.$$

In the expression for the definite integral, namely, $\int_a^b f(x) dx$, x is said to be the variable of integration. The lower limit is a and the upper limit is b .

The sign \int is an elongated S; it is the integral sign, and stands for *sum*.

Since x is only the variable of integration, a dummy variable, it does not in itself matter. It may be replaced by any other variable, say t , and the result is the same. Thus

$$\int_a^b f(t) dt = \int_a^b f(x) dx.$$

To further examine the definite integral, let us put

$$b = x,$$

and define the function

$$G(x) = \int_a^x f(t) dt.$$

This is the area under the curve $y = f(t)$ from $t = a$ to $t = x$.

By putting $x = a$, we have

$$G(a) = \int_a^a f(t) dt = 0;$$

that is, the area under the curve $y = f(t)$ from $t = a$ to $t = a$ is 0.

Consider the derivative of the function $G(x)$. We have by the definition of the derivative

$$G'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta G}{\Delta x},$$

where

$$\frac{\Delta G}{\Delta x} = \frac{G(x + \Delta x) - G(x)}{\Delta x}.$$

But $G(x + \Delta x) - G(x)$ is the area under the curve from x to $x + \Delta x$. As seen from the graph, the magnitude of this area is between the mag-

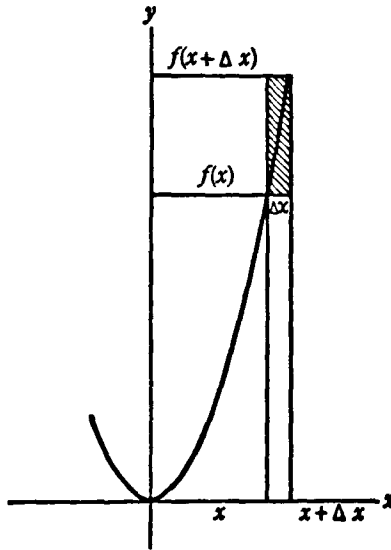


Figure 29

nitudes of the areas of 2 rectangles, both with base Δx ; one with height $f(x)$ and the other with height $f(x + \Delta x)$; in symbols,

$$f(x)(\Delta x) < G(x + \Delta x) - G(x) < f(x + \Delta x)(\Delta x).$$

The difference between the 2 rectangular areas is shaded in Fig. 29. We divide the members of the inequality by Δx , and get

$$f(x) < \frac{G(x + \Delta x) - G(x)}{\Delta x} < f(x + \Delta x).$$

If we let $\Delta x \rightarrow 0$, the term in the middle becomes the derivative $G'(x)$. Also, $f(x + \Delta x) \rightarrow f(x)$; so, in the limit,

$$G'(x) = f(x).$$

This result is equivalent to the fundamental theorem of the calculus already stated. We have shown that the derivative of the definite integral considered as a function of the upper limit is the function representing the curve under which the area is sought.

■ **EXAMPLE 1**

Find the area under the curve

$$y = f(x) = x^3,$$

from $x = 1$ to $x = 3$.

The indefinite integral is

$$\int x^3 dx = \frac{x^4}{4} + c.$$

To check this,

$$\frac{d}{dx} \left(\frac{x^4}{4} + c \right) = x^3.$$

Hence

$$\int_1^3 x^3 dx = \left[\frac{x^4}{4} \right]_1^3 = \frac{(3)^4}{4} - \frac{(1)^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

This is the area under the curve from $x = 1$ to $x = 3$.

■ **EXAMPLE 2**

Find the area under the curve $y = x$ from $x = 1$ to $x = 2$. The desired area is given by

$$\int_1^2 x dx = \left[\frac{x^2}{2} \right]_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}.$$

● **EXERCISES 74**

Evaluate each of the following. In each case describe the area that is obtained.

- | | |
|---|-----------------------------------|
| 1. $\int_1^5 x^2 dx.$ | 5. $\int_1^{16} 2\sqrt{x} dx.$ |
| 2. $\int_0^3 (1 - 2x + x^2) dx.$ | 6. $\int_0^{27} 5\sqrt[3]{x} dx.$ |
| 3. $\int_{-1}^3 (1 + 3x - x^3) dx.$ | 7. $\int_0^2 (e^x + 10) dx.$ |
| 4. $\int_{-2}^2 (x^2 - 2x - 5) dx.$ | 8. $\int_1^2 10 dx/x.$ |
| 9. $\int_0^1 (a + bx) dx$, where a and b are constants. | |
| 10. $\int_0^1 x^m dx$, where m is a constant not equal to -1 . | |

75. Consumers' Surplus

A demand curve shows the amount people would buy at a given price. Let the market price be p_0 . At this market price D_0 units are sold. Then everyone who would be willing to pay more than the market price gains from the fact that the price is only p_0 . The gain is called *consumers' surplus* (Marshall). It represents the total money gain if the situation with a market price p_0 is compared with perfect discrimination. Under perfect discrimination a monopolist would extract from each customer for each unit of the commodity the maximum price he would be willing to pay for it.

If we make the assumption of constant marginal utility of money and also that all people have the same utility function, then the area measured by $\int_0^{D_0} p dD - p_0 D_0$ can be interpreted as *gain in utility*. It is the area under the demand curve minus the total revenue and is called the *consumers' surplus*

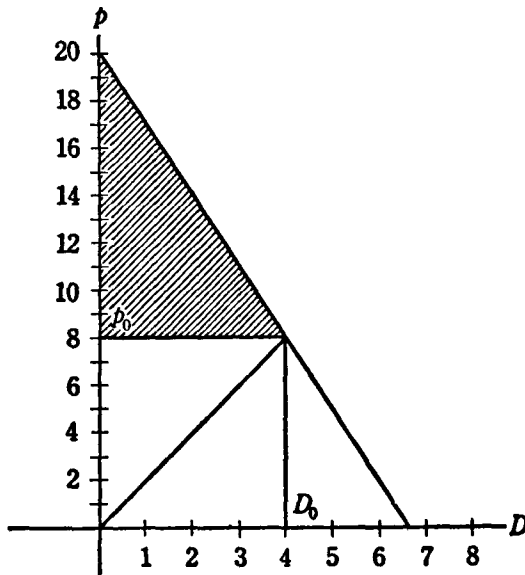


Figure 30

EXAMPLE 1

Assume, for instance, a demand function of the form

$$20 - 3D = p.$$

The quantity demanded is a function of the price

$$D = \frac{20}{3} - \frac{p}{3}.$$

It should be recalled from the definition of a collective-demand function that this represents the amount which people will buy on the market at various prices. From the graph it appears that the higher the price, the smaller the demand.

Let the supply function on this market be $p = 2D$. The quantity supplied is a function of the price $D = \frac{1}{2}p$. The market equilibrium is established at the point where demand equals supply; that is, where $20/3 - p/3 = \frac{1}{2}p$.

The solution of this equation is $p_0 = 8$, and from the demand or supply function, $D_0 = 4$. At a price of \$8, the demand is equal to the supply, and the quantity exchanged is 4 units.

But it appears from our graph that there are people who are willing to pay as much as \$20 for the commodity in question. All these and others who would have been willing to pay more than the market price $p_0 = \$8$ profit from the fact that they have only the market price to pay.

Their total monetary gain could have been extracted by a clever monopolist who might be able to practice perfect discrimination. It is represented by the shaded area in the graph.

This shaded area is the area under the demand curve from 0 to D_0 minus the product p_0D_0 , that is, the rectangle which represents the actual outlay of the consumers. We have

$$\begin{aligned} \int_0^{D_0} p dD - p_0 D_0 &= \int_0^4 (20 - 3D) dD - (8)(4) \\ &= \left[20D - \frac{3D^2}{2} \right]_0^4 - 32 = 56 - 32 = 24. \end{aligned}$$

Hence the consumers' surplus in our case is \$24. This represents the monetary savings of the public because of the existence of a free market on which everybody, rich or poor, can buy at the prevailing-market price.

EXAMPLE 2

Let $p = 1 - D$. Assume that $p_0 = 1/3$; then $D_0 = 2/3$. The consumers' surplus is

$$\int_0^{2/3} (1 - D) dD - \frac{2}{9} = \left[D - \frac{D^2}{2} \right]_0^{2/3} - \frac{2}{9} = \frac{2}{9}.$$

EXERCISES 75

1. Let the demand curve be $p = 10 - 2D$. Evaluate the consumers' surplus for (a) $p_0 = 1$; (b) $p_0 = 3$; (c) $p_0 = 5$; (d) $p = 0$ (free good). (e) Make a graph.
2. The demand function for a commodity is $p = 36 - D^2$. Find the consumers' surplus for (a) $p_0 = 11$; (b) $p_0 = 5$; (c) $p_0 = 6$; (d) $p_0 = 0$. (e) Make a graph.
3. The demand function for a commodity is $p = 100 - 2D - D^2$. Find the

consumers' surplus for (a) $p_0 = 1$; (b) $p_0 = 5$; (c) $p_0 = 0$ (free good). (d) Illustrate by a graph.

4. The demand function of a commodity is $p = 81 - D^4$. Find the consumers' surplus for (a) $p = 0$ (free good); (b) $p_0 = 10$; (c) $p_0 = 36$. (d) Indicate them graphically.

5. The demand function of a commodity is $p = 100 - 5D - D^2$. Find the consumers' surplus for (a) $p_0 = 0$; (b) $p_0 = 11$; (c) $p_0 = 25$. (d) Illustrate with a graph.

6. Consider the demand curve for sugar in Problem 9, Exercises 56. Find the consumers' surplus (a) under monopoly; (b) under free competition; (c) if sugar were a free good. (d) Compare the 3 situations graphically

7. Consider the demand curve for steel in Problem 10, Exercises 56. Find the consumers' surplus (a) under monopoly; (b) under free competition; (c) if steel were a free good. (d) Compare the 3 situations graphically.

8. Consider the situation described in Problem 1, Exercises 9. Evaluate the consumers' surplus before and after the imposition of various specific taxes and subsidies. Illustrate by a graph.

9. Use the data in Problem 2, Exercises 9, to evaluate the consumers' surplus before and after the imposition of various taxes or subsidies. Illustrate by a graph.

10. Use the data in Problem 3, Exercises 9, and evaluate the consumers' surplus before and after the imposition of the taxes and subsidies indicated. Show also, by a graph, the effects of taxes and subsidies on consumers' rent.

11. Use the data in Problem 6, Exercises 9, to evaluate the consumers' surplus before and after the imposition of the indicated taxes or subsidies for sugar. Illustrate by a graph.

12. Use the data in Problem 8, Exercises 9. Consider the effect on consumers' surplus of the tax or subsidy indicated for agricultural products. Illustrate by a graph.

III

**PROBABILITY AND
STATISTICS**

PROBABILITY

76. Frequency Definition of Probability

It is important to remember that probability and statistics are concerned with mass phenomena, and never with a single event. It makes no sense to talk about probability if there are no mass phenomena to which the probability refers.

Various definitions of probability have been given. The so-called *classical definition* of probability is as follows:

If there are N exhaustive, mutually exclusive and equally likely cases, and among these N cases there are M favorable to some event, then the probability of this event is the ratio M/N .

Consider, for instance, throws with a coin. There are 2 possible events, heads or tails. If the coin is true, these 2 events have the same likelihood. Consider the event of obtaining a head. Among the 2 equally likely cases, heads or tails, there is 1 favorable, that is, the case of obtaining a head. Hence the probability of obtaining a head with a true coin is $1/2$.

Next, consider the toss of a die. There are 6 possible events ($N = 6$), that is, to obtain 1, 2, 3, 4, 5, or 6. In a true die these events are equally likely. Consider the event of throwing a 2. This is one favorable case ($M = 1$) among 6 equally likely cases. So the probability of throwing a 2 with a true die is $1/6$.

There are various objections against the classical definition of probability. It is circular since it involves the notion of probability (that is, equal probability) within the definition of probability itself. More important from our point of view is the fact that it is impossible to state in economic and social matters what events are equally probable.

We will use a different definition of probability, which seems to be more

adequate, at least for the field of social and economic statistics. It is the so-called *frequency definition* of probability. This definition starts with an empirical notion, the idea of *relative frequency*. Assume that we make a number of trials, say n trials. We observe a certain event in whose probability we are interested. Let us assume that in those n trials the event happens m times. Then the *relative frequency* of the event is m/n .

For example, assume that a coin is tossed 100 times ($n = 100$). We want to find the relative frequency of heads, and we observe that heads turn up 53 times in these 100 trials ($m = 53$). Hence the relative frequency of heads is $53/100$.

Next, suppose we throw a die 600 times ($n = 600$). We are interested in the relative frequency of obtaining 2. Among the 600 throws of the die we observe that 2 turns up 95 times ($m = 95$). Hence the relative frequency of 2 in our experiment is $95/600$.

We define *probability* as the limiting value of the relative frequency as the number of trials approaches infinity; that is,

$$p = \lim_{n \rightarrow \infty} \frac{m}{n}.$$

Here we denote by p the probability of the event whose relative frequency is m/n .

There is another condition which must be fulfilled if we are to be permitted to talk about probability and apply the laws of probability. The sequence of experiments or trials which becomes infinite must be a *random sequence*.

A sequence is a random sequence if the following condition holds: Suppose we make an arbitrary selection from the trials. We take, for example, every second trial, or every fifth, or make any other arbitrary selection from our collection of trials. In general, we select n_1 trials out of our n trials, according to any arbitrary principle. Assume further that we observe that the event whose probability we want to compute occurs m_1 times among the n_1 trials. Then we must have

$$p = \lim_{n_1 \rightarrow \infty} \frac{m_1}{n_1}.$$

In this formula p is the probability. *This same relationship must be true also for any other arbitrary selection from our original series of trials.*

Let, for instance, n_2 be a set of trials selected by another principle from the original series of n trials. There are now m_2 cases of the occurrence of our event. We have again

$$p = \lim_{n_2 \rightarrow \infty} \frac{m_2}{n_2}.$$

The same is also true for a third selection, and so on.

We conclude that in order for probability to exist in the frequency sense it is necessary that the following condition holds: *The limiting value of the relative frequency in any arbitrary selection from the sequence of our original trials must have the same value as the limiting value of the relative frequency of the event in the series of original trials. This limit is called the probability of the event.*

EXAMPLE 1

A die is tossed 60 times. We note below the trial number and the result recorded:

Trial Number	Result	Trial Number	Result	Trial Number	Result
1	2	21	3	41	3
2	1	22	5	42	5
3	3	23	4	43	4
4	1	24	4	44	2
5	5	25	1	45	4
6	3	26	5	46	1
7	5	27	5	47	1
8	1	28	3	48	3
9	6	29	6	49	5
10	2	30	5	50	2
11	4	31	2	51	5
12	5	32	5	52	4
13	4	33	5	53	2
14	1	34	2	54	3
15	5	35	6	55	2
16	4	36	2	56	6
17	5	37	1	57	1
18	5	38	1	58	1
19	1	39	5	59	4
20	1	40	1	60	4

We are interested in the probability of throwing a 2 with this particular die. The total number of trials is $n = 60$. Among these 60 trials 2 is thrown 9 times; that is, $m = 9$. Hence we have

$$\frac{m}{n} = \frac{9}{60} = 0.15.$$

We should expect that this ratio m/n would converge toward a number p if we made more and more trials, say 6,000 or 60,000 trials instead of just 60. With a true die this limit, which is the probability, would be $1/6 = 0.166667$.

To test the randomness of our sequence we make a selection. Suppose we take only the odd trials, that is, the trials number 1, 3, 5, 7, \dots , 59. There are, in our selection, 30 trials, that is, $n_1 = 30$. Among these 30 trials

there are four 2's ($m_1 = 4$). The relative frequency of 2's in such a selection is

$$\frac{m_1}{n_1} = \frac{4}{30} = 0.133.$$

This is not too different from the relative frequency in the original series which was $m/n = 0.15$.

Next we make a second selection. We take only the even trials, that is, numbers 2, 4, 6, 8, \dots , 60. There are again 30 trials in our selection ($n_2 = 30$). Among the 30 even trials 2 occurs 5 times. The relative frequency of 2's is

$$\frac{m_2}{n_2} = \frac{5}{30} = 0.167.$$

Again, this is not too different from the relative frequency in our original trials, namely, $m/n = 0.15$.

We make another selection, by taking only the trials which correspond to prime numbers. A prime number is a number which is divisible, without remainder, only by 1 and by itself. The prime numbers up to 60 are 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59. There are altogether 18 prime numbers in this sequence. The trials corresponding to them constitute our third selection; that is, $n_3 = 18$. Among the 18 trials corresponding to prime numbers, we find three 2's. Hence $m_3 = 3$.

The relative frequency of 2's in the selection of trials corresponding to prime numbers is

$$\frac{m_3}{n_3} = \frac{3}{18} = 0.167.$$

This is again not very different from the relative frequency of 2's in our original 60 trials.

Next we make a selection in the following way: We take all the trials which come after a 6 has turned up. This gives us trial numbers: 10, 30, 36, 57. There are only 4 selected trials, or $n_4 = 4$. Among these 4 trials, we have two 2's, so $m_4 = 2$. Hence the relative frequency is

$$\frac{m_4}{n_4} = \frac{2}{4} = 0.5.$$

In view of the very small number in this selection this result may also be considered not too different from the original relative frequency, $m/n = 0.15$.

■ EXAMPLE 2

Toss a coin 50 times. Note the number of the trial and the result, head or tail. Compute the relative frequency of heads. Make the following selections: (a) all even trials; (b) all trials whose numbers are divisible by 3; (c) all trials whose numbers are prime numbers; (d) all trials which occur after a tail. Compute the relative frequencies of heads in all these selections and compare with the original relative frequency in the 50 trials.

■ EXAMPLE 3

A true coin is thrown many times. Denote a head by H and a tail by T . Then the series of trials is

No. of Trial:	1	2	3	4	5	6	7	8	9	10	...
Outcome:	H	T	T	H	T	H	T	T	H	H	...

If this series is continued, we obtain, in the limit, $p = 1/2$ for the probability of a head. Let us make a selection from our trials. Take for instance all even-numbered trials: $T H H T H \dots$. The limit of the number of heads thrown divided by the total number of trials in the selection must again be $p = 1/2$. Now take all throws which come after throwing a head: $T T T H \dots$. Again we must have in the limit, $p = 1/2$, and similarly for all other selections.

● EXERCISES 76

1. A true die has its center of gravity correctly placed so that no side is favored. Suppose that this true die is thrown a great many times. Care is taken that the outcome of each throw is independent of all the previous throws. (a) What is the probability of throwing a 6? (b) What is the probability of throwing a 6 after a 1 has been thrown? (c) What is the probability of throwing a 6 after two 3's? (d) What is the probability of throwing a 6 after ten 1's?

2. Let us assume an infinite highway. There is a white milestone every mile for 9 miles and a red milestone every 10 miles. Take as a sample the first 100 milestones. (a) What is the relative frequency of a red milestone among all milestones? (b) What is the limit of this relative frequency? (c) Consider the relative frequency of red milestones in a sample of all milestones which follow a red milestone. (d) What is the limit of this relative frequency? (e) Consider the relative frequency of red milestones in a sample of all milestones which follow 2 white milestones. (f) What is the limit of this relative frequency? (g) Consider the relative frequency of a red milestone in a sample of all milestones which follow 3 white milestones. (h) What is the limit of this relative frequency? (i) Consider the relative frequency of red milestones in a sample of all milestones which follow a white milestone. (j) What is the limit of this relative frequency? (k) Compare all the computed limits. Is there a probability of red milestones among all milestones in the sense defined above?

3. A skillful gambler throws a die 30 times. The results are: 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6. He can repeat this sequence indefinitely. (a) What is the relative frequency of obtaining a 2 among all throws in the sample? (b) What is the limit of this relative frequency? (c) Consider the relative frequency of a 2 in a sample of all throws which follow a 1. (d) What is the limit of this relative frequency? (e) What is the relative frequency of a 2 in a sample of all throws which follow a 5? (f) Find the limit of this relative frequency. (g) What is the relative frequency of a 2 in a sample consisting of all odd throws? (h) What is the limit of the relative frequency? (i) Find the relative frequency of a 2

in a sample which consists of all even throws. (j) What is the limit of this relative frequency? (k) Compare all the computed limits. Can we talk about the probability of obtaining a 2 in the sense defined above?

4. An urn contains red and white balls. A ball is drawn, its color noted, and it is replaced in the urn. There is an irregular sequence of red and white balls. The limit of the relative frequency of red balls is $1/2$. The same limit appears approximately if we take as our samples all throws which follow a white ball, all throws which follow 2 red balls, all even throws, all odd throws, etc. All these selections give a result very close to $1/2$. Can we assert under these conditions that the probability of obtaining a red ball is $1/2$?

5. A gambler throws a coin 12 times and obtains the sequence $H, T, T, H, T, T, H, T, T, H, T, T$. He can continue this sequence indefinitely. (a) Find the limiting relative frequency of a head among all throws. (b) Find the limiting relative frequency of a head in a sample of all even throws. (c) Find the limiting relative frequency of a head in a sample of all odd throws. (d) Consider the limiting relative frequency of a head among a sample of all throws which follow a head. (e) Find the limiting relative frequency of a head among a sample of all throws which follow 2 tails. (f) Consider all the limits computed and discuss the existence of the probability of obtaining a head.

6. Consider the data in the experiment given in Example 1. (a) Compute the relative frequency of a 1 in the total sequence. (b) Make a selection of all trials whose number is divisible by 3 and compute the relative frequency of 1's. (c) Take all trials which occur after a 2 and compute the relative frequency of 1's in this new sequence. (d) Flip a coin 60 times and take all the trials from the original series which correspond to heads turning up. Compute the relative frequency of 1's in the new selection.

7. Use the same methods as indicated in Problem 6 to find the relative frequency of an even number. Compare the result with the probability of an even number turning up with a true die.

8. Use the same methods as indicated in Problem 6 to find the relative frequency of an odd number. Compare the result with the probability of an odd number turning up with a true die.

9. The following table gives the number of male and female births in the United States:

Year	Male Births	Female Births
1935	1,105,489	1,049,616
1936	1,099,465	1,045,325
1937	1,130,641	1,072,696
1938	1,172,541	1,114,421
1939	1,162,600	1,102,988
1940	1,211,684	1,148,715
1941	1,289,734	1,223,693
1942	1,444,365	1,364,631
1943	1,506,959	1,427,901
1944	1,435,301	1,359,499

Compute the relative frequency of male birth using the following data: (a) all years given in the table; (b) only the even years; (c) the odd years. (d) Flip a coin 10 times, 1 time for each year, and take each year for which a head turns up. (e) What do you conclude about the probability of a male birth in the United States?

77. Laws of Probability

Probability is defined as the limit of the relative frequency of an event; thus

$$p = \lim_{n \rightarrow \infty} \frac{m}{n},$$

assuming that the sequence of events is a random sequence. This is the case if in any number of arbitrary selections among the original sequence the relative frequencies tend to the same limit; that is,

$$\lim_{n \rightarrow \infty} \frac{m}{n} = \lim_{n_1 \rightarrow \infty} \frac{m_1}{n_1} = \lim_{n_2 \rightarrow \infty} \frac{m_2}{n_2} = \dots = p.$$

No relative frequency can be larger than 1. It will be 1 if the event occurs always in the n trials, that is, if $m = n$. The relative frequency cannot be smaller than 0. It will be 0 if the event does not happen at all in the n trials, that is, $m = 0$.

Since probability is the limit of relative frequency m/n , where $0 \leq m \leq n$, the probability must be a positive number between 0 and 1. A probability of one corresponds to *certainty*. For instance, it is certain that a coin must come up heads or tails. On the other hand, a probability of 0 corresponds to *impossibility*. It is impossible, for instance, to throw a 7 with a single die.

Addition theorem

Let us consider 2 mutually exclusive events, say A and B . There are altogether n trials. Among the n trials the event A occurs m_A times. The relative frequency of the event A is m_A/n . The event B , on the other hand, occurs m_B times. The relative frequency of the event B is m_B/n .

Now consider the event, which happens if *either* A or B occurs. This will happen in $m_A + m_B$ of the n trials. The relative frequency of the event (either A or B) is

$$\frac{m_A + m_B}{n} = \frac{m_A}{n} + \frac{m_B}{n}.$$

Thus the relative frequency of the event, either A or B , is the sum of the relative frequencies of the mutually exclusive events A and B .

The probability of A is the limit of the relative frequency of A . The probability of B is the limit of the relative frequency of B . Denote the probability of A by p_A and the probability of B by p_B . Consequently, the prob-

ability of the event "either A or B " is

$$p = p_A + p_B.$$

This theorem can be generalized for any number of exclusive events. *The probability that either one of a number of mutually exclusive events happens is the sum of their probabilities.*

■ EXAMPLE 1

Consider a perfect die. The probability of throwing 1 is $1/6$; the probability of throwing 2 is also $1/6$. The probability of throwing *either* 1 or 2 with an unbiased die is

$$p = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Multiplication theorem

Again let us consider n trials. Among the n trials there are m_A trials in which the event A occurs. Hence m_A/n is the relative frequency of A and its limit is the probability of A . Now consider another event B . Among the m_A trials in which the event A occurs, there are \bar{m}_B in which we *also* have B . The relative frequency of the combined event, A and B , is \bar{m}_B/n . Its limit is the probability that A and B will happen together.

The relative frequency of A and B can be written

$$\frac{\bar{m}_B}{n} = \frac{\bar{m}_B}{m_A} \cdot \frac{m_A}{n}.$$

On the right side the second fraction is simply the relative frequency of the event A . Its limit is the probability of A . But the first term on the right side is the relative frequency of the event B among the trials in which the event A has occurred. Its limit is the so-called *conditional probability*.

Denote now the probability of A as p_A , the probability of getting A and B by p , and the conditional probability of B , if it is known that A has occurred, by $p_{B(A)}$.

By passing to the limit as $n \rightarrow \infty$ in the above formula, we obtain

$$p = p_{B(A)}p_A.$$

■ EXAMPLE 2

Consider an urn in which there are 2 red balls and 3 black balls. The probability of drawing a red ball at random will be denoted by p_A ; we have

$$p_A = 2/5$$

since there are 2 red balls among the 5 balls in the urn.

Consider now the probability of drawing a black ball, *after a red ball has been drawn* (and not replaced). This conditional probability will be denoted

by $p_{B(A)}$. We have

$$p_{B(A)} = 3/4.$$

When 1 red ball has been drawn, 4 balls are left in the urn; 3 of these 4 balls are black. Hence the conditional probability of the event B , after A is known to have happened, is $3/4$.

Consider finally the probability of drawing a red ball (event A) and then a black ball (event B). We have from the above formula

$$p = (2/5)(3/4) = 6/20 = 3/10.$$

A very important special case of the last consideration is the case of *independent* events A and B . In this case the relative frequency of B is independent of the fact whether A has occurred or not, so we have

$$\frac{\bar{m}_B}{m_A} = \frac{m_B}{n}.$$

Denoting the limit of the relative frequency of the trials with the event B among all the trials by p_B , we have from the above formula

$$p = p_A p_B.$$

The probability that 2 *independent* events will happen together is the product of their probabilities.

The theorem can be extended to any number of independent events. That is, *the probability that a number of independent events will happen together is the product of their probabilities.*

■ EXAMPLE 3

Consider the independent throwing of 2 unbiased coins, say a dime and a nickel. What is the probability of obtaining 2 heads?

The probability of obtaining a head with the dime is $1/2$, since the coin is unbiased. The probability of obtaining a head with the nickel is $1/2$, since this coin is also unbiased. The coins are thrown independently. It follows that the probability of obtaining 2 heads is the product of the 2 probabilities; that is,

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

● EXERCISES 77

1. A true die is thrown. Find the probabilities of obtaining (a) either 1 or 2; (b) either 2 or 5 or 6; (c) an even number; (d) an odd number; (e) any number but 5; (f) either 1 or 2 or 3 or 4 or 5 or 6.
2. Three exclusive events have the probabilities $p_A = 1/3$ for A , $p_B = 1/6$ for B , $p_C = 1/2$ for C . What are the probabilities for obtaining (a) either A or B ? (b) either A or C ? (c) either B or C ? (d) either A , B , or C ?
3. The probability of obtaining a head with a coin is p , the probability of ob-

taining a tail is q . What is the probability of obtaining either head or tail? What conclusions can be drawn about the sum of the 2 probabilities?

4. Three exclusive events have the probabilities $p_A = 0.5$, $p_B = 0.2$, $p_C = 0.3$. Find the probabilities of the following contingencies: (a) either A or B ; (b) either A or C ; (c) either B or C ; (d) either A , B , or C .

5. A perfect die is cast 2 times. Find the probability of obtaining the sum 8. (HINT: Consider all possible contingencies.)

6. If A , B , C , \dots are a series of exclusive events which exhaust all possibilities, it follows that $p_A + p_B + p_C + \dots = 1$. Explain.

7. A half dollar, a dime, and a nickel are thrown independently. All are true coins. Find the probability (a) of obtaining heads with all of them; (b) of getting tails with all of them; (c) of getting a head with the half dollar and tails with the other 2 coins; (d) of getting a head with the half dollar, a tail with the dime, and a head with the nickel in 3 independent throws of the 3 coins.

8. Events A , B , C , D are independent. Their probabilities are, respectively $p_A = 1/8$, $p_B = 1/2$, $p_C = 1/4$, $p_D = 1/10$. Find the probabilities that the following events will happen together: (a) A and B ; (b) A and C ; (c) A and D ; (d) B and C ; (e) B and D ; (f) C and D ; (g) A , B and C ; (h) A , B and D ; (i) A , C and D ; (j) B , C and D ; (k) A , B , C and D .

9. Five perfect coins are thrown independently. What is the probability (a) of obtaining only heads; (b) only tails; (c) heads with the first 2 coins and tails with the rest; (d) heads with the first coin and tails with the others?

10. Two perfect dice are cast independently. Find the probabilities of obtaining (a) the sum 6; (b) the sum 2; (c) the sum 12. (HINT: Consider all contingencies.)

11. A total of n perfect dice are cast independently. Determine the probability of obtaining as the sum the number (a) n ; (b) $6n$.

12. An urn contains 2 red balls, 5 black balls, and 3 white balls. Find the following probabilities assuming that none of the drawn balls are replaced: (a) the conditional probability of obtaining a black ball, after a red ball has been drawn; (b) the probability of drawing a red ball and then a black ball; (c) the conditional probability of drawing a white ball after a red ball has been drawn; (d) the probability of drawing a red ball and white ball in succession; (e) the conditional probability of drawing a black ball after a white ball has been drawn; (f) the probability of drawing a white and a black ball in succession; (g) the conditional probability of drawing a red ball, after a white ball has been drawn; (h) the probability of drawing in succession a white and a red ball.

78. Probability Distributions

A random variable is a variable which can assume a number of values with given probabilities. Random variables are also called *stochastic* or *chance variables*. For instance, the outcome of the throws with a die is a random variable, since it can assume the values 1, 2, 3, 4, 5, 6 with certain probabilities.

The array of the values which the random variable can have together with the probabilities of these values is called *the probability distribution*. It is apparent that the sum of all the probabilities must be 1. This follows from the addition theorem of probabilities.

EXAMPLE 1

Consider a throw with 2 unbiased coins. The coins are thrown independently. The random variable x is the number of heads. The following contingencies are equally probable:

Coin 1	Coin 2
H	H
H	T
T	H
T	T

The probability of each contingency listed in the previous table is $1/4$. Hence we obtain the following probability distribution for x , the number of heads:

x	p
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$
	1

The sum of probabilities is 1. This follows from the fact that, with 2 coins, either no heads, 1 head, or 2 heads must come up, so the sum of the probabilities of all these contingencies is *certainty*: that is, $p = 1$.

We show in Figure 31 a so-called histogram. This is a graphical representation of the probability distribution given above. Note that the various rectangles are drawn in such a way that their *centers* are at the values of x given in the table; the height of each rectangle corresponds to the probability associated with its respective value of x .

It should be noted that the sum of the areas of all the rectangles in Figure 31 is 1. This corresponds to the fact that the sum of all the probabilities is 1.

From the above table we can compute the so-called *cumulative probability distribution*. This is derived by adding the probabilities successively:

	Cumulative Probability
x	
0	$\frac{1}{4}$
1	$\frac{3}{4}$
2	1

Here $1/4$ is the probability that x is less than or equal to 0; $3/4$ is the probability that x is less than or equal to 1; 1 is the probability that x is less than or equal to 2.

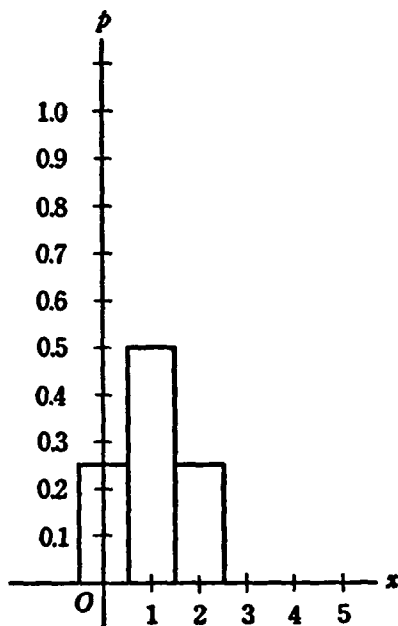


Figure 31

For a random variable which can vary continuously between the values $x = a$ and $x = b$, we define a *probability density*: $p(x)$. The function $p(x)$ is such that the integral

$$\int_c^d p(x) dx$$

(where $c \geq a$, $d \leq b$) gives the probability that x will be somewhere between c and d . Since it is certain that the random variable x must assume values between a and b , we have

$$\int_a^b p(x) dx = 1.$$

This follows from the fact that the probability that x will be somewhere in the permissible interval a to b is certainty.

The cumulative probability is now given by

$$\int_a^{x_0} p(x) dx.$$

This is the probability that x will be smaller than or equal to a given value $x_0 \leq b$.

EXAMPLE 2

Consider a random variable x which can assume all possible values between 2 and 6 with *equal* probability. We have $a = 2$, $b = 6$. This circumstance yields the probability density, $p(x) = 1/4$, $2 \leq x \leq 6$. This is con-

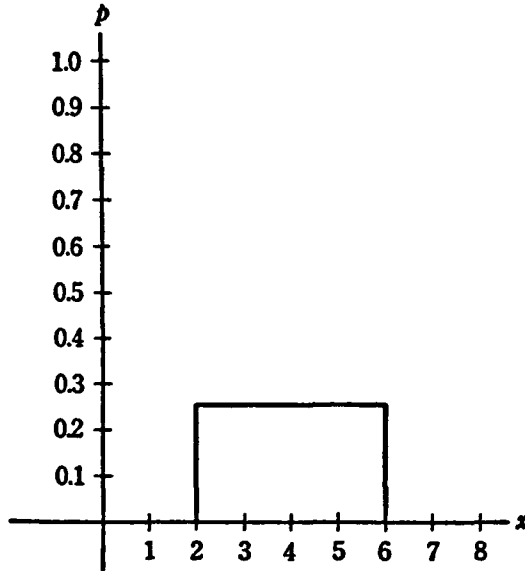


Figure 32

firmed by the fact that the integral of the probability density function over the whole range from $x = 2$ to 6 is

$$\int_a^b p(x) dx = \int_2^6 \left(\frac{1}{4}\right) dx = \left[\frac{1}{4}x\right]_2^6 = \left(\frac{1}{4}\right)6 - \left(\frac{1}{4}\right)(2) = 1.$$

This result corresponds to the fact that the area under the curve shown in Figure 32 is 1. Probability distributions of this type are called *rectangular probability distributions*.

The cumulative distribution for the previous illustration is given by

$$\int_a^{x_0} p(x) dx = \int_2^{x_0} \frac{1}{4} dx = \left[\frac{1}{4}x\right]_2^{x_0} = \frac{x_0}{4} - \frac{1}{2}.$$

For instance, the probability that x will be smaller than or equal to 5, $x_0 = 5$, is

$$\frac{5}{4} - \frac{1}{2} = \frac{3}{4}.$$

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The probability that x will be equal to or smaller than 4, $x_0 = 4$, is

$$\frac{4}{4} - \frac{1}{2} = \frac{1}{2}.$$

By integration, we have also for the probability that $3 \leq x \leq 4$

$$\int_3^4 \frac{1}{4} dx = \left[\frac{1}{4} x \right]_3^4 = \frac{4}{4} - \frac{3}{4} = \frac{1}{4}.$$

EXAMPLE 3

The probability distribution of the various results of throws with a true die is as follows:

x	p
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

The sum of the probabilities is 1.

EXAMPLE 4

The probability density of a rectangular probability distribution is defined by $p(x) = 1/4$, $-2 \leq x \leq 2$. This is sometimes said to be the probability that x will fall between x and $x + dx$, where dx is the differential of x . We have

$$\int_{-2}^2 \left(\frac{1}{4} \right) dx = \left[\frac{x}{4} \right]_{-2}^2 = 1.$$

EXERCISES 78

1. The probability distribution of a variable X is

X	p
0	0.5
1	0.2
2	?

Find the probability of $X = 2$. Check. Make a histogram.

2. A rectangular probability distribution is defined as $p(X) = k$, $0 \leq X \leq 5$. Determine k . Check. Make a graph of the probability density function. Find the cumulative distribution.

3. A given probability distribution is as follows:

X	p
1	a
2	0.5
3	a

Find (a) the probability of $X = 1$ and $X = 3$. Check. (b) Find the cumulative distribution. (c) Make a histogram.

4. A probability distribution is given by $p(x) = e^{-x}$, $0 \leq x \leq \infty$. (a) Prove that the integral is 1. (HINT: Evaluate $\int_0^{\infty} e^{-x} dx = G(u)$ and consider $\lim_{u \rightarrow \infty} G(u)$.)

(b) Find the cumulative-probability distribution. (c) What is the probability that $0 \leq x \leq 3$? (d) that $1 \leq x \leq 5$?

5. $p(x) = (1/2)(1/2)^x$, $x = 1, 2, \dots$. (a) Prove that the sum of the probabilities is 1. (b) What is $p(3)$? (c) $p(6)$? (d) $p(1)$? (e) Make a graph.

6. Normal distribution follows the law $p(x) = e^{-x^2/2\pi}/\sqrt{2\pi}$, where $\pi = 3.14159$. Make a graph.

7. A probability distribution follows an arithmetic progression for $x = 1, 2, 3, 4$. We are given that $p(1) = 0.4$, $p(2) = 0.3, \dots$. (a) Establish the complete distribution. (b) Prove that the sum of the probabilities is 1.

8. Take the normal distribution in Problem 6. Where does the probability attain its maximum? The value of x which maximizes the probability is called *the mode*. What is the value of the maximum probability?

9. Consider the probability distribution of throwing x heads independently with 3 perfect coins. Establish the probability distribution, the cumulative-probability distribution, and make a histogram. (HINT: See Example 1.)

10. Consider the probability of throwing x heads independently with 4 perfect coins. Establish the probability distribution, the cumulative-distribution, and make a histogram. (HINT: See Example 1.)

*11. Consider the probability of throwing the sum x with 2 perfect dice. Establish the probability distribution, the cumulative-probability distribution, and make a histogram.

RANDOM VARIABLES

79. Mathematical Expectation

A random variable has been defined as a variable which can assume a number of values with definite probabilities. The values which the random variable can assume together with the probabilities are called the *probability distribution*.

We describe in the following pages a number of ways in which a probability distribution can be characterized. One of the most important characteristics of the probability distribution is the mathematical expectation, or mean value, of the random variable. This is the arithmetic mean of the probability distribution. We will however reserve the term arithmetic mean for the sample, and call the mean value the mathematical expectation, or true mean, when dealing with probabilities. It is also called the population mean.

The mathematical expectation is computed by multiplying all possible values of the random variable by their respective probabilities, and then summing the products. It characterizes the general location or central tendency of the distribution.

Assume first that we have a discrete random variable, which can assume the values x_1, x_2, \dots, x_m with the respective probabilities p_1, p_2, \dots, p_m . The mathematical expectation of the random variable x is denoted by $Ex = \mu$, and is computed by the formula

$$\mu = Ex = x_1p_1 + x_2p_2 + x_3p_3 + \dots + x_m p_m = \sum_{i=1}^m x_i p_i.$$

If we have a continuous variable x , let us assume that it can vary between a and b ; that is, $a \leq x \leq b$. Moreover, let the probability density of the distribution of x be given by $f(x)$. In the formula above obtained for a discrete variable, we replace summation by integration and get for the mathe-

mathematical expectation of the random variable x :

$$\mu = Ex = \int_a^b xp(x) dx.$$

EXAMPLE 1

Consider a discrete variable having the following probability distribution:

x	p	xp
0	0.5	0.0
1	0.3	0.3
2	0.2	0.4
$\mu = Ex =$	$\overline{0.7}$	

We show in Figure 33 a histogram of the above probability distribution. We have indicated the mathematical expectation μ by a heavy vertical line.

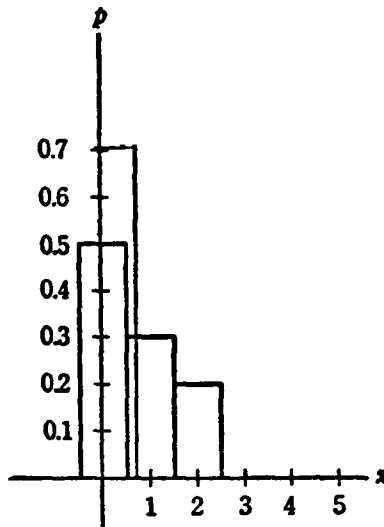


Figure 33

It is apparent from the graph that the mathematical expectation, or population mean, is a good way of characterizing the general location, or central tendency, of the probability distribution.

EXAMPLE 2

Consider the case of a continuous variable that possesses the probability-density function, $p(x) = 1/3, 0 \leq x \leq 3$. We have

$$\mu = Ex = \int_0^3 \left(\frac{x}{3}\right) dx = \left[\frac{x^2}{6}\right]_0^3 = \frac{3}{2}.$$

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We show in Figure 34 the probability density and the corresponding mathematical expectation, or population mean, μ . It will be obvious again that μ provides a good measure of the general location of the distribution.

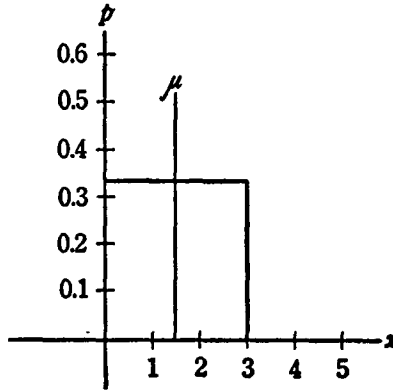


Figure 34

An important proposition which is demonstrated in advanced books on probability is the *law of large numbers*. It may be formulated as follows:

It is almost certain that the difference between the theoretical mean value (mathematical expectation) and the empirical arithmetic mean of a number of random variables becomes as small as desired, if the number of random variables is sufficiently large.

● EXERCISES 79

1. Consider the following probability distribution:

x	p	
1	0.4	0.4
2	0.4	0.4
3	0.2	0.6

Find $Ex = \mu$. Make a histogram and indicate the population mean.

2. Let $p(x) = 1/10, -5 \leq x \leq 5$. Find $Ex = \mu$. Make a graph of the distribution and indicate the population mean.

3. Take the following probability distribution:

x	p
0	0.2
1	0.2
2	0.2
3	0.2
4	0.2

Find $Ex = \mu$.

4. Given the probability distribution $p(x) = 1/a, 0 \leq x \leq a$. Find $Ex = \mu$.
5. Consider the probability distribution:

x	p
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

Find $Ex = \mu$. Make a histogram and indicate the population mean.

6. Given the probability distribution $p(x) = 1/2a, -a \leq x \leq a$. Find $Ex = \mu$.
7. Take the following probability distribution:

x	p
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

Find $Ex = \mu$.

8. Given the probability distribution $p(x) = 1/(b - a), a \leq x \leq b$. Find $Ex = \mu$.
9. Assume the probability distribution:

x	p
0	$\frac{1}{27}$
1	$\frac{2}{9}$
2	$\frac{4}{9}$
3	$\frac{8}{27}$

Find $Ex = \mu$.

10. Given the probability distribution $p(x) = 2 - 2x, 0 \leq x \leq 1$. Find $Ex = \mu$. Make a graph of the distribution and indicate the population mean.

80. Computations with Mathematical Expectations

The concept of mathematical expectation is important in the derivation of various theorems in modern statistics. We will indicate briefly some rules for employing the notion of mathematical expectation in various computations.

(a) *The mathematical expectation of a random variable times a constant is the constant times the mathematical expectation of this random variable.*

We give the proof for a *discrete* random variable. Let us define a random variable x which can assume the values x_1, x_2, \dots, x_n with the respective

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probabilities p_1, p_2, \dots, p_m . Its probability distribution is

x	p
x_1	p_1
x_2	p_2
x_3	p_3
.	.
.	.
.	.
x_m	p_m

As indicated previously, the mathematical expectation, or the population mean, of x is given by

$$\mu_x = Ex = x_1p_1 + x_2p_2 + \dots + x_mp_m = \sum_{i=1}^m x_i p_i$$

Let us introduce a new random variable, say $v = cx$, where c is a constant. The probability distribution of v is

v	p
$v_1 = cx_1$	p_1
$v_2 = cx_2$	p_2
.	.
.	.
.	.
$v_m = cx_m$	p_m

We have for the mathematical expectation of v (which is x times the constant c):

$$\begin{aligned} \mu_v = Ev &= Ecx = v_1p_1 + v_2p_2 + \dots + v_mp_m = \sum_{i=1}^m p_i v_i \\ &= cx_1p_1 + cx_2p_2 + \dots + cx_mp_m = \sum_{i=1}^m cx_i p_i \end{aligned}$$

Evidently c can be factored out of the summation, and we have

$$\mu_v = Ev = c(x_1p_1 + x_2p_2 + \dots + x_mp_m) = c\left(\sum_{i=1}^m x_i p_i\right) = cEx = c\mu_x$$

This establishes our conclusion in the case of a discrete variable: *The mathematical expectation of a random variable times a constant is the constant times the mathematical expectation of the random variable.*

■ EXAMPLE 1

Assume the probability distribution of a random variable x given as follows:

x	p
2	0.25
4	0.75

The mathematical expectation of x is

$$\mu_x = Ex = (2)(0.25) + (4)(0.75) = 3.5.$$

Define a new variable, say $v = 5x$. We have for its probability distribution

v	p
10	0.25
20	0.75

Its mathematical expectation is

$$\mu_v = Ev = (10)(0.25) + (20)(0.75) = 17.5.$$

Hence

$$Ev = 17.5 = (5)(3.5) = (5)Ex,$$

which is the conclusion predicted by the rule given.

(b) *The mathematical expectation of a sum of random variables is the sum of the mathematical expectations.*

■ EXAMPLE 2

Assume 2 independent random variables whose probability distributions are given below:

x	p	y	q
1	0.5	1	0.25
2	0.5	2	0.5
		3	0.25

The mathematical expectations are

$$\mu_x = Ex = (1)(0.5) + (2)(0.5) = 1.5,$$

$$\mu_y = Ey = (1)(0.25) + (2)(0.5) + (3)(0.25) = 2.$$

Let us find the mathematical expectation of the sum $w = x + y$. Since, by assumption, the random variables x and y are independent, we can construct a table which gives the probability of the simultaneous occurrence of the possible values of x and y . The probabilities are computed by multiply-

ing the probabilities of x and y . The following table represents a *joint-probability distribution* of the random variables x and y :

	y	1	2	3
x				
1		0.125	0.25	0.125
2		0.125	0.25	0.125

For instance, from the original tables, we find that the probability that $x = 1$ is 0.5. The probability that $y = 1$ is 0.25. The probability that $x = 1$ and $y = 1$, simultaneously, is $(0.5)(0.25) = 0.125$; this value is found at its appropriate position in the new table.

From the table above we can derive the following probability distribution for $w = x + y$:

w	P
2	0.125
3	0.375
4	0.375
5	0.125

This is derived in the following way: To make the sum $w = x + y = 2$, we must have $x = 1$ and $y = 1$. The probability for this circumstance from the table above is 0.125. To get a sum $w = 3$, we have either $x = 1$ and $y = 2$, or $x = 2$ and $y = 1$. The probability of the first of these 2 contingencies is 0.25; the probability of the second is 0.125. Now the addition theorem of probabilities applies; that is, the probability of $w = 3$ is $0.25 + 0.125 = 0.375$, and so forth.

By direct calculation, employing the values in the new table, it is determined that the mathematical expectation of the sum w is 3.5. But this value is

$$Ex + Ey = \mu_x + \mu_y = 1.5 + 2.$$

This emphasizes that the mathematical expectation of the sum of 2 random variables x and y is the sum of their mathematical expectations.

(c) *The mathematical expectation of the product of 2 independent random variables is the product of their mathematical expectations.*

■ EXAMPLE 3

Using the same random variables as in Example 2, we derive from the table of the simultaneous distribution of x and y the distribution of the

product $z = xy$

z	Q
1	0.125
2	0.375
3	0.125
4	0.25
6	0.125

The mathematical expectation of z is

$$\mu_z = Ez = (1)(0.125) + (2)(0.375) + (3)(0.125) + (4)(0.25) + (6)(0.125) = 3.$$

It is true also that

$$Ez = Exy = 3 = (1.5)(2) = (Ex)(Ey) = \mu_x \cdot \mu_y.$$

This result emphasizes that the mathematical expectation of the product of 2 independent random variables is the product of their mathematical expectations.

(d) *The mathematical expectation of a constant is equal to the constant.*

We can consider the constant as a random variable which assumes the one particular value c with certainty (probability 1). Hence the mathematical expectation of a constant is equal to the constant itself.

It should be noted that the mathematical expectation μ itself is a constant. Hence we have

$$E(Ex) = Ex = \mu.$$

EXAMPLE 4

Consider the random variables x, y, z . Assume that they are independent, and that their mathematical expectations are

$$\mu_x = Ex = 1,$$

$$\mu_y = Ey = -2,$$

$$\mu_z = Ez = 5.$$

What is the mathematical expectation of the function

$$w = 3 + 2x - 4xy + 4yz?$$

From our previous rules we have

$$\begin{aligned} \mu_w &= Ew = 3 + 2Ex - 4(Ex)(Ey) + (Ey)(Ez), \\ &= 3 + 2(1) - 4(1)(-2) + 4(-2)(5) = -27. \end{aligned}$$

EXERCISES 80

1. Assume we have 2 independent random variables x and y . Let $Ex = 6$, $Ey = 8$. Find (a) $E2x$; (b) $E3y$; (c) $E(x + 3)$; (d) $E(3y - 6)$; (e) $E(x + y)$; (f)

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$E(y - x)$; (g) $E(2x - 4y)$; (h) $E(2x + 5y - 3)$; (i) Exy ; (j) $E(5xy)$; (k) $E(xy - 4x + 5y - 6)$; (l) $E(3x + y - xy + 6)$.

2. Assume 3 independent random variables x , y and z . Let the mathematical expectations be $E_x = 1$, $E_y = -5$, $E_z = 6$. Find (a) $E2x$; (b) $E4y$; (c) $E5z$; (d) $E(x + y + z)$; (e) $E(x - 2y + z)$; (f) $E(2x - y)$; (g) $E(5y - 3x)$; (h) Exy ; (i) Exz ; (j) Eyz ; (k) $E(3xy - 6yz + 4xz)$.

3. Assume 2 independent random variables. The probability density of x is $p(x) = 1/4$, $1 \leq x \leq 5$. The probability density of y is $p(y) = 1/3$, $3 \leq y \leq 6$. (a) Find E_x ; (b) E_y ; (c) $E(x + y)$; (d) $E(x - y)$; (e) $E(2x - 5y)$; (f) $E(3x - y + 8)$; (g) Exy .

4. Assume the following probability distributions of 2 independent random variables:

x	p	y	q
2	0.25	0	0.25
4	0.25	1	0.5
6	0.25	2	0.25
8	0.25		

(a) Compute the mathematical expectations μ_x and μ_y . (b) Compute the mathematical expectations of $E(x + y)$; $E(x - y)$; Exy by formula and directly by the methods in Examples 2 and 3.

5. Use the data in Example 2 to compute the following: (a) $E(2x - y)$; (b) $E(3x + 2y)$; (c) $E(3x - 2y - 6)$. Use both the formula and the method utilized in the example.

6. Use the data in Example 2 to compute the following expectations: (a) $E(3xy)$; (b) $E(-xy)$; (c) $E(10 - 2xy)$. Use both the formula and the direct method of Example 3.

7. Call the random variable x in Problem 1, Exercises 79, and call the random variable y in Problem 2, Exercises 79. Find the following expectations, assuming independent distributions: (a) $E(x - y)$; (b) Exy ; (c) $E(3x - 5y + 10)$; (d) $E(100 - 5xy)$.

MOMENTS

81. Moments about the Origin

Assume a random variable x and its probability distribution as given. Instead of computing the mathematical expectation of x we can also compute the mathematical expectation of any arbitrary function of x , for instance, x^2 , $\log x$, e^x , $1/x$, and so forth. A particularly useful class of these functions comprises the *moments*.

In general, the k th moment about the origin of a random variable x is defined as the mathematical expectation of the k th power of x ; that is,

$$\mu_k' = Ex^k, \quad k = 0, 1, 2, \dots$$

For a discrete variable x , which assumes the values x_1, x_2, \dots, x_m with the respective probabilities p_1, p_2, \dots, p_m , we define the k th moment about the origin as

$$\mu_k' = Ex^k = \sum_{i=1}^m x_i^k p_i, \quad k = 0, 1, 2, \dots$$

Assume that the random variable x varies continuously between a and b . The probability density of x is $p(x)$. The k th moment about the origin is

$$\mu_k' = Ex^k = \int_a^b x^k p(x) dx, \quad k = 0, 1, 2, \dots$$

It should be noted that the summation employed for discrete variables has been replaced by integration.

From the definition of the k th moment about the origin, we have

$$\mu_0' = Ex^0 = E1 = 1.$$

This says that the 0th moment about the origin is 1. Also,

$$\mu_1' = Ex^1 = Ex = \mu.$$

Thus the first moment about the origin is μ , the mathematical expectation of the random variable x itself. This is the *mean value of the distribution of x , or the mean of the population.*

Higher order moments about 0 have no immediate interpretation.

EXAMPLE 1

Find the third moment about 0 of the probability distribution

x	p	x^3p
0	0.5	0.0
1	0.3	0.3
2	0.2	1.6

$$\mu_3' = Ex^3 = 0 + 0.3 + 1.6 = 1.9.$$

EXAMPLE 2

Let $p(x) = 1/3, 0 \leq x \leq 3$. We have

$$\mu_4' = Ex^4 = \int_0^3 \left(\frac{x^4}{3}\right) dx = \left[\frac{x^5}{15}\right]_0^3 = \frac{243}{15} = \frac{81}{5}.$$

EXERCISES 81

- Given the probability distribution in Problem 1, Exercises 79. Find $\mu_5' = Ex^5$; $\mu_3' = Ex^3$.
- Given the probability distribution in Problem 2, Exercises 79. Find $\mu_2' = Ex^2$; $\mu_3' = Ex^3$.
- Given the probability distribution in Problem 3, Exercises 79. Find $\mu_2' = Ex^2$; $\mu_4' = Ex^4$.
- Given the probability distribution in Problem 4, Exercises 79. Find $\mu_2' = Ex^2$; $\mu_4' = Ex^4$.
- Given the probability distribution in Problem 5, Exercises 79. Find $\mu_2' = Ex^2$; $\mu_5' = Ex^5$.
- Given the probability distribution in Problem 6, Exercises 79. Find $\mu_3' = Ex^3$; $\mu_6' = Ex^6$.
- Given the probability distribution in Problem 7, Exercises 79. Find μ_2' ; μ_3' .
- A probability distribution is of the form $p(x) = 1/a, 0 \leq x \leq a$. Find a if $\mu_1' = Ex = \mu = 2$. Check.
- Given the probability distribution

x	p
0	a
1	b
2	c
3	d

Find the probabilities a, b, c, d , if $\mu_1' = Ex = \mu = 1.5, \mu_2' = Ex^2 = 2.9, \mu_3' = Ex^3 = 6.3$. Check.

****10.** Given the probability distribution $p(x) = 1, 0 \leq x \leq 1$. Find the formula for the k th moment about the origin.

****11.** Given the probability distribution $p(x) = 1/(b - a), a \leq x \leq b$. Find the formula for the k th moment about the origin.

82. Moments about the Mathematical Expectation

A very important class of moments are the moments about the mathematical expectation. The k th moment about the mathematical expectation is the mathematical expectation of the k th power of the deviation of x from its mathematical expectation $Ex = \mu$; that is,

$$\mu_k = E(x - \mu)^k = \sum_{i=1}^m (x_i - \mu)^k p_i.$$

Thus, to obtain μ_k , we compute first the mathematical expectation μ . This is subtracted from the values of x , and the difference thus obtained is raised to the k th power. The result is multiplied in each case by the corresponding probability; then the products are summed.

As a second illustration, consider a continuous random variable x , whose probability density p is $p(x)$. We have for the k th moment about the mathematical expectation

$$\mu_k = E(x - \mu)^k = \int_a^b (x - \mu)^k p(x) dx.$$

Let us now consider the moments about the mathematical expectation for various values of k :

$$\mu_0 = E(x - \mu)^0 = E1 = 1.$$

Thus the moment of order 0 about the mathematical expectation is 1. To study the first moment, we recall that the mathematical expectation of a constant is equal to the constant; moreover, the mathematical expectation $Ex = \mu$ is itself such a constant. Hence, $E(Ex) = Ex = \mu$. So we have as a conclusion,

$$\mu_1 = E(x - \mu) = Ex - E(\mu) = \mu - \mu = 0.$$

It follows that the first moment about the mathematical expectation is 0. Next we consider the second moment:

$$\mu_2 = E(x - \mu)^2 = \sigma^2.$$

The second moment about the mathematical expectation is very important in the field of statistics, and is known as the *dispersion of the distribution*. It shows the manner in which the various items x are distributed about the mean value $Ex = \mu$. It is called also the *population variance*, and designated

by σ^2 . The square root of the second moment about the mathematical expectation is denoted by σ , and is called the *population standard deviation*; that is,

$$\sqrt{\mu_2} = \sqrt{E(x - \mu)^2} = \sigma.$$

The third moment about the mathematical expectation,

$$\mu_3 = E(x - \mu)^3,$$

provides a measure of the skewness of the distribution. If a distribution is symmetrical about the mean value or mathematical expectation μ , then $\mu_3 = 0$. For instance, the distribution in Figure 34 is symmetrical, so $\mu_3 = 0$; the distribution in Figure 33 is asymmetrical.

The fourth moment about the mathematical expectation,

$$\mu_4 = E(x - \mu)^4,$$

is connected with the flatness or peakedness of the probability distribution. The technical name for this is *kurtosis*.

The normal distribution in Figure 38 has no kurtosis. The distributions in Figure 42 are more peaked than the normal distribution. Formulas for the computation of the second, third, and fourth moments about the mathematical expectation from the moments about the origin will be given below in Problems 6, 9, and 10, Exercises 82.

■ EXAMPLE 1

Consider the following probability distribution:

x	p_1	xp	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 p_1$
1	$\frac{1}{3}$	$\frac{1}{3}$	-1	1	$\frac{1}{3}$
2	$\frac{2}{3}$	$\frac{4}{3}$	0	0	0
3	$\frac{1}{3}$	$\frac{3}{3}$	1	1	$\frac{1}{3}$

We have $\mu = Ex = 1/3 + 2/3 + 3/3 = 6/3 = 2$. From the last column we have also $\sigma^2 = 1/3 + 0 + 1/3 = 2/3$.

Now consider a second probability distribution:

x	p_2	xp	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 p_2$
1	$\frac{1}{6}$	$\frac{1}{6}$	-1	1	$\frac{1}{6}$
2	$\frac{2}{3}$	$\frac{4}{3}$	0	0	0
3	$\frac{1}{6}$	$\frac{3}{6}$	1	1	$\frac{1}{6}$

From the above computations; the mean $\mu = Ex = 2$ and the variance = $1/3$.

The 2 probability distributions are represented in Figure 35. The first distribution is drawn with heavy lines, the second with broken lines. They have the same population mean, $\mu = Ex = 2$. But the variance of the first

distribution is twice that of the second; it is apparent from the figure how the variance measures the spread, or dispersion, of the distribution about its mean.

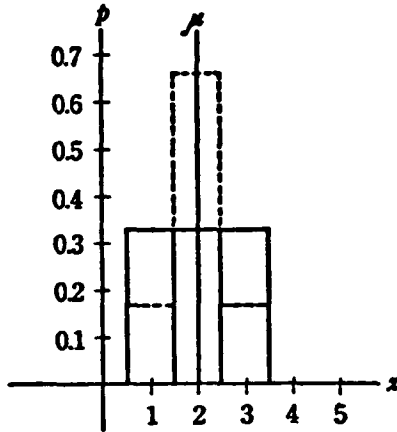


Figure 35

■ EXAMPLE 2

Consider the probability distribution that follows:

x	p	$x - \mu$	$(x - \mu)^2 p$
0	0.5	-0.7	0.245
1	0.3	0.3	0.027
2	0.2	1.3	0.338

We have $\mu = Ex = 0.7$. $\mu_2 = E(x - \mu)^2 = 0.610 = \sigma^2$. This is the population variance.

■ EXAMPLE 3

Take the probability distribution $p(x) = 1/3$, $0 \leq x \leq 3$. Then $Ex = \mu = 1.5$.

$$\begin{aligned} \mu_3 &= E(x - \mu)^3 = \int_0^3 [(x - 1.5)^3/3] dx \\ &= \int_0^3 1/3(x^3 - 4.5x^2 + 6.75x - 3.375) dx = 0. \end{aligned}$$

Since the third moment about the mathematical expectation is 0, the distribution is symmetrical.

● EXERCISES 82

1. Given the distribution in Problem 1, Exercises 79. Find μ_2 ; μ_3 ; μ_4 .

2. Given the probability distribution in Problem 2, Exercises 79. Find μ_2 ; μ_3 ; μ_4 .
3. Given the probability distribution in Problem 3, Exercises 79. Find μ_2 ; μ_3 ; μ_4 .
4. Given the probability distribution in Problem 4, Exercises 79. Find μ_2 ; μ_3 ; μ_4 .
5. Given the probability distribution in Problem 5, Exercises 79. Find μ_2 ; μ_3 .
- *6. Show that the variance $\mu_2 = \mu_2' - (\mu_1')^2 = Ex^2 - (Ex)^2 = Ex^2 - \mu^2 = \sigma^2$; where σ is the standard deviation. (HINT: Develop $(x - \mu)^2$ by the binomial theorem and evaluate the expectation.)
7. Use the formula from Problem 6 to compute the variance for the probability distribution in Problem 1, Exercises 79. Find the standard deviation.
8. Use the formula in Problem 6 to compute the variance and standard deviation of the probability distribution of Problem 2, Exercises 79.
- *9. Prove that $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = Ex^3 - 3Ex^2Ex + 2(Ex)^3 = Ex^3 - 3\mu Ex^2 + 2\mu^3$.
- *10. Prove that $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = Ex^4 - 4Ex^3Ex + 6Ex^2(Ex)^2 - 3(Ex)^4 = Ex^4 - 4\mu Ex^3 + 6\mu^2 Ex^2 - 3\mu^4$.
11. Use the formulas in Problem 6, Problem 9, and Problem 10 to compute μ_2 , μ_3 , and μ_4 for the distribution in Problem 3, Exercises 79.
12. Use the formulas in Problems 6, 9 and 10 to compute μ_2 , μ_3 , and μ_4 for the distribution in Problem 4, Exercises 79.

BINOMIAL AND NORMAL DISTRIBUTIONS

83. Repeated Trials and Binomial Distribution

Assume that the probability of an event happening is p . The probability that it does not happen is $q = 1 - p$. We will say we have a success if the event occurs and a failure if it does not happen.

We have altogether n trials in which we observe the event. What is the probability that the event will occur exactly r times in n independent trials?

One way in which this can happen is to have r successes first and then the $(n - r)$ failures. These events are independent and happen simultaneously. Hence by the multiplication theorem of probabilities, the probability of obtaining r successes first and then $(n - r)$ failures is $p^r q^{n-r}$.

But we do not care about the arrangement of the successes and failures, as long as there are altogether r successes and $n - r$ failures. For instance, we would be equally satisfied if the $n - r$ failures came first and the r successes last. The probability of this particular arrangement would be the same as above. This is also true of any other arrangement of the r successes and the $n - r$ failures. Let us add the probabilities of all these arrangements.

We know from the theory of combinations that the number of arrangements of the r successes and $n - r$ failures is the number of combinations of n things taken r at a time. This number of combinations is given by the binomial coefficient; thus

$${}_n C_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}.$$

So the probability must be multiplied by this binomial coefficient. Hence

the probability of the event happening r times out of n trials is

$$P_r = {}_n C_r p^r q^{n-r}.$$

We remember that ${}_n C_0 = 1$. Furthermore, since the only possible values for r are $0, 1, 2, \dots, n$, then $P_0 + P_1 + P_2 + \dots + P_n = 1$.

■ EXAMPLE 1

A true coin is tossed 3 times. What is the probability of heads coming up 2 times?

Since we have a true coin, the probability of heads is $p = 1/2$. Also, the probability of heads *not* coming up is $q = 1 - p = 1/2$. The number of trials is $n = 3$. The number of successes, according to the problem, is $r = 2$. Hence

$$P_2 = {}_3 C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = \frac{3 \cdot 2}{1 \cdot 2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = 3/8.$$

To check this we write down all the possible outcomes of 3 throws with a true coin, we shall let H denote heads and T tails. Altogether there are 8 possible arrangements of 3 throws. Of these there are only 3 which have 2 heads. Since all these 8 arrangements have the same probability, $1/8$, we have $P_2 = 3/8$ as before.

■ EXAMPLE 2

Let an event have the probability $p = 1/3$. Hence the probability that it will not happen is $q = 1 - p = 2/3$. There are 4 trials.

Let us construct the whole probability distribution, that is, P_0, P_1, P_2, P_3 , and P_4 , which is called a *binomial distribution*, because of the binomial coefficients occurring in it. The distribution is

$$\begin{aligned} P_0 &= {}_4 C_0 (1/3)^0 (2/3)^4 = (1)(1)(16/81) = 16/81. \\ P_1 &= {}_4 C_1 (1/3)^1 (2/3)^3 = (4)(1/3)(8/27) = 32/81. \\ P_2 &= {}_4 C_2 (1/3)^2 (2/3)^2 = (6)(1/9)(4/9) = 24/81. \\ P_3 &= {}_4 C_3 (1/3)^3 (2/3)^1 = (4)(1/27)(2/3) = 8/81. \\ P_4 &= {}_4 C_4 (1/3)^4 (2/3)^0 = (1)(1/81)(1) = 1/81. \end{aligned}$$

As a matter of interest, and as a check, it is observed that the sum of these probabilities is 1. We show in Figure 36 a histogram of this binomial distribution.

■ EXAMPLE 3

Let the probability of an event be $p = 1/4$. What is the probability that it will happen 2 times in 4 trials? We have $q = 1 - p = 3/4$. Then

$$P_2 = {}_4 C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^2 = \frac{4 \cdot 3}{1 \cdot 2} \left(\frac{1}{16}\right) \left(\frac{9}{16}\right) = \frac{27}{128}.$$

What is the probability that it will happen *at least* 1 time in 3 trials? The event can happen in 3 trials either not at all, or 1, 2, or 3 times. Hence the desired probability is $P = P_1 + P_2 + P_3$. Since the sum of all the

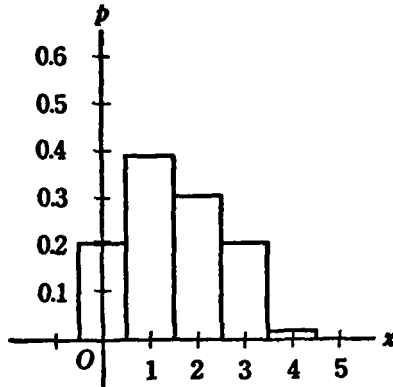


Figure 36

probabilities is 1, the sum is readily obtained as $1 - P_0 = 37/64$, since

$$P_0 = {}_3C_0(1/4)^0(3/4)^3 = 27/64.$$

What is the probability that it will happen *not more* than 2 times in 3 trials? It is

$$P = P_0 + P_1 + P_2 = 1 - P_3 = 63/64,$$

since $P_3 = {}_3C_3(1/4)^3(3/4)^0 = 1/64$.

● EXERCISES 83

1. An event has the probability $p = 1/3$. Find the probability that it will happen (a) 1 time in 3 trials; (b) 2 times in 6 trials; (c) 5 times in 5 trials; (d) 2 times in 4 trials; (e) *at least* 2 times in 7 trials; (f) *not more* than 3 times in 6 trials?

2. A true die is thrown. Find the probabilities that 6 will appear (a) 2 times in 3 trials; (b) 10 times in 10 trials (use logarithms); (c) 3 times in 6 trials; (d) at least 1 time in 5 trials; (e) *not more* than 2 times in 12 trials.

3. A true coin is thrown. Find the probability that we get a head (a) 1 time in 2 trials; (b) 2 times in 6 trials; (c) 4 times in 4 trials; (d) at least 1 time in 20 trials; (e) *not more* than 2 times in 15 trials.

4. An event has the probability $p = 0.2$. Find the probability that it happens (a) 2 times in 6 trials; (b) 3 times in 4 trials; (c) 6 times in 12 trials; (d) at least 1 time in 50 trials; (e) *not more* than 2 times in 200 trials.

5. An event has the probability $p = 0.8$. What is the probability that it will happen (a) 3 times in 6 trials? (b) 5 times in 9 trials? (c) at least 3 times in 4 trials? (d) *not more* than 9 times in 11 trials?

6. What is the probability of obtaining 10 heads with a true coin in 10 trials?

7. An event has the probability $p = 0.99$. What is the probability that it happens (a) 1 time in 4 trials? (b) 2 times in 7 trials? (c) at least 5 times in 6 trials? (d) not more than 2 times in 3 trials? (HINT: Use logarithms.)

8. An event has the probability $p = 0.999$. What is the probability that it happens (a) 1 time in 2 trials? (b) not at all in 10 trials? (c) at least 1 time in 5 trials? (d) not more than 1 time in 100 trials? (HINT: Use logarithms.)

9. Two true dice are thrown. What is the probability of obtaining the sum of 12 (a) 6 times in 6 trials? (b) 1 time in 100 trials? (c) at least 1 time in 5 trials? (d) not more than 2 times in 60 trials?

10. A true coin is tossed. What is the probability of obtaining (a) n heads in n trials? (b) 0 heads in n trials? (c) at least 1 head in n trials?

11. An event has the probability $p = 2/5$. Compute the complete binomial distribution for the outcomes in the case of 6 trials. Plot the histogram of the distribution.

12. An event has the probability $p = 3/8$. Find the complete binomial distribution for $n = 5$ trials. Plot the histogram of the distribution.

84. The Normal Distribution

As we have seen, the binomial distribution gives the probability that an event with probability p will happen r times in n trials; the formula for the computation is

$$P_r = {}_n C_r p^r q^{n-r}.$$

The computation of the binomial coefficient ${}_n C_r$ becomes very laborious if both n and r are large. Consider for instance an event whose probability of happening is $p = 1/3$. The probability that it will not happen is $q = 2/3$. The probability that it will happen 40 times in 100 trials is given by

$$\begin{aligned} P_{40} &= {}_{100} C_{40} \left(\frac{1}{3}\right)^{40} \left(\frac{2}{3}\right)^{60}, \\ &= \frac{(100)(99)(98)(97)\cdots(61)}{(1)(2)(3)(4)\cdots(40)} \left(\frac{1}{3}\right)^{40} \left(\frac{2}{3}\right)^{60}. \end{aligned}$$

The computation of the binomial coefficient involves 40 multiplication in the numerator and the same number in the denominator. If n and r are still larger, the computations become even more involved. Hence it is desirable to have a convenient approximation to the binomial distribution for large n and r .

We will show in this section that a good approximation to the binomial distribution for large n is, under certain conditions, the so-called normal distribution. This distribution is also called the normal-error curve, or the Gaussian distribution. A rigorous proof of the fact that as n becomes large the binomial distribution approximates the normal one is beyond the scope of this book. It requires very powerful mathematical methods.

We will compute first some moments of the binomial distribution. We recall that the random variable r , the number of successes in n trials, has the following distribution:

r	p_r
0	${}_nC_0 p^0 q^n$
1	${}_nC_1 p^1 q^{n-1}$
2	${}_nC_2 p^2 q^{n-2}$
3	${}_nC_3 p^3 q^{n-3}$
.....	
n	${}_nC_n p^n q^0$

From this we compute the first moment about the origin; this is the mean value of mathematical expectation of r . We obtain

$$\begin{aligned} \mu_1' = \mu = Er &= (0) {}_nC_0 p^0 q^n + (1) {}_nC_1 p^1 q^{n-1} + (2) {}_nC_2 p^2 q^{n-2} \\ &\quad + (3) {}_nC_3 p^3 q^{n-3} + \dots + (n) {}_nC_n p^n q^0 \\ &= (1) \frac{n}{1} p^1 q^{n-1} + (2) \frac{n(n-1)}{1 \cdot 2} p^2 q^{n-2} + (3) \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} p^3 q^{n-3} \\ &\quad + \dots + (n) \frac{n(n-1)(n-2) \dots 1}{1 \cdot 2 \cdot 3 \dots n} p^n q^0. \end{aligned}$$

We can factor out the common factor np , and get

$$\begin{aligned} Er = (np) \left[1 \cdot p^0 q^{n-1} + \frac{(n-1)}{1} p^1 q^{n-2} + \frac{(n-1)(n-2)}{1 \cdot 2} p^2 q^{n-3} + \dots \right. \\ \left. + \frac{(n-1)(n-2) \dots 1}{1 \cdot 2 \dots (n-1)} p^{n-1} q^0 \right]. \end{aligned}$$

Consider now the expression in the square brackets. The coefficients are the binomial coefficients in the expansion

$$(p + q)^{n-1} = {}_{n-1}C_0 p^0 q^{n-1} + {}_{n-1}C_1 p^1 q^{n-2} + \dots + {}_{n-1}C_{n-1} p^{n-1} q^0.$$

Hence we can write $\mu = Er = np(p + q)^{n-1}$.

But we recall that $p + q = 1$. Thus, $\mu = Er = np$. This is the mean value or mathematical expectation of the binomial distribution.

EXAMPLE 1

Consider an event which has the probability $p = 2/3$, $q = 1/3$. Let the number of trials be $n = 4$. What is the mean value or mathematical expectation of the number of successes r ?

Using the formula for the binomial distribution we compute the following table:

r	P_r
0	1/81
1	8/81
2	24/81
3	32/81
4	16/81

By direct computation we have for the mathematical expectation of r

$$\begin{aligned}\mu = Er &= (0)(1/81) + (1)(8/81) + (2)(24/81) + (3)(32/81) + (4)(16/81) \\ &= 216/81 = 8/3.\end{aligned}$$

The same result can also be obtained rather simply by the formula

$$\mu = Er = np = (4)(2/3) = 8/3.$$

By a similar method, we can compute the second moment about the origin for the binomial distribution

$$\mu_2' = Er^2 = npq + n^2p^2.$$

We recall the formula for computing the variance from the 2 first moments about the origin (Problem 6, Exercises 82):

$$\sigma^2 = \mu_2 = \mu_2' - (\mu_1')^2 = Ex^2 - \mu^2.$$

Hence we have for the variance of the binomial distribution

$$\sigma^2 = \mu_2 = Er^2 - (Er)^2 = npq + n^2p^2 - (np)^2 = npq.$$

The standard deviation is the square root of the variance; so

$$\sigma = \sqrt{\mu_2} = \sqrt{npq}.$$

■ EXAMPLE 2

Let us use the same binomial distribution as in Example 1. We recall that the mathematical expectation or mean value of the binomial distribution was found to be $\mu = Er = 8/3$.

From the table given above we get by direct computation for the variance of the binomial distribution

$$\begin{aligned}\mu_2 = \sigma^2 &= E(r - Er)^2 \\ &= \left(0 - \frac{8}{3}\right)^2 \left(\frac{1}{81}\right) + \left(1 - \frac{8}{3}\right)^2 \left(\frac{8}{81}\right) + \left(2 - \frac{8}{3}\right)^2 \left(\frac{24}{81}\right) \\ &\quad + \left(3 - \frac{8}{3}\right)^2 \left(\frac{32}{81}\right) + \left(4 - \frac{8}{3}\right)^2 \left(\frac{16}{81}\right) \\ &= \frac{648}{729} = \frac{8}{9}.\end{aligned}$$

Using the formula just derived, we obtain also

$$\sigma^2 = \mu_2 = npq = (4)(2/3)(1/3) = 8/9.$$

The standard deviation of our binomial distribution is

$$\sigma = \sqrt{\mu_2} = \sqrt{8/9} = 0.943.$$

We state now, without proof, the following theorem: *If, in a binomial distribution, the number of trials n becomes large, and the probabilities p or q are not very small, the binomial distribution can be approximated by the so-called normal distribution, whose probability density is given by*

$$p(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}, \quad \pi = 3.1416.$$

$$u = \frac{r - \mu}{\sigma}, \quad \text{and} \quad \mu = Er = np, \sigma^2 = npq.$$

The normal distribution is not only a convenient approximation to the binomial distribution. It occurs frequently in empirical distributions of natural and social phenomena because of the *central limit theorem*:

Assume that we have n random variables which are independent. Their distribution is arbitrary. Consider the distribution of their sum. It can be shown that this sum is, under certain conditions, normally distributed if n becomes infinite.

Many empirical statistical distributions are normal. This is especially true of the distribution of errors of observations. A graph of the normal distribution is shown in Figure 37.

In Figure 37 is shown the convergence of a binomial distribution to the normal, as n becomes larger and larger. This convergence is shown by studying the binomial $(1/2 + 1/2)^n$ for $n = 2, 4, 10$. The curves are so constructed that they all have the same mean and variance. It is apparent that the larger n (the number of trials) becomes, the more the histograms have an appearance not too different from the normal distribution. The normal distribution is shown at the bottom of the figure.

The probability with a normally distributed variable u with mean 0 and variance 1, $a \leq u \leq b$, is given by the integral,

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

The integral of the normal probability distribution is tabulated by exhibiting the area under the normal curve from 0 to u ; thus

$$\int_0^u \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

This is the probability that the normally distributed variable will fall between 0 and u .

It is seen from Figure 37 that the normal distribution is symmetrical about its mean $\mu = 0$. Hence, the probability that a normally distributed

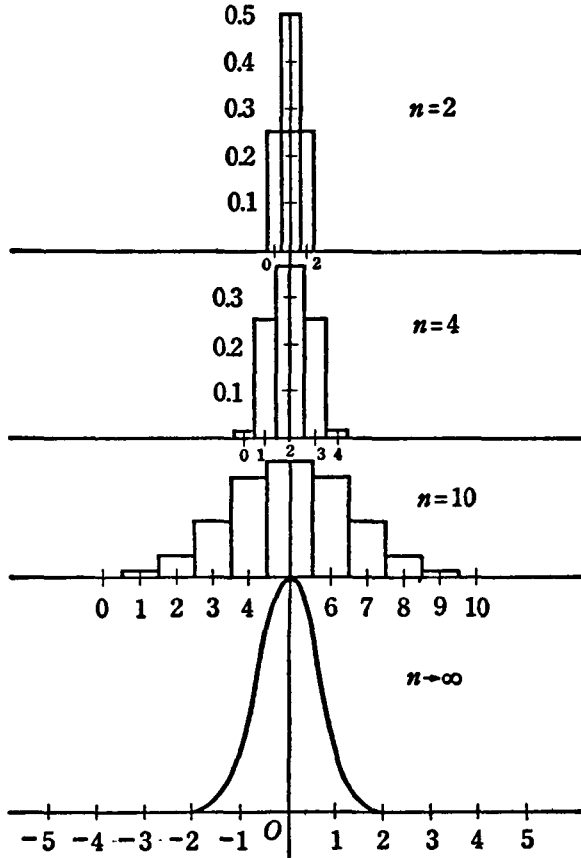


Figure 37

variable will fall between 0 and u is the same as that it will fall between $-u$ and 0. The normal distribution goes from $-\infty$ to $+\infty$, as seen in Figure 37.

If we have a normal distribution of a random variable x , having the mean μ and variance σ^2 , we must make the transformation,

$$u = \frac{x - \mu}{\sigma},$$

In order to use our tables (note Table 4 in the Appendix). Since the table gives areas between 0 and u , we find the probability that x lies between x_1 and x_2 in the following way. Obtain the values of u_1 and u_2 by use of the following formulas:

$$u_1 = \frac{x_1 - \mu}{\sigma}, \quad u_2 = \frac{x_2 - \mu}{\sigma}.$$

Then look in the table under u_1 and u_2 . If u_1 and u_2 have the same sign, subtract the smaller area from the larger one. If u_1 and u_2 have different signs, add the areas. This will give the probability that the normally distributed variable x with mean μ and variance σ^2 lies between x_1 and x_2 .

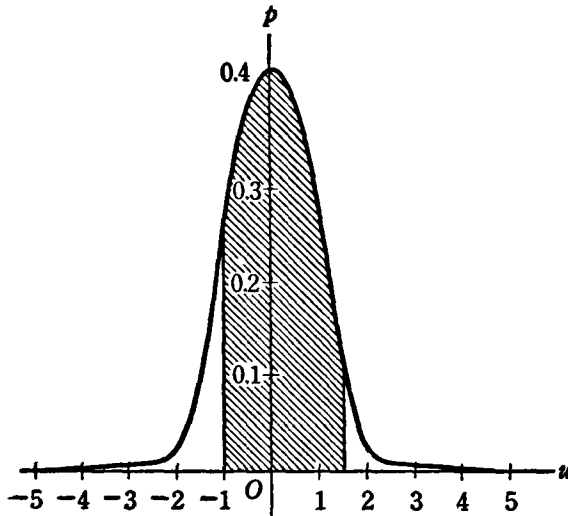


Figure 38

As an example we show in Figure 38 the area under the normal curve from $u_1 = -1$ to $u_2 = 1.5$. The u 's have different signs; hence, we must add the areas obtained from the table; that is, since the area for $u = 1$ is given in Table 4 as 0.3413 and for $u = 1.5$ is 0.4332, we have $0.3413 + 0.4332 = 0.7745$. This is the shaded area shown in the graph. It corresponds to the probability that the normally distributed variable u with 0 mean and unit variance will fall in the interval $-1 \leq u \leq 1.5$.

EXAMPLE 3

Assume that x is normally distributed. Its population mean is $\mu = 5$ and its population variance is $\sigma^2 = 4$. Hence its population standard deviation is $\sigma = 2$.

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(a) Find the probability that $6 \leq x \leq 8$. We have

$$u_1 = \frac{6 - 5}{2} = 0.5,$$

$$u_2 = \frac{8 - 5}{2} = 1.5.$$

From Table 4 we obtain 0.1915 for the area under the normal curve at $u = 0.5$ and obtain 0.4332 at $u = 1.5$. Since u_1 and u_2 have the same sign, we must subtract. So we obtain $0.4332 - 0.1915 = 0.2417$ for the probability that x will fall between 6 and 8.

(b) Find the probability that $2 \leq x \leq 3$. This time we have

$$u_1 = \frac{2 - 5}{2} = -1.5,$$

$$u_2 = \frac{3 - 5}{2} = -1.$$

Since the normal distribution is symmetrical, we can neglect the minus signs. We have the area 0.4332 for $u = 1.5$ and the area 0.3413 for $u = 1$. Since we have like signs for u_1 and u_2 , we subtract; thus $0.4332 - 0.3413 = 0.0919$ is the probability that x will fall between 2 and 3.

(c) What is the probability that $1 \leq x \leq 7$? We have

$$u_1 = \frac{1 - 5}{2} = -2,$$

$$u_2 = \frac{7 - 5}{2} = 1.$$

We have 0.4773 in the table for the area under the normal curve for $u = 2$ (neglecting again the negative sign because of the symmetry of the normal distribution), and we have 0.3413 for $u = 1$. Since we have unlike signs, we must add the areas. The sum, $0.4773 + 0.3413 = 0.8186$, gives the probability that x will fall between 1 and 7.

(d) What is the probability that $5 \leq x \leq 8$? We have

$$u_1 = \frac{5 - 5}{2} = 0,$$

$$u_2 = \frac{8 - 5}{2} = 1.5.$$

From the table we have 0 area for $u = 0$ and 0.4332 for $u = 1.5$. Hence the probability that x will fall between 5 and 8 is 0.4332.

(e) What is the probability that $9 \leq x \leq 20$? We have

$$u_1 = \frac{9 - 5}{2} = 2,$$

$$u_2 = \frac{20 - 5}{2} = 7.5.$$

We have the area 0.4773 for $u = 2$; $u = 7.5$ is beyond the range of the table. But we note that for u greater than 3.88 the area is 0.5000, correct to four decimals. Hence we take this area for any u beyond the range of the table. Since u_1 and u_2 have like signs, we subtract and obtain $0.5000 - 0.4773 = 0.0227$. This is the probability that x lies between 9 and 20.

(f) What is the probability that $x \geq 11$? This can be thought of as $11 \leq x \leq \infty$. We have

$$u_1 = \frac{11 - 5}{2} = 3,$$

$$u_2 = \infty.$$

For $u = 3$, the area is 0.4987. For $u = \infty$ we take the area as 0.5000. The signs are alike, so we subtract and get $0.5000 - 0.4987 = 0.0013$. This is the probability that x is greater than 11.

■ EXAMPLE 4

Compute the normal approximation to the binomial distribution in Example 1 for $r = 2$.

Since the normal distribution is continuous, we take the area between $r = 1.5$ and $r = 2.5$. Remember that $\mu = Er = 2.67$ and $\sigma = 0.943$. We have

$$u_1 = \frac{1.5 - 2.67}{0.943} = -1.24,$$

$$u_2 = \frac{2.5 - 2.67}{0.943} = -0.18.$$

We have from the tables the area 0.0714 for $u = 0.18$ and the area 0.3925 for $u = 1.24$. The signs of u_1 and u_2 are the same, so we subtract and get $0.3925 - 0.0714 = 0.3211$.

This is the normal approximation to the value for $r = 2$ in our binomial. This value was $24/81 = 0.2963$. The error of the normal approximation is $0.3211 - 0.2963 = 0.0248$.

● EXERCISES 84

1. An event has the probability $p = 0.4$. What is the probability that it will happen not less than 35 times and not more than 55 times in 100 trials? (HINT: Use the normal approximation to the binomial distribution.)

2. An event has the probability $p = 0.49$. What is the probability that it will happen not less than 30 and not more than 40 times in 50 trials?

3. An event has the probability $p = 0.475$. What is the probability that it will happen not less than 935 times and not more than 945 times in 2,000 trials?

4. An event has the probability $1/2$. Find the probability that it will happen *at least* 6 times in 10 trials (a) by the exact evaluation of the binomial distribution; (b) by the normal approximation. Compare the results.

5. An event has the probability $1/3$. What is the probability in 1,000 trials that it will happen (a) between 300 and 400 times? (b) between 330 and 335 times? (c) between 300 and 320 times? (d) between 350 and 355 times? (e) at least 290 times? (f) at least 370 times? (g) not more than 315 times? (h) not more than 355 times?

6. A variable x is normally distributed with $\mu = Ex = 10$, $\sigma^2 = 9$. Find the probability that x will fall (a) between 10 and 11; (b) between 12 and 19; (c) between 8 and 9.5; (d) between 7 and 8.2; (e) between 0 and 6; (f) between 12.5 and 35; (g) between 9.5 and 11.3.

7. A variable x is normally distributed with $\mu = Ex = 50$, $\sigma^2 = 25$. Find the probability that it will fall between the following limits: (a) 45 and 48; (b) 52 and 60; (c) 0 and 40; (d) 55 and 100; (e) that it is larger than 54; (f) smaller than 57; (g) larger than 43; (h) smaller than 39.

8. An event has the probability $1/2$. Compute the probabilities that it happens 0, 1, 2, 3, 4, 5, 6, 7, 8 times in 8 trials. Compare these probabilities with the normal approximations. Make a histogram and plot the normal curve.

9. Compute the normal approximations to the binomial distribution in Problem 11, Exercises 83. Make a histogram of the binomial and plot the normal distribution.

10. Compute the normal approximation to the binomial distribution in Problem 12, Exercises 83. Plot the histogram of the binomial and the normal distribution.

****11.** Find the value of x which maximizes the probability P of the normal distribution with mean μ and variance σ^2 .

$$P = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

The value which maximizes the probability is the *mode*. Show that for the normal distribution the mode is equal to the population mean μ .

ELEMENTS OF SAMPLING

85. Estimation

Modern statistics distinguishes between the population and the sample. The sample may be used to gain some information about the unknown population from which it is drawn. We will deal only with *random samples*, but it should be emphasized that under certain conditions other methods of sampling may be more efficient in providing information about the unknown population.

A random sample is constructed in such a way that every item in the population has an equal chance to be chosen in the sample. Suppose, for instance, we want some idea of the family income in a town of 10,000 families. There is only enough money available to investigate 100 families. We want to construct a random sample of 100 out of the total population of 10,000 families.

One method of constructing the sample is the following: Write the names and addresses of all 10,000 families on slips of paper. The 10,000 paper slips are put into a container and well mixed. Out of the container 100 slips are drawn at random. The 100 families selected in this fashion constitute a random sample of the total population of 10,000 families.

Great caution must be exerted in selecting the random sample. For instance, it may be more convenient to select the 100 families from the families of the faculty of a college located in the city. This, however, would not give a random sample, for not every family in the city has the same chance to be chosen in the sample. Probably the average income estimated from this selection of faculty families would be too high. Or, we may be tempted to select the 100 families out of the telephone directory of the city. We could, for instance, take the first 100 names in the telephone directory.

This would violate an important principle of random sampling. Not all families have telephones, especially the poorer ones. Hence, the postulate of random sampling that every family must have an equal chance to be chosen is violated. Our estimate of the average income from the sample out of the telephone directory would probably overestimate the average income.

Another method of sampling which will not be discussed here at length is the method of *stratified sampling*. Under certain conditions this method may be preferable to random sampling. To sample in this way, we divide our whole population into strata. For instance, the 10,000 families in the city may be classified according to professions. These strata form our subsamples. The size of the random sample to be chosen from each stratum is proportionate to the number of families in the particular stratum. For instance, if 25 per cent of the families in the city belong to a certain profession, then 25 per cent of the sample, that is, 25 families out of a total sample of 100 families, ought to be taken at random from this particular profession. This method of sampling, which has definite advantages, is called *representative sampling*.

In economic statistics, we generally do not know all we wish to about the population from which our data are a sample. We may consider the data as random samples of a hypothetically infinite population. For instance, if a set of wheat prices is recorded on a market it may be considered as a random sample of all (unrecorded) wheat prices from transactions on the market; or even as a sample of the (hypothetically infinite) population of all possible wheat prices.

Statistical *estimation* is the method by which we extract information about the population from the sample. This is the only object of sampling. We are not, in general, interested in the sample for its own sake.

The estimates derived from the sample are functions of the various observations contained in the sample. Since these observations are random variables, the estimates are themselves also random variables. An estimate is also called a *statistic*.

For instance, in the example given above, we drew a random sample of 100 families from a total (finite) population of 10,000 families in a city in order to learn something about the average income *in the population*, that is, among the 10,000 families.

Various principles have been advanced for estimation. We can state here only a few: One important property is *unbiasedness*.

By an unbiased estimate we understand an estimate whose mean value (mathematical expectation) is equal to the true value in the population. For instance: *The sample mean is an unbiased estimate of the population mean if its mathematical expectation is equal to the population mean.* The sample variance is an unbiased estimate of the population variance if the mathematical expectation of the sample variance is equal to the population variance, and so on.

To show that the *sample mean* is an unbiased estimate of the population

mean, assume a sample consisting of the items

$$x_1, x_2, x_3, \dots, x_n.$$

All the items in the sample are supposed to have been drawn at random from a population with mean $Ex = \mu$ and population variance σ^2 . Since we have a random sample, the random variables x_1, x_2, \dots, x_n are independent.

We define the sample mean as the arithmetic mean of the n items

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n} = \frac{\sum_{i=1}^n x_i}{n}.$$

We have noted that all observations are drawn from a population with true mean (population mean) μ . Hence the mathematical expectation of all the items x_1, x_2, \dots, x_n is $Ex = \mu$; that is,

$$\begin{aligned} Ex_1 &= Ex = \mu, \\ Ex_2 &= Ex = \mu, \\ Ex_3 &= Ex = \mu, \\ &\dots\dots\dots \\ Ex_n &= Ex = \mu. \end{aligned}$$

We want to show that the mathematical expectation of \bar{x} is also equal to μ .
Take

$$E\bar{x} = E\left[\left(\frac{1}{n}\right)x_1 + \left(\frac{1}{n}\right)x_2 + \dots + \left(\frac{1}{n}\right)x_n\right]$$

We use 2 propositions established earlier when dealing with computations involving mathematical expectations (Section 80); namely, *the mathematical expectation of a sum is the sum of the mathematical expectations.*

The mathematical expectation of a constant times a random variable is the constant times the mathematical expectation of the random variable.

From these 2 propositions it follows that

$$\begin{aligned} E\bar{x} &= \left(\frac{1}{n}\right)Ex_1 + \left(\frac{1}{n}\right)Ex_2 + \dots + \left(\frac{1}{n}\right)Ex_n \\ &= \left(\frac{1}{n}\right)(Ex + Ex + \dots + Ex) \\ &= \left(\frac{1}{n}\right)(n\mu) = \mu. \end{aligned}$$

The mathematical expectation of the sample mean \bar{x} is the population mean μ . Hence \bar{x} is an unbiased estimate of the population mean.

By the method of expectations we can also compute the *variance of the sample mean* \bar{x} ; that is,

$$\sigma_{\bar{x}}^2 = E(\bar{x} - E\bar{x})^2.$$

We have

$$\begin{aligned} \bar{x} - \mu &= \frac{x_1 + x_2 + \cdots + x_n}{n} - \mu \\ &= \frac{x_1 + x_2 + \cdots + x_n - n\mu}{n} \\ &= \frac{(x_1 - \mu) + (x_2 - \mu) + \cdots + (x_n - \mu)}{n}. \end{aligned}$$

Since the sample x_1, x_2, \dots, x_n is drawn from the population with mean $E_x = \mu$ and variance σ^2 , we have

$$\begin{aligned} E(x_1 - \mu)^2 &= \sigma^2, \\ E(x_2 - \mu)^2 &= \sigma^2, \\ &\dots\dots\dots \\ E(x_n - \mu)^2 &= \sigma^2. \end{aligned}$$

The items x_1, x_2, \dots, x_n are taken at random. Hence they are independent. We recall that the mathematical expectation of 2 independent random variables is the product of their mathematical expectations; hence we have, for example,

$$\begin{aligned} E(x_1 - \mu)(x_2 - \mu) &= E[x_1x_2 - x_1\mu - x_2\mu + (\mu)^2] \\ &= Ex_1Ex_2 - (\mu)Ex_1 - (\mu)(Ex_2) + E(\mu)^2 = 0. \end{aligned}$$

By a similar argument we have

$$\begin{aligned} E(x_1 - \mu)(x_3 - \mu) &= 0, \\ E(x_1 - \mu)(x_4 - \mu) &= 0, \\ &\dots\dots\dots \\ E(x_1 - \mu)(x_n - \mu) &= 0. \end{aligned}$$

In general, we have

$$E(x_i - \mu)(x_j - \mu) = 0, \quad \text{if } i \neq j.$$

Squaring the expression for $\bar{x} - \mu$ derived previously and taking the mathematical expectation, we have

$$\begin{aligned} \sigma_{\bar{x}}^2 &= E(\bar{x} - \mu)^2 \\ &= E \left[\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_n - \mu)^2 + 2(x_1 - \mu)(x_2 - \mu) \right. \\ &\quad \left. + \cdots + 2(x_i - \mu)(x_j - \mu) + \cdots}{n^2} \right], \end{aligned}$$

where $i \neq j$ in the product terms. We know that the mathematical expectation of all the products is 0. The mathematical expectation of the n squares is σ^2 for each square. Hence we get for the variance of the sample mean

$$\sigma_{\bar{x}}^2 = E(\bar{x} - \mu)^2 = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

That is, the variance of the sample mean of a sample of n is the population variance σ^2 divided by n .

If we extract the square root of the variance of the sample mean, we get the so-called *standard error* of the sample mean, namely,

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}.$$

In words: *The standard error of the sample mean \bar{x} in a sample of n is the standard deviation of the population σ divided by the square root of n .*

Now we consider the *sample variance*. Let us again assume that we have a sample of n , namely,

$$x_1, x_2, \dots, x_n.$$

All these items are drawn at random from a population with population mean μ and population variance σ^2 . Hence we have

$$\begin{array}{ll} Ex_1 = \mu, & E(x_1 - \mu)^2 = \sigma^2, \\ Ex_2 = \mu, & E(x_2 - \mu)^2 = \sigma^2, \\ \dots\dots\dots & \dots\dots\dots \\ Ex_n = \mu. & E(x_n - \mu)^2 = \sigma^2. \end{array}$$

We define the sample variance as follows:

$$\begin{aligned} V &= \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \dots + (x_n - \bar{x})^2}{n - 1}, \\ &= \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}. \end{aligned}$$

In words: *The sample variance is computed by dividing the sum of squares of the deviations from their arithmetic mean by $n - 1$, that is, by one less than the number of items in the sample. It may be considered as the arithmetic mean of the squares of deviations, which are random variables.*

We want to show that the mathematical expectation of the sample variance is equal to population variance; that is,

$$EV = \sigma^2.$$

If this is true, then the sample variance is an unbiased estimate of the population variance.

**Since the mathematical expectation of a sum is the sum of the mathematical expectations, we have

$$EV = \frac{E(x_1 - \bar{x})^2 + E(x_2 - \bar{x})^2 + \dots + E(x_n - \bar{x})^2}{n - 1}.$$

Let us compute one of the mathematical expectations in the numerator; for instance, $E(x_1 - \bar{x})^2$.

Evidently we can add and subtract the constant $\mu = Ex$ in the parentheses without changing the expression; that is,

$$(x_1 - \bar{x}) = (x_1 - \mu) - (\bar{x} - \mu).$$

Squaring this expression, and obtaining the mathematical expectation, we have

$$\begin{aligned} E(x_1 - \bar{x})^2 &= E[(x_1 - \mu)^2 - 2(x_1 - \mu)(\bar{x} - \mu) + (\bar{x} - \mu)^2] \\ &= E(x_1 - \mu)^2 - 2E(x_1 - \mu)(\bar{x} - \mu) + E(\bar{x} - \mu)^2. \end{aligned}$$

We have from previously derived formulas

$$E(x_1 - \mu)^2 = \sigma^2,$$

$$E(\bar{x} - \mu)^2 = \frac{\sigma^2}{n}.$$

There still is the problem of computing the mathematical expectation of the middle term. Ignoring the factor -2 , the mathematical expectation of this term is

$$\begin{aligned} &E(x_1 - \mu)(\bar{x} - \mu) \\ &= E \left[(x_1 - \mu) \left(\frac{x_1 + x_2 + \cdots + x_n}{n} - \mu \right) \right] \\ &= E \left[(x_1 - \mu) \left(\frac{x_1 + x_2 + \cdots + x_n - n\mu}{n} \right) \right] \\ &= E \left[(x_1 - \mu) \left(\frac{(x_1 - \mu) + (x_2 - \mu) + \cdots + (x_n - \mu)}{n} \right) \right] \\ &= E \left[\frac{(x_1 - \mu)^2 + (x_1 - \mu)(x_2 - \mu) + \cdots + (x_1 - \mu)(x_n - \mu)}{n} \right] \\ &= \frac{E(x_1 - \mu)^2 + E(x_1 - \mu)(x_2 - \mu) + \cdots + E(x_1 - \mu)(x_n - \mu)}{n}. \end{aligned}$$

Since the mathematical expectation of the square $(x_1 - \mu)^2$ is, by definition σ^2 , and since the mathematical expectation of all the other terms is 0 (this has been shown above), this fraction becomes σ^2/n .

Using this result and the previously established formulas, we have

$$\begin{aligned} E(x_1 - \bar{x})^2 &= E(x_1 - \mu)^2 - 2E(x_1 - \mu)(\bar{x} - \mu) + E(\bar{x} - \mu)^2 \\ &= \sigma^2 - \frac{2\sigma^2}{n} + \frac{\sigma^2}{n} \\ &= \frac{(n-1)\sigma^2}{n}. \end{aligned}$$

By a similar argument we derive also for the other terms in the numerator of EV

$$\begin{aligned}
 E(x_2 - \bar{x})^2 &= \frac{(n-1)\sigma^2}{n}, \\
 E(x_3 - \bar{x})^2 &= \frac{(n-1)\sigma^2}{n}, \\
 &\dots\dots\dots \\
 E(x_n - \bar{x})^2 &= \frac{(n-1)\sigma^2}{n}.
 \end{aligned}$$

Hence we have finally for the mathematical expectation of the variance

$$\begin{aligned}
 EV &= \frac{E(x_1 - \bar{x})^2 + E(x_2 - \bar{x})^2 + E(x_3 - \bar{x})^2 + \dots + E(x_n - \bar{x})^2}{n-1} \\
 &= \frac{n(n-1)/n}{n-1} \sigma^2 = \sigma^2.
 \end{aligned}$$

Hence the mathematical expectation of the sample variance is the population variance σ^2 . The sample variance as given by the above formula is an unbiased estimate of the population variance. **

It should be noted that in order to compute V we divide the sum of squares of the deviations from the mean by $n - 1$. That is, we divide the number of observations by a number which is 1 less than n . The quantity $(n - 1)$ is frequently called the *number of degrees of freedom* used for the computation of the variance. This term has a somewhat complicated meaning. Speaking rather loosely, we may express the idea of degrees of freedom as follows: The amount of information available in a sample depends upon its size. A sample of 100 contains more information than a sample of only 10 items, and so on. We use some of this information in a sample of n items to compute the sample mean. This uses up 1 degree of freedom. Hence, when we compute the sample variance we have only $(n - 1)$ degrees of freedom left, since for the computation of the variance we need to know the sample mean.

The method of maximum likelihood, which is used very frequently to obtain estimates, will be illustrated in Problem 5, Exercises 85. The method of least squares, which is also used to obtain estimates will be discussed in Section 96.

Consistency is also a desirable attribute for an estimate. A consistent statistic will tend more and more to the population value estimated, as the sample increases in size. The sample mean and the sample median (Problem 11, Exercises 87) are examples of consistent statistics if the population is normal.

Some estimates tend to be normally distributed as the sample increases in size. They are called *efficient* if they have in the limit the smallest possible

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variance. The sample mean is an efficient statistic if the population is normal. Its variance is smaller than, for example, the variance of the sample median for large samples.

Finally there are statistics which include all the relevant information available even in small samples. Such statistics are called *sufficient*. The sample mean of a sample taken from a normal population is such a sufficient statistic.

● EXERCISES 85

1. Show that the mathematical expectation of the quantity $z = \sum_{i=1}^n x_i$; $Ez = n\mu$ if the random sample x_1, x_2, \dots, x_n is drawn from a population with true mean $Ex = \mu$.

**2. Let x_1, x_2, \dots, x_n be a random sample drawn from a population with true mean μ and population variance σ^2 . Consider the expression

$$w = \sum_{i=1}^n k_i x_i,$$

where the k_i are constants and $\sum_{i=1}^n k_i = 1$. (a) Show that the mathematical expectation

$Ew = Ex = \mu$. (b) Demonstrate that $E(w - Ew)^2 = \sigma^2 \sum_{i=1}^n k_i^2$. (c) Use the previous results to establish the formulas given above for the mathematical expectation and variance of the sample mean \bar{x} . (HINT: Take $k_i = 1/n$.)

*3. Show that the formula for the sample variance, namely,

$$V = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1},$$

may be simplified to

$$V = \frac{\sum_{i=1}^n x_i^2 - n\bar{x}^2}{n - 1}.$$

(HINT: Remember the definition of the sample mean: $\bar{x} = \sum_{i=1}^n x_i/n$.)

4. Using the results in the last example, show that the sample variance in the case of grouped items is

$$V = \frac{\sum_{i=1}^m (x_i - \bar{x})^2 f_i}{N - 1} = \frac{\sum_{i=1}^m x_i^2 f_i - N\bar{x}^2}{N - 1},$$

where the symbols have the following interpretation: There are m classes, the fre

quencies of which are f_1, f_2, \dots, f_m ; the x_i is the class mean; $\sum_{i=1}^m f_i = N$ is the total number in the sample, and the sample mean is $\bar{x} = \sum_{i=1}^m x_i f_i / N$.

****5.** Let the probability density of x_1 be

$$p_1 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_1 - a)^2};$$

let the probability density of x_2 be

$$p_2 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_2 - a)^2}$$

$$\dots;$$

and finally let the probability density of x_n be

$$p_n = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_n - a)^2}.$$

All the x_i are independently, normally distributed with variance 1. (a) Find the probability that the independent events x_1, x_2, \dots, x_n will happen together. (b) Maximize this probability (*method of maximum likelihood*) by choosing the appropriate value of a . (c) Show that the maximum likelihood estimate is the sample mean \bar{x} .

****6.** Show that the mathematical expectation of

$$W = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}$$

is not equal to the population variance σ^2 . What is the mathematical expectation of W ?

****7.** Show that the mathematical expectation of

$$Z = \frac{\sum_{i=1}^n (x_i - \mu)^2}{n}$$

is equal to the population variance. What is the difference between Z and the W in Problem 6?

86. Frequency Distributions

If a sample consists of many items, it is virtually necessary to arrange it in a frequency distribution. The classes of this distribution are given in advance. If an item falls exactly on the limit of 2 adjacent classes, it ought to be distributed between them evenly; that is, one half ought to be put into

the upper class and one half into the lower. The class mid-points are the arithmetic means of the class limits.

■ **EXAMPLE**

A sample consists of the following items: 2, 8, 6, 3, 5, 1, 4. Arrange a frequency distribution with the following classes: 0-2, 2-4, 4-6, 6-8, 8-10. Between 0 and 2 there is the item 1, and we will count one half of item 2, since 2 coincides with a limit; so there is a frequency of 1.5. Between 2 and 4 there is the item 3, and we will count one half of item 2 and one half of item 4; so there is a frequency of 2; and so on. This may be displayed in the following table:

<i>Class Limits</i>	<i>Class Mid-Point</i>	<i>Frequency</i>
0-2	1	1.5
2-4	3	2
4-6	5	2
6-8	7	1
8-10	9	0.5

The graph of a frequency distribution is called a *histogram*.

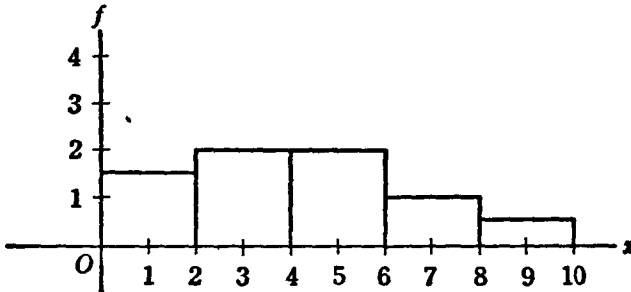


Figure 39

We show in Figure 39 the histogram of the frequency distribution just constructed. Note that the limits of the blocks in the graph are the limits of the classes in the frequency distribution.

● **EXERCISES 86**

1. Given the following sample of items: 4, 10, 23, 20, 19, 8, 6, 5, 25, 4, 17, 3. Arrange the set of values (a) in a frequency distribution with classes 0-4, 4-8, ... (b) with classes 0-5, 5-10, 10-15, ...; (c) with classes 0-10, 10-20, ... (d) Make histograms.

2. Given a sample consisting of the items 2, 4, 6, 3, 9, 6, 4, 9, 8, 5, 2, 9, 5, 1. Arrange the sample in a frequency distribution with (a) classes 0-2, 2-4, ...; (b) classes 0-3, 3-6, ... (c) Graph the distributions.

3. Given the data 0, 8, 18, 45, 23, 45, 9, 17, 19, 25, 39, 41, 19, 25, 6, 10, 36. (a) Arrange in a frequency distribution with the classes 0-10, 10-20, . . . (b) Make a histogram.

*4. Given the data 1, 4, 10, 17, 5, 6, 10. (a) The *range* is defined as the difference between the largest and the smallest item; find the range. (b) Form a frequency distribution by dividing the range into 5 parts; (c) by dividing the range into 7 parts. (d) Make graphs of the distributions.

5. Use the data in Problem 1. (a) Find the range. (b) Form a frequency distribution by dividing the range into 3 parts; (c) into 4 parts. (d) Graph.

6. Use the data in Problem 2. (a) Find the range. (b) Form a frequency distribution by dividing the range into 4 parts. (c) Make graphs.

7. Use the data in Problem 3. (a) Find the range. (b) Form a frequency distribution by dividing the range into 6 parts. (c) Make a histogram.

8. Use the data in Problem 4. (a) Find the range. (b) Form a frequency distribution by dividing the range into 3 parts. (c) Make a histogram.

87. Sample Mean and Variance

Let us set up a table that displays a frequency distribution and certain associated products.

Class Mean	Frequency		
x_i	f_i	$x_i f_i$	$x_i^2 f_i$
x_1	f_1	$x_1 f_1$	$x_1^2 f_1$
x_2	f_2	$x_2 f_2$	$x_2^2 f_2$
.....
x_m	f_m	$x_m f_m$	$x_m^2 f_m$
Sums:	$N = \sum_{i=1}^m f_i$	$\sum_{i=1}^m x_i f_i$	$\sum_{i=1}^m x_i^2 f_i$

As previously indicated, the class means are the arithmetic means of the class limits.

We have defined earlier the sample mean as $\bar{x} = (1/n) \sum_{i=1}^n x_i$. In the frequency distribution under consideration we have m classes with certain frequencies. Under such a circumstance the sample mean becomes

$$\bar{x} = \frac{1}{N} \sum_{i=1}^m x_i f_i$$

if the sum of all the frequencies is

$$N = \sum_{i=1}^m f_i$$

So N is computed as the sum of the second column of the above table, and

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\bar{x} is computed as the ratio of the sum of the third column and the sum of the second column. The sample mean is an unbiased estimate of the population mean.

We gave earlier a formula for the sample variance of ungrouped data in Problem 3, Exercises 85; that is,

$$V = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1} = \frac{\sum_{i=1}^n x_i^2 - n\bar{x}^2}{n - 1}.$$

With grouped data this becomes

$$V = \frac{\sum_{i=1}^m (x_i - \bar{x})^2 f_i}{N - 1} = \frac{\sum_{i=1}^m x_i^2 f_i - N\bar{x}^2}{N - 1}.$$

The expression $\sum_{i=1}^m x_i^2 f_i$ is the sum obtained from column 4 in the above table. The sample variance is an unbiased estimate of the population variance.

The sample standard deviation is the square root of the sample variance; that is, $s = \sqrt{V}$.

EXAMPLE 1

Assume that we have the sample 3, 5, 2. We have 3 items; hence, $n = 3$. We compute $\sum_{i=1}^3 x_i = 3 + 5 + 2 = 10$; $\sum_{i=1}^3 x_i^2 = 9 + 25 + 4 = 38$. From the formulas above

The sample mean is $\bar{x} = 10/3 = 3.33$.

The sample variance is $V = \frac{38 - (3)(3.33)^2}{2} = 2.37$.

The sample standard deviation is $s = \sqrt{2.37} = 1.54$

EXAMPLE 2

By contrast with Example 1, let us consider an illustration involving grouped data, as follows:

Class Limits	Class Mean	Frequency		
	x	f	xf	
0-4	2	1	2	
4-8	6	3	18	10
8-12	10	1	10	10
		$N = 5$	$\sum xf = 30$	$\sum x^2 f = 21$

We have

$$\bar{x} = 30/5 = 6; \quad V = [212 - (5)(36)]/4 = 32/4 = 8; \quad s = \sqrt{8} = 2.83.$$

We show in Figure 40 a histogram of the frequency distribution given in Example 2. The class means are indicated by arrows. We have also indi-

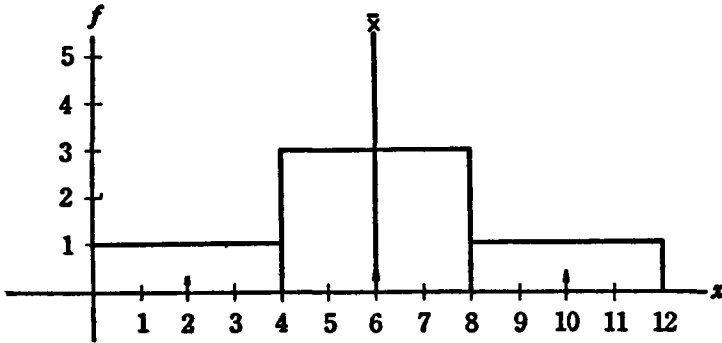


Figure 40

cated the sample mean by a heavy vertical line. It provides a measure of the general location of the frequency distribution or its central tendency.

■ **EXAMPLE 3**

Make the following experiment. Write the name and some characteristic (height, age) of each student in the class on a slip of paper. Make a frequency distribution, if necessary, and establish the average. Mix the slips well in a hat and draw successively samples of 2, 3, 4, ... items. Find the sample mean and variance of each sample.

● **EXERCISES 87**

1. Given the following sample:

Class Limits	Frequency
0-10	1
10-20	2
20-30	3
30-40	2
40-50	2

Make a histogram. Find the sample mean, variance, and standard deviation.

2. Given the following sample:

Class Limits	Frequencies
1-3	10
3-5	50
5-7	40

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Make a histogram. Find the sample mean, variance, and standard deviation.

3. Given the following sample:

Class Limits	Frequencies
0-4	1
4-8	4
8-12	5
12-16	3
16-20	2
20-24	1

Make a histogram. Find the sample mean, variance, and standard deviation.

4. Given the following sample:

Class Limits	Frequencies
2-5	2
5-8	3
8-11	5
11-14	4
14-17	3
17-20	3

Make a histogram. Find the sample mean, variance, and standard deviation.

5. Given the following sample:

Class Limits	Frequencies
0-5	1
5-10	3
10-15	4
15-20	1
20-25	1

Make a histogram. Find the sample mean, variance, and standard deviation.

6. Given the following sample:

Class Limits	Frequencies
0-20	5
20-40	20
40-60	30
60-80	15

Make a histogram. Determine the sample mean, variance, and standard deviation.

7. The distribution of families and single consumers in the United States, by money income levels, 1942, is given by the following table:

Income Class Limits	Frequency
\$ 0-500	3,488,000
500-1,000	6,652,000
1,000-1,500	6,601,000
1,500-2,000	6,008,000
2,000-2,500	4,618,000
2,500-3,000	3,372,000
3,000-4,000	2,620,000
4,000-5,000	2,633,000
5,000-7,500	1,901,000
7,500-10,000	628,000

Find the sample mean, variance, and standard deviation.

8. Income tax returns of corporations in the United States, 1942, by total assets are listed in the following table:

Total Assets Class Limits	Number of Returns Frequency
\$ 0-50,000	196,642
50,000-100,000	58,338
100,000-250,000	57,365
250,000-500,000	27,300
500,000-1,000,000	18,109
1,000,000-5,000,000	19,582
5,000,000-10,000,000	2,905
10,000,000-50,000,000	2,467

Compute the mean, variance, and standard deviation.

9. The population of the United States, 1945, by ages, is as follows:

Age Class Limits	Frequency (Millions)
0-5	11
5-10	11
10-15	12
15-20	12
20-25	12
25-30	11
30-35	10
35-40	10
40-45	9
45-50	8
50-55	7
55-60	6

Age Class Limits	Frequency (Millions)
60-65	5
65-70	4
70-75	3
75-80	3

(a) Make a histogram. (b) Compute the sample mean, variance, and standard deviation.

10. Compute the sample mean and variance without grouping the data: Exercises 86: (a) Problem 1; (b) Problem 2; (c) Problem 3; (d) Problem 4.

**11. The sample median is computed by ordering the items in a sample according to size. The median is the item in the middle (for odd sample numbers) or the arithmetic mean of the 2 items in the middle (for even sample numbers). Compute the median for (a) the data in Problem 2, Exercises 86; (b) the data in Problem 1, Exercises 86.

88. Sheppard's Correction

If we compute the sample variance from grouped data, we should correct for grouping. An error is evidently committed by substituting the mid-point for all the items in a given class.

This error of grouping can be corrected quite satisfactorily by subtracting from the variance the square of the class interval, divided by 12. This is called *Sheppard's correction*.

■ EXAMPLE

In Example 2 of Section 87, the class interval was 4 and the uncorrected variance was 8. Using Sheppard's correction, we have for the corrected value of the variance $8 - (4)^2/12 = 8 - 16/12 = 8 - 1.33 = 6.67$. The corrected standard deviation is $\sqrt{6.67} = 2.58$. This should be compared with the uncorrected value of 2.83.

● EXERCISES 88

1. Using the data in Problem 1, Exercises 87, compute the variance and the standard deviation, using Sheppard's correction for grouping.

2. Using the data in Problem 2, Exercises 87, compute the variance and standard deviation, with Sheppard's correction.

3. Using the data in Problem 3, Exercises 87, compute the sample variance and standard deviation, using Sheppard's correction.

4. Using the data in Problem 4, Exercises 87, compute the variance and standard deviation, using Sheppard's correction.

5. Using the data in Problem 5, Exercises 87, compute the variance and standard deviation, using Sheppard's correction.

6. Using the data in Problem 6, Exercises 87, compute the variance and standard deviation, with Sheppard's correction.

7. Using the data in Problem 9, Exercises 87, compute the sample variance and standard deviation, with the use of Sheppard's correction.

8. Using the data in Problem 10, Exercises 87, compute the sample variance and standard deviation, by the use of Sheppard's correction.

89. Confidence Limits

We have shown in Section 85 that the arithmetic mean of a sample of n items has the mathematical expectation $E\bar{x} = \mu$ and the variance $\sigma_{\bar{x}}^2 = \sigma^2/n$. The following is proved in more advanced books on statistics: *If the sample comes from a normal population, the quantity*

$$u = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$$

is normally distributed with mean 0 and variance 1. It follows the normal distribution described above. If the population is not normally distributed, u will under certain conditions tend to be normally distributed if the sample becomes large.

For large samples we can assume that the square root of the sample variance will provide a reasonably good estimate of the population standard deviation. This follows from the law of large numbers. It is almost certain that the sample variance, which is the mean of certain random variables, differs from the population variance (the mathematical expectation of these variables) as little as desired, if only the sample becomes sufficiently large. Hence, in this case, we will compute the quantity

$$s_{\bar{x}} = \sqrt{\frac{V}{n}} = \frac{s}{\sqrt{n}},$$

as an empirical estimate for the standard error of the arithmetic mean $\sigma_{\bar{x}}$. The quantity

$$u = \frac{\bar{x} - \mu}{s_{\bar{x}}}$$

will be approximately normally distributed with mean 0 and variance 1. The larger the sample, the better will be the approximation.

This theorem can be used to establish a large sample pair of confidence limits for the (unknown) true mean of the population from which the sample is taken. Fiducial limits are computed in the same way but have a somewhat different interpretation. From the table of the normal distribution we see that the probability that u lies between -1.96 and 1.96 is 0.05 . This gives us the 95 per cent confidence interval.

The quantities \bar{x} and $s_{\bar{x}}$ are computed according to the known formulas.

Then we compute the limits

$$\bar{x} + 1.96s_{\bar{x}} \quad \text{and} \quad \bar{x} - 1.96s_{\bar{x}}$$

Now consider the statement that the (unknown) true mean of the population μ will be contained within these confidence, or fiducial, limits. This statement will be true in 95 per cent of the cases and untrue in 5 per cent if the total number of cases is large. It is almost certain that the difference between the empirical relative frequency and the theoretical probability becomes as small as desired in large samples. That is, in the long run, on the average, we will be right in 95 out of 100 cases if we claim that the true or population mean is contained within the limits computed above.

EXAMPLE 1

Assume that we have a sample of 100 items ($N = 100$). Consider that the empirical arithmetic mean in the sample is $\bar{x} = 20$, and the sample variance is $V = 4$. Find a set of confidence limits for the population mean for a confidence coefficient of 95 per cent.

We compute the empirical approximation to the standard error of the sample mean; thus

$$s_{\bar{x}} = \sqrt{\frac{4}{100}} = \frac{2}{10} = 0.2.$$

The 2 limits are

$$\bar{x} + 1.96s_{\bar{x}} = 20 + 1.96(0.2) = 20.392,$$

$$\bar{x} - 1.96s_{\bar{x}} = 20 - 1.96(0.2) = 19.608.$$

These limits have the meaning: The statement that the population mean lies between 19.608 and 20.392 and between similarly computed limits will sometimes be true and sometimes false. But if we use the above formula, then in a large number of samples the chances are that, in the long run, the interval will contain the true mean in about 95 per cent of all cases, on the average.

EXAMPLE 2

Assume that we have the following sample:

Class Limits	Class Mean x	Frequency		
		f	$x f$	$x^2 f$
0-10	5	10	50	250
10-20	15	30	450	6,750
20-30	25	20	500	12,500
		60	1,000	19,500

The number of items N in the sample is 60. We have from the sums

indicated in the table

$$\bar{x} = \frac{1,000}{60} = 16.67,$$

$$V = \frac{19,500 - (60)(16.67)^2}{59} = 47.91,$$

$$s = \sqrt{46.89} = 6.92.$$

From this we get an approximation for the standard error of the sample mean; namely,

$$s_{\bar{x}} = \frac{6.92}{\sqrt{60}} = 0.893.$$

We want to establish the 95 per cent confidence, or fiducial, limits for the population mean. They are

$$16.67 + (1.96)(0.893) = 18.42,$$

$$16.67 - (1.96)(0.893) = 14.92.$$

Thus, in repeated samples, the (unknown) population mean μ will lie, in the long run, within similarly computed limits in about 95 per cent of the cases, on the average.

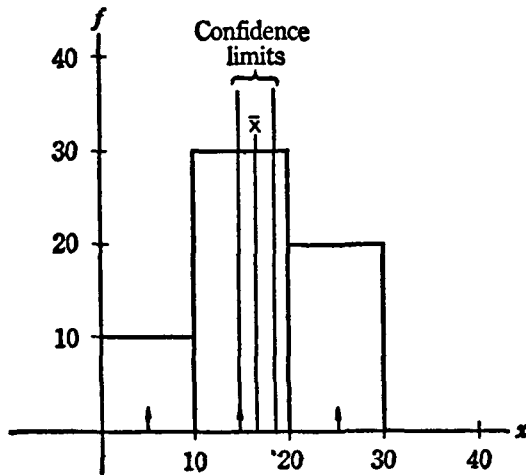


Figure 41

We indicate in Figure 41 the above frequency distribution, the sample mean, and the fiducial, or confidence, limits of the population mean established above.

● **EXERCISES 89**

Compute the 95 per cent confidence limits for the population mean of the samples listed below:

1. Problem 1, Exercises 86.
2. Problem 2, Exercises 86.
3. Problem 3, Exercises 86.
4. Problem 4, Exercises 86.
5. Problem 1, Exercises 87.
6. Problem 3, Exercises 87.
7. Problem 4, Exercises 87.
8. Problem 7, Exercises 87.
9. Problem 8, Exercises 87.
10. Problem 9, Exercises 87.

TESTS OF HYPOTHESES

90. Tests of Statistical Hypotheses (Large Samples)

A very important branch of modern statistics deals with the tests of statistical hypotheses. A *statistical hypothesis* is a hypothesis about a property of the (unknown) population.

Suppose that we have a sample with the help of which we want to test the hypothesis. We compute the probability that the particular sample and its characteristics (mean, and so on) would have arisen if the hypothesis were true. This probability will not in general be 0, but it may be very small.

We will choose a *level of significance*, for instance, 5 per cent. This means that a hypothesis is to be rejected if our sample could have arisen by pure chance in less than 5 per cent of the cases, if the hypothesis is true. If the probability is less than 5 per cent, say 0.001 per cent, we will reject the hypothesis; if the probability is larger than 5 per cent, say 20 per cent, we will not reject it.

We give here the large sample theory of tests of hypotheses concerning the population mean μ of a normal population. The small-sample theory is presented in Section 91. We have shown above that for large samples the sample standard deviation s is a good approximation to the population standard deviation σ . This follows from the law of large numbers. Hence the quantity

$$u = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

will be normally distributed with 0 mean and unit variance, where μ is the

hypothetical population mean and

$$s_{\bar{x}} = \frac{s}{\sqrt{n}}$$

is an empirical approximation to its standard error.

From these data we can compute the probability that a positive or negative deviation equal to or greater than u might have arisen by pure chance. This probability is

$$P = 1 - 2p,$$

where p is the probability listed for u in Table 4. To be more specific, P is the probability of obtaining the empirical sample mean \bar{x} , or another sample mean with the same or greater positive or negative deviation from the hypothetical population mean μ .

Two types of errors are committed in testing hypotheses. The first type occurs if a true hypothesis is rejected. The probability of the error of the first kind is given by the level of significance. If the level of significance is 5 per cent, then a true hypothesis will be rejected in about 5 per cent of the cases, in the long run, on the average.

But there is another type of error, called *error of the second kind*. This error occurs if a false hypothesis is not rejected. Statistical procedures ought to be devised in such a way that the probability of committing an error of the second kind is minimized. The following tests have the property of minimizing the probability of errors of the second kind for a given probability of errors of the first kind.

EXAMPLE 1

Let us assume a sample of 400 items. The sample mean is $\bar{x} = 21$, the sample variance $V = 100$. We want to test the hypothesis that the (unknown) population mean $\mu = 20$. The level of significance is 5 per cent. We have

$$s_{\bar{x}} = \sqrt{\frac{V}{n}} = \sqrt{\frac{100}{400}} = 0.5,$$

$$u = \frac{\bar{x} - \mu}{s_{\bar{x}}} = \frac{21 - 20}{0.5} = 2.$$

For $u = 2$ we have from Table 4, $p = 0.4773$. Hence the probability of obtaining as large a value, or a larger value, of u (disregarding the sign) by chance is

$$P = 1 - (2)(0.4773) = 0.0454.$$

This value, only about $4\frac{1}{2}$ per cent, is less than the level of significance specified (5 per cent). Hence the hypothesis that $\mu = 20$ must be rejected.

EXAMPLE 2

In a sample of $N = 100$ take $\bar{x} = 19.5$, $V = 25$, $s = 5$. Test the hypothesis that $\mu = 20$ where the level of significance is 5 per cent.

We have $s_{\bar{x}} = 5/10 = 0.5$; $u = (19.5 - 20)/0.5 = -1$. From Table 4, we have 0.3413 for $u = 1$. The probability of a positive or negative divergence as large as or larger than the one observed is $1 - (2)(0.3413) = 0.3174$. This value is larger than 5 per cent; so the hypothesis is not rejected.

EXERCISES 90

1. Take the sample $\bar{x} = 25$, $N = 2,500$, $V = 100$. Test the hypothesis that $\mu = 24.3$, level of significance 5 per cent.
2. Consider the sample $N = 100$, $\bar{x} = 19.5$, $V = 36$. Test the hypothesis that $\mu = 20$, level of significance 1 per cent.
3. Given the frequency distribution in Problem 2, Exercises 87. Test the hypothesis that $\mu = 4.1$, level of significance 5 per cent.
4. Given the sample in Problem 6, Exercises 87. Test the hypothesis that $\mu = 43$, level of significance 1 per cent.
5. Given the sample in Problem 5, Exercises 87. Test the hypothesis (a) $\mu = 12$; (b) $\mu = 13$; (c) $\mu = 14$; (d) $\mu = 16$; level of significance 1 per cent.
6. A sample of 2,500 items has a sample mean of 41 and a sample variance of 100. Test the hypothesis that the population mean $\mu = 40$, level of significance 5 per cent.
7. A sample of 49 items has a sample mean of 8 and a sample variance of 16. Test the hypothesis that the true mean $\mu = 10$, level of significance 1 per cent.
- *8. Show that the probability that the mean in a sample will fall between $\bar{x} - 2.58s_{\bar{x}}$ and $\bar{x} + 2.58s_{\bar{x}}$ is 0.99. (HINT: Remember that \bar{x} in large samples is normally distributed with mean μ and variance σ^2/N . The limits are called the 99 per cent confidence, or fiducial, limits.)
9. Use the results in Problem 8 to establish the 99 per cent fiducial, or confidence, limits for the mean of the sample in Problem 2, Exercises 87.
10. Use the results in Problem 8 to establish the 99 per cent confidence, or fiducial limits, for the mean of the sample in Problem 6, Exercises 87.

91. Tests of Statistical Hypotheses (Small Samples)

The tests of hypotheses given above are only valid for large samples. They are based upon the fact that for large samples the sample variance V is a good approximation for the population variance σ^2 . This follows from the law of large numbers. The chances are that the approximation will be closer the larger the sample.

If the sample is small the above procedures are not adequate. But it can be shown by methods beyond the scope of this book that for small

samples from a normal population. The quantity,

$$t = \frac{\bar{x} - \mu}{s_{\bar{x}}}$$

follows not the normal but the so-called *t distribution*, or *Student's distribution*. The probability density of *t* is given by

$$p(t) = \frac{\left(\frac{n-1}{2}\right)!}{\sqrt{n\pi} \left(\frac{n-2}{2}\right)! \left(1 + \frac{t^2}{n}\right)^{(n+1)/2}}$$

This distribution is shown graphically in Figure 42, and has been tabulated. Note Table 5. In this Table, *n* is the number of degrees of freedom.

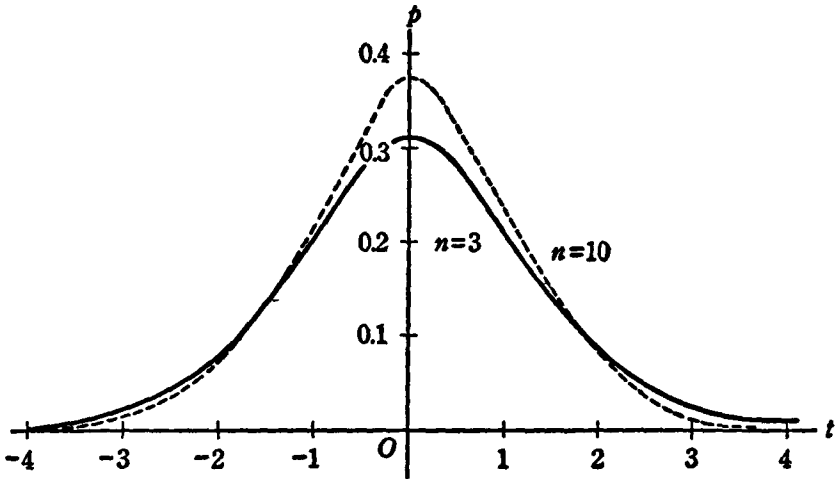


Figure 42

We show in Figure 42 the *t* distribution for 3 and 10 degrees of freedom. The area tabulated is the probability that

$$1 - \int_{-t}^t p(x) dx = P.$$

The quantity *P* is the probability that the deviation will be numerically equal to or larger than *t*.

The *t* distribution may also be used to find confidence limits for the population mean in small samples from a normal population.

The number of degrees of freedom *n* at which the table of the *t* distribution has to be entered is *N* - 1, one less than the number of items in the sample.

It appears from Figure 42 that the larger the number of degrees of

freedom, the more will the t distribution approximate the normal distribution. Hence, if the number of degrees of freedom becomes as large as or larger than 30, we may use instead of t the quantity u , which is normally distributed with 0 mean and unit variance, tabulated in Table 4.

EXAMPLE

In a sample of $N = 9$ items we have the sample mean $\bar{x} = 13$ and the sample variance $V = 25$. Hence $s = 5$, and $s_{\bar{x}} = s/\sqrt{N} = 5/\sqrt{9} = 5/3 = 1.67$. Test the hypothesis that the true or population mean is $\mu = 10$. Take the level of significance as 5 per cent. Thus $t = (x - \mu)/s_{\bar{x}} = (13 - 10)/1.67 = 3/1.67 = 1.8$. For $n = N - 1 = 8$ degrees of freedom, we have $t = 2.306$ at the 5 per cent level of significance. Our empirical t is only 1.8. Hence its probability is greater than 5 per cent, and the hypothesis that the true or population mean is 10 is not rejected.

EXERCISES 91

1. Use the sample of Problem 1, Exercises 87, to test the hypothesis that $\mu = 24$, level of significance 1 per cent.
2. Use the data in Problem 2, Exercises 87, to test the hypothesis that $\mu = 4$, level of significance 5 per cent.
3. Use the data in Problem 3, Exercises 87, to test the hypothesis that $\mu = 10$, level of significance 1 per cent.
4. Use the data in Problem 4, Exercises 87, to test the hypothesis that $\mu = 11$, level of significance 5 per cent.
5. Use the data in Problem 5, Exercises 87, to test the hypothesis that $\mu = 10$, level of significance 5 per cent.
6. Use the data in Problem 6, Exercises 87, to test the hypothesis that $\mu = 38$, level of significance 1 per cent.
7. Use the data in Problem 7, Exercises 87, to test the hypothesis that $\mu = 2,100$, level of significance 1 per cent.
8. Use the data in Problem 9, Exercises 87, to test the hypothesis that $\mu = 35$, level of significance 5 per cent.
- *9. Consider the problem of establishing the 95 per cent confidence, or fiducial, limits for the sample mean in small samples. (HINT: The number of degrees of freedom has to be taken into account.)
- **10. Use the results of Problem 9 to establish the 95 per cent confidence, or fiducial, limits for the data in Problem 1, Exercises 87.
- **11. Use the results of Problem 9 to establish the 95 per cent and the 99 per cent fiducial, or confidence, limits for the sample in Problem 3, Exercises 87.

92. Test of Significance for Two Samples

In a test of significance, we test a hypothesis which we call the *null* hypothesis. First, we fix *a priori* a certain level of significance. Then we find

the probability that the positive or negative deviation between the empirical results of the sample and the postulated hypothesis should have arisen by pure chance. If the probability is less than the level of significance chosen, we reject the null hypothesis. We say that the quantity we are testing is *significant*.

We will use the test of significance to test the difference between the mean in 2 samples. Let there be N_1 observations in the first sample, and N_2 in the second sample. Let \bar{x}_1 and \bar{x}_2 be the means in the 2 samples and V_1 and V_2 their sample variances. Both samples come from normal populations.

We have the null hypothesis that in the population the 2 population means are equal, that is, that $Ex_1 - Ex_2 = 0$.

To test this hypothesis we compute

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_{\bar{x}_1 - \bar{x}_2}}.$$

The denominator of this fraction can be computed by recalling that

$$s_{\bar{x}_1 - \bar{x}_2}^2 = \frac{[(N_1 - 1)V_1 + (N_2 - 1)V_2](N_1 + N_2)}{N_1 N_2 (N_1 + N_2 - 2)},$$

which is the square of the standard error of the difference of the means. For large samples, $t = u$, and is normally distributed. For small samples we have t distributed according to the t distribution with $n = N_1 + N_2 - 2$ degrees of freedom.

EXAMPLE 1

Assume that we have 2 samples with $N_1 = 100$, $N_2 = 400$. Take the sample means as $\bar{x}_1 = 11.5$ and $\bar{x}_2 = 12$, and the sample variances as $V_1 = 9$ and $V_2 = 4$. We want to test the hypothesis that in the populations we have no difference between their means. The level of significance is 1 per cent. Since we have large samples we must evaluate

$$u = \frac{\bar{x}_1 - \bar{x}_2}{s_{\bar{x}_1 - \bar{x}_2}}.$$

For the square of the standard error of the difference between the sample means we obtain

$$s_{\bar{x}_1 - \bar{x}_2}^2 = \frac{[(99)(9) + (399)(4)](100 + 400)}{(100)(400)(100 + 400 - 2)} = 0.0624.$$

Then the standard error of the difference of the means is

$$s_{\bar{x}_1 - \bar{x}_2} = \sqrt{0.0624} = 0.25.$$

Finally,

$$u = \frac{11.5 - 12}{0.25} = -2.$$

From Table 4, we have 0.4773 for $u = 2$; we neglect the sign because of the symmetry of the normal distribution. This is the probability of getting a u less than 2. We want the probability of obtaining a u greater in absolute value than 2. Hence our probability is $1 - 2(0.4773) = 0.0454$. This is more than 0.01, the given level of significance. Hence, the null hypothesis cannot be rejected. The difference in the means is not significant.

EXAMPLE 2

Let us illustrate the situation for small samples. Assume in the first sample $N_1 = 9$, $\bar{x}_1 = 5$, $V_1 = 25$; in the second sample $N_2 = 4$, $\bar{x}_2 = 13$, $V_2 = 16$. We want to test the null hypothesis that in the populations there is no difference between the means; that is, we test the significance of $\bar{x}_1 - \bar{x}_2$, level of significance 5 per cent.

We have for the standard error of the difference of the 2 means:

$$s_{\bar{x}_1 - \bar{x}_2} = 2.85.$$

Hence

$$t = \frac{(5 - 13)}{2.85} = -2.81.$$

Because of the symmetry of the t distribution we can neglect the negative sign. For $n = N_1 + N_2 - 2 = 11$ degrees of freedom, we have $t = 2.201$ at the level of significance specified. Our empirical t , namely, 2.81, is larger. Hence the null hypothesis that there is no difference in the population between the 2 means is to be rejected. The 2 population means are in all probability significantly different.

EXERCISES 92

1. Consider the 2 samples:

Sample 1		Sample 2	
Class Limits	Frequencies	Class Limits	Frequencies
0-4	1	0-5	2
4-8	4	5-10	5
8-12	3	10-15	2

Test the significance of the difference of the means, level of significance 5 per cent.

2. Consider the 2 samples:

Sample 1		Sample 2	
Class Limits	Frequencies	Class Limits	Frequencies
0-10	10	0-20	10
10-20	10	20-40	30
20-30	10	40-60	20
30-40	10		
40-50	10		

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Test the significance of the difference of the 2 sample means, level of significance 1 per cent.

3. Consider the 2 samples:

Sample 1		Sample 2	
Class Limits	Frequencies	Class Limits	Frequencies
10-20	7	0-20	5
20-30	4	20-40	5

Test the significance of the difference between the sample means, level of significance 1 per cent.

4. Consider the 3 samples:

Sample 1		Sample 2		Sample 3	
Class Limits	Frequencies	Class Limits	Frequencies	Class Limits	Frequencies
0-2	1	2-6	5	0-4	2
2-4	2	6-10	5	4-8	7
4-6	3				
6-8	3				

Test the significance of (a) $\bar{x}_1 - \bar{x}_2$; (b) $\bar{x}_1 - \bar{x}_3$; (c) $\bar{x}_2 - \bar{x}_3$. Take the level of significance as 5 per cent.

5. Given 3 samples:

Sample 1		Sample 2		Sample 3	
Class Limits	Frequencies	Class Limits	Frequencies	Class Limits	Frequencies
0-10	49	0-7	30	0-12	51
10-20	51	7-14	37	12-24	50
		14-21	35		

Test the significance of (a) $\bar{x}_1 - \bar{x}_2$; (b) $\bar{x}_1 - \bar{x}_3$; (c) $\bar{x}_2 - \bar{x}_3$; the level of significance is 1 per cent.

6. Given 2 samples:

Sample 1		Sample 2	
Class Limits	Frequencies	Class Limits	Frequencies
0-4	1	0-5	1
4-8	0	5-10	4
8-12	3	10-15	3
12-16	3	15-20	2
16-20	1		

Test the significance of the difference between the sample means, level of significance 1 per cent.

7. Given the 3 samples:

Sample 1		Sample 2		Sample 3	
Class Limits	Frequencies	Class Limits	Frequencies	Class Limits	Frequencies
0-10	20	0-20	60	0-5	20
10-20	50	20-40	20	5-10	20
20-30	20			10-15	20
				15-20	15
				20-25	10

(a) Test the significance of $\bar{x}_1 - \bar{x}_2$; (b) of $\bar{x}_1 - \bar{x}_3$; (c) of $\bar{x}_2 - \bar{x}_3$, level of significance 5 per cent.

8. Given the 2 samples:

Sample 1: 1, 5, 3, 0, 2.

Sample 2: 9, 7, 5, 0, 3, 4, 7.

Test the significance of the difference between the sample means, level of significance 1 per cent.

****9.** Call the difference between 2 sample means in large samples d . Establish limits for d such that the probability that d will fall between them is (a) 95 per cent; (b) 99 per cent; (c) 100 per cent; (d) 90 per cent.

10. The following table gives the family wage or salary, United States, 1939.

Wage Class Limits	Frequency in Millions of Families		
	Urban	Rural Nonfarm	Rural Farm
\$ 0-200	4.1	1.8	4.1
200-500	0.5	0.4	0.7
500-1,000	1.4	1.0	1.0
1,000-1,500	3.1	1.4	0.7
1,500-2,000	3.3	1.0	0.3
2,000-2,500	2.9	0.7	0.2
2,500-3,000	2.0	0.4	0.1
3,000-5,000	1.1	0.2	0.0

Test the significance of the difference between the means of these distributions. The level of significance is 1 per cent.

11. The following table gives the age distribution, United States, 1940.

Age Class Limits	Frequency in Millions of Persons			
	Male	Female	White	Colored
0-5	5	6	9	1
5-10	5	6	9	1
10-15	6	6	10	1
15-20	6	6	11	1
20-25	6	6	10	1
25-30	5	6	10	1

Age Class Limits	Frequency in Millions of Persons			
	Male	Female	White	Colored
30-35	5	6	9	1
35-40	5	5	9	1
40-45	4	5	8	1
45-50	4	4	8	1
50-55	4	4	7	1
55-60	3	3	5	
60-65	2	2	4	
65-70	2	2	4	
70-75	1	1	2	

Test the significance of the difference of the means when the level of significance is 5 per cent (a) between male and female; (b) between white and colored.

FITTING OF DISTRIBUTIONS

**93. Fitting of the Normal Distributions

We have reason to believe sometimes that a random sample comes from a normal population. Then we may try to fit the normal curve to the sample values in order to achieve an approximation to the normal distribution of the population.

■ EXAMPLE

Given the following sample:

Class Limits	x	f	Upper Limits (Standardized)	Cumulative Probability	p	Np
0-2	1	1	-2.36	0.0091	0.0077	0.39
2-4	3	1	-1.74	0.0409	0.0318	1.59
4-6	5	4	-1.11	0.1335	0.0926	4.63
6-8	7	8	-0.48	0.3156	0.1821	9.11
8-10	9	13	0.15	0.5596	0.2440	12.20
10-12	11	15	0.78	0.7823	0.2227	11.14
12-14	13	5	1.41	0.9207	0.1384	6.92
14-16	15	2	2.04	0.9793	0.0586	2.93
16-18	17	0	2.67	0.9962	0.0169	0.85
18-20	19	1	3.30	0.9995	0.0033	0.17
		<u>50</u>				
		$N = 50$				

To fit a normal distribution we convert the upper limits of the various classes in the frequency distribution into standard measure by computing

$$u = \frac{X - \bar{x}}{s},$$

where X is the upper class limit, \bar{x} is the sample mean, and s the sample standard deviation.

The probabilities for u are given in Table 4. For negative u , the value of the area under the normal curve has to be subtracted from 0.5. For positive u , 0.5 has to be added to the value given in the table. The probabilities p are computed by taking the differences of 2 successive items in the column of cumulative probabilities.

As indicated we find that $N = 50$. The sample mean $\bar{x} = 9.52$, the sample variance $V = 10.09$, and the sample standard deviation $s = 3.18$. The 4th column gives the upper limit in each class in standardized form. We have, for example, for the 1st class $(2 - 9.52)/3.18 = -2.36$, and so on.

The cumulative probabilities are derived from Table 4. We have, for example, for the 3rd class, corresponding to $u = -1.11$, $0.5 - 0.3665 = 0.1335$, and so on. For positive standardized variables we have, for example, in the 7th class, corresponding to $u = 1.41$, $0.5 + 0.4207 = 0.9207$. By forming the differences we get the actual probabilities; for instance, in the 3rd class, we have $0.1335 - 0.0409 = 0.0926$. Multiplying by $N = 50$ we get the last column.

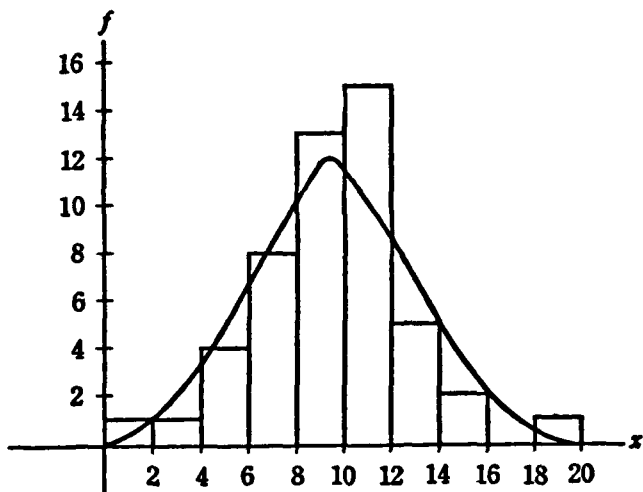


Figure 43

We show in Figure 43 the histogram of the frequency distribution in the above example and also the fitted normal curve.

● EXERCISES 93

1. Given the data in Problem 2, Exercises 87. (a) Fit a normal curve. (b) Graph the original frequency curve and the fitted normal curve.

2. Given the data in Problem 3, Exercises 87. (a) Fit a normal curve. (b) Graph the original frequency curve and the fitted normal curve.

3. Given the data in Problem 4, Exercises 87. (a) Fit a normal curve. (b) Graph the original frequency distribution and the fitted normal curve.

4. Given the data in Problem 5, Exercises 87. (a) Fit a normal curve. (b) Graph the original frequency distribution and the fitted normal curve.

5. Given the data in Problem 6, Exercises 87. (a) Fit a normal curve. (b) Graph the original frequency curve and the fitted normal curve.

6. Given the data in Problem 7, Exercises 87. (a) Fit a normal curve. (b) Graph the original frequency distribution and the fitted normal curve.

7. Given the data in Problem 9, Exercises 87. (a) Fit a normal curve. (b) Graph the original frequency distribution and the corresponding normal curve.

8. Given the following data for the United States, 1941:

Acres	No. of Farms
0 to 3	36,000
3 to 10	470,000
10 to 20	559,000
20 to 50	1,221,000
50 to 100	1,291,000
100 to 500	2,225,000

(a) Fit a normal curve. (b) Graph the original frequency curve and the fitted curve.

9. Given the following data for the United States, 1942:

Income Classes	No. of Families
500-1,000	2,319,000
1,000-1,500	4,604,000
1,500-2,000	4,837,000
2,000-2,500	4,920,000
2,500-3,000	3,953,000

(a) Fit a normal distribution. (b) Graph the data, and the normal frequency curve.

10. Given the data:

Class Limits	Frequencies
0-2	10
2-4	40
4-6	60
6-8	40
8-10	10

(a) Fit a normal distribution. (b) Graph the data, and the fitted normal curve.

****94. Tests of Goodness of Fit**

To test the goodness of fit for fitting the normal distribution we compute the quantity χ^2 , defined by

$$\chi^2 = \sum \frac{(f_i - Np_i)^2}{Np_i}.$$

In this summation we have the squares of the deviations between the empirical relative frequencies and the theoretical probabilities, divided by the latter. If we have a perfect fit, relative frequencies and probabilities are identical, so $\chi^2 = 0$.

The distribution of this quantity has been established. Its probability density is

$$p(\chi^2) = \frac{(\chi^2)^{(k-2)/2} e^{-(\chi^2)/2}}{2^{(k)/2} \left(\frac{k-2}{2}\right)!},$$

where k is the number of degrees of freedom.

Care must be taken that in all classes the quantity Np_i is at least as large as 5. If this is not the case neighboring classes have to be combined into 1 single class.

The number of degrees of freedom is the number of classes upon which the test is based, less 3. The area under the curve has been tabulated. The tabulated probability is

$$P = 1 - \int_0^{\chi^2} p(s) ds.$$

We get from Table 6 the probability P that for a given number of degrees of freedom the empirical χ^2 will be equal to or larger than the χ^2 in the table.

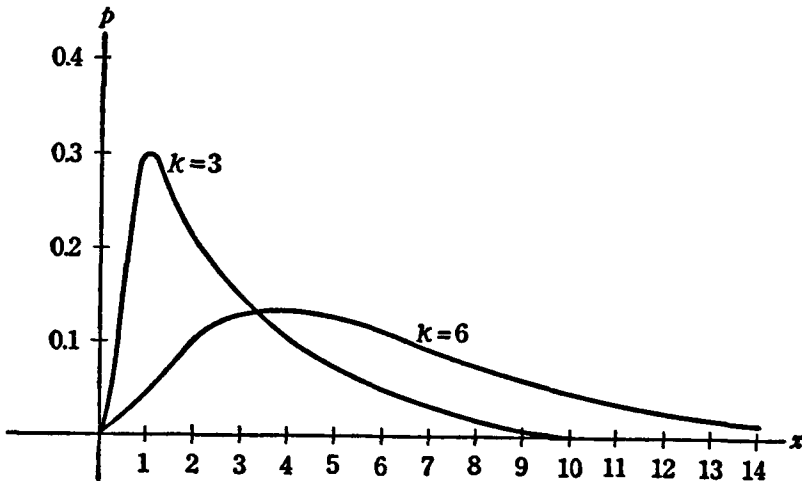


Figure 44

The number of degrees of freedom is k . This is the number of classes in the distribution, minus the number of constants used in computing χ^2 .

We show in Figure 44 the χ^2 distribution for 3 and 6 degrees of freedom.

If the number of degrees of freedom becomes large, we can assume that

the quantity

$$u = \sqrt{2\chi^2} - \sqrt{2k - 1}$$

follows the normal distribution, with mean zero and variance one. (Table 4).

It is apparent from the graph that the χ^2 distribution for $k = 6$ is nearer to a normal distribution than the one for χ^2 with 3 degrees of freedom. We may use the normally distributed quantity u , defined above, if we have 30 or more degrees of freedom.

The null hypothesis tested in a test of significance for goodness of fit is that we obtain a perfect fit, that is, that in the population the value of χ^2 is 0.

EXAMPLE

We want to test the goodness of fit of the normal curve fitted in the previous example. We assume a level of significance 5 per cent. We have to combine the first 3 and the last 4 classes, since for them the values of Np are less than 5. The data are

Class Limits	f	Np	$(f - Np)^2/Np$
below 6	6	6.61	0.06
6-8	8	9.11	0.14
8-10	13	12.20	0.05
10-12	15	11.14	1.34
over 12	8	10.87	0.76
			$\chi^2 = 2.35$

The terms in the last column come from the formula $(f - Np)^2/Np$. We get, for instance, in the 1st class: $(6 - 6.61)^2/6.61 = 0.06$. The sum of all these quantities is χ^2 .

To test the significance of χ^2 we note that the number of classes is 5. The number of degrees of freedom is $5 - 3 = 2$, because 3 constants (N , \bar{x} , V) have been estimated from the sample. Hence we lose 3 degrees of freedom. At the 5 per cent level, χ^2 can be as large as 5.99 for 2 degrees of freedom. Our empirical χ^2 is much smaller; in fact, it is only 2.35. It is not significant, which means that we get a reasonably good fit.

EXERCISES 94

1. Given the data in Problem 1, Exercises 87. (a) Fit a normal curve. (b) Test the goodness of fit, level of significance 1 per cent.
2. Given the data in Problem 6, Exercises 87. (a) Fit a normal curve. (b) Test the goodness of fit, level of significance 5 per cent.
3. Given the data in Problem 7, Exercises 87. (a) Fit a normal curve. (b) Test the goodness of fit, level of significance 1 per cent.
4. Given the data in Problem 8, Exercises 93. Test the goodness of fit of the normal curve, level of significance 5 per cent.

5. Given the data in Problem 9, Exercises 93. Test the goodness of fit of the normal curve, level of significance 1 per cent.

6. Give the data in Problem 10, Exercises 93. Fit a normal curve and test the goodness of fit, level of significance 5 per cent.

7. Use the data in Problem 1, Exercises 92, Sample 1, to fit a normal frequency curve. Test the goodness of fit, level of significance 1 per cent.

8. Use the data in Problem 4, Exercises 92, Sample 3, to fit a normal frequency curve. Test the goodness of fit, level of significance 5 per cent.

****95. Contingency Tables**

Assume we have a random sample. The individual items have been classified according to 2 characteristics, those of *A* and *B*, and *C* and *D*. We obtain a 2 × 2 contingency table as follows:

	<i>C</i>	<i>D</i>	Total
<i>A</i>	<i>a</i>	<i>b</i>	<i>a</i> + <i>b</i>
<i>B</i>	<i>c</i>	<i>d</i>	<i>c</i> + <i>d</i>
Total	<u><i>a</i> + <i>c</i></u>	<u><i>b</i> + <i>d</i></u>	<u><i>a</i> + <i>b</i> + <i>c</i> + <i>d</i></u>

If the sample comes from a normal population we can test the independence of the classification by computing

$$\chi^2 = \frac{(ad - bc)^2(a + b + c + d)}{(a + b)(c + d)(a + c)(b + d)}.$$

This is distributed like χ^2 with 1 degree of freedom.

This distribution is only approximate. It will be valid if the expected values, which would prevail under independence, are not less than 5.

■ EXAMPLE

Consider the following contingency table:

		11	9	Total
		8	12	20
	Total	<u>19</u>	<u>21</u>	<u>40</u>

The formula gives

$$\chi^2 = \frac{[(11)(12) - (8)(9)]^2(40)}{(20)(20)(19)(21)} = \frac{120}{133} = 0.902.$$

Suppose we choose for our level of significance 1 per cent. For 1 degree of freedom, χ^2 may be as large as 6.63. Since our empirical value is smaller, there is no reason to reject the null hypothesis that the characteristics are independent.

EXERCISES 95

1. Given the contingency table:

25	5
15	35.

Test for independence, level of significance 5 per cent.

2. Given the contingency table:

25	25
15	35.

Test for independence, level of significance 1 per cent.

3. The following data are for the United States, 1940, in millions:

	White	Colored
Male	59.4	6.6
Female	58.8	6.8.

Test the independence of sex and color, level of significance 1 per cent.

4. The following data, in millions, were true for the United States, 1930:

	White	Colored
Male	55.9	6.2
Female	54.4	6.2.

Test the independence of sex and color, level of significance 5 per cent.

5. The following data, in millions, are for the United States' labor force, 1930:

	Employed	Unemployed
Male	35.0	2.0
Female	10.0	0.4

Test the independence of employment status and sex, level of significance 5 per cent.

6. The United States' labor force, 1940, in millions, is characterized below:

	Employed	Unemployed
Male	34.0	6.2
Female	11.2	1.8.

Test the independence of employment status and sex, level of significance 1 per cent.

7. The distribution of the labor force for the United States, 1940, in millions, is given below:

	Agriculture Forestry Fisheries	Other Industries
Employed	8.5	36.7
Unemployed	0.6	6.2.

Test the independence of employment status and the distribution in industries, level of significance 5 per cent.

****8.** Show that the expected values in a 2×2 table in case of independence are

$$\frac{(a+b)(a+c)}{a+b+c+d} \qquad \frac{(a+b)(b+d)}{a+b+c+d}$$

$$\frac{(a+c)(c+d)}{a+b+c+d} \qquad \frac{(b+d)(c+d)}{a+b+c+d}.$$

Compute the expected values for the Example on page 270.

REGRESSION AND CORRELATION

96. Method of Least Squares

One method of estimation, employing the method of least squares, is particularly appropriate with economic data, where frequently we do not have normal distributions. *The method of least squares consists in the choice of an estimate so that the sum of the squares of the deviations of the data from the estimate is a minimum.*

The method of least squares gives the best, unbiased linear estimate. The word best means here that the least-squares estimate has a smaller variance than any other linear estimate. By unbiased is meant that the mean value (mathematical expectation) of the estimate is equal to the true population value. That is, the average of a great many least-square estimates is likely to differ from the true value as little as we like. The adjective *linear* is employed because the estimates must be linear functions of the parameters estimated. All these results are independent of any assumption of normality of the observations. (Markoff theorem.)

Assume we have a number of observations x_1, x_2, \dots, x_n . Let us use the method of least squares in order to estimate the "true" value of x in a sample of n independent observations.

Let a be the quantity we want to find. According to the fundamental principle of least squares, we want to minimize the sum of squares:

$$Q = (x_1 - a)^2 + (x_2 - a)^2 + (x_3 - a)^2 + \dots + (x_n - a)^2.$$

This is the sum of the squares of the deviations of the observations from the estimate a .

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We want to adjust a in such a fashion that Q become a minimum. Hence we must have $dQ/da = 0$, $d^2Q/da^2 > 0$. Thus

$$\begin{aligned} dQ/da &= -2(x_1 - a) - 2(x_2 - a) - 2(x_3 - a) - \cdots - 2(x_n - a) = 0, \\ d^2Q/da^2 &= 2 + 2 + 2 + \cdots + 2 = 2n > 0. \end{aligned}$$

Simplifying the first equation, we have

$$na = x_1 + x_2 + x_3 + \cdots + x_n$$

$$a = \left(\frac{1}{n}\right) \sum_{i=1}^n x_i = \bar{x}.$$

We see that the least square estimate $a = \bar{x}$ is the arithmetic mean. It follows from the fact that the second derivative is positive (n being the number of observations) that we have a true minimum of Q .

● EXERCISES 96

1. Let a sample consist of the items 2, 5, 9. (a) Compute the sum of squares of the deviations from the estimated value a ; that is, $Q = (2 - a)^2 + (5 - a)^2 + (9 - a)^2$, by squaring the expressions in brackets. (b) Find the derivative of Q with respect to a . (c) Show that the least squares estimate of a is the arithmetic mean of the three numbers given above.

2. Given the 6 numbers 2, 7, 4, 1, 0, -3 . Proceed as in Example 1.

*3. Use the method of least squares to find the estimate of the true value of $\log x_1, \log x_2, \dots, \log x_n$. This least squares estimate is $\log G = (1/n) \sum_{i=1}^n \log x_i$, so G is the geometric mean.

4. Use the formula established in Problem 3 to find the geometric mean of (a) 2, 3, 9; (b) 1, 16, 19; (c) 2, 10, 15, 105; (d) 2, 5, 5, 10, 18; (e) 1, 6, 5; (f) 0, 5, 2, 3, 5.5, 7.6, 10.7.

**5. Find the minimum value of Q which is assumed in the example above for $a = \bar{x}$. Interpret it statistically.

6. Use the formula established in Problem 3 to find the geometric mean of the data given in Problem 1, Exercises 86.

7. Use the result of Problem 3 to establish the geometric mean of the data in Problem 2, Exercises 86.

8. Use the data in Problem 3, Exercises 86, to compute the geometric mean according to the result in Problem 3.

9. Compute the geometric mean of the data in Problem 4, Exercises 86, according to the results of Problem 3.

**10. Consider the situation described in Problem 5, Exercises 85. Show that the least-square estimate and the maximum-likelihood estimate are the same, namely, \bar{x} , which is the arithmetic mean of the observations in the sample.

97. Curve Fitting

The method of least squares is useful in curve fitting. Assume we have the series of items, x_1, x_2, \dots, x_n , and the associated series, y_1, y_2, \dots, y_n . We want to estimate or predict y if x is given, and if we assume that there is a linear relationship $\hat{y} = a + bx$, where \hat{y} is the estimated value. It follows that

$$\begin{aligned} \hat{y}_1 &= a + bx_1 \\ \hat{y}_2 &= a + bx_2 \\ &\dots\dots\dots \\ \hat{y}_n &= a + bx_n. \end{aligned}$$

So the errors are $(y_i - \hat{y}_i) = y_i - (a + bx_i)$.

The sum of squares of the errors to be minimized is $F = (y_1 - a - bx_1)^2 + (y_2 - a - bx_2)^2 + \dots + (y_n - a - bx_n)^2$. This is to be minimized by the appropriate choice of the values of a and b . We have the necessary conditions $\partial F/\partial a = 0$ and $\partial F/\partial b = 0$. These give the 2 *normal equations* for a and b , namely,

$$\begin{aligned} \sum y &= na + b\sum x, \\ \sum xy &= a\sum x + b\sum x^2, \end{aligned}$$

where $\sum x$, for example, is adopted for purposes of simplicity to denote the sum of all the x 's.

If the variable x is time, the fitted line is called a *linear trend*.

EXAMPLE

Assume the development of an economic time series:

Year	x	y	xy	x^2
1910	-3	1	-3	9
1920	-1	2	-2	1
1930	1	2	2	1
1940	3	3	9	9
Sums	0	8	6	20.

We have chosen the x so that their sum becomes 0. It is observed that $\sum x = 0, \sum y = 8, \sum xy = 6, \sum x^2 = 20$.

Since $n = 4$, the normal equations are

$$\begin{aligned} 8 &= 4a + 0 \cdot b = 4a, \\ 6 &= 0 \cdot a + 20b = 20b. \end{aligned}$$

This gives the solution $a = 2$ and $b = 0.3$. Thus the equation for prediction is

$$\hat{y} = a + bx = 2 + 0.3x.$$

This result can be used for interpolation, since the relationship is linear. For instance, estimate the value of y in 1935: The corresponding x is 2, and

the estimated value of y is $2 + 0.3(2) = 2.6$. If conditions are the same for the future as in the past we can also use the formula for extrapolation. For example, what under these conditions is the predicted value of y in 1950? Evidently $x = 5$, so $\hat{y} = 2 + 0.3(5) = 3.5$.

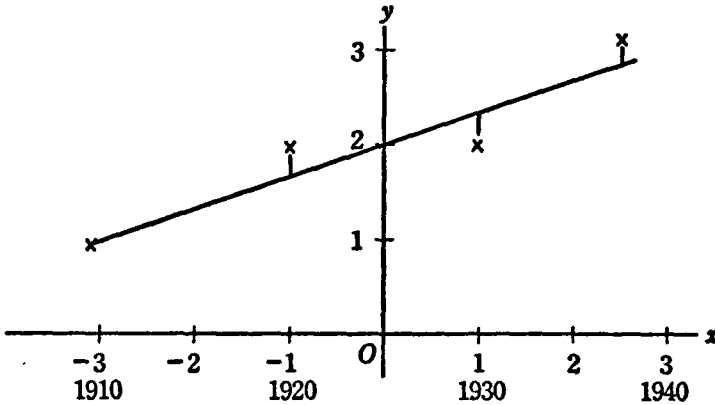


Figure 45

This example is illustrated by Figure 45. We show the observations as crosses in their relation to the fitted straight line. The deviations from the fitted line, the sum of the squares of which is minimized, are also indicated.

● EXERCISES 97

1. Given the following values for consumption of wheat in the United States:

Year	Consumption
1910	36
1915	43
1920	34
1925	41
1930	47.

(a) Fit by the method of least squares the best linear trend. (HINT: Choose $x = -2, -1, 0, 1, 2$.) (b) Interpolate for 1926; (c) for 1927; (d) for 1929. (e) Extrapolate for 1940; (f) for 1945; (g) for 1947.

2. Given the following data for the price of rye in the United States:

Year	Price
1890	66
1900	56
1910	90
1920	78.

(a) Fit a linear trend by the method of least squares. (HINT: Choose $x = -3, -1, 1, 3$.) (b) Interpolate for 1915; (c) 1916; (d) 1919. (e) Extrapolate for 1925; (f) 1932.

3. Given the following data for consumption of cotton in the United States:

Year	Consumption
1895	25
1900	36
1905	49
1910	45
1915	65
1920	50
1925	61.

(a) Fit a linear trend by the method of least squares. (b) Graph the data and the trend. (c) Interpolate for 1921; (d) 1923; (e) 1924. (f) Extrapolate for 1930; (g) for 1943.

4. Given the following data for the price of potatoes in the United States:

Year	Price
1918	120
1921	110
1924	62
1927	97.

(a) Fit a linear trend by the method of least squares. (b) Graph the data and trend. (c) Interpolate for 1925; (d) 1926. (e) Extrapolate for 1930; (f) for 1935.

5. Given the following data for the consumption of pig iron in the United States:

Year	Consumption
1880	140
1890	121
1900	101
1910	105.

(a) Fit a linear trend by the method of least squares. (b) Graph the data and trend. (c) Interpolate for 1905; (d) for 1909. (e) Extrapolate for 1914; (f) for 1916.

6. Given the following production data for steel in the United States:

Year	Production
1930	12
1932	4
1934	6
1936	11
1938	8.

(a) Fit a linear trend by the method of least squares. (b) Graph the data and trend. (c) Interpolate for 1935; (d) for 1937. (e) Extrapolate for 1939; (f) 1940.

7. Given the following data on Wholesale Price Indexes, 1929 = 100.

Year	All Commodities	Raw Materials	Semi-Manufactured	Manufactured Production
1930	91	86	87	93
1931	77	67	74	82
1932	68	56	63	74
1933	69	58	70	75
1935	84	79	78	87
1937	91	87	91	92
1939	81	72	82	85
1940	82	74	84	86
1941	92	86	92	94
1942	104	103	99	104
1943	108	115	99	106
1944	109	116	100	107.

(a) Fit straight-line trends by the method of least squares. (HINT: Call 1930 year 1, 1931 year 2, and so forth.) (b) Graph original data and trends. (c) Interpolate for 1934. (d) Interpolate for 1938. (e) Extrapolate for 1945.

8. Given the following data pertaining to the cost of living index, United States, where 1935-1939 = 100:

Year	Index
1930	119
1931	109
1932	98
1933	92
1934	96
1935	98
1936	99
1937	103
1938	101
1939	99
1940	100
1941	105
1942	116.

(a) Fit a straight-line trend. (b) Graph the data and trend. (c) Extrapolate for 1943; (d) 1944; (e) 1945.

****9.** Prove that the normal equations in the example above actually give a minimum of F . Consider necessary and sufficient conditions.

***10.** Assume the relationship between x and \hat{y} to be $\hat{y} = a + bx + cx^2$ (parabola), and show that under these conditions the method of least squares leads to the

normal equations:

$$\begin{aligned}\sum y &= na + b\sum x + c\sum x^2, \\ \sum xy &= a\sum x + b\sum x^2 + c\sum x^3, \\ \sum x^2y &= a\sum x^2 + b\sum x^3 + c\sum x^4.\end{aligned}$$

11. Use the results of Problem 10 to fit a parabolic trend to the data of the example above.

12. Use the results of Problem 10 to establish a parabolic trend for the data in Problem 2. Work parts (b) to (f) in Problem 2 if a parabolic trend is assumed, and compare results with the linear trend.

13. Use the results of Problem 10 to fit a parabolic trend to the data of Problem 5. Use this trend to work (b) to (f) in Problem 5 and compare with the results of the linear trend.

***14.** Establish the normal equations for an exponential trend in the logarithmic form: $\log \hat{y} = a + bx$. (HINT: Substitute $\log y$ for y in the normal equation for a straight-line trend. The normal equations are, then,

$$\begin{aligned}\sum \log y &= an + b\sum x, \\ \sum x \log y &= a\sum x + b\sum x^2.\end{aligned}$$

Show that the normal equations follow from the necessary conditions of a minimum and that the sufficient conditions are fulfilled.

15. Given the data:

Year	Population United States (millions)
1900	76
1910	92
1920	106
1930	123
1940	132.

(a) Fit an exponential trend (Problem 14). (b) Graph the data and trend. (c) Interpolate for 1925; (d) for 1937; (e) 1939. (f) Extrapolate for 1945.

16. Use the results of Problem 14 to fit an exponential trend to the data in the example above.

17. Use the results of Problem 14 to fit an exponential trend to the data of Problem 2. Work parts (b) to (f) of the problem. (HINT: The estimates are now $\log y$ and have to be converted into ordinary numbers.)

18. Use the results of Problem 14 to fit an exponential trend to the data in Problem 5. Work parts (b) to (f) of Problem 5 with an exponential trend.

19. Use the results of Problem 14 to fit an exponential trend to the data of Problem 3. Work parts (b) to (g) of Problem 3 with this trend.

****20.** Assume that the values y_1, y_2, \dots, y_n are normally and independently distributed according to

$$p_1 = (1/\sqrt{2\pi}) e^{-\frac{1}{2}(y_1 - a - by_1)^2}, \quad p_2 = (1/\sqrt{2\pi}) e^{-\frac{1}{2}(y_2 - a - by_2)^2}$$

$$p_n = (1/\sqrt{2\pi}) e^{-\frac{1}{2}(y_n - a - by_n)^2}.$$

(a) Establish the probability that the independent events y_1, y_2, \dots, y_n will happen together. (b) Maximize this probability (method of maximum likelihood). Show that the resulting normal equations for a and b are the same as in the method of least squares.

***21.** Given 2 series, x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n . The estimated value of x is $\hat{x} = A + By$. Establish the normal equation for A and B which minimize $G = (x_1 - A - By_1)^2 + (x_2 - A - By_2)^2 + \dots + (x_n - A - By_n)^2$. Show that the normal equations are the same as in the example above with the role of x and y exchanged.

***22.** Let x be income and y the number of people receiving income x or higher. Assume a Pareto curve for the cumulative distribution of income (Section 14). Fit this curve in the logarithmic form $\log \hat{y} = a + b \log x$, if the data are x_1, x_2, \dots, x_n ; y_1, y_2, \dots, y_n . (HINT: Show that the normal equations for a and b result from those obtained by fitting $\hat{y} = a + bx$, but with $\log x$ substituted for x and $\log y$ substituted for y ; that is,

$$\sum \log y = na + b \sum \log x,$$

$$\sum (\log x)(\log y) = a \sum \log x + b \sum (\log x)^2.$$

Show that the necessary and sufficient conditions for a minimum are fulfilled.

23. Given the data for income distribution in the United States for 1918:

Income Classes	Frequencies
500 to 1,000	12,531,000
1,000 to 3,000	19,875,000
3,000 to 10,000	1,971,000
10,000 to 1,000,000	255,000.

(a) Find the cumulative distribution. (b) Fit the Pareto distribution using the results of Problem 22. (c) Estimate the number of people receiving \$5,000 or more, (d) \$7,250 or more, (e) \$1,785 or more, (f) between \$1,100 and \$1,250, (g) between \$5,000 and \$8,000.

24. Take the following income distribution, United States, 1926:

Income Classes	Frequencies
\$1,000 to 5,000	31,796,000
5,000 to 25,000	1,111,000
25,000 to 250,000	111,000
250,000 to 1,000,000	4,000.

(a) Find the cumulative frequency distribution. (b) Fit a Pareto curve (Problem 22). (c) Estimate the number of people having an income of more than \$4,000; (d) more

than \$10,000; (e) more than \$70,000; (f) more than \$2,000; (g) between \$4,000 and \$4,500; (h) between \$500,000 and \$750,000.

25. Family wages or salaries, United States, 1939:

Class Limits	Number of Families
\$500 to 1,000	5,169,000
1,000 to 1,500	4,675,000
1,500 to 2,000	3,746,000
2,000 to 3,000	3,684,000
3,000 to 5,000	1,779,000.

(a) Find the cumulative frequency distribution. (b) Fit a Pareto curve. (c) Estimate the number of families having an income of more than \$2,500; (d) more than \$4,000; (e) between \$1,250 and \$1,275; (f) between \$2,500 and \$2,750.

26. Use the data in Problem 10, Exercises 92, to estimate the Pareto distributions of incomes of urban, rural farm, and rural non-farm workers.

27. Use the data in Problem 7, Exercises 87, to derive the Pareto distribution of incomes for families.

28. Use the data in Problem 8, Exercises 87, to find the Pareto distribution of incomes for corporations.

98. Regression

Assume that we have 2 series of n values, each of 2 variables x and y : x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n . The variables x and y are both taken as random variables. We assume that they are both normally distributed. The marginal distributions are also normal. We have already shown that the best linear equation for the prediction of y , if x is given, is found by the method of least squares. The value of y obtained in this way, namely, $\hat{y} = a + bx$, is called *the regression of y on x*. The normal equations for the determination of a and b are

$$\sum y = na + b\sum x,$$

$$\sum xy = a\sum x + b\sum x^2.$$

On the other hand, if we know y and want to predict x , we use the linear equation, $\hat{x} = A + By$, called *the regression of x on y*. The method of least squares this time gives the following normal equations for the determination of A and B :

$$\sum x = nA + B\sum y,$$

$$\sum xy = A\sum y + B\sum y^2.$$

It will be noted that the roles of x and y are exchanged.

EXAMPLE

Consider the following series of observations:

x	y	x^2	xy	y^2
1	4	1	4	16
3	2	9	6	4
0	2	0	0	4
4	4	16	16	16
8	12	26	26	40.

If we want to predict y when we know x , we compute the regression of y on x ; that is,

$$\hat{y} = a + bx.$$

Using the above formulas we have the following normal equations from the table:

$$\begin{aligned} 12 &= 4a + 8b, \\ 26 &= 8a + 26b. \end{aligned}$$

Solving this system of equations, we have $a = 2.6$, $b = 0.2$. Hence the regression of y on x is

$$\hat{y} = 2.6 + 0.2x.$$

For instance, for $x = 2$, we have $\hat{y} = 3$. For $x = 10$, we have $\hat{y} = 4.6$.

Next we compute the regression of x on y :

$$\hat{x} = A + By.$$

The normal equations become, in this case,

$$\begin{aligned} 8 &= 4A + 12B, \\ 26 &= 12A + 40B. \end{aligned}$$

Solving this system of equations, we have $A = 0.5$, $B = 0.5$. Hence the regression of x on y is

$$\hat{x} = 0.5 + 0.5y.$$

Suppose for instance we want to estimate x , if we know that $y = 1$; we obtain $\hat{x} = 1$. Or, if we want to estimate x if $y = 6$, there results $\hat{x} = 3.5$.

We show in Figure 46 the so-called scatter diagram of our 2 variables x and y . The observations are denoted by points. The 2 regression lines are also given and labeled \hat{x} and \hat{y} .

EXERCISES 98

1. Given the data:

x	y
1	0
2	1
3	5
0	1.

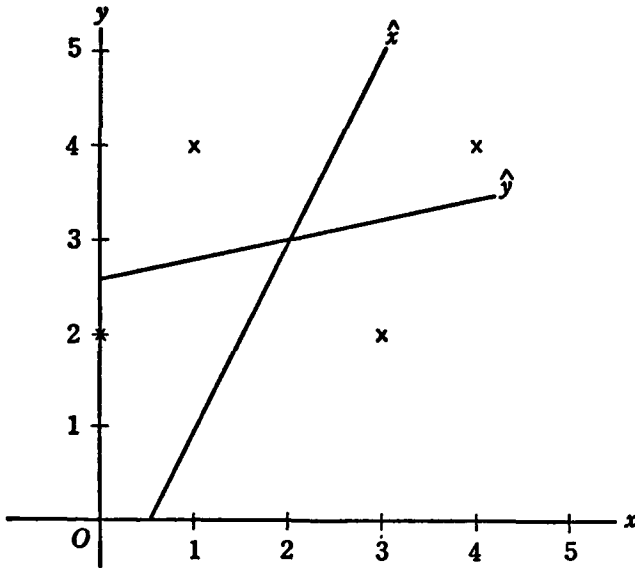


Figure 46

- (a) Find both regression lines. (b) Make a scatter diagram. (c) Predict y if $x = 4$.
 (d) Predict x if $y = 3$.

2. Given the data:

x	y
1	5
0	0
0	1
2	3
4	5
1	4.

- (a) Determine both regression lines. (b) Make a scatter diagram. (c) Predict y if $x = 5$. (d) Predict x if $y = 2$.

3. Given the data:

x	y
1	6
5	1
3	0
2	0
1	1
1	2
7	1
3	5.

- (a) Find both regression lines. (b) Make a scatter diagram. (c) Predict y if $x = 10$.
 (d) Predict x if $y = 2.5$.

4. Given the data:

x	y
8	7
3	0
5	10.

(a) Find both regression lines. (b) Make a scatter diagram. (c) Predict y if $x = 6$. (d) Predict x if $y = 4$.

5. The following table gives a wholesale price index (x) and the index of industrial production in the United States (y):

Year	Wholesale Prices x	Industrial Production y
1935	80	87
1936	81	103
1937	86	113
1938	79	89
1939	77	109.

(a) Determine both regression lines. (b) Make a scatter diagram. (c) Assuming that conditions remain as in the period under consideration, predict the index of industrial production for a wholesale price index of 90. (d) Under the same conditions as given, predict the wholesale price index if the index of industrial production is 120.

6. The following table gives wheat prices and wheat-production in the United States:

Year	Production, Million Bushels x	Price per Bushel y
1932	2.9	0.6
1933	2.4	0.9
1934	1.4	1.2
1935	2.3	1.3
1936	1.5	1.5
1937	2.6	1.3
1938	2.6	0.8
1939	2.6	1.0.

(a) Find both regression lines. (b) Make a scatter diagram. (c) Assuming the same fundamental conditions to prevail on the wheat market as during the period of the data, predict the price of wheat if the production is 2 million bushels. (d) Under the same conditions, predict the production of wheat if the price is \$1 per bushel.

7. The following table gives the total deposits and the interest rate in the United States for various years:

Year	Total Deposits in Billions of Dollars	Interest Rates, per cent
	<i>x</i>	<i>y</i>
1935	45	3.0
1936	50	2.7
1937	52	2.6
1938	51	2.5
1939	55	2.8.

(a) Make a scatter diagram. (b) Determine both regression lines. (c) Assuming the fundamental conditions the same as in the period covered by the data, predict the interest rate if there are 60 billion deposits. (d) Under the same conditions, predict the deposits if the interest rate is 1.5 per cent.

8. The following table gives agricultural production and price indexes in the United States:

Year	Index of Production	Index of Prices
	<i>x</i>	<i>y</i>
1937	106	122
1938	103	97
1939	106	95
1940	110	100.

(a) Make a scatter diagram. (b) Find both regression lines. (c) Assuming conditions to be the same in agriculture as in the period covered, predict the price index if the index of agricultural production is 100. (d) Under the same conditions, predict the index of production if the price index is 130.

9. The following table gives the total national income and the index of industrial production in the United States:

Year	National Income, in Billions of Dollars	Index of Industrial Production
	<i>x</i>	<i>y</i>
1929	83	110
1930	69	91
1931	54	75
1932	40	58
1933	42	69
1934	49	74
1935	56	87
1936	65	103
1937	72	113
1938	64	89.

(a) Make a scatter diagram. (b) Compute both regression lines. (c) Assuming economic conditions to be the same as in the period 1929–1938, predict the index of

production if there is a national income of 100 billion. (d) Under the same conditions, predict the national income if the index of industrial production is 50.

10. The following table gives disposable income and gross investment in the United States for certain years:

Year	Disposable Income per Capita, Deflated	Gross Investment per Capita, Deflated
	x	y
1930	\$478	\$97
1931	440	81
1932	372	45
1933	381	45
1934	419	66
1935	449	78
1936	511	103
1937	520	98.

(a) Make a scatter diagram. (b) Find both regression lines. (c) Assuming the same economic conditions to prevail as in the period covered by the data, predict per capita deflated gross investment if the per capita disposable deflated income is \$600. (d) Under the same conditions, predict the per capita deflated disposable income if the per capita deflated gross investment is \$35.

11. The following table gives information on leather belts:

Year	Output	Total Cost
	x	y
1934	3	234
1935	4	316
1936	11	880
1937	6	670.

(a) Find both regression lines. (b) Predict the total cost if the output is 10. (c) Predict the output if total cost is 1,000. (d) Make a scatter diagram.

12. The following table gives information on steel, United States:

Year	Production	Total Cost
	x	y
1930	12	84
1932	4	44
1934	6	51
1936	11	82
1938	8	61.

(a) Find both regression lines. (b) Predict the cost if production is 10. (c) Predict the production if the cost is 50. (d) Make a scatter diagram with both regression lines.

99. Simple Correlation

A measure for the linear relationship between the two variables x and y is described as the *sample correlation coefficient*; it is given by the formula:

$$r = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sqrt{\sum(x - \bar{x})^2 \sum(y - \bar{y})^2}}$$

This may be simplified to

$$r = \frac{\sum xy - n\bar{x}\bar{y}}{\sqrt{(\sum x^2 - n\bar{x}^2)(\sum y^2 - n\bar{y}^2)}}$$

where \bar{x} and \bar{y} are the sample means of x and y . The sample correlation coefficient r provides an empirical estimate of the population correlation coefficient ρ .

If $r = 0$ the variables are said to be uncorrelated. But this does not mean that they are independent in the probability sense, for it is possible, for instance, that the variance of y depends on x , etc.

If we assume that we have a joint-normal distribution of the random variables x and y , we can test the significance of r . We compute the quantity:

$$t = \frac{r\sqrt{(n-2)}}{\sqrt{1-r^2}}$$

This statistic t follows Student's t distribution with $n - 2$ degrees of freedom. The null hypothesis is that in the population corresponding to the sample there is no linear relationship between x and y , that is, the population correlation coefficient $\rho = 0$. If this null hypothesis is rejected from the point of view of a given level of significance, we say that r is *significant*.

The number of degrees of freedom is $n' = n - 2$. This follows from the fact that the regression equation involves 2 constants, a and b , or A and B , which have to be estimated from the data. This uses 2 degrees of freedom out of a total of n observations.

**To interpret r , the sample correlation coefficient, we start with the identities proved earlier. We have for the 2 sums of squares:

$$\begin{aligned} \sum(x - \bar{x})^2 &= \sum x^2 - n\bar{x}^2, \\ \sum(y - \bar{y})^2 &= \sum y^2 - n\bar{y}^2. \end{aligned}$$

These sums of squares have been used previously to compute the sample variances. We have a similar relationship for the sum of cross products:

$$\sum(x - \bar{x})(y - \bar{y}) = \sum xy - n\bar{x}\bar{y}.$$

Now consider the 2 normal equations which give a and b for the regression of y on x , ($\hat{y} = a + bx$):

$$\begin{aligned} \sum y &= na + b\sum x, \\ \sum xy &= a\sum x + b\sum x^2. \end{aligned}$$

Multiply the members of the first equation by $\sum x$ and the second by n , in order to eliminate a ; there results

$$\begin{aligned}(\sum x)(\sum y) &= na\sum x + b(\sum x)^2, \\ n\sum xy &= na\sum x + nb\sum x^2.\end{aligned}$$

Subtracting the members of the first equation from the corresponding members of the second to eliminate a ; it follows that

$$n\sum xy - (\sum x)(\sum y) = b[n\sum x^2 - (\sum x)^2].$$

After dividing both sides of the equation by n , there results

$$\sum xy - n\left(\frac{\sum x}{n}\right)\left(\frac{\sum y}{n}\right) = b\left[\sum x^2 - n\left(\frac{\sum x}{n}\right)^2\right],$$

or
$$\sum xy - n\bar{x}\bar{y} = b(\sum x^2 - n\bar{x}^2).$$

This follows from the definition of the sample means; namely,

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n}, \\ \bar{y} &= \frac{\sum y}{n}.\end{aligned}$$

The term on the left side is the sum of cross products. The term on the right side is the sum of the squares of the deviations of x from its mean \bar{x} ; thus

$$\sum(x - \bar{x})(y - \bar{y}) = b\sum(x - \bar{x})^2,$$

and, finally,

$$b = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sum(x - \bar{x})^2}.$$

Now consider the first normal equation,

$$\sum y = na + b\sum x.$$

We may divide by n , and obtain, recalling the values of \bar{y} and \bar{x} ,

$$\bar{y} = a + b\bar{x}.$$

Hence, deducting the last expression from $\hat{y} = a + bx$,

$$\hat{y} - \bar{y} = a + bx - (a + b\bar{x}) = b(x - \bar{x}).$$

So the deviation of y from the fitted line is

$$y - \hat{y} = y - \bar{y} - (\hat{y} - \bar{y}) = (y - \bar{y}) - b(x - \bar{x}).$$

The sum of squares of these deviations is

$$\sum(y - \hat{y})^2 = \sum(y - \bar{y})^2 - 2b\sum(x - \bar{x})(y - \bar{y}) + b^2\sum(x - \bar{x})^2.$$

Substituting the value of b derived above, this can be simplified to

$$\sum(y - \hat{y})^2 = \sum(y - \bar{y})^2 - \frac{[\sum(x - \bar{x})(y - \bar{y})]^2}{\sum(x - \bar{x})^2}.$$

Now divide both sides of the last equation by $\sum(y - \bar{y})^2$; there results

$$\frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 1 - \frac{[\sum(x - \bar{x})(y - \bar{y})]^2}{\sum(x - \bar{x})^2 \sum(y - \bar{y})^2}.$$

The last term on the right is the square of the sample correlation coefficient, as defined above. Hence we get the relationship: **

$$\frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 1 - r^2.$$

In this formula we have on the left the ratio of 2 sums of squares. In the numerator appears the sum of the squares of the deviation of the variable y from the fitted, or estimated, values \hat{y} . In the denominator is the sum of the squares of the deviations of y from its mean \bar{y} .

By dividing numerator and denominator of the fraction on the left by $n - 1$, we get the ratio of two variances; that is,

$$\frac{\frac{\sum(y - \hat{y})^2}{n - 1}}{\frac{\sum(y - \bar{y})^2}{n - 1}} = 1 - r^2,$$

Now we have on the left in the numerator the portion of the variance of y which has not been explained by the regression of y on x . In the denominator we have the variance of y . Hence the square of the correlation coefficient may be interpreted as the proportion of the variance of y which can be explained or accounted for by the regression of y on x . The square of the correlation coefficient is called the *coefficient of determination*.

The expression in the numerator cannot be larger than the expression in the denominator. If there are no deviations from the linear regression line, the expression in the numerator will become 0. In that case we have $r^2 = 1$. We have a perfect linear correlation if either $r = 1$ or $r = -1$. Evidently r cannot be larger than 1 or smaller than -1 . The expression on the left, the ratio of 2 non-negative quantities (sums of squares), cannot become negative.

Next consider the case in which the numerator in the expression on the left assumes its greatest possible value. In this case all the variability of y remains unaccounted for by the regression of y on x ; the expression in the numerator is the same as the one in the denominator. Hence in this case we get 1 on the left side of the equation, and it follows that $r = 0$. A 0 correlation coefficient corresponds to the case when there is no linear relationship

between x and y . Of course, there may still be a nonlinear relationship. If $r = 0$ we say that the two random variables x and y are *uncorrelated*.

We show, in Figure 47, 3 scatter diagrams as illustrations for various values of the correlation coefficient. On the top we have the case $r = 1$, so the coefficient of determination is $r^2 = 1$. There are no deviations between the fitted line and the variable. The 2 regression lines coincide. Thus, there is a perfect positive linear relationship between x and y .

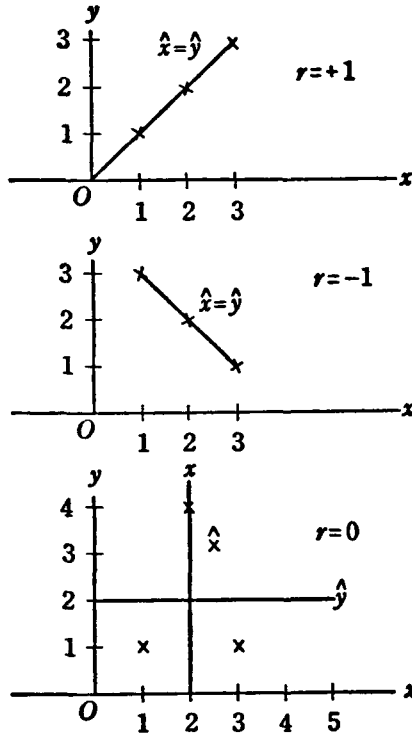


Figure 47

The diagram in the middle shows the case where $r = -1$; the coefficient of determination is 1, as before. The 2 regression lines coincide again, but now they slope downward. There is a perfect negative linear relationship between x and y .

Finally, the diagram on the bottom shows a case where $r = 0$. The coefficient of determination $r^2 = 0$. This means that there is no linear relationship between the random variables x and y . The regression lines are seen to be perpendicular.

In order to test the significance of the sample regression coefficient b ,

we compute

$$t = \frac{b \sqrt{(n-2)(\sum x^2 - n\bar{x}^2)}}{\sqrt{\sum y^2 - n\bar{y}^2 - b^2(\sum x^2 - n\bar{x}^2)}}$$

For tests of hypotheses we have

$$t = \frac{(b - \beta) \sqrt{(n-2)(\sum x^2 - n\bar{x}^2)}}{\sqrt{\sum y^2 - n\bar{y}^2 - b^2(\sum x^2 - n\bar{x}^2)}}$$

where β is the hypothetical regression coefficient in the population. Both these quantities t are distributed like Student's t with $n - 2$ degrees of freedom.

EXAMPLE

Given the following data:

x	y	x^2	xy	y^2	
1	2	1	2	4	
2	2	4	4	4	
2	1	4	2	1	
1	0	1	0	0	
0	2	0	0	4	
Sums:	6	7	10	8	13.

We have $n = 5$, $\bar{x} = 1.2$, $\bar{y} = 1.4$. The correlation coefficient is

$$\begin{aligned} r &= \frac{8 - (5)(1.2)(1.4)}{\sqrt{[10 - 5(1.2)^2][13 - 5(1.4)^2]}} = \frac{-0.4}{\sqrt{(2.8)(3.2)}} = \frac{-0.4}{\sqrt{8.96}} \\ &= \frac{-0.4}{2.99} = -0.134. \end{aligned}$$

Since $r = -0.134$, we have a negative relationship between x and y . Thus large values of x are associated with small values of y , and vice versa.

The coefficient of determination $r^2 = (-0.134)^2 = 0.018$. This shows that if we know x we can explain 0.018 (almost 2 per cent) of the variance of y . And knowledge of y enables us to explain almost 2 per cent of the variance of x .

To make a test of significance let us adopt a level of significance of 5 per cent. The statistic $t = (-0.134)\sqrt{(5-2)}/\sqrt{1 - (0.134)^2} = -0.234$. For $5 - 2 = 3$ degrees of freedom we see that t must be at least 3.182 to be significant. Hence the null hypothesis (that ρ in the population is 0) is not rejected and our empirical correlation coefficient of -0.134 is not significant.

To test our empirical regression coefficient $b = -0.143$ on the 1 per

cent significance level, we compute

$$t = \frac{(-0.143) \sqrt{(5-2) [10 - (5)(1.2)^2]}}{\sqrt{13 - (5)(1.4)^2 - (-0.143)^2 [10 - (5)(1.2)^2]}}$$

$$= -0.234$$

For $n - 2 = 5 - 2 = 3$ degrees of freedom we must have $t = 5.841$ on the 1 per cent significance level. Our empirical t (-0.234) is much smaller and not significant. We conclude that it is likely that in the population there is no linear relationship between x and y .

To find confidence limits with a 95 per cent confidence coefficient we use the above formula with $t = \pm 3.182$ (5 per cent level, 3 degrees of freedom):

$$\frac{(-0.143 - \beta) \sqrt{(5-2) [10 - (5)(1.2)^2]}}{\sqrt{13 - (5)(1.4)^2 - (-0.143)^2 [10 - (5)(1.2)^2]}} = \pm 3.182.$$

This gives the following 95 per cent confidence limits for the population regression coefficient β : -2.089 and 1.803 .

● EXERCISES 99

Compute the correlation coefficient and the coefficient of determination for the following problems. Test the significance of the correlation coefficient, assuming that the data come from a normal population. The level of significance is 5 per cent. Compute the 95 per cent confidence limits of the regression coefficient.

1. Problem 1, Exercises 98.
2. Problem 2, Exercises 98.
3. Problem 3, Exercises 98.
4. Problem 4, Exercises 98.
5. Problem 5, Exercises 98.
6. Problem 6, Exercises 98.
7. Problem 7, Exercises 98.
8. Problem 8, Exercises 98.
9. Problem 9, Exercises 98.

**10. Show that the 2 regression lines coincide if $r = 1$, or $r = -1$.

**11. (*Multiple regression*) Given the relationship between 3 variables x , y and z : $\hat{z} = a + by + cz$, derive by the method of least squares the normal equations:

$$\begin{aligned}\sum z &= na + b\sum x + c\sum y \\ \sum xz &= a\sum x + b\sum x^2 + c\sum xy \\ \sum yz &= a\sum y + b\sum xy + c\sum y^2\end{aligned}$$

100. Fitting of Demand and Supply Curves

Great care has to be exercised in the fitting of empirical demand and supply curves from time series data. We can deal here only with the simplest possible cases and indicate some of the problems involved.

If we consider the interaction of supply and demand from the point of view of economic theory, we note the following: The price paid and the quantity sold are *both* determined by mutual interaction of the demand and supply functions on the market (see Section 8). *The intersection of the supply and demand curves determines the price and the quantity sold.*

From these truths follows the fact that we cannot, in general, compute both the demand and the supply function from time-series data, frequently called *the problem of identification*. Neglecting more complicated cases, we note 3 important, special situations:

(a) *Stable Demand Function and Fluctuating Supply Function*, which is probably the typical situation for agricultural commodities. It has been observed that the per capita demand for agricultural products is relatively

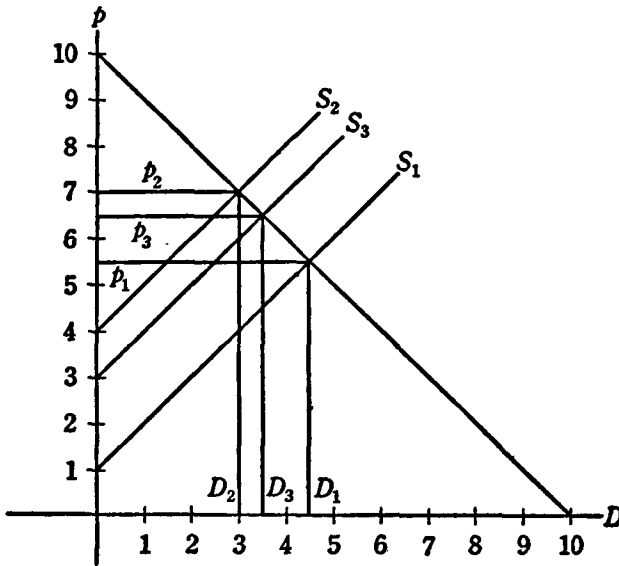


Figure 48

stable in time. But the supply of agricultural commodities fluctuates very violently, especially under the influence of weather conditions. It is apparent from the graph that the successive yearly observations will lie on the demand curve. Hence we can estimate the demand curve but not the supply curve from time-series data.

We show in Figure 48 a somewhat idealized situation in which the demand curve D is constant; S_1 , S_2 , S_3 represent supply curves at various points of time, say in 3 consecutive years. They shift, for instance, because of weather conditions.

Assume that we have a very good harvest in year 1, a very bad one in year 2, and a medium harvest in year 3. Since the demand curve is stable

and the supply curve subject to shifts, all the quantities and prices observed on the market will lie on the demand curve. A statistical regression analysis should in this case reveal the demand curve.

(b) *Stable Supply Function and Fluctuating Demand Function.* This may be typical for certain industrial products, such as steel. The supply conditions in old-established industries are relatively stable. But the demand for industrial products (for example steel) fluctuates violently in the business cycle.

It appears from the graph that all the observations will lie on the supply curve. Hence it appears reasonable that we can fit the supply function but not the demand function from time-series data.

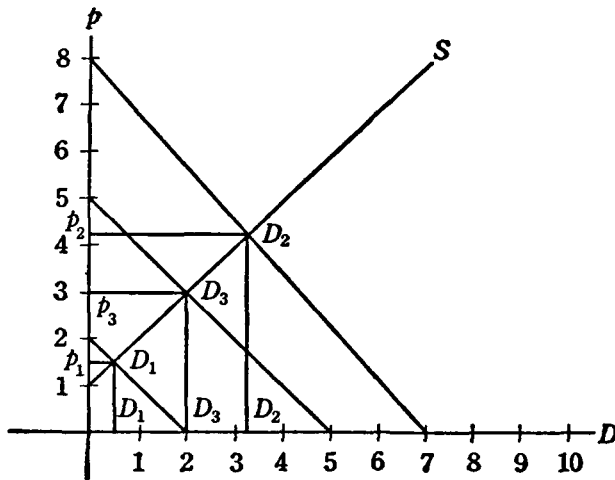


Figure 49

We show in Figure 49 a situation with a stable-supply curve and a fluctuating-demand curve; S is the stable-supply curve. Let D_1 , for instance, be the demand curve in the depth of the depression, D_2 the demand function at a time of boom, D_3 the demand function at a time of mediocre business. The intersections of the supply curve with the various demand curves lie here on the supply curve. Hence a statistical analysis of the recorded quantities sold and prices established on the market will reveal the supply curve, but not the demand curve.

(c) *Neither the Demand nor the Supply Curve Remain Stable.* This is probably the most frequent case in practice. The observed quantities sold on the market and the market prices do not lie either on the demand or on the supply curve. Neither of the 2 curves can be estimated by statistical analysis of prices and quantities alone.

We show in Figure 50 the case that is probably most common, in which neither the demand nor the supply curve stay constant. We show 3 shifting demand and supply curves.

The prices and quantities actually established on the market lie neither on the demand nor on the supply curve. But, since the shifts in demand are more violent than the shifts in supply, a statistical analysis of the quantities sold and bought and the prices charged and paid on the market will give a relationship, which will more nearly represent the supply curve than the demand curve. It should be emphasized that neither the demand nor the supply curve can actually be estimated statistically in this case.

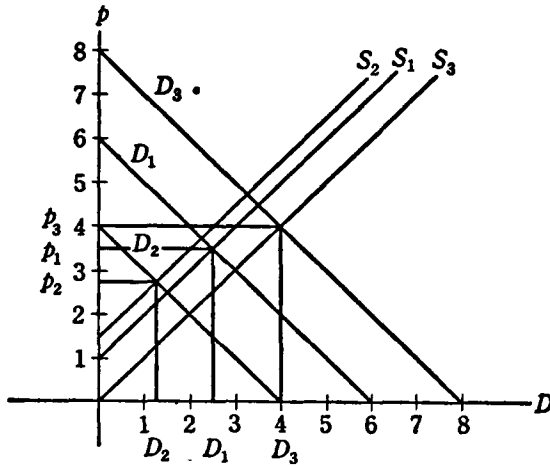


Figure 50

Cobweb theorem: Suppose a commodity has a more or less fixed period of production. That is, a decision about production has to be made for a fixed period (for example 1 year) before the final product comes forth. This is the case with many agricultural commodities, for example hogs. (See Problem 12, Exercises 27.)

The demand for this commodity in each year will depend upon the price existing in the particular year. Hence regression analysis of the quantities and prices of the same year will yield an approximation to the demand function.

But the supply of the commodity in any given year will depend upon the price of last year. Hence regression analysis of the quantities of any given year and the prices of the year before gives an approximation to the supply function. Such a correlation, where one series lags behind the other, is called a lag correlation.

We show in Figure 51 an illustration of the cobweb theorem. We assume a fixed demand function and a stable supply curve. If the quantity sold was 3 and the price 7 (at the intersection of the demand and supply curves) then the price and quantity would remain stable. We could not get any data to determine the demand or supply function. Apart from random fluctuations

and shifts in the curves which we have excluded, and apart from outside influences, all the prices recorded at various points in time will be 7. Also, all the quantities recorded as sold on the market will be 3.

But assume, that by accident or for some other reason the quantity 1 is produced. We see from the demand curve that with this small quantity the price will be very high, that is 9.

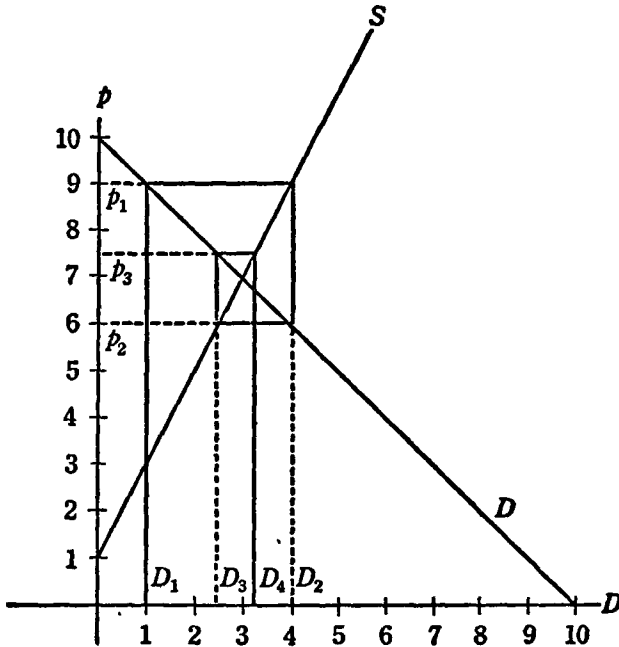


Figure 51

We have assumed that the period of production of the commodity is fixed, say one year. At a high price of 9 the suppliers of the commodity plan to produce 4 units, as shown on the supply curve. Hence in the next year the quantity 4 is supplied.

But if 4 units are supplied, the buyers will not pay a price of 9 but only of 6. Again, the suppliers take the price of 6 for granted as next year's price and plan to produce only 2.5 units in the 3rd year. But at the low quantity of 2.5 units the consumers will pay a price as high as 7.5 in the 3rd year, and so forth.

It appears from the graph that the points (D_1, p_1) , (D_2, p_2) , (D_3, p_3) , all lie on the demand curve D . Hence the ordinary regression analysis of the quantities sold on the market and the prices which prevail simultaneously should give an estimate of the demand curve.

(a) Find the 2 regression lines. (b) Predict the price if the consumption is 50. (c) Predict the consumption if the price is 90. (d) Find the correlation coefficient; (e) the coefficient of determination. (f) Make a scatter diagram with both regression lines. (g) Test the significance of the correlation coefficient (level 1 per cent). (h) Test the regression coefficient of consumption on price, level of significance 1 per cent. (i) Replace the data by their logarithms, and fit a demand curve with constant elasticity. (j) Find the 95 per cent confidence limits of the regression coefficient of log consumption on log price (elasticity).

2. Corn in the United States

Year	Consumption	Price
1926	27	64
1927	28	72
1928	28	75
1929	26	78

(a) Find the 2 regression lines. (b) Predict the price if the consumption is 25. (c) Predict the consumption if the price is 65. (d) Also, find the correlation coefficient; (e) the coefficient of determination. (f) Make a scatter diagram and indicate both regression lines. (g) Test the significance of the correlation coefficient (level 5 per cent). (h) Take the logarithms of the data and fit the two regression lines. (i) Find the correlation coefficient for the logarithms, (j) the coefficient of determination. (k) Test the correlation coefficient for its significance (level 5 per cent). (l) Test the regression coefficient of log consumption on log price (elasticity), level of significance 5 per cent.

3. Wheat (world wide)

Year	Consumption	Price
1885	23	77
1890	24	84
1895	27	50
1900	27	62
1905	34	75
1910	36	91
1915	43	96
1920	34	183
1925	41	144
1930	47	67

(a) Find both regression lines. (b) Predict the price if consumption is 40. (c) Predict the consumption if the price is 80. (d) Find the correlation coefficient; (e) the coefficient of determination. (f) Make a scatter diagram with both regression lines. (g) Test the significance of the correlation coefficient (level of significance 1 per cent). (h) Find the 95 per cent confidence limits for the regression coefficient of consumption on price. (i) Take the logarithms of the data and find both regression lines for such values. (j) Find the correlation coefficient between the logarithms. (k) Test the correlation coefficient, level of significance 5 per cent.

Rye, United States

Year	Consumption	Price
1880	54	74
1890	77	66
1900	90	56
1910	162	90
1920	162	78

- (a) Find both regression lines. (b) Predict the consumption if the price is 60. (c) Predict the price if the consumption is 80. (d) Also, find the correlation coefficient, (e) the coefficient of determination. (f) Make a scatter diagram with both regression lines. (g) Test the significance of the correlation coefficient (level of significance 1 per cent). (h) Take the logarithms of the data and compute both regression lines. (i) Find the correlation coefficient for the logarithms; (j) the coefficient of determination. (k) Test the significance of the correlation coefficient (level of significance 1 per cent). (l) Find the 95 per cent confidence limits for the regression coefficient of log consumption on log price (elasticity).

Cotton, United States

Year	Consumption	Price
1875	12	11
1880	18	10
1885	20	8
1890	26	9
1895	25	8
1900	36	9
1905	49	11
1910	45	14
1915	65	11
1920	50	16
1925	61	20

- (a) Find both regression lines. (b) Predict the consumption if the price is 5. (c) Predict the price if the consumption is 50. (d) Find the coefficient of correlation; (e) the coefficient of determination. (f) Make a scatter diagram with both regression lines. (g) Test the significance of the correlation coefficient (level of significance 5 per cent).

Potatoes, United States

Year	Consumption	Price
1915	36	62
1918	41	120
1921	36	110
1924	42	62
1927	40	97

- (a) Compute both regression lines. (b) Predict the price if the consumption is 35. (c) Predict the consumption if the price is 50. (d) Find the coefficient of correlation, (e) the coefficient of determination. (f) Make a scatter diagram with both regression lines. (g) Test the significance of the correlation coefficient (level of significance 1 per cent). (h) Take the logarithms of the data. (i) Find both regression lines; (j) the correlation coefficient; (k) the determination coefficient. (l) Make a scatter diagram with both regression lines. (m) Test the significance of the correlation coefficient (level of significance 5 per cent). (n) Find the 99 per cent confidence limits of the regression coefficient of log consumption on log price (elasticity).

Pig Iron, United States

Year	Price	Consumption
1880	129	140
1890	103	121
1900	102	101
1910	98	105

- (a) Find both regression lines. (b) Predict the consumption if the price is 100. (c) Predict the price if the consumption is 150. (d) Find the correlation coefficient, (e) the coefficient of determination. (f) Make a scatter diagram with both regression lines. (g) Test the significance of the correlation coefficient (level of significance 1 per cent). (h) Find the 95 per cent confidence limits of the regression coefficient of consumption on price.

Hogs, United States

Year	Total Quantity	Average Price Wholesale
	(Millions)	Cents per Pound
1939	67	6.6
1940	78	5.8
1941	71	9.6
1942	78	13.9
1943	95	14.6
1944	97	13.5
1945	69	14.8

- (a) Compute both regression lines. (b) Predict the quantity if the price is 5. (c) Predict the quantity if the price is 15. (d) Make a scatter diagram. (e) Compute the correlation coefficient. (f) Test the significance, level 5 per cent; (g) coefficient of determination. (h) Tabulate each quantity sold and *last year's price*. Compute the regression of quantity on price of the year before. (i) Predict the quantity, if the price in the year before was 10. (j) Predict the quantity sold, if the price in the year before was 12. (k) Find the lag correlation coefficient. (l) Test its significance, level 5 per cent. (m) Compute the coefficient of determination.

tion and the price index of last year, and find the linear regression of the index of production on last year's price index. (i) Predict the index of production if the price index of last year was 110; (j) if it was 160. (k) Find the lag correlation coefficient; (l) test its significance, level 5 per cent. (m) Find the coefficient of determination. (n) Find the 95 per cent confidence limits for the regression coefficient of production on the price of the year before.

****11. (Identification)** Let the demand function be $y_t = a + bx_t$, where y_t is the quantity and x_t is the price for the year t . Similarly, let $y_t = c + dx_t$ be the supply function. (a) Multiply the supply function by an arbitrary constant K , and add the result to the demand function. Show that the resulting relationship cannot be distinguished from the original demand function because it involves the same variables. Hence the demand function is not identified. (b) Multiply the demand function by an arbitrary constant L , and add to the supply function. Show that the resulting function involves the same variables as the supply function. Hence the latter is not identified.

****12. (Identification).** Let the demand function be $y_t = a + bx_t$ and the supply function: $y_t = c + dx_{t-1}$. (The same notation is used as in Problem 11.) Proceed as in Problem 11, but show that now both the demand and the supply function are identified (HINT: Consider which variables enter into the functions constructed; distinguish between x_t and x_{t-1} .)

****13. (Multiple regression).** Given the following Canadian data:

Year	Price of tea (x)	Price of coffee (y)	Consumption of tea (z)
1929	32	20	3.9
1930	31	17	4.5
1931	27	14	4.1
1932	30	15	3.7
1933	32	18	3.6

(a) Use the results of Problem 11, Exercises 99, to fit a multiple regression equation of the form $\hat{z} = a + bx + cy$. (b) Predict z if we have $x = 30$, $y = 16$.

INDEX NUMBERS

101. Index Numbers

Index numbers are supposed to measure the value of money. The way in which they are constructed, which commodities are included, etc., depends largely upon the purpose for which they are to be used.

One important purpose of index numbers is the measurement of the standard of living. If the consumption of a well defined group of people is known, theoretical limits can be established (under certain conditions), and, under more restrictive conditions, the change in the cost of living can actually be estimated.

FORMULAS:

If the quantities consumed or sold are not known, there are 2 important formulas that are employed rather frequently. Denote the price at the base period as p_0 and at the period for comparison by p_1 . The number of commodities in the index is n . Then

$$S = \frac{\sum p_1/p_0}{n} \quad (\text{Sauerbeck}),$$

$$G = \sqrt[n]{\prod \left(\frac{p_1}{p_0} \right)}$$

where p_1/p_0 is the price ratio for each commodity, and \prod denotes multiplication. The summation \sum and the multiplication \prod are extended over all the commodities. Moreover, S is the arithmetic mean and G is the geometric mean of the n price ratios.

If the quantities consumed are known, we denote by q_0 the quantity consumed or bought in the base period and by q_1 the quantity in the period of comparison. We have the following formulas:

$$L = \frac{\sum p_1 q_0}{\sum p_0 q_0} \quad (\text{Laspeyre}).$$

$$P = \frac{\sum p_1 q_1}{\sum p_0 q_1} \quad (\text{Paasche}).$$

$$I = \sqrt{LP} \quad (\text{Fisher's ideal index}).$$

The summations are extended over all commodities. It can be shown under somewhat restrictive conditions, and if there is no change in taste, that Laspeyre's index number gives the upper limit and Paasche's the lower limit of the "true" change in the cost of living. Under still greater restrictions Fisher's ideal index number I , which is the geometric mean of L and P , gives an approximation to the "true" cost of living index.

■ **EXAMPLE**

We give an example for only 2 commodities:

Commodity	A	B
p_0	1	2
q_0	5	10
p_1	2	3
q_1	1	5.

We have by the use of the above formulas:

$$S = \frac{\sum (p_1/p_0)}{n} = \frac{(2/1) + (3/2)}{2} = \frac{2 + 1.5}{2} = \frac{3.5}{2} = 1.75 = 175 \text{ per cent.}$$

$$G = \sqrt[n]{\prod \left(\frac{p_1}{p_0}\right)} = \sqrt{\left(\frac{2}{1}\right) \left(\frac{3}{2}\right)} = \sqrt{3} = 1.73 = 173 \text{ per cent.}$$

Taking the quantities q into account, as required by the formulas of Laspeyre, Paasche, and Fisher, we get

$$L = \frac{\sum p_1 q_0}{\sum p_0 q_0} = \frac{(2)(5) + (3)(10)}{(1)(5) + (2)(10)} = \frac{10 + 30}{5 + 20} = \frac{40}{25} = 1.6 = 160 \text{ per cent.}$$

$$P = \frac{\sum p_1 q_1}{\sum p_0 q_1} = \frac{(2)(1) + (3)(5)}{(1)(1) + (2)(5)} = \frac{2 + 15}{1 + 10} = \frac{17}{11} = 1.55 = 155 \text{ per cent.}$$

Finally: $I = \sqrt{LP} = \sqrt{(1.6)(1.55)} = 1.57 = 157 \text{ per cent.}$

These values of L and P are much closer than is common. As a general principle, the "true" index of the cost of living is regarded as between P

and L ; it may be approximated by the geometric mean of these two indexes, which is I .

● EXERCISES 101

1. Given the following data:

Commodity	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>
p_0	1	2	7	6	1	6
q_0	10	8	6	7	1	9
p_1	1.5	3	8	8	8	7
q_1	5	6	3	6	0	5.

Compute the index number according to (a) Sauerbeck's formula; (b) geometric mean; (c) Laspeyre's formula; (d) Paasche's formula; (e) Fisher's ideal formula. (f) Establish limits for the "true" index of cost of living. (g) Estimate the "true" index of cost of living.

2. Given the data:

Commodities	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>
p_0	2	5	10	1	2.7	1	5	6.9
q_0	20	9	4	6	10	8	9	7
p_1	1.5	4	9	0.8	2.5	0.6	4	6.8
q_1	25	10	5	8	11	9	9.5	7.2.

(a) Compute the change in the value of money according to Sauerbeck; (b) geometric mean; (c) Laspeyre; (d) Paasche; (e) Fisher. (f) Give limits for the "true" cost of living index. (g) Estimate the "true" cost of living index.

3. Given the data:

Commodity	Corn	Wheat	Oats	Potatoes	Sugar	Tobacco
Price 1926	0.75	1.45	0.41	1.42	0.043	0.182
Quantity 1926	2,692	831	1,247	354	12,952	1,298
Price 1930	0.84	0.87	0.39	0.90	0.034	0.144
Quantity 1930	2,060	858	1,278	333	13,169	1,635.

Compute the increase in the cost of living (a) by Sauerbeck's index number; (b) geometric mean; (c) Laspeyre; (d) Paasche; (e) Fisher. (f) Estimate the limits for the "true" increase in the cost of living. (g) Estimate the increase in the cost of living.

4. Given the data:

Commodity	<i>A</i>	<i>B</i>	<i>C</i>
p_0	1	5	8
q_0	10	12	5
p_1	1.5	6	10
q_1	8	10	2.

Compute the increase in the cost of living using (a) Sauerbeck's index; (b) geometric mean; (c) Laspeyre's; (d) Paasche's; (e) Fisher's ideal index formula. (f) Establish limits for the increase in the cost of living. (g) Estimate the "true" cost of living index.

****5.** Consider the definition of Laspeyre's index number. Assume that we compute $L = 1.20$. Show that a person whose income in time 0 was \$100 will be not worse off in time 1 with an income of \$120. (HINT: Consider the commodities which he bought in time 0 and which he could buy with his increased income in time 1.)

6. Given the data:

Commodities	<i>A</i>	<i>B</i>
p_0	1	1
q_0	10	5
p_1	2	x
q_1	5	2.

Find x if the following indexes have the value 1.5: (a) Sauerbeck's; (b) geometric mean; (c) Laspeyre's; (d) Paasche's; (e) Fisher's.

7. Use the data in Problem 3 for the following commodities: corn, wheat, oats, potatoes to compute the increase in the cost of living between 1926 and 1930 according to the following formulas: (a) Sauerbeck's; (b) geometric mean; (c) Laspeyre's; (d) Paasche's; (e) Ideal.

8. Use the data in Problem 3 for the commodities, sugar and tobacco, to compute the increase in the cost of living according to the following formulas: (a) Sauerbeck's; (b) geometric mean; (c) Laspeyre's; (d) Paasche's; (e) Ideal.

9. Given the data:

<i>Beef</i>		
Year	Consumption, Million lb.	Retail Price, Cents per lb.
1939	7,159	36.0
1942	8,104	29.3
1943	6,434	30.2.

<i>Pork</i>		
Year	Consumption, Million lb.	Retail Price, Cents per lb.
1939	8,474	30.4
1942	8,139	41.4
1943	9,380	40.3.

<i>Lamb</i>		
Year	Consumption, Million lb.	Retail Price, Cents per lb.
1939	868	28.2
1942	948	35.3
1943	865	40.3.

Butter

Year	Consumption, Million lb.	Retail Price, Cents per lb.
1939	1,782	32.5
1942	1,764	34.8
1943	1,673	37.7.

Cheese

Year	Consumption, Million lb.	Retail Price, Cents per lb.
1939	537	25.3
1942	603	34.8
1943	753	37.7.

Oranges

Year	Production Million Boxes	Retail Price, Cents per lb.
1939	70	28.9
1942	272	35.7
1943	297	44.3.

Potatoes

Year	Production, Million bu.	Retail Price, Cents per lb.
1939	342	2.5
1942	433	3.4
1943	611	4.6.

Sugar

Year	Disappearance 1,000 Short Tons	Retail Price - Cents per lb.
1939	7,078	5.4
1942	5,974	6.8
1943	6,664	6.9.

Use the formulae for Laspeyre's, Paasche's, and Fisher's index to compute the increase or decrease in food price (a) from 1939 to 1942; (b) from 1939 to 1943; (c) from 1942 to 1943.

****10.** Investigate the various price-index formulas for homogeneity with respect to the prices. Check by using the Euler theorem. (See Sections 64, 65.)

Suggestions for Further Study

It has been the purpose of this book to introduce the student of economic theory to some of the fundamental ideas of algebra, analytic

geometry, trigonometry, calculus, probability and statistics. Illustrations of the mathematics include examples and exercises taken from mathematical economic theory and from econometric research. If the student wants to gain greater proficiency in the various fields treated in the present text, the following suggestions are made.

An excellent comprehensive survey of modern algebra, which covers topics not touched upon in this book, is Garrett Birkhoff and Saunders MacLane, *A Survey of Modern Algebra* (New York: Macmillan, 1941). The theory of matrices, important in modern statistics and many other applied fields, is presented in R. A. Frazer, W. J. Duncan, and A. R. Collar, *Elementary Matrices* (New York: Macmillan, 1946), and also in an appendix to the author's *Econometrics* (New York, Wiley, 1952). Many topics in point set theory, group theory, and so forth are introduced in an elementary and easily understandable way in the book: John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior* (Princeton: Princeton University Press, 1944). This book presents an entirely novel approach to the fundamental problems of economic theory.

A more rigorous treatment of many topics included in this book may be found in K. O. May, *Elementary Analysis* (New York: Wiley, 1952). A more comprehensive treatment of many calculus problems touched upon in this text is to be found in R. G. D. Allen, *Mathematical Analysis for Economists* (New York: Macmillan, 1939). This book contains chapters on differential equations and the calculus of variations. A more advanced book is David V. Widder, *Advanced Calculus* (New York: Prentice Hall, 1947). Methods useful in dynamic economics are presented in W. J. Baumol, *Economic Dynamics* (New York: Macmillan, 1951).

Griffith C. Evans' book, *Mathematical Introduction to Economics* (New York: McGraw-Hill, 1930) treats many topics in mathematical economics, especially in the field of nonstatic economics. An advanced treatise in modern economic theory is P. A. Samuelson, *Foundations of Economic Analysis* (Cambridge, Mass.: Harvard University Press, 1947). An introduction to linear programming is given in R. Dorfman, *Applications of Linear Programming to the Theory of the Firm* (Berkeley, Cal.: University of California Press, 1951). A comprehensive survey of the field of econometrics is presented in Harold T. Davis, *The Theory of Econometrics* (Bloomington: Principia Press, 1941); J. Tinbergen, *Econometrics* (New York: Blakiston, 1951); and in the author's book, *Econometrics* (New York: Wiley, 1952).

Probability is presented essentially from the same point of view as in this text in Richard von Mises, *Probability, Statistics and Truth* (New York: Macmillan, 1939). A very comprehensive treatment of the subject is available in J. V. Uspensky, *Introduction to Mathematical Probability* (New York: McGraw-Hill, 1937).

The two volumes, R. L. Anderson and T. A. Bancroft, *Statistical Theory in Research* (New York: McGraw-Hill, 1952) and A. M. Mood, *Introduction*

to the Theory of Statistics (New York: McGraw-Hill, 1950), give a good elementary introduction to modern statistical theory. A survey of modern mathematical statistics is available in S. S. Wilks, *Mathematical Statistics* (Princeton: Princeton University Press, 1943), and in the two volumes by Maurice G. Kendall, *The Advanced Theory of Statistics* (London: Charles Griffin, 1945, 1946). An introduction to multiple regression and correlation is given in M. Ezekiel, *Methods of Correlation Analysis* (New York: Wiley, 1941).

ANSWERS TO ODD-NUMBERED PROBLEMS

Exercises 1

1. 3, 11, 18, 6
3. (a) 2, Nonpermissible value, -6 , $66/5$, $-34/3$.
 (b) $-22/4$, $x = -2$ is permissible. (c) $x = 0$ is not permissible.
5. 1, 9, 25, $49/25$ 7. 0, -8 , 64, 1,000, $-1,000$
9. (a) a (b) $a - 2b$ (c) $a + b$ (d) $a(1 + b)$
 (e) $a(1 - b)$ (f) $2a$ (g) 0
11. (a) $5a$ (b) $5b$ (c) $5(a + b)$ (d) $5(a - b)$
 (e) $f(a + b) = 5(a + b) = 5a + 5b$ (f) $f(a - b) = 5(a - b) = 5a - 5b$
 $f(a) + f(b) = 5a + 5b$ $f(a) - f(b) = 5a - 5b$

Exercises 4

1. $-2/3$ 3. $17/5$ 5. $7/5$ 7. $1/6$
9. If $x_2 = 0$ we have $x_1 = 10$, $r = 10$. If $x_1 = 0$ we have $x_2 = 5$, $r = 5$. Hence the first activity will be selected, since it gives the higher revenue.

Exercises 5

1. (a) $y = -2x + 2$ (b) $y = -2$ (c) $y = 5x$ (d) $y = -x + 5$
3. (a) $y = -x + 2$ (b) $y = 2x + 5$ (c) $y = 5x + 8$ (d) $y = 3$
5. (a) $-1, 0$ (b) $-4, -3$ (c) $3, -7$ (d) $4, 6$

Exercises 6

1. (a) 10 (b) 16.8 (c) 15 (d) 5
 (e) 50 (f) 95 (g) $\sqrt{100}$ (h) 20 ✓
3. (a) 9 (b) 6 (c) 18.75 (d) $\sqrt{4}$ ✓
 (e) 12 (f) 32 (g) 40 (h) 10
5. (a) 0.333 (b) 1.111 (c) 5.051 (d) 4.625
 (e) 10.25 (f) 11.375 (g) 5.56 (h) 12.5
7. (a) 3.4 (b) 2.725 (c) 4.9 (d) $4\frac{3}{8}$
 (e) 13.2 (f) 10.13 (g) 6.4 (h) 21.33
9. (a) 0.68 (b) 0.53 (c) 0.455 (d) 48 ✓
 (e) 38 (f) 28 (g) 0.88 (h) 88
11. (a) $5/3$ (b) $5/2$ (c) $D = \frac{5 - 3p}{2}$

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Exercises 7

- | | | | |
|------------|-----------|---------------------|----------------------|
| 1. (a) 3 | (b) 19 | (c) 199 | (d) $5\frac{1}{2}$ |
| (e) 13 | (f) 50.5 | (g) $1/2$ | |
| 3. (a) 4 | (b) 6 | (c) 22 | (d) 32 |
| (e) 5 | (f) 17.50 | (g) 40 | (h) 2 |
| 5. (a) 1.0 | (b) 0.818 | (c) 0.546 | (d) $\overline{8.7}$ |
| (e) 6.5 | (f) 4.41 | (g) 0.091 | |
| 7. (a) m | (b) $10m$ | (c) m^2 | (d) $1/3$ |
| (e) $8/m$ | (f) $6/m$ | (g) 1 | (h) $5/m^2$ |
| (i) 0 | | | |
| 9. (a) 75 | (b) 15 | (c) $5\frac{1}{15}$ | (d) 11 |

Exercises 8

- | | |
|--------------------------|-------------------------|
| 1. (a) $2\frac{1}{6}$ | (b) $3\frac{1}{6}$ ✓ |
| 3. (a) $4\frac{1}{6}$ | (b) $3\frac{1}{6}$ |
| 5. (a) 7 | (b) 14 |
| 7. (a) $1\frac{1}{3}$ | (b) 5 |
| 9. (a) $(A - N)/(B + M)$ | (b) $(AM - BN)/(B + M)$ |
| 11. (a) $11/13$ | (b) 1.01 |

Exercises 9

- | | | |
|---|--|----------------------|
| 1. (a) $10\frac{1}{6}, 4\frac{1}{6}$ | (b) $5\frac{1}{6}, 9\frac{1}{6}, 9\frac{1}{6}$ | (c) 6, 8, 16 |
| (d) 5.1, 9.8, 4.9 | (e) 4.2, 11.6, 11.6 | (f) 4.65, 10.7, 2.68 |
| 3. (a) 9.02, 6.96, 0.35 | (b) 9.03, 6.93, 0.69 | (c) 9.17, 6.67, 3.34 |
| (d) 8.99, 7.01, 0.14 | (e) 8.92, 7.16, 1.79 | |
| 5. (a) $5/8$ subsidy | (b) 2.5 tax | (c) 5 tax |
| 7. (a) $(a + m)/(b + n), (an - bm)/(b + n)$ | | |
| (b) $(an - bm - bnt)/(b + n), (a + m + nt)/(b + n)$ | | |
| with $-t$ for a subsidy, t for a tax | | |
| $(ant - bmt - bnt^2)/(b + n)$ for a tax, | | |
| $(ant - bmt + bnt^2)/(b + n)$ for a subsidy | | |
| 9. (a) Price rises 0.9, quantity decreases by 1.8 | | |
| (b) Price decreases by 1.2, quantity increases by 2.4 | | |

Exercises 10

- | | | |
|------------------|--------------------------|---------|
| 1. 47, -19 | 3. $1\frac{1}{23}, 6/23$ | 5. 0, 0 |
| 7. $-1/2, -1, 1$ | 9. 1, -1, 2 | |

Exercises 11

- | | |
|--|----------------------------------|
| 1. (a) $1/2, 4$ | (b) $1\frac{1}{2}, 1\frac{1}{2}$ |
| 3. (a) 1, 2 | (b) 9, 14 |
| 5. (a) 5.399, 1.524, 13.308 | (b) 60.0, 70.0, 7.0 |
| 7. $(h - a)(g - n) - c(m - e)/(b - k)(g - n) - fc,$
$f(h - a) - (m - e)(b - k)/fc - (g - n)(b - k)$ | |
| 9. 2, 1, 3 and 18, 27, 8 | |

Exercises 12

- | | | |
|---|------------|-----------------|
| 1. 1, 3 | 3. 3, 2, 4 | 5. 8, 32, 4 |
| 7. Only 1 independent equation and 2 unknowns | | 9. 10, 10, 5, 3 |

Exercises 13

- | | | |
|---------------|----------------|----------------|
| 1. 2.08, 9.4 | 3. 2.62, 17.36 | 5. 4.73, 157.9 |
| 7. 3.2, 219.3 | 9. 5, 20 | 11. 2.24, 6.67 |

Exercises 14

- | | | | |
|--------------------|------------------------|-----------------------------------|----------------|
| 1. (a) 7,248 | (b) 1,050 | (c) 3,359,300 | (d) 38,092 |
| (e) 120,870,000 | (f) 8,267,600 | (g) 382,240 | (h) 86,214,000 |
| (i) 462,170,000 | | | |
| 3. (a) 3,053 | (b) 1,516 | (c) 50 | (d) 3,779,500 |
| (e) 20 | (f) 20,833,000 | (g) 2,303,400 | (h) 42,571 |
| 5. (a) 6091 | (b) 991,780 | (c) 4,247,700 | (d) 34 |
| (e) 395 | (f) 3,903,500 | (g) 1,121,500 | (h) 260,010 |
| (i) 11,199,000 | (j) 29,660 | (k) 4,808 | |
| 7. (a) a/K^b | (b) $a/10^{ab}$ | (c) $(a/100)^{1/b}$ | |
| (d) $(a/10)^{1/b}$ | (e) $a(b^b - b^a)/b^b$ | (f) $a(3^b - 1)/(15 \times 10^b)$ | |
| 9. (a) 1,000 | (b) 444 | (c) 31.62 | |
| (d) 141.42 | (e) 3 | (f) 22 | |

Exercises 15

- | | | | |
|----------------------|---------------|-------------|----------------------------|
| 1. (a) 0.098 | (b) 1.024 | (c) 1,286.1 | (d) $3,125 \times 10^{12}$ |
| (e) 1.585 | (f) 0.833 | (g) 2.886 | (h) 8.027 |
| 3. (a) 0.177 | (b) 0.380 | (c) 1.06 | (d) 55.49 |
| (e) 0.657 | (f) 0.04 | (g) 0.833 | (h) 66.267 |
| 5. (a) 32.64 | (b) 27.68 | (c) 20.84 | (d) 20.06 |
| (e) 81.50 | (f) 26.02 | (g) 18.84 | (h) 12.56 |
| 7. (a) 27.46 | (b) 65.42 | (c) 108.05 | (d) 4,071.5 |
| (e) 473.21 | (f) 4.57 | (g) 4.402 | (h) 4.200 |
| (i) 4.733 | (j) 4.975 | | |
| 9. (a) $(1/5)^{1/b}$ | (b) $a^{1/b}$ | (c) a | (d) a^{1-b} |
| 11. (a) 20 | (b) 100 | (c) 4 | |
| (d) 100 | (e) 50 | (f) 100 | |

Exercises 16

- | | | |
|--------------|---------|-----------|
| 1. (a) 14 | (b) 23 | (c) 452 |
| 3. (a) 40 | (b) 10 | (c) -9890 |
| 5. (a) 0, 10 | (b) 90 | (c) 240 |
| 7. (a) -2.5 | (b) 7.5 | (c) 13 |
| 9. 4 | | |

Exercises 17

- | | | |
|--|----------|------------|
| 1. (a) 140 | (b) 150 | (c) -900 |
| 3. (a) 94 | (b) 325 | (c) 16,750 |
| 5. $2an + n(n - 1) d/2$ | | |
| 7. (a) 120, -40 | (b) -240 | (c) 120 |
| 9. $a = Am(m - 1) - Bn(n - 1)/mn(m - n)$ | | |
| $d = 2(Am - Bn)/mn(n - m)$ | | |

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Exercises 18

1. (a) 400 (b) 650 (c) 2,450 (d) 1,100
 3. (a) 200 (b) 8th year (c) 2,700
 5. (a) 50 (b) 700 (c) 950
 7. (a) 960 (b) -240 (c) 240 (d) 2,160
 9. 159

Exercises 19

1. (a) 80 (b) 5,120 (c) 163,840
 3. (a) 125 (b) 3,125 (c) 1,953,125
 5. (a) 10 (b) 100,000 (c) 10,000,000,000

7. $a = \frac{A^{m-1/m-n}}{B^{n-1/m-n}}, r = \left(\frac{B}{A}\right)^{1/(m-n)}$

9. $a = 8, r = 1$

Exercises 20

1. (a) 21 (b) 341 (c) 21,845
 3. (a) 35 (b) 155 (c) 635
 5. (a) $5\frac{3}{4}$ (b) $5/28$ (c) $11\frac{3}{4}$
 7. ar^n . This is y_{n+1} .

9. (a) $\left[\frac{B-A}{A}\right]^{1/m}$

(b) $\frac{\left[\left(\frac{B-A}{A}\right)^{1/m} - 1\right]}{B-2A} A^2$

Exercises 21

1. (a) 1,060 (b) 1,124 (c) 1,690
 (d) 7.96 years (e) 1.166 years (f) 12.9 years
 3. (a) 228,220 (b) 4.66% (c) 376,560
 (d) 543,230 (e) 10.4 years (f) 18.2 years
 (g) 5.6 years
 5. (a) 0.7% (b) 123,880,000 (c) 128,330,000
 (d) 131,080,000 (e) 141,680,000 (f) 157,530,000
 7. (a) 3,438,800,000 (b) 535,080,000 (c) 203,990,000
 (d) 113,140,000 (e) 92,814,000
 9. (a) 3.8% (b) 27,380,000 (c) 33,060,000
 11. 156,200,000

Exercises 22

1. (a) \$10.816 (b) \$14.801 (c) \$71.06
 3. (a) \$525.50 (b) \$638.14 (c) \$805.24 (d) \$500.00
 5. (a) 4.1% (b) 1.6% (c) 0.4%
 7. (a) \$1,104,600 (b) \$1,280,000 (c) \$1,628,900
 9. (a) 35 years (b) 81.3 years (c) 116.3 years
 11. (a) \$284,000,000,000 (b) \$314,000,000,000 (c) \$419,000,000,000

Exercises 23

7. $ad = bc$
 9. The determinant appearing in the numerator equals zero and the determinant in the denominator is not equal to zero.
 11. (a) $\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 4 \cdot 1 = 0$
 (b) $x = k$
 $y = -k/2$ and k is any number, so an infinite number of solutions is possible.
 13. (a) $\lambda = 4, 1$
 (b) $y = 2k$ when $\lambda = 4$
 $y = -k$ when $\lambda = 1$
 where k is an arbitrary number

Exercises 24

1. (a) 46 (b) 43 (c) -39
 3. $6x + 45$ 5. abc 7. 40, 40

Exercises 25

1. -5, -5.5, -10.5 3. 1, -1, 2 5. $-2^{13/22}, -2^{10/11}, 5^{5/22}$
 7. (a) $\begin{vmatrix} -17.0 & 2.5 & 14.5 \\ 13.7 & -133.0 & 119.3 \\ 4.2 & 64.6 & -68.8 \end{vmatrix} = 0$
 (b) $P_1 = P_3 = 1; (4.2)(1) + (64.6)(1) - (68.8)(1) = 0$
 9. $\begin{vmatrix} a & a \\ b & b \end{vmatrix} = ab - ab = 0,$ $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$

Exercises 26

1. (a) $y(x) = (10)(5)^x$ 3. (a) $y(x) = (10,000)(1/10)^x$
 (b) 6,250 (b) 100
 (c) 1,550,000 (c) 1/10
 5. $A(x + 1) - 1.05A(x) = 0$ 7. b/a is positive
 9. $-1 < \frac{b}{a} < 0.$

Exercises 27

1. 8 (b) 8
 (c) 8 (d) $8 - 1/2(8) = 4$
 3. (a) $y(x) = (-2)^x(15) + 1$ (b) 61, -119
 5. (a) 13,315.36 (b) 17,546.68
 7. (a) $a \left[km^{x+1} + \frac{c}{c+b} \right] + b \left[km^x + \frac{c}{c+b} \right] = c$
 (b) $akm^{x+1} + bkm^x = 0$
 (c) $k + n = K$
 9. 175, 199, 231/256

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$$9. \lim_{n \rightarrow \infty} \left[\frac{ar^n}{r-1} - \frac{a}{r-1} \right] = 0 - \frac{a}{r-1} = \frac{a}{1-r}$$

11. (a) $1/1 - a$ (b) $4/5$ (c) $1/3$ (d) $9/10$
 13. (a) 0.5516 (b) $22,301,517$ (c) $22,243,376$

Exercises 33

1. (a) $6x + 3\Delta x - 5$ (b) $6x - 5$
 (c) 25 (d) $87^\circ 43'$
 3. (a) $3x^2 + 3x\Delta x + (\Delta x)^2$ (b) $3x^2$
 (c) 12 (d) $85^\circ 14'$
 (e) ± 0.76
 5. (a) $6x - 6 + 3x\Delta x$ (b) $6x - 6$
 (c) $-6, 99^\circ 27'$ (d) $-36, 91^\circ 35'$
 (e) $6, 80^\circ 33'$ (f) 1.04466
 (g) 0.90379
 7. (a) $-3x^2 - 3x\Delta x - (\Delta x)^2$ (b) $-3x^2$
 (c) $-3, 108^\circ 26'$ (d) $-0.76, 0.76, 0.438976, -0.438976$
 9. (a) $1/12, 1/24$ (b) $-1/12, 1/24$

Exercises 34

1. (a) $C' = 1$ (b) 1 (c) 1 (d) 1
 3. (a) $C' = 2D + 2$ (b) 22 (c) 2 (d) 6
 (e) 32 (f) 8
 5. $C' = 2 + 2D$ (a) 4 (b) 12 (c) 22
 7. $C' = 56$ (a) 56 (b) 56
 9. (a) $C' = m + 2nD$ (b) $m + 2n$ (c) m
 (d) $m + 10n$ (e) $m + 2mn$ (f) $m + 2n^2$
 11. $C' = 0.8$ (a) 0.8 (b) 0.8
 (c) 0.8 (d) 0.8 (e) 0.8

Exercises 35

1. (a) $10D - 3D^2$ (b) $10 - 6D$ (c) 4
 (d) 10 (e) -8 (f) -10
 3. (a) $16D - D^2/2$ (b) $16 - 3D^2/2$ (c) 14.5
 (d) -21.5 (e) 14.25156
 5. (a) $p = 12 - 0.1D$ (b) $12D - 0.1D^2$ (c) $12 - 0.2D$
 (d) 1 (e) 4 (f) -2
 7. (a) $p = 26.5 - 5D$ (b) $26.5D - 5D^2$ (c) $26.5 - 10D$
 (d) -3.5 (e) -13.5 (f) -8.5
 (g) 5.5
 9. (a) $kD/b - aD^2/b$ (b) $k/b - 2aD/b$ (c) k/b
 (d) $k - 2a/b$ (e) $k - 2a^2/b$ (f) $k(1 - 2a)/b$

Exercises 36

1. (a) $5x^4$ (b) 5
 3. $7x^2$
 5. (a) $6x^6$ (b) 0 (c) $(-0.69883, 0.11647)$
 7. $20x^{10}$
 9. (a) $4x^2$ (b) $-32, 91^\circ 47'$

Exercises 37

1. (a) $24x^5$ (b) -24 (c) 0
 3. (a) $16x^3$ (b) 0 (c) $(-0.39685, 0.09922)$
 5. $-12x^2$
 7. $-30x^4$ 9. $-8x, 92^\circ 23'$

Exercises 38

1. (a) $4 - 4x$ (b) 4 (c) -100
 3. (a) $4x - 15x^2 + 4x^3$ (b) 0 (c) $-20; 92^\circ 52'$
 5. (a) $4x^3 - 6x$ (b) $-20; 92^\circ 52'$
 7. (a) $1,000D - 2D^2 - 3D^3$ (b) $1,000 - 4D - 9D^2$
 (c) 987 (d) -344 (e) 652 (f) 674.04
 9. (a) $8D - D^2$ (b) $8 - 2D$ (c) -2
 (d) 6 (e) 0 (f) -5
 11. (a) $76D - 73D^2$ (b) $76 - 146D$ (c) 17.60
 (d) -4.30 (e) 22.71 (f) 38.77
 13. (a) $15D - 2D^2$ (b) $15 - 4D$ (c) 13
 (d) 11 (e) 14.2 (f) 12
 (g) 11.20

Exercises 39

1. $-2 + 10x$
 3. (a) $2 + 2x$ (b) 4 (d) -6
 (e) -0.81802 (f) $(-1.8661, 0.85013)$
 5. (a) 10 (b) 0 (c) 0
 (d) 0 (e) 0 (f) 0
 7. (a) 3 (b) 3 (c) 3
 9. (a) $6.75 - 0.0006D$ (b) 3.75
 (c) 2.55 (d) 2.85
 11. (a) $1.052 - 0.004D$ (b) 0.932 (c) 0.452
 (d) 0.652 (e) 0.752

Exercises 40

1. $6x^2 - 20x - 11$ 3. $-5 + 6x + 21x^2 - 20x^3 - 20x^4$
 5. $7x^6 - 24x^5 - 10x^4 + 72x^3 - 87x^2 + 92x + 30$
 7. (a) $-7x^6 + 48x^5 - 85x^4 + 128x^3 - 81x^2 + 70x - 33$
 (b) $40, 88^\circ 34'$ (c) $-4865, 90^\circ 1'$
 9. (a) $2D^4 - 10D^3 - 40D^2 + 200D$ (b) $8D^3 - 30D^2 - 80D + 200$
 (c) -78 (d) -94
 (e) 98 (f) -62.5

Exercises 41

1. $12x^2 + 12x/(2x + 1)^2$ 3. $-4x^2 - 2/(x^2 + 2x)^2$
 5. (a) 0.366, $-1.366; 1.63394, 3.36605$ (b) 0.69067, $-1.69067; 1.01644, 3.12989$
 (c) $-1.51447, 0.51447; 2.16985, 1.7607$
 7. (a) $10D/1 + 5D$ (b) $10/(1 + 5D)^2$
 (c) $10/36$ (d) $10/2601$
 9. (a) $40/(20 - 3D)^2$ (b) $40/289$
 (c) $8/5$

Exercises 42

1. $15(3x - 2)^4$
3. (a) $50D(1 + 5D^2)^4$ (b) 64,800
(c) 19,448,100 (d) 671,618, 400
5. $\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}$ $q \cdot u^{q-1} \frac{du}{dx} = px^{p-1}$
 $\therefore \frac{du}{dx} = \frac{px^{p-1}}{qu^{q-1}} = \frac{p}{q} \left[\frac{x^{p-1}}{(x^{p/q})^{q-1}} \right] = \frac{p}{q} \left[\frac{x^{p-1}}{x^{p-p/q}} \right]$
 $= \frac{p}{q} (x^{p/q-1}) = nx^{n-1}$ where $n = \frac{p}{q}$
7. (a) $\frac{1}{\sqrt{1+2D}}$ (b) $\frac{1}{\sqrt{3}}$
(c) $1/2$ (d) $1/3$
9. (a) $D\sqrt{10-2D}$ (b) $\frac{10-3D}{\sqrt{10-2D}}$
(c) $1/2$ (d) -3.5
11. (a) $-1/x^2$ (b) $-7/x^2$
(c) $-20/(2x+1)^2$ (d) $\frac{-12}{\sqrt{x^2}}$
13. (a) $\frac{12,800}{D^{0.8}}; -\frac{3,840}{D^{1.8}}$ (b) -78.16
(c) -37.76 (d) -27.24
(e) -23.75 (f) -46.14
15. (a) $R = 2,180/D, R' = -2,180/D^2$ (b) -60.56
(c) -200.18 (d) -51.60
(e) -482.77 (f) -196.57
17. (a) $x = g(y) = \sqrt{y}$
 $x^2 = y$
 $\therefore y = x^2 = f(x)$
 (b) $2x, \frac{1}{2\sqrt{y}}$ (c) (8) $\left(\frac{1}{2\sqrt{16}}\right) = 1$
 (d) (6) $\left(\frac{1}{2\sqrt{9}}\right) = 1$ (e) (20) $\left(\frac{1}{2\sqrt{100}}\right) = 1$
 (f) $f'(x) \cdot g'(y) = \frac{dy}{dx} \cdot \frac{dx}{dy} = 1$
19. $\frac{dx}{dy} = 2y, \frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$
21. $x = y^n,$
 $\frac{dx}{dy} = ny^{n-1},$
 $\frac{dy}{dx} = \frac{1}{ny^{n-1}} = \frac{1}{n[x^{1/n}]^{n-1}} = \frac{1}{nx^{n-1/n}} = \frac{1}{n(\sqrt[n]{x^{n-1}})} = \frac{1}{n}x^{1/n-1}$

Exercises 43

1. (a) 2.48832 (b) 2.5936

Exercises 44

1. (a) 1 (b) 0 (c) 6 (d) 1/3
 (e) 5/4 (f) -1 (g) -6 (h) -2
 (i) -1/4 (j) -3/2 (k) -9/6
 4. (a) 2 (b) 10 (c) 6 (d) 60
 (e) 115 (f) 6.9 (g) 3.5

Exercises 45

1. $10/x$; 10; 2.5. 3. $2 + 2 \log_e x$; 2; 4; 6.96982.
 5. $4(\log_e x)^3/x$; 0; $32/e^2$; 4.71453.
 7. (a) $100D/\log_e D$ (b) $100 \log_e D - 100/(\log_e D)^2$
 (c) 23.53 (d) -63.87
 9. (a) $5/D$ (b) 5
 (c) 1 (d) $5/e^2$
 11. (a) $(20 - 3D + 3D \log_e D)/D(20 - 3D)^2$ (b) 0.06920
 (c) $20/(20 - 3e)^2$ (d) 1.16567

Exercises 46

1. $-4/1 - 4x$. 3. $2x + 3/x^2 + 3x - 7$.
 5. $5x^2 - 12x - 5/(5x - 6)^2$.
 7. $2(1 + x)(1 - 2x)^2/(x^2 - 5)^2 - 6(1 + x)^2(1 - 2x)^2/(x^2 - 5)^2$
 $- 4x(1 + x)^2(1 - 2x)^2/(x^2 - 5)^2$.
 9. (a) $18/1 + 3D$ (b) 18 (c) 4.5 (d) 18/19

Exercises 47

1. $-2e^{-2x}$. 3. $-5/4e^{2x}$.
 5. $5e^x/(2 + 3e^x)^2$. 7. $(10^x)(2.30259)$.
 9. (a) $100D - De^D$ (b) $100 - e^D(D + 1)$
 (c) $100 - 2e$ (d) $100 - 4e^2$
 11. (a) $50e^{6D}$ (b) $50e^6$ (c) $50e^{60}$ (d) $50e$

Exercises 48

1. (a) $2x - 2x^2/1 + 2x - x^2$ (b) 0
 (c) ∞ (d) $2\sqrt{9}$
 (e) $2\sqrt[2]{9}$
 3. (a) 3 (b) 3 (c) 3
 5. (a) $5x$ (b) 5 (c) 0
 (d) 6 (e) 25 (f) 10
 7. $bx/a + bx$ 9. $(x/u)(du/dx) = Eu/Ex$
 11. $\frac{uEu/Ex + vEv/Ex}{u + v}$.
 13. $(au/au + bv)Eu/Ex + (bv/au + bv)Ev/Ex$.
 15. $Eu/Ex - Ev/Ex$.

Exercises 49

1. (a) $10 - 3D/-3D$ (b) $-2\frac{3}{8}$ (c) $-1/3$
 (d) $-1\frac{3}{8}$ (e) $-5\frac{3}{8}$
3. $Dp^a = b$, $p = [b/D]^{1/a}$,

$$ED/Ep = \frac{[b/D]^{1/a}}{D} \cdot \frac{1/-b^{1/a} \cdot D^{-(a+1/a)}}{a} = -1/D^{a+1/a} \cdot aD^{a+1/a} = -a$$
5. $ED/Ep = a/D[1/0] = a/0 = \infty$
7. dp/dD is the slope of the tangent (p, D) and for demand curves which fall from left down to the right this slope is negative.
9. $(8 - D)/(-D)$ (a) -7 (b) -0.60
 (c) -0.143 (d) -3
11. $(76 - 73D)/(-73D)$ (a) -1.082 (b) -0.388
 (c) -1.603 (d) -1.314 (e) -1.975
13. $-1/3$ 15. -0.5
 14. $(15 - 2D)/(-2D)$ (a) -6.5 (b) -8.375 (c) -10.538

Exercises 50

1. $ED/Ep = 1,000 - 2D - 3D^2/D(1 - 2 - 6D)$
 $R' = (1,000 - 2D - 3D^2)[1 + (-2D - 6D^2)/(1,000 - 2D - 3D^2)]$
 $= 1,000 - 2D - 3D^2 - 2D - 6D^2$
 $= 1,000 - 4D - 9D^2$
3. $ED/Ep = 6/D(2 + 3D)(2 + 3D)^2 - 18 = (6/-18D)(2 + 3D)$
 $R' = 6/2 + 3D[1 + (-18D)/6(2 + 3D)]$
 $= 6/2 + 3D[12 + 18D - 18D/6(2 + 3D)]$
 $= 12/(2 + 3D)^2$
5. $ED/Ep = 100 - 3 \log_e D/D[D/-3] = 100 - 3 \log_e D/-3$
 $R' = 100 - 3 \log_e D[1 + (-3/100 - 3 \log_e D)]$
 $= 97 - 3 \log_e D$
7. $R' = p + p/\infty = p + 0$
 $\therefore R' = p$
9. $ED/Ep = (76 - 73D)(-73D)$
 $R' = (76 - 73D)[1 - 1/(76 - 73D)/(-73D)]$
 $= 76 - 146D$
11. $ED/Ep = -1/2$
 $R' = -2180D^{-3}$

Exercises 51

1. (a) Increasing (b) Increasing
 (c) Increasing (d) Increasing
3. Monotonically increasing
5. Increasing for $b > 0$ Decreasing for $b < 0$
7. (a) $R' = 1 - 2D$ (b) $R' = 10 - 6D$
 Increasing for $D < 1/2$ Increasing for $D < 1\frac{2}{3}$
 Decreasing for $D > 1/2$ Decreasing for $D > 1\frac{2}{3}$
- (c) $R' = 20 - 10D$ (d) $R' = 100 - 6D^2$
 Increasing for $D < 2$ Increasing for $D < 10/\sqrt{6}$
 Decreasing for $D > 2$ Decreasing for $D > 10/\sqrt{6}$
- (e) $R' = 20D - 3D^2$
 Increasing for $D < 20/3$
 Decreasing for $D > 20/3$

9. $R' = 76 - 146D$
 Increasing for $D < 0.521$
 Decreasing for $D > 0.521$

Exercises 52

1. (a) 10, 0, 0
 (b) $5 - 2x + 5x^4, -2 + 20x^3, 60x^2$
 (c) $xe^x + e^x, xe^x + 2e^x, xe^x + 3e^x$
 (d) $1 - \log_e x/x^2, -3x + 2x \log_e x/x^4, 11 - 6 \log_e x/x^4$
3. (a) $1/x, -1/x^2, 2/x^3, -6/x^4$
 (b) $(-1)^{m+1}(m-1)!x^{-m}$
5. $y^{(n)} = (-1)^n[m + (n-1)]!/(m-1)!x^{m-n}$
7. $y^{(n)} = a^n e^{ax}$
9. $y' = 8x^7 - 14x^6 + 3x^2,$
 $y'' = 56x^6 - 84x^5 + 6x,$
 $y''' = 336x^5 - 420x^4 + 6,$
 $y^{(4)} = 1,680x^4 - 1,680x^3,$
 $y^{(5)} = 6,720x^3 - 5,040x^2,$
 $y^{(6)} = 20,160x^2 - 10,080x,$
 $y^{(7)} = 40,320x - 10,080,$
 $y^{(8)} = 40,320,$

Exercises 53

1. (a) Concave upward (b) Concave upward
 (c) Concave downward (d) Concave downward
 (e) Since $y'' = 6x$ the curve is concave upward for positive x and concave downward for negative x .
3. (a) Concave downward (b) Concave downward
 (c) Concave downward (d) Concave downward
 (e) Concave downward
 (f) Since $y'' = -2$ the curve is concave downward for all values of x .
5. (a) Concave upward for all values of x .
 (b) Concave upward for all values of x .
 (c) Concave downward for all values of x .
 (d) Concave upward for $x > -2$, concave downward for $x < -2$.
7. Concave upward for $x > 2/3$, concave downward for $x < 2/3$

Exercises 54

1. (a) $50,000 - 600D + 3D^2$ (b) $-600 + 6D$
 (c) Increasing marginal costs for $D > 100$, decreasing marginal costs for $D < 100$ and constant marginal cost for $D = 100$
3. (a) $40 - 24D + 3D^2$ (b) $-24 + 6D$
 (c) Increasing marginal costs for $D > 4$, decreasing marginal costs for $D < 4$ and constant marginal costs for $D = 4$
5. (a) $1,000 - 48D + 12D^2 + 4D^3$ (b) $-48 + 24D + 12D^2$
 (c) Increasing marginal costs for $D > 1.236068$, decreasing marginal costs for $D < 1.236068$ and constant marginal costs for $D = 1.236068$
7. Constant marginal costs for all values of D .
9. Constant marginal costs for all values of D .

Exercises 55

- Critical points: 0.264, -1.264.
Minimum at $x = 0.264$, maximum at $x = -1.264$
- Critical points, $2/3$, $-2/3$. Maximum at $x = 2/3$, minimum at $x = -2/3$
- Critical points: 0, -1, -3. Minimum at $x = 0$, maximum at $x = -1$, minimum at $x = -3$.
- Critical point $1/e$ a minimum
- Critical points: 3, -3, 2, -2. A minimum at $x = 3$, a maximum at $x = -3$, a maximum at $x = 2$, a minimum at $x = -2$.

Exercises 56

- (a) $3D$ (b) 3
(c) $10D - 5D^2$ (d) $10 - 10D$
(e) $7D - 5D^2$ (f) $D = 7/10$
(g) $-10 < 0$ (j) 6.5, 4.55, 2.1, 1.95
- (a) $D = 1$ (b) $D > -1/2$
(c) 9; 9; 2; 7.
- (a) $D = 1/2$ (b) $R'' = -20/e \cdot C'' = 0 \therefore R'' < C''$
- (a) $D = 1/3$ (b) $-150 < 0$
- (a) $2.34D - 1.34D^2, 1 - 0.83D + 0.85D^2, -1 + 3.17D - 2.19D^2$
(b) $D = 0.724$
(c) $R'' = -2.68, C'' = 1.70 \therefore R'' < C''$
(d) 1.370, 0.992, 0.845, 0.147.
(e) Competitive: $D = 1.043, p = 0.942$
Monopoly: $D = 0.724, p = 1.370$

Exercises 57

- (a) $D = 3$ (b) $C' = 11 = A$
- (a) $D = 1$ (b) $C' = 9 = A$
- (a) $D = 0$ (b) $C' = 2 = A$
- (a) $D = 1/e$ (b) $C' = 1 - 1/e = A$
- (a) $5 - 3D + 2D^2$ (b) $D = 3/4$
(c) $5 - 6D + 6D^2$ (d) $C' = 3\frac{7}{8} = A$

Exercises 58

- Concave upward for $x > 1/6$, concave downward for $x < 1/6$
(b) Point of inflection at $(1/6, -1/54)$ (c) $-1/6$
- This curve has no inflection point, but a maximum at $x = 3/2$, so concave downwards.
- $y' = 4x^3$
 $y'' = 12x^2$
Since $y'' = 12x^2$ this is positive for all values of x other than 0 and the concavity of the curve does not change so there is no inflection point.
(b) $y' = 5x^4$
 $y'' = 20x^3$
 $y = x^5$ then has an inflection point at $x = 0$ since the concavity changes at that point.
- $50D - 8D^2 + D^3$
(a) $D = 2\frac{3}{4}$ (b) $C' = 50 - 16D + 3D^2$
 $C'' = -16 + 6D = 0$
 $\therefore D = 2\frac{3}{4}$
 $C''' = 6$, so a minimum at $D = 2\frac{3}{4}$, which is the inflection point on C.

9. $C = f(D)$
 $C' = f'(D)$
 $C'' = f''(D) = 0$ is the inflection point on C , but at the same time gives critical points on C' , in our case minima.

Exercises 59

- | | | | |
|-----------|---------|-----------------|---------------|
| 1. (a) 0 | (b) 6 | (c) 3/2 | (d) 3 |
| 3. (a) -6 | (b) 11 | (c) -3 | (d) $-6a + 1$ |
| 5. (a) 1 | (b) 0 | (c) $-a$ | |
| 7. (a) 0 | (b) 11 | (c) $3a^2 + 4a$ | |
| 9. (a) -1 | (b) -31 | (c) $-a^2$ | |

Exercises 60

- | | |
|---|-------------------------------|
| 1. (a) $2x + y^2 - 2, 2xy + 5$ | (b) 4, 9 |
| 3. (a) $2 + 3y - z, 3x, -x$ | (b) 7, 0, $0 - xye^{-z}$ |
| 5. (a) $1 + \log_e xy, x/y$ | (b) 1, 1 |
| 7. (a) $-4.0848p^{-1.040} \pi^{0.939} Q^{-1.022},$
$-184.704 Q^{-0.022} \pi^{0.939} p^{-2.040},$
$166.7664 Q^{-0.022} p^{-1.040} \pi^{-0.061}.$ | (b) 43.334; 0.95836; 0.94215. |
| 9. (a) $54.0144 Q^{-0.928} p^{-0.513} \pi^{0.513},$
$-384.8526 Q^{0.972} p^{-1.513} \pi^{0.513},$
$384.8526 Q^{0.972} p^{-0.513} \pi^{-0.487}.$ | (b) 0.78226; 0.23696; 4.4518. |

Exercises 61

- | | |
|---|-----------------------|
| 1. (a) $10 - 2a + b; a.$ | (b) 12; 2. |
| 3. (a) $100 - 2a - 5b + 3c; 200 - 4b - 5a - c; 50 - 6c + 3a - b.$ | |
| | (b) 103; 182; 21. |
| 5. (a) $0.64 L^{-0.28} C^{0.28}; 0.36 L^{0.44} C^{-0.44}.$ | |
| | (b) 0.57239; 0.43905. |
| 7. (a) $0.58 L^{0.42} C^{-0.42}; 0.43 L^{-0.47} C^{0.47}.$ | |
| | (b) 0.55079; 0.45938. |
| 9. (a) 31.056; 12.755; 4.4366; 12.571; 24.401. | |
| 11. (a) $2.13a^{1.12} b^{0.24}, 0.34a^{2.12} b^{-0.68}.$ | |
| | (b) 4.9107; 0.66628. |

Exercises 62

- | | |
|--|-----------------------------------|
| 1. (a) $\frac{-10p_A}{100 - 10p_A - 2p_B}, \frac{2p_B}{100 - 10p_A - 2p_B}.$ | (b) $\frac{-2}{7}, \frac{-1}{7}.$ |
| 3. (a) $\frac{-1.9p_A}{63.3 - 1.9p_A + 0.2p_B + 0.5p_C}, \frac{0.2p_B}{63.3 - 1.9p_A + 0.2p_B + 0.5p_C},$
$\frac{0.5p_C}{63.3 - 1.9p_A + 0.2p_B + 0.5p_C}.$ | |
| | (b) -0.28571; 0.02406; 0.05263. |
| 5. (a) $\frac{0.1p_A}{10.3 + 0.1p_A + 0.1p_B - 0.3p_C}, \frac{0.1p_B}{10.3 + 0.1p_A + 0.1p_B - 0.3p_C},$
$\frac{-0.3p_C}{10.3 + 0.1p_A + 0.1p_B - 0.3p_C}.$ | |
| | (b) 0.094; 0.118; -0.424. |

$$7. (a) \frac{-0.02p_A}{49.07 - 0.02p_A - 0.36p_B - 0.03p_C + 0.03p_D}$$

$$\frac{-0.36p_B}{49.07 - 0.02p_A - 0.36p_B - 0.03p_C + 0.03p_D}$$

$$\frac{-0.03p_C}{49.07 - 0.02p_A - 0.36p_B - 0.03p_C + 0.03p_D}$$

$$\frac{0.03p_D}{49.07 - 0.02p_A - 0.36p_B - 0.03p_C + 0.03p_D}$$

(b) -0.0004; -0.0149; -0.0019; 0.0006.

$$9. (a) \frac{0.1p_B}{24.2 + 0.1p_B - 0.6p_C - 0.3p_D},$$

$$\frac{-0.6p_C}{24.2 + 0.1p_B - 0.6p_C - 0.3p_D},$$

$$\frac{-0.3p_D}{24.2 + 0.1p_B - 0.6p_C - 0.3p_D}.$$

(b) 0.004; -0.026; -0.013.

11. 0.538; -0.717; 0.717.

13. 0.317; -0.38; 0.327.

$$15. (a) \frac{11.2K}{11.2K - 8.6P - 379} \cdot (b) \frac{-8.6p}{11.2K - 8.6P - 379} \cdot (c) 1.46214; -0.50522 ;$$

17. (a) -1.32; 0.4.

Exercises 63

1. (a) $\frac{-184}{69}$ (b) -1 (c) -1 (d) -0.62350

3. (a) -3/2

$$(b) D^3 = p^{-3}, \quad 2D \frac{\partial D}{\partial p} = -3p^{-4}, \quad \frac{ED}{Ep} = \frac{-3p}{2D} \cdot \frac{p}{D} = \frac{-3p^{-3}}{2p^{-3}} = \frac{-3}{2}.$$

5. (a) -x/z; -y/z.

(b) -x/z; -y/z.

7. (a) -3/2; -1/2.

(b) -3/2; -1/2.

9. -1; -1.

$$11. \frac{dp}{da} = f_D \frac{dD}{da} + f_a, \quad \frac{dp}{da} = g_D \frac{dD}{da},$$

Solving for $\frac{dD}{da}$;

$$\frac{dD}{da} = \frac{f_a}{f_D - g_D} \text{ which is } < 0, \quad \frac{dp}{da} = g_D \frac{dD}{da} \text{ which is } < 0.$$

Exercises 64

1. $k = 3.$

3. $k = 1.$

5. (a) $D_A = \frac{50(p_B)}{(p_A)} = t^0 \frac{50p_B}{p_A}$.

$k = 0$. Similarly for D_B .

(d) The demand for A and B is unchanged if all prices change by the same per cent.

(e) 30; 166⅔%.

7. Homogeneous of degree 1.

9. Homogeneous of degree 0.

Exercises 65

1. (a) Homogeneous of degree 4.

(b) $x(8x^2) - y(16y^2) = 4u$.

3. (a) Homogeneous of degree 2.

(b) $x(2x) + y(2y) = 2u$.

6. Yes,

$$L [(1.01)(0.75) L^{-0.25} C^{0.25}] + C [(1.01)(0.25) L^{0.75} C^{-0.75}] \\ = 1.01 L^{0.75} C^{0.25} (0.75 + 0.25) = 1.01 L^{0.75} C^{0.25}.$$

2x.

7. No,

$$L [0.43 L^{-0.57} C^{0.58}] + C [0.58 L^{0.43} C^{-0.42}] \\ = L^{0.43} C^{0.58} (0.43 + 0.58) = 1.01 L^{0.43} C^{0.58}.$$

9. Not linear homogeneous.

a [(100) (0.28) $a^{-0.72} b^{0.23} c^{0.04} d^{0.17} f^{0.18}$] +

b [(100) (0.23) $a^{0.28} b^{-0.77} c^{0.04} d^{0.17} f^{0.18}$] +

c [(100) (0.04) $a^{0.28} b^{0.23} c^{-0.98} d^{0.17} f^{0.18}$] +

d [(100) (0.17) $a^{0.28} b^{0.23} c^{0.04} d^{-0.83} f^{0.18}$] +

f [(100) (0.18) $a^{0.28} b^{0.23} c^{0.04} d^{0.17} f^{-0.83}$]

$$= x[0.28 + 0.23 + 0.04 + 0.17 + 0.18] = 0.90x.$$

11. No.

Exercises 66

1. $12x^2; -30xy; -15y^2 + 6$.

3. 0; $-x/y^2; 1/y$

5. $6x, 0, -2, -1, 4, 0$.

7. (a) $0; \frac{2x}{y^3}; \frac{-1}{y^2}$.

(b) $0; \frac{-1}{54}; \frac{-1}{36}$.

9. (a) $6x, 0, 0, 12u^2z, -z, -y, 0, -x, 0, 4u^2$.

(b) $6, 0, 0, 0, 0, 1, 0, -1, 0, 32$.

Exercises 67

1. Minimum at $(-1, -\frac{1}{2}, -1\frac{1}{4})$.

3. Minimum at $(6, 13\frac{1}{2})$; -218.25 .

5. The determinant is < 0 so neither a maximum nor minimum.

7. Determinant = 0 so neither a maximum nor a minimum.

9. (a) $a < 0, c < 0, b^2 < ac$. (b) $a > 0, c > 0, b^2 < ac$. (c) $b^2 > ac$.

11. Minimum at $(0, 0, 0, 0, 0)$.

Exercises 68

1. (a) $D_A = 3; D_B = 1$ and $-8 < 0; -10 < 0$.

(b) 19, 20, 24, 53.

3. (a) $D_A = 2, D_B = 3; -2 < 0$ and $-6 < 0$.

(b) 7, 20, 37, 37.

5. (a) $D_A = 5/6, D_B = 1\frac{1}{6}, D_C = 3\%$.

(b) $7\frac{1}{2}, 11, 33, 40.9095, 112.4214$.

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7. (a) $D_A = 9\ 80517$, $D_B = 6.69637$, $D_C = 1.4098$, $D_D = 4.10416$.
 (b) 121.94830; 56.51815; 51.5412; 21.79168; 345.72347; 1390.5661.

$$9. \text{ (a) } D_A = \frac{a(2d+2n) - cq}{(2d+2n)(2b+2m) - q^2}, \quad D_B = \frac{c(2b+2m) - aq}{(2d+2n)(2b+2m) - q^2}$$

$$b > -m, d > -n.$$

$$\text{(b) } p_A = \frac{2a(d+n)(b+2m) - aq^2 + bcq}{4(d+n)(b+m) - q^2}, \quad p_B = \frac{2c(b+m)(d+2n) - cq^2 + adq}{4(d+n)(b+m) - q^2}$$

Exercises 69

1. $(2\frac{1}{2}, 5/6), 25/12$.

3. $(16\frac{2}{3}, 8\frac{1}{3}, 8\frac{1}{3}) \cdot 833\frac{1}{3}$.

5. $\left(\frac{-5\sqrt{5}}{3}, \frac{-2\sqrt{5}}{3}\right) 9\sqrt{5}$.

7. $\left(5/3, \frac{-10}{9}, \frac{10}{3}\right) \cdot \frac{-500}{81}$.

$$9. x = \frac{-k(2mc - bn)}{2(bmn - m^2c - an^2)},$$

$$\frac{k^2(b^2 - 4ac)}{4(bmn - m^2c - an^2)}.$$

$$y = \frac{-k(2an - bm)}{a(bmn - m^2c - an^2)},$$

11. $f_x - \lambda g_x = 0, \quad f_y - \lambda g_y = 0,$

$$\therefore \frac{f_x}{g_x} = \lambda = \frac{f_y}{g_y}$$

$$\therefore \frac{f_x}{f_y} = \frac{g_x}{g_y},$$

$$\frac{f_x}{f_y} = \frac{g_x}{g_y}.$$

By implicit differentiation:

$$0 = f_x + f_y \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y},$$

Similarly $\frac{dy}{dx} = \frac{-g_x}{g_y}$ from $g(x, y) = 0$.

Exercises 70

1. $5; 2\frac{1}{2}$.

3. $x = 1152.155, y = 1225.307$. 5. $4; 1\frac{1}{2}$.

7. (a) $5; \frac{5}{q}$.

(b) -1 .

(c) -1 .

9. (a) $\frac{J}{2}, \frac{J}{4}$.

(b) $1; 1$.

(c) $1, 1$.

11. $\frac{y}{1} = \frac{x}{2},$

But $x = 5, y = 2\frac{1}{2}$.

$\therefore 5/2 = 5/2.$

13. $\frac{\partial V}{\partial x} = \frac{2U\partial U}{\partial x} = 2Uy + \lambda = 0,$

$\frac{\partial V}{\partial y} = \frac{2U\partial U}{\partial y} = 2Ux + 2\lambda = 0.$

Eliminating λ and using the restriction gives the same result as in Problem 1.

$\frac{\partial W}{\partial x} = \frac{5U^4\partial U}{\partial x} = 5U^4y + \lambda = 0,$

$\frac{\partial W}{\partial y} = \frac{5U^4\partial U}{\partial y} = 5U^4x + 2\lambda = 0,$

which gives the same result.

$\frac{\partial z}{\partial x} = \frac{1}{U} \frac{\partial U}{\partial x} = \frac{1}{U} y + \lambda = 0,$

$\frac{\partial z}{\partial y} = \frac{1}{U} \frac{\partial U}{\partial y} = \frac{1}{U} x + 2\lambda$ which gives the same result.

15. $U(x,y) + \lambda [xp_x + yp_y - I],$

$\frac{\partial U}{\partial x} + \lambda p_x = 0,$

$\frac{\partial U}{\partial y} + \lambda p_y = 0$

$\therefore \frac{\frac{\partial U}{\partial x}}{p_x} = \frac{-\lambda}{1} = \frac{\frac{\partial U}{\partial y}}{p_y}$

$\therefore -\lambda$ is the marginal utility of money.

Exercises 71

1. $1, \frac{1}{4}, 3\frac{1}{2}, 4.$ 3. $3, 3/2, 11, 90.$

5. $-2a^{-2} < 0, -2b^{-2} < 0, 4a^{-3}b^{-2} > 0, -5/2 - 5/2,$

$\frac{-3a^{-6/2}}{4} < 0, \frac{-3b^{-5/2}}{4} < 0, (9/36) a b' > 0,$

$-6a^{-4} < 0, -6b^{-4} < 0, 4a^{-2}b^{-2} > 0,$

$-2a^{-2} < 0, -2b^{-2} < 0, 4a^{-2}b^{-2} > 0.$

7. $\pi = xp_x - ap_a - bp_b + \lambda g(x,a,b)$

$\pi_a = -p_a + \lambda g_a = 0$

$\pi_b = -p_b + \lambda g_b = 0$

$\pi_x = p_x + \lambda g_x = 0$

$\therefore \frac{g_a}{p_a} = \frac{g_b}{p_b} = \frac{-g_x}{p_x} = \frac{1}{\lambda}.$

9. (a) $a = 1/2, b = \sqrt[3]{\frac{1}{4}}, -2a^{-2} < 0, -6b^{-4} < 0, 12a^{-2}b^{-4} > 0.$

(b) $1/2, \sqrt[3]{\frac{1}{4}}, 15.48, 42.20.$

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Exercises 72

1. $x - x^2 + \frac{x^4}{4} + C.$ 3. $10 \log_e x + C.$ 5. $\frac{51}{2} x^{2/3} + C.$
 7. $x + \frac{2x^3}{3} + \frac{4x^6}{3}.$ 9. $x + \frac{1}{2x^4} + C.$

Exercises 73

1. (a) $150\frac{2}{3} + D + 50D^2 - \frac{5D^3}{3}.$ (b) $\frac{150\frac{2}{3}}{D} + 1 + 50D - \frac{5D^2}{3}.$
 3. (a) $2D + 5e^D + 95.$ (b) $2 + \frac{5e^D}{D} + \frac{95}{D}.$
 (c) $5e^{60} + 215; 2 + 5e^{60}; \frac{43 + e^{60}}{12}.$
 5. (a) $\frac{200}{3} D^{2/3} - 1780.$ (b) $\frac{200}{3} D^{1/2} - \frac{1780}{D}.$
 (c) $2486\frac{2}{3}; 25; 155\frac{5}{12}.$
 7. (a) $200\sqrt{D} - 700.$ (b) $\frac{\sqrt{200}}{D} - \frac{700}{D}.$
 (c) $1300; 10; 13.$
 9. $6.75D - 0.0003D^2 - 10,485,$ $6.75 - 0.0003D - \frac{10,485}{D}.$
 11. $1.052D - 0.002D^2 + 16.8,$ $1.052 - 0.002D + \frac{16.8}{D}.$

Exercises 74

1. $41\frac{1}{3}.$ The area under the curve from $x = 1$ to $x = 5.$
 3. $-r.$ The area under the curve from $x = -1$ to $x = 3.$
 5. $84.$ The area under the curve from $x = 1$ to $x = 16.$
 7. $26.3891.$ The area under the curve from $x = 0$ to $x = 2.$
 9. $a + \frac{b}{2}.$ The area under the curve from $x = 0$ to $x = 1.$

Exercises 75

1. (a) $20\frac{1}{4}$ (b) $12\frac{1}{4}$ (c) $6\frac{1}{4}$ (d) 75
 3. (a) 567 (b) 531.40271 (c) 576.02495
 5. (a) 469.71661 (b) 4661 (c) 453.1821
 7. (a) 0.351 (b) 0.729 (c) 2.043
 9. (a) 1.92667 (b) 0.80667 (c) 1.30667
 (d) 1.79307 (e) 3.52667 (f) 2.82907
 11. (a) 1.210 (b) 1.204 (c) 1.178
 (d) 1.229 (e) 1.242

Exercises 76

1. (a) $1/6$ (b) $1/6$ (c) $1/6$ (d) $1/6$

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Exercises 79

1. 1.8. 3. 2.0. 5. 1.0. 7. $1\frac{1}{2}$.
 9. 2.0.

Exercises 80

1. (a) 12 (b) 24 (c) 9 (d) 18
 (e) 14 (f) 2 (g) -20 (h) 49
 (i) 48 (j) 240 (k) 58 (l) -16
 3. (a) 3 (b) $4\frac{1}{2}$ (c) $7\frac{1}{2}$ (d) $-1\frac{1}{2}$
 (e) $-16\frac{1}{2}$ (f) 12.5 (g) $13\frac{1}{2}$
 5. (a) 1 (b) 8.5 (c) -5.5
 7. (a) 1.8 (b) 0 (c) 15.4 (d) 100

Exercises 81

1. 73.8; 9.0. 3. 6.0; 70.8. 5. 1.5; 8.5.
 7. 3.0; 6.75. 9. 0.1; 0.4; 0.4; 0.1. 11. $\frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}$.

Exercises 82

1. 0.56; 0.5536; 0.57920. 3. 2.0; 0; 6.8.
 5. $1/2$; 0. 7. 0.56; 0.748.
 9. $E(x - \mu)^3 = Ex^3 - 3\mu Ex^2 + 3(\mu)^2 Ex - \mu^3$
 $= \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$
 $= Ex^3 - 3Ex \cdot Ex^2 + 2(Ex)^3$
 $= Ex^3 - 3\mu Ex^2 + 2(\mu)^3$.
 11. 2.0; 0; 6.8.

Exercises 83

1. (a) $4/9$ (b) $80/243$ (c) $1/243$ (d) $8/27$
 (e) $1611/2187$ (f) $1808/2187$
 3. (a) $1/2$ (b) $15/64$ (c) $1/16$ (d) $1 - (25)10^{-10}$
 (e) 0.00369
 5. (a) 0.08192 (b) 0.06606 (c) 0.8192 (d) 0.32212
 7. (a) 0.000003960 (b) 0.000000002058 (c) 0.99860 (d) 0.02967
 9. (a) 0.45941/10 (b) 0.17097 (c) 0.72531 (d) 0.000001217
 11. $p_0 = 0.0466$. $p_4 = 0.1382$.
 $p_1 = 0.1866$. $p_5 = 0.0369$.
 $p_2 = 0.3110$. $p_6 = 0.0041$.
 $p_3 = 0.2765$.

Exercises 84

1. 0.8450 3. 0.1615
 5. (a) 0.8950 (b) 0.3272 (c) 0.2082 (d) 0.0569
 (e) 0.9525 (f) 0.0764 (g) 0.2420 (h) 0.8023
 7. (a) 0.1859 (b) 0.3219 (c) 0.0227 (d) 0.1587
 (e) 0.2119 (f) 0.9192 (g) 0.9192 (h) 0.0139

- | | |
|--------------------|-----------------|
| 9. $p_0 = 0.0492.$ | $p_4 = 0.1361.$ |
| $p_1 = 0.1696.$ | $p_5 = 0.0352.$ |
| $p_2 = 0.3053.$ | $p_6 = 0.0046.$ |
| $p_3 = 0.2893.$ | |

11.
$$P = \left(\frac{1}{\sigma\sqrt{2\pi}} \right) e^{-(x-\mu)^2/2\sigma^2}$$

$$\frac{dP}{dx} = \left(\frac{1}{\sigma\sqrt{2\pi}} \right) e^{-(x-\mu)^2/2\sigma^2} \cdot \left[\frac{-2(x-\mu)}{2\sigma^2} \right] = 0$$

$-x + \mu = 0, \quad x = \mu.$

$$\frac{d^2P}{dx^2} = -1/\sigma^2 \sqrt{2\pi} < 0 \text{ so } P \text{ is a maximum.}$$

Exercises 85

1. $EZ = E \left[\sum_{i=1}^n x_i \right] = \sum_{i=1}^n E(x_i) = n\mu$
3.
$$\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1} = \frac{1}{n-1} \left[\sum_{i=1}^n \left(x_i - \frac{\sum_{i=1}^n x_i}{n} \right)^2 \right]$$

$$= \frac{1}{n-1} \left[\sum_{i=1}^n \left\{ x_i^2 + \left(\frac{\sum_{i=1}^n x_i}{n} \right)^2 - 2x_i \frac{\sum_{i=1}^n x_i}{n} \right\} \right]$$

$$= \frac{1}{n-1} \left[\sum_{i=1}^n x_i^2 + n \left(\frac{\sum_{i=1}^n x_i}{n} \right)^2 - 2n \left(\frac{\sum_{i=1}^n x_i}{n} \right)^2 \right]$$

$$= \frac{1}{n-1} \left[\sum_{i=1}^n x_i^2 - n\bar{x}^2 \right]$$
5. (a) $L = \frac{1}{(2\pi)^{n/2}} \exp. \left\{ -1/2 \sum_{i=1}^n (x_i - a)^2 \right\}$
- (b) $\frac{dL}{da} = \left[\frac{1}{(2\pi)^{n/2}} \exp. \left\{ -1/2 \sum_{i=1}^n (x_i - a)^2 \right\} \right] \sum_{i=1}^n (x_i - a) = 0$
- $\therefore \sum_{i=1}^n x_i = na \quad a = \sum_{i=1}^n x_i/n = \bar{x}.$
7. $E(Z) = 1/n \left[\sum_{i=1}^n E(x_i - \mu)^2 \right] = \frac{n\sigma^2}{n} = \sigma^2$

W is the estimated variance, Z is the actual variance, and from 6 W is a biased estimate of σ^2 .

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Exercises 86

1.	Class		Class		Class	
	Limits	Frequency	Limits	Frequency	Limits	Frequency
(a)	0-4	1.0	(b) 0-5	2.5	(c) 0-10	5.5
	4-8	3.5	5-10	3.0	10-20	3.0
	8-12	1.5	10-15	0.5	20-30	2.5
	12-16	0.0	15-20	2.5	30-40	1.0
	16-20	2.5	20-25	2.0		
	20-24	1.5	25-30	0.5		
	24-28	1.0	30-35	0.0		
	28-32	0.0	35-40	1.0		
	32-36	1.0				

3.	Class	
	Limits	Frequency
	0-10	4.5
	10-20	4.5
	20-30	3.0
	30-40	2.0
	40-50	3.0

5. (a)	32		(b) Class		(c) Class	
			Limits	Frequency	Limits	Frequency
			4-14.67	6.0	4-12	6.0
			14.67-25.34	5.0	12-20	2.5
			25.34-36.01	1.0	20-28	2.5
					28-36	1.0

7. (a)	45		(b) Class	
			Limits	Frequency
			0-7.5	2.0
			7.5-15	3.0
			15-22.5	4.0
			22.5-30	3.0
			30-37.5	1.0
			37.5-45	4.

Exercises 87

- | | |
|------------------------------|------------------------------|
| 1. 27; 173.3; 13.16. | 3. 11.0; 28.8; 5.37. |
| 5. 11.5; 32.2; 5.67. | 7. 2,146.5; 2,902,857; 1704. |
| 9. (b) 31.72; 416.63; 20.41. | |
| 11. (a) 5. | (b) 13.5. |

Exercises 88

- | | |
|---------------|------------------|
| 1. 165, 12.85 | 3. 27.47, 5.24 |
| 5. 30.1, 5.49 | 7. 414.55, 20.36 |

Exercises 89

- | | | |
|-----------------|----------------------|------------------|
| 1. 8.97; 20.53. | 3. 15.90; 29.38. | 5. 18.85; 35.15. |
| 7. 9.24; 13.36. | 9. 485,988; 495,972. | |

Exercises 90

1. Rejected. 3. Rejected.
 5. (a) Accepted. (b) Accepted. (c) Accepted. (d) Accepted.
 7. Rejected. 9. 4.27; 4.93.

Exercises 91

1. Accepted. 3. Accepted.
 5. Accepted. 7. Rejected.
 9. $\frac{\bar{x} - \mu}{s/\sqrt{n}}$ is distributed as t with $(n - 1)$ degrees of freedom.

The values of " t " being from the " t " tables with $(n - 1)$ degrees of freedom.
 The confidence limits are:

$$\bar{x} - s/\sqrt{n} t_{(n-1)(0.05)}$$

$$\bar{x} + s/\sqrt{n} t_{(n-1)(0.05)}$$

11. 8.14, 13.86; 7.05, 14.95.

Exercises 92

1. Nonsignificant. 3. Nonsignificant.
 5. (a) Nonsignificant. (b) Nonsignificant. (c) Nonsignificant.
 7. (a) Nonsignificant. (b) Significant. (c) Significant.
 9. $\frac{d - \mu_d}{\sigma_d}$ is normal with mean 0 and variance 1, where $\mu_d = (\mu_1 - \mu_2)$
 (a) $d - 1.96\sigma_d, d + 1.96\sigma_d$;
 (b) $d - 2.57\sigma_d, d + 2.57\sigma_d$;
 (c) $-\infty, +\infty$
 (d) $d - 1.65\sigma_d, d + 1.65\sigma_d$
 11. (a) Nonsignificant. (b) Nonsignificant.

Exercises 93

1. (a) $\bar{x} = 27, s = 13.16.$

Upper Limits (Standardized)	Cumulative Probability	p	Np
-1.24	0.1075	0.1075	10.75
0.31	0.6217	0.5142	51.42
1.86	0.9686	0.3469	34.69.

3. (a) 11.3; 4.71.

Upper Limits (Standardized)	Cumulative Probability	p	Np
-1.34	0.0901	0.0901	1.80
-0.70	0.2420	0.1519	3.04
-0.06	0.4761	0.2341	4.68
0.57	0.7157	0.2396	4.79
1.21	0.8869	0.1712	3.42
1.85	0.9678	0.0809	1.62,

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5. (a) 45.7; 17.34.

Upper Limits (Standardized)	Cumulative Probability	p	Np
-1.48	0.0694	0.0694	4.86
-0.33	0.3707	0.3013	21.09
0.82	0.7939	0.4232	29.62
1.98	0.9762	0.1823	12.76

7. (a) 31.72; 20.41.

Upper Limits (Standardized)	Cumulative Probability	p	Np
-1.31	0.0951	0.0951	12.74
-1.06	0.1446	0.0495	6.63
-0.82	0.2061	0.0615	8.24
-0.57	0.2843	0.0782	10.48
-0.33	0.3707	0.0864	11.58
-0.08	0.4681	0.0974	13.05
0.16	0.5636	0.0955	12.80
0.41	0.6591	0.0955	12.80
0.65	0.7422	0.0831	11.14
0.90	0.8159	0.0737	9.88
1.14	0.8729	0.0570	7.64
1.39	0.9177	0.0448	6.00
1.63	0.9485	0.0308	4.13
1.88	0.9700	0.0215	2.88
2.12	0.9830	0.0130	1.74
2.52	0.9941	0.0111	1.49.

9. (a) 1837; 641.3.

Upper Limits (Standardized)	Cumulative Probability	p	Np
-1.31	0.0951	0.0951	1,962,198
-0.53	0.2981	0.2030	4,188,499
0.25	0.5987	0.3006	6,202,280
1.03	0.8485	0.2498	5,154,123
1.81	0.9649	0.1164	2,401,681.

Exercises 94

1. (a) $\bar{x} = 27$, $s = 13.16$.

Upper Limits (Standardized)	Cumulative Probability	p	Np
-1.29	0.0985	0.0985	0.98
0.53	0.2981	0.1996	2.00
0.23	0.5910	0.2929	2.93
0.99	0.8389	0.2479	2.45
1.75	0.9599	0.1210	1.21.

Nonsignificant.

3. Significant.

5. Nonsignificant.

7. $\bar{x} = 7, s = 2.83.$

Upper Limits (Standardized)	Cumulative Probability	p	Np
-1.06	0.1446	0.1446	1.16
0.35	0.6368	0.4922	3.94
1.78	0.9625	0.3257	2.61.

Nonsignificant.

Exercises 95

- | | |
|--------------------|--------------------|
| 1. Significant. | 3. Nonsignificant. |
| 5. Nonsignificant. | 7. Nonsignificant. |

Exercises 96

1. (a) $3a^2 - 32a + 110.$ (b) $a = 5\frac{1}{2}.$ (c) $\bar{x} = 5\frac{1}{2}.$

3. $Q = \sum_{i=1}^n (\log x_i - \log G)^2,$

$$\frac{dQ}{dG} = -2 \sum_{i=1}^n (\log x_i - \log G)/G = 0,$$

$$\sum_{i=1}^n \log x_i = n \log G,$$

$$\log G = \frac{\sum_{i=1}^n \log x_i}{n}.$$

5. $s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}$ is the sample variance,

so $Q = (n - 1)s^2$, where s^2 is the sample variance.

7. 3.9831. 9. 5.7349.

Exercises 97

- | | | | |
|-------------------------------------|--|-------------|-----------|
| 1. (a) $\hat{y} = 40.2 + 2x.$ | (b) 42.6. | (c) 43. | (d) 43.8. |
| (e) 48.2. | (f) 50.2. | (g) 51.0. | |
| 3. (a) $\hat{y} = 47.285 + 5.43x.$ | (b) 59.231. | (d) 61.403. | |
| (c) 62.489. | (f) 69.005. | (g) 83.123. | |
| 5. (a) $\hat{y} = 116.75 - 6.25x.$ | (c) 104.25. | (d) 99.25. | |
| (e) 93.0. | (f) 90.5. | | |
| 7. (a) $\hat{y} = 70.774 + 2.088x,$ | (c) 81.214; 73.520; 78.685; 84.957. | | |
| $\hat{y} = 58.55 + 2.994x,$ | (d) 89.57; 85.496; 86.353; 91.677. | | |
| $\hat{y} = 69.1 + 1.917x,$ | (e) 104.182; 106.454; 99.772; 103.437. | | |
| $\hat{y} = 76.557 + 1.68x.$ | | | |

$$9. \frac{\partial F}{\partial a} = -2 \left[\sum_{i=1}^n y_i - an - b \sum_{i=1}^n x_i^2 \right] = 0,$$

$$\frac{\partial^2 F}{\partial a^2} = 2n > 0,$$

$$\frac{\partial F}{\partial b} = 2 \left[- \sum_{i=1}^n x_i y_i + a \sum_{i=1}^n x_i + b \sum_{i=1}^n x_i^2 \right] = 0,$$

$$\frac{\partial^2 F}{\partial b^2} = 2 \sum_{i=1}^n x_i^2 > 0.$$

Necessary and sufficient conditions for a minimum are met.

11. $\hat{y} = 2.0 + 0.3x$.
13. (a) $\hat{y} = 109.5625 - 6.25x + 1.4375x^2$. (c) 103.81.
 (d) 103.33. (e) 106.57.
 (f) 108.67.
15. (a) $\log \hat{y} = 2.01608 + 0.06056x$. (c) 111.25.
 (d) 131.53. (e) 135.25.
 (f) 150.48.
17. (a) $\log \hat{y} = 1.85352 + 0.02119x$. (b) 78.687.
 (c) 79.458. (d) 81.818.
 (e) 86.758. (f) 92.886.
19. (a) $\log \hat{y} = 1.65641 + 0.05608x$. (c) 60.227,
 (d) 63.419. (e) 65.077.
 (f) 75.985. (g) 106.31

$$21. G = \sum_{i=1}^n (x_i - A - By_i)^2,$$

$$\frac{\partial G}{\partial A} = 2 \sum_{i=1}^n (x_i - A - By_i)(-1) = 0,$$

$$\therefore \sum_{i=1}^n x_i = An + B \sum_{i=1}^n y_i,$$

$$\frac{\partial G}{\partial B} = 2 \sum_{i=1}^n (x_i - A - By_i)(-y_i) = 0,$$

$$\sum_{i=1}^n x_i y_i = A \sum_{i=1}^n y_i + B \sum_{i=1}^n y_i^2.$$

23. (a) Cumulative Probability
 0.36278
 0.93556
 0.99262
 1.00000.
 (g) 96.000.
- (b) $\log \hat{y} = 8.93768 - 0.6223 \log x$.
 (c) 4,323,200.
 (d) 3,340,800.
 (e) 8,207,000.
 (f) 848,000.
25. (a) Cumulative Probability
 0.27130
 0.51667
 0.71328
 0.90664
 1.00000.
- (b) $\log \hat{y} = 8.46854 - 0.59016 \log x$.
 (c) 2,905,400.
 (d) 2,201,600.
 (e) 50,900.
 (f) 158,800.

Exercises 98

1. (a) $\hat{y} = 0.4 + 0.9x$, (c) 4.0. (d) 1.89.
 $\hat{z} = 0.97 + 0.305y$.

3. (a) $\hat{y} = 2.86 - 0.30x$, (c) -0.14 . (d) 2.74.
 $\hat{z} = 3.44 - 0.28y$.
5. (a) $\hat{y} = -14.25 + 1.42x$, (c) 113.6.
 $\hat{z} = 68.58 + 0.12y$.
 (d) 82.98.
7. (b) $\hat{y} = 4.14 - 0.028x$, (c) 2.46. (d) 62.8.
 $\hat{z} = 77.8 - 10.0y$.
9. (b) $\hat{y} = 13.84 + 1.23x$, (c) 136.84. (d) 33.57.
 $\hat{z} = -1.43 + 0.70y$.
11. (a) $\hat{y} = 40.92 + 80.68x$, (b) 847.72. (c) 11.225.
 $\hat{z} = 0.225 + 0.011y$.

Exercises 99

1. 0.524; 0.275; nonsignificant; $-0.70, 3.42$.
 3. $-0.291; 0.085$; nonsignificant; $-1.305, 0.697$.
 5. 0.409; 0.167; nonsignificant; $-4.412, 7.268$.
 7. $-0.532; 0.283$; nonsignificant; $-0.110, 0.056$.
 9. 0.932; 0.869; significant; 0.84, 1.62.
11. $F = \sum_{i=1}^n (z_i - a - bx_i - cy_i)^2$

$$\frac{\partial F}{\partial a} = -2 \sum_{i=1}^n (z_i - a - bx_i - cy_i) = 0.$$

$$\frac{\partial F}{\partial b} = -2 \sum_{i=1}^n (z_i - a - bx_i - cy_i)x_i = 0.$$

$$\frac{\partial F}{\partial c} = -2 \sum_{i=1}^n (z_i - a - bx_i - cy_i)y_i = 0.$$

Exercises 100

1. (a) $\hat{y} = 133.4 - 1.20x$, (b) 73.4. (c) 48.3.
 $\hat{z} = 59.09 - 0.12y$.
 (d) -0.382 . (e) 0.146. (g) Nonsignificant.
 (h) Nonsignificant. (i) $\log \hat{y} = 2.1393 - 0.1771 \log x$.
 (j) $-4.916, 2.522$.
3. (a) $\hat{y} = 48.88 + 1.31x$, (b) 101.28.
 $\hat{z} = 28.40 + 0.056y$.
 (c) 32.88. (d) 0.2697.
 (e) 0.0727. (g) Nonsignificant. (h) $-0.116, 0.424$.
 (i) $\log \hat{y} = 1.1216 + 0.5389 \log x$, (j) 0.3487.
 $\log \hat{z} = 1.0754 + 0.2259 \log y$.
 (k) Nonsignificant. (l) Nonsignificant
5. (a) $\hat{y} = 6.755 + 0.1293x$, (b) 17.008.
 $\hat{z} = 1.7234 + 3.0569y$.
 (c) 14.514. (d) 0.629.
 (e) 0.3956. (g) Significant.
7. (a) $\hat{y} = -5.0632 + 1.1279x$, (b) 107.727.
 $\hat{z} = 24.092 + 0.7187y$.
 (c) 131.897. (d) 0.9005.
 (e) 0.8109. (g) Nonsignificant.
 (h) $-0.527, 2.783$.

340 *Answers to Odd-numbered Problems*

9. (a) $\hat{y} = 73.3955 + 0.2898x$. (c) 108.1715.
 (d) 98.0285. (e) 103.8245.
 (f) 0.3811. (g) Nonsignificant.
 (h) 0.1452. (i) $\hat{y} = 87.77581 + 0.14096x$.
 (k) 101.87181. (l) 105.39581.
 (m) 99.47549. (n) 0.2010.
 (o) Nonsignificant. (p) 0.0404.

11. (a) $y_t = \frac{a + Kc}{1 + K} + \frac{b + Kd}{1 + K} x_t$

(b) $y_t = \frac{c + La}{1 + L} + \frac{d + Lb}{1 + L} x_t$

13. (a) $\hat{z} = 6.007 - 0.098x + 0.055y$.
 (b) 3.947.

Exercises 101

1. (a) 244%. (b) 178%. (c) 130%. (d) 127%.
 (e) 129%. (f) 127% - 130%. (g) 129%.
 3. (a) -19%. (b) -21%. (c) -13%. (d) -15%.
 (e) -14%. (f) -13% to -15%. (g) -14%.

5. $L = \frac{\sum p_1q_0}{\sum p_0q_0} = 1.2$.

But $\sum p_0q_0 = 100$,
 and $\sum p_1q_0 = 120$, so that with an income of 120 in time 1 a person may buy the same amounts of commodities, at the new prices, as he bought with 100 at the prices in time 0; so is no worse off.

7. (a) -17%. (b) -20%. (c) -11%. (d) -13%. (e) -12%.
 9. (a) 11%; 9%; 10%. (b) 12%; 15%; 13%. (c) 2%; 1%; 1.4%.

TABLE 1

10.0—Four-Place Common Logarithms of Numbers—54.9

N	0	1	2	3	4	5	6	7	8	9
10	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374
11	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755
12	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106
13	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430
14	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732
15	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014
16	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279
17	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529
18	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765
19	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989
20	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201
21	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404
22	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598
23	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784
24	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962
25	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133
26	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298
27	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456
28	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609
29	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757
30	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900
31	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038
32	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172
33	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302
34	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428
35	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551
36	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670
37	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786
38	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899
39	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010
40	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117
41	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222
42	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325
43	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425
44	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522
45	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618
46	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712
47	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803
48	.6812	.6821	.6830	.6839	.6848	.6857	.6866	.6875	.6884	.6893
49	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981
50	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067
51	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152
52	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235
53	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316
54	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396
N	0	1	2	3	4	5	6	7	8	9

10.0—Four-Place Common Logarithms of Numbers—54.9

Table 1

55.0—Four-Place Common Logarithms of Numbers—99.9

<i>N</i>	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
<i>N</i>	0	1	2	3	4	5	6	7	8	9

55.0—Four-Place Common Logarithms of Numbers—99.9

TABLE 2

Natural Trigonometric Functions for Decimal Fractions of a Degree

Deg.	Sin	Tan	Ctn	Cos	Deg.	Deg.	Sin	Tan	Ctn	Cos	Deg.
0-	.00000	.00000	—	1.00000	90-	0-	.10453	.10510	9 5144	.99452	84-
.1	.00175	.00175	572.96	1.00000	.9	.1	.10626	.10687	9 3572	.99434	.9
.2	.00349	.00349	286 48	0.99999	.8	.2	.10800	.10863	9.2052	.99415	.8
.3	.00524	.00524	190.98	.99999	.7	.3	.10973	.11040	9 0579	.99396	.7
.4	.00698	.00698	143.24	.99998	.6	.4	.11147	.11217	8.9152	.99377	.6
.5	.00873	.00873	114 59	.99996	.5	.5	.11320	.11394	8.7769	.99357	.5
.6	.01047	.01047	95 489	.99996	.4	.6	.11494	.11570	8 6427	.99337	.4
.7	.01222	.01222	81 847	.99993	.3	.7	.11667	.11747	8 5126	.99317	.3
.8	.01396	.01396	71 615	.99990	.2	.8	.11840	.11924	8 3863	.99297	.2
.9	.01571	.01571	63.657	.99988	.1	.9	.12014	.12101	8 2636	.99276	.1
1-	.01745	.01745	57.290	.99985	89-	7-	.12187	.12278	8.1443	.99255	83-
.1	.01920	.01920	52 081	.99982	.9	.1	.12360	.12456	8 0285	.99233	.9
.2	.02094	.02094	47 740	.99978	.8	.2	.12533	.12633	7 9158	.99211	.8
.3	.02269	.02269	44 066	.99974	.7	.3	.12706	.12810	7 8062	.99189	.7
.4	.02443	.02444	40.917	.99970	.6	.4	.12880	.12988	7.6996	.99167	.6
.5	.02618	.02619	38.188	.99966	.5	.5	.13053	.13165	7 5958	.99144	.5
.6	.02792	.02793	35 801	.99961	.4	.6	.13226	.13343	7 4947	.99122	.4
.7	.02967	.02968	33 694	.99956	.3	.7	.13399	.13521	7.3962	.99098	.3
.8	.03141	.03143	31 821	.99951	.2	.8	.13572	.13698	7 3002	.99075	.2
.9	.03316	.03317	30.145	.99945	.1	.9	.13744	.13876	7 2066	.99051	.1
2-	.03490	.03492	28 636	.99939	88-	8-	.13917	.14054	7 1154	.99027	82-
.1	.03664	.03667	27.271	.99933	.9	.1	.14090	.14232	7 0264	.99002	.9
.2	.03839	.03842	26 031	.99926	.8	.2	.14263	.14410	6.9395	.98978	.8
.3	.04013	.04016	24.898	.99919	.7	.3	.14436	.14588	6 8548	.98953	.7
.4	.04188	.04191	23 859	.99912	.6	.4	.14608	.14767	6 7720	.98927	.6
.5	.04362	.04366	22.904	.99905	.5	.5	.14781	.14945	6 6912	.98902	.5
.6	.04536	.04541	22.022	.99897	.4	.6	.14954	.15124	6.6122	.98876	.4
.7	.04711	.04716	21.205	.99889	.3	.7	.15126	.15302	6 5350	.98849	.3
.8	.04885	.04891	20 446	.99881	.2	.8	.15299	.15481	6 4596	.98823	.2
.9	.05059	.05066	19 740	.99872	.1	.9	.15471	.15660	6 3859	.98796	.1
3-	.05234	.05241	19.081	.99863	87-	9-	.15643	.15838	6 3138	.98769	81-
.1	.05408	.05416	18 464	.99854	.9	.1	.15816	.16017	6 2432	.98741	.9
.2	.05582	.05591	17.886	.99844	.8	.2	.15988	.16196	6 1742	.98714	.8
.3	.05756	.05766	17.343	.99834	.7	.3	.16160	.16376	6 1066	.98686	.7
.4	.05931	.05941	16 832	.99824	.6	.4	.16333	.16555	6 0405	.98657	.6
.5	.06105	.06116	16.350	.99813	.5	.5	.16505	.16734	5 9758	.98629	.5
.6	.06279	.06291	15 895	.99803	.4	.6	.16677	.16914	5.9124	.98600	.4
.7	.06453	.06467	15 464	.99792	.3	.7	.16849	.17093	5.8502	.98570	.3
.8	.06627	.06642	15 056	.99780	.2	.8	.17021	.17273	5 7894	.98541	.2
.9	.06802	.06817	14.669	.99768	.1	.9	.17193	.17453	5.7297	.98511	.1
4-	.06976	.06993	14 301	.99756	86-	10-	.17365	.17633	5.6713	.98481	80-
.1	.07150	.07168	13.951	.99744	.9	.1	.17537	.17813	5.6140	.98450	.9
.2	.07324	.07344	13.617	.99731	.8	.2	.17708	.17993	5 5578	.98420	.8
.3	.07498	.07519	13 300	.99719	.7	.3	.17880	.18173	5 5026	.98389	.7
.4	.07672	.07695	12.996	.99705	.6	.4	.18052	.18353	5 4486	.98357	.6
.5	.07846	.07870	12.706	.99692	.5	.5	.18224	.18534	5 3955	.98325	.5
.6	.08020	.08046	12 429	.99678	.4	.6	.18396	.18714	5 3435	.98294	.4
.7	.08194	.08221	12.163	.99664	.3	.7	.18567	.18895	5 2924	.98261	.3
.8	.08368	.08397	11.909	.99649	.2	.8	.18738	.19076	5.2422	.98229	.2
.9	.08542	.08573	11.664	.99635	.1	.9	.18910	.19257	5.1929	.98196	.1
5-	.08716	.08749	11.430	.99619	85-	11-	.19081	.19438	5 1446	.98163	79-
.1	.08889	.08925	11 205	.99604	.9	.1	.19252	.19619	5 0970	.98129	.9
.2	.09063	.09101	10 988	.99588	.8	.2	.19423	.19801	5.0504	.98096	.8
.3	.09237	.09277	10.780	.99572	.7	.3	.19595	.19982	5.0045	.98061	.7
.4	.09411	.09453	10.579	.99556	.6	.4	.19766	.20164	4.9594	.98027	.6
.5	.09585	.09629	10 385	.99540	.5	.5	.19937	.20345	4.9152	.97992	.5
.6	.09758	.09805	10 199	.99523	.4	.6	.20108	.20527	4.8716	.97958	.4
.7	.09932	.09981	10 019	.99506	.3	.7	.20279	.20709	4 8288	.97922	.3
.8	.10106	.10158	9.8448	.99488	.2	.8	.20450	.20891	4 7867	.97887	.2
.9	.10279	.10334	9.6768	.99470	.1	.9	.20620	.21073	4 7453	.97851	.1
6-	.10453	.10510	9.5144	.99452	84-	12-	.20791	.21256	4.7046	.97815	78-
Deg.	Cos	Ctn	Tan	Sin	Deg.	Deg.	Cos	Ctn	Tan	Sin	Deg.

Natural Trigonometric Functions for Decimal Fractions of a Degree

Table 2

Natural Trigonometric Functions for Decimal Fractions of a Degree

Deg.	Sin	Tan	Ctn	Cos	Deg.	Deg.	Sin	Tan	Ctn	Cos	Deg.
12-	.20791	.21256	4.7046	.97815	78-	17.5	.30071	.31530	3.1716	.95372	72.5
.1	.20962	.21438	4.6646	.97778	.9	.6	.30237	.31722	3.1524	.95319	.4
.2	.21132	.21621	4.6252	.97742	.8	.7	.30403	.31914	3.1334	.95266	.3
.3	.21303	.21804	4.5864	.97705	.7	.8	.30570	.32106	3.1146	.95213	.2
.4	.21474	.21986	4.5483	.97667	.6	.9	.30736	.32299	3.0961	.95159	.1
.5	.21644	.22169	4.5107	.97630	.5	18-	.30902	.32492	3.0777	.95106	72-
.6	.21814	.22353	4.4737	.97592	.4	.1	.31068	.32685	3.0596	.95052	.9
.7	.21985	.22536	4.4373	.97553	.3	.2	.31233	.32878	3.0415	.94997	.8
.8	.22155	.22719	4.4015	.97515	.2	.3	.31399	.33072	3.0237	.94943	.7
.9	.22325	.22903	4.3662	.97476	.1	.4	.31565	.33266	3.0061	.94888	.6
13-	.22495	.23087	4.3315	.97437	77-	.5	.31730	.33460	2.9887	.94832	.5
.1	.22666	.23271	4.2972	.97398	.9	.6	.31896	.33654	2.9714	.94777	.4
.2	.22835	.23455	4.2635	.97358	.8	.7	.32061	.33848	2.9544	.94721	.3
.3	.23005	.23639	4.2303	.97318	.7	.8	.32227	.34043	2.9375	.94665	.2
.4	.23175	.23823	4.1976	.97278	.6	.9	.32392	.34238	2.9208	.94609	.1
.5	.23345	.24008	4.1653	.97237	.5	19-	.32557	.34433	2.9042	.94552	71-
.6	.23514	.24193	4.1335	.97196	.4	.1	.32722	.34628	2.8878	.94495	.9
.7	.23684	.24377	4.1022	.97155	.3	.2	.32887	.34824	2.8716	.94438	.8
.8	.23853	.24562	4.0713	.97113	.2	.3	.33051	.35020	2.8556	.94380	.7
.9	.24023	.24747	4.0408	.97072	.1	.4	.33216	.35216	2.8397	.94322	.6
14-	.24192	.24933	4.0108	.97030	70-	.5	.33381	.35412	2.8239	.94264	.5
.1	.24362	.25118	3.9812	.96987	.9	.6	.33545	.35608	2.8083	.94206	.4
.2	.24531	.25304	3.9520	.96945	.8	.7	.33710	.35805	2.7929	.94147	.3
.3	.24700	.25490	3.9232	.96902	.7	.8	.33874	.36002	2.7776	.94088	.2
.4	.24869	.25676	3.8947	.96858	.6	.9	.34038	.36199	2.7625	.94029	.1
.5	.25038	.25862	3.8667	.96815	.5	20-	.34202	.36397	2.7475	.93969	70-
.6	.25207	.26048	3.8391	.96771	.4	.1	.34366	.36596	2.7326	.93909	.9
.7	.25376	.26235	3.8118	.96727	.3	.2	.34530	.36793	2.7179	.93849	.8
.8	.25545	.26421	3.7848	.96682	.2	.3	.34694	.36991	2.7034	.93789	.7
.9	.25713	.26608	3.7583	.96638	.1	.4	.34857	.37190	2.6889	.93728	.6
15-	.25882	.26795	3.7321	.96593	75-	.5	.35021	.37388	2.6746	.93667	.5
.1	.26050	.26982	3.7062	.96547	.9	.6	.35184	.37588	2.6605	.93606	.4
.2	.26219	.27169	3.6806	.96502	.8	.7	.35347	.37787	2.6464	.93544	.3
.3	.26387	.27357	3.6554	.96456	.7	.8	.35511	.37986	2.6325	.93483	.2
.4	.26556	.27545	3.6305	.96410	.6	.9	.35674	.38186	2.6187	.93420	.1
.5	.26724	.27732	3.6059	.96363	.5	21-	.35837	.38386	2.6051	.93358	69-
.6	.26892	.27921	3.5816	.96316	.4	.1	.36000	.38587	2.5916	.93296	.9
.7	.27060	.28109	3.5576	.96269	.3	.2	.36162	.38787	2.5782	.93232	.8
.8	.27228	.28297	3.5339	.96222	.2	.3	.36325	.38988	2.5649	.93169	.7
.9	.27396	.28486	3.5105	.96174	.1	.4	.36488	.39190	2.5517	.93106	.6
16-	.27564	.28675	3.4874	.96126	74-	.5	.36650	.39391	2.5386	.93042	.5
.1	.27731	.28864	3.4646	.96078	.9	.6	.36812	.39593	2.5257	.92978	.4
.2	.27899	.29053	3.4420	.96029	.8	.7	.36975	.39795	2.5129	.92913	.3
.3	.28067	.29242	3.4197	.95981	.7	.8	.37137	.39997	2.5002	.92849	.2
.4	.28234	.29432	3.3977	.95931	.6	.9	.37299	.40200	2.4876	.92784	.1
.5	.28402	.29621	3.3759	.95882	.5	22-	.37461	.40403	2.4751	.92718	68-
.6	.28569	.29811	3.3544	.95832	.4	.1	.37622	.40606	2.4627	.92653	.9
.7	.28736	.30001	3.3332	.95782	.3	.2	.37784	.40809	2.4504	.92587	.8
.8	.28903	.30192	3.3122	.95732	.2	.3	.37946	.41013	2.4383	.92521	.7
.9	.29070	.30382	3.2914	.95681	.1	.4	.38107	.41217	2.4262	.92455	.6
17-	.29237	.30573	3.2709	.95630	73-	.5	.38268	.41421	2.4142	.92388	.5
.1	.29404	.30764	3.2506	.95579	.9	.6	.38430	.41626	2.4023	.92321	.4
.2	.29571	.30955	3.2305	.95528	.8	.7	.38591	.41831	2.3906	.92254	.3
.3	.29737	.31147	3.2106	.95476	.7	.8	.38752	.42036	2.3789	.92186	.2
.4	.29904	.31338	3.1910	.95424	.6	.9	.38912	.42242	2.3673	.92119	.1
17.5	.30071	.31530	3.1716	.95372	72.5	23-	.39073	.42447	2.3559	.92050	67-
Deg.	Cos	Ctn	Tan	Sin	Deg.	Deg.	Cos	Ctn	Tan	Sin	Deg.

Natural Trigonometric Functions for Decimal Fractions of a Degree

Table 2

Natural Trigonometric Functions for Decimal Fractions of a Degree

Deg.	Sin	Tan	Ctn	Cos	Deg.	Deg.	Sin	Tan	Ctn	Cos	Deg.
23-	.39073	.42447	2.3559	.92050	67-	28.5	.47716	.54296	1.8418	.87882	61.5
.1	.39234	.42654	2.3445	.91982	.9	.6	.47869	.54522	1.8341	.87798	.4
.2	.39394	.42860	2.3332	.91914	.8	.7	.48022	.54748	1.8265	.87715	.3
.3	.39555	.43067	2.3220	.91845	.7	.8	.48175	.54975	1.8190	.87631	.2
.4	.39715	.43274	2.3109	.91775	.6	.9	.48328	.55203	1.8115	.87546	.1
.5	.39875	.43481	2.2998	.91706	.5	29-	.48481	.55431	1.8040	.87462	61-
.6	.40035	.43689	2.2889	.91636	.4	.1	.48634	.55659	1.7966	.87377	.9
.7	.40195	.43897	2.2781	.91566	.3	.2	.48786	.55888	1.7893	.87292	.8
.8	.40355	.44105	2.2673	.91496	.2	.3	.48938	.56117	1.7820	.87207	.7
.9	.40514	.44314	2.2566	.91425	.1	.4	.49090	.56347	1.7747	.87121	.6
24-	.40674	.44523	2.2460	.91355	66-	.5	.49242	.56577	1.7675	.87036	.5
.1	.40833	.44732	2.2355	.91283	.9	.6	.49394	.56808	1.7603	.86949	.4
.2	.40992	.44942	2.2251	.91212	.8	.7	.49546	.57039	1.7532	.86863	.3
.3	.41151	.45152	2.2148	.91140	.7	.8	.49697	.57271	1.7461	.86777	.2
.4	.41310	.45362	2.2045	.91068	.6	.9	.49849	.57503	1.7391	.86690	.1
.5	.41469	.45573	2.1943	.90996	.5	30-	.50000	.57735	1.7321	.86603	60-
.6	.41628	.45784	2.1842	.90924	.4	.1	.50151	.57968	1.7251	.86515	.9
.7	.41787	.45995	2.1742	.90851	.3	.2	.50302	.58201	1.7182	.86427	.8
.8	.41945	.46206	2.1642	.90778	.2	.3	.50453	.58435	1.7113	.86340	.7
.9	.42104	.46418	2.1543	.90704	.1	.4	.50603	.58670	1.7045	.86251	.6
25-	.42262	.46631	2.1445	.90631	65-	.5	.50754	.58905	1.6977	.86163	.5
.1	.42420	.46843	2.1348	.90557	.9	.6	.50904	.59140	1.6909	.86074	.4
.2	.42578	.47056	2.1251	.90483	.8	.7	.51054	.59376	1.6842	.85985	.3
.3	.42736	.47270	2.1155	.90408	.7	.8	.51204	.59612	1.6775	.85896	.2
.4	.42894	.47483	2.1060	.90334	.6	.9	.51354	.59849	1.6709	.85806	.1
.5	.43051	.47698	2.0965	.90259	.5	31-	.51504	.60086	1.6643	.85717	59-
.6	.43209	.47912	2.0872	.90183	.4	.1	.51653	.60324	1.6577	.85627	.9
.7	.43366	.48127	2.0778	.90108	.3	.2	.51803	.60562	1.6512	.85536	.8
.8	.43523	.48342	2.0686	.90032	.2	.3	.51952	.60801	1.6447	.85446	.7
.9	.43680	.48557	2.0594	.89956	.1	.4	.52101	.61040	1.6383	.85355	.6
26-	.43837	.48773	2.0503	.89879	64-	.5	.52250	.61280	1.6319	.85264	.5
.1	.43994	.48989	2.0413	.89803	.9	.6	.52399	.61520	1.6255	.85173	.4
.2	.44151	.49206	2.0323	.89726	.8	.7	.52547	.61761	1.6191	.85081	.3
.3	.44307	.49423	2.0233	.89649	.7	.8	.52696	.62003	1.6128	.84989	.2
.4	.44464	.49640	2.0145	.89571	.6	.9	.52844	.62245	1.6066	.84897	.1
.5	.44620	.49858	2.0057	.89493	.5	32-	.52992	.62487	1.6003	.84806	58-
.6	.44776	.50076	1.9970	.89415	.4	.1	.53140	.62730	1.5941	.84712	.9
.7	.44932	.50295	1.9883	.89337	.3	.2	.53288	.62973	1.5880	.84619	.8
.8	.45088	.50514	1.9797	.89259	.2	.3	.53435	.63217	1.5818	.84526	.7
.9	.45243	.50733	1.9711	.89180	.1	.4	.53583	.63462	1.5757	.84433	.6
27-	.45399	.50953	1.9626	.89101	63-	.5	.53730	.63707	1.5697	.84339	.5
.1	.45554	.51173	1.9542	.89021	.9	.6	.53877	.63953	1.5637	.84245	.4
.2	.45710	.51393	1.9458	.88942	.8	.7	.54024	.64199	1.5577	.84151	.3
.3	.45865	.51614	1.9375	.88862	.7	.8	.54171	.64446	1.5517	.84057	.2
.4	.46020	.51835	1.9292	.88782	.6	.9	.54317	.64693	1.5458	.83962	.1
.5	.46175	.52057	1.9210	.88701	.5	33-	.54464	.64941	1.5399	.83867	57-
.6	.46330	.52279	1.9128	.88620	.4	.1	.54610	.65189	1.5340	.83772	.9
.7	.46484	.52501	1.9047	.88539	.3	.2	.54756	.65438	1.5282	.83676	.8
.8	.46639	.52724	1.8967	.88458	.2	.3	.54902	.65688	1.5224	.83581	.7
.9	.46793	.52947	1.8887	.88377	.1	.4	.55048	.65938	1.5166	.83485	.6
28-	.46947	.53171	1.8807	.88295	62-	.5	.55194	.66189	1.5108	.83389	.5
.1	.47101	.53395	1.8728	.88213	.9	.6	.55339	.66440	1.5051	.83292	.4
.2	.47255	.53620	1.8650	.88130	.8	.7	.55484	.66692	1.4994	.83195	.3
.3	.47409	.53844	1.8572	.88048	.7	.8	.55630	.66944	1.4938	.83098	.2
.4	.47562	.54070	1.8495	.87965	.6	.9	.55775	.67197	1.4882	.83001	.1
28.5	.47716	.54296	1.8418	.87882	61.5	34-	.55919	.67451	1.4826	.82904	56-
Deg.	Cos	Ctn	Tan	Sin	Deg.	Deg.	Cos	Ctn	Tan	Sin	Deg.

Natural Trigonometric Functions for Decimal Fractions of a Degree

Table 2

Natural Trigonometric Functions for Decimal Fractions of a Degree

Deg.	Sin	Tan	Ctn	Cos	Deg.	Deg.	Sin	Tan	Ctn	Cos	Deg.
34-	.55919	.67451	1.4826	.82904	50-	39.5	.63608	.82434	1.2131	.77162	50.5
.1	.56064	.67705	1.4770	.82806	.9	.6	.63742	.82727	1.2088	.77051	.4
.2	.56208	.67960	1.4715	.82708	.8	.7	.63877	.83022	1.2045	.76940	.3
.3	.56353	.68215	1.4659	.82610	.7	.8	.64011	.83317	1.2002	.76828	.2
.4	.56497	.68471	1.4605	.82511	.6	.9	.64145	.83613	1.1960	.76717	.1
.5	.56641	.68728	1.4550	.82413	.5	40-	.64279	.83910	1.1918	.76604	50-
.6	.56784	.68985	1.4496	.82314	.4	.1	.64412	.84208	1.1875	.76492	.9
.7	.56928	.69243	1.4442	.82214	.3	.2	.64546	.84507	1.1833	.76380	.8
.8	.57071	.69502	1.4388	.82115	.2	.3	.64679	.84806	1.1792	.76267	.7
.9	.57215	.69761	1.4335	.82015	.1	.4	.64812	.85107	1.1750	.76154	.6
35-	.57358	.70021	1.4281	.81915	55-	.5	.64945	.85408	1.1708	.76041	.5
.1	.57501	.70281	1.4229	.81815	.9	.6	.65077	.85710	1.1667	.75927	.4
.2	.57643	.70542	1.4176	.81714	.8	.7	.65210	.86014	1.1626	.75813	.3
.3	.57786	.70804	1.4124	.81614	.7	.8	.65342	.86318	1.1585	.75700	.2
.4	.57928	.71066	1.4071	.81513	.6	.9	.65474	.86623	1.1544	.75585	.1
.5	.58070	.71329	1.4019	.81412	.5	41-	.65606	.86929	1.1504	.75471	40-
.6	.58212	.71593	1.3968	.81310	.4	.1	.65738	.87236	1.1463	.75356	.9
.7	.58354	.71857	1.3916	.81208	.3	.2	.65869	.87543	1.1423	.75241	.8
.8	.58496	.72122	1.3865	.81106	.2	.3	.66000	.87852	1.1383	.75126	.7
.9	.58637	.72388	1.3814	.81004	.1	.4	.66131	.88162	1.1343	.75011	.6
36-	.58779	.72654	1.3764	.80902	54-	.5	.66262	.88473	1.1303	.74896	.5
.1	.58920	.72921	1.3713	.80799	.9	.6	.66393	.88784	1.1263	.74780	.4
.2	.59061	.73189	1.3663	.80696	.8	.7	.66523	.89097	1.1224	.74664	.3
.3	.59201	.73457	1.3613	.80593	.7	.8	.66653	.89410	1.1184	.74548	.2
.4	.59342	.73726	1.3564	.80489	.6	.9	.66783	.89725	1.1145	.74431	.1
.5	.59482	.73996	1.3514	.80386	.5	42-	.66913	.90040	1.1106	.74314	40-
.6	.59622	.74267	1.3465	.80282	.4	.1	.67043	.90357	1.1067	.74198	.9
.7	.59763	.74538	1.3416	.80178	.3	.2	.67172	.90674	1.1028	.74080	.8
.8	.59902	.74810	1.3367	.80073	.2	.3	.67301	.90993	1.0990	.73963	.7
.9	.60042	.75082	1.3319	.79968	.1	.4	.67430	.91313	1.0951	.73846	.6
37-	.60182	.75355	1.3270	.79864	53-	.5	.67559	.91633	1.0913	.73728	.5
.1	.60321	.75629	1.3222	.79758	.9	.6	.67688	.91955	1.0875	.73610	.4
.2	.60460	.75904	1.3175	.79653	.8	.7	.67816	.92277	1.0837	.73491	.3
.3	.60599	.76180	1.3127	.79547	.7	.8	.67944	.92601	1.0799	.73373	.2
.4	.60738	.76456	1.3079	.79441	.6	.9	.68072	.92926	1.0761	.73254	.1
.5	.60876	.76733	1.3032	.79335	.5	43-	.68200	.93252	1.0724	.73135	47-
.6	.61015	.77010	1.2985	.79229	.4	.1	.68327	.93578	1.0686	.73016	.9
.7	.61153	.77289	1.2938	.79122	.3	.2	.68455	.93906	1.0649	.72897	.8
.8	.61291	.77568	1.2892	.79016	.2	.3	.68582	.94235	1.0612	.72777	.7
.9	.61429	.77848	1.2846	.78908	.1	.4	.68709	.94565	1.0575	.72657	.6
38-	.61566	.78129	1.2799	.78801	52-	.5	.68835	.94896	1.0538	.72537	.5
.1	.61704	.78410	1.2753	.78694	.9	.6	.68962	.95229	1.0501	.72417	.4
.2	.61841	.78692	1.2708	.78586	.8	.7	.69088	.95562	1.0464	.72297	.3
.3	.61978	.78975	1.2662	.78478	.7	.8	.69214	.95897	1.0428	.72176	.2
.4	.62115	.79259	1.2617	.78369	.6	.9	.69340	.96232	1.0392	.72055	.1
.5	.62251	.79544	1.2572	.78261	.5	44-	.69466	.96569	1.0355	.71934	40-
.6	.62388	.79829	1.2527	.78152	.4	.1	.69591	.96907	1.0319	.71813	.9
.7	.62524	.80115	1.2482	.78043	.3	.2	.69717	.97246	1.0283	.71691	.8
.8	.62660	.80402	1.2437	.77934	.2	.3	.69842	.97586	1.0247	.71569	.7
.9	.62796	.80690	1.2393	.77824	.1	.4	.69966	.97927	1.0212	.71447	.6
39-	.62932	.80978	1.2349	.77715	51-	.5	.70091	.98270	1.0176	.71325	.5
.1	.63068	.81268	1.2305	.77605	.9	.6	.70215	.98613	1.0141	.71203	.4
.2	.63203	.81558	1.2261	.77494	.8	.7	.70339	.98958	1.0105	.71080	.3
.3	.63338	.81849	1.2218	.77384	.7	.8	.70463	.99304	1.0070	.70957	.2
.4	.63473	.82141	1.2174	.77273	.6	.9	.70587	.99652	1.0035	.70834	.1
39.5	.63608	.82434	1.2131	.77162	50.5	45-	.70711	1.00000	1.0000	.70711	45-
Deg.	Cos	Ctn	Tan	Sin	Deg.	Deg.	Cos	Ctn	Tan	Sin	Deg.

Natural Trigonometric Functions for Decimal Fractions of a Degree

TABLE 3
1.00 — Four-Place Natural Logarithms — 5.59

<i>N</i>	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.0	0.0000	0.0100	0.0198	0.0296	0.0392	0.0488	0.0583	0.0677	0.0770	0.0862
1.1	0.0953	0.1044	0.1133	0.1222	0.1310	0.1398	0.1484	0.1570	0.1655	0.1740
1.2	0.1823	0.1906	0.1989	0.2070	0.2151	0.2231	0.2311	0.2390	0.2469	0.2546
1.3	0.2624	0.2700	0.2776	0.2852	0.2927	0.3001	0.3075	0.3148	0.3221	0.3293
1.4	0.3365	0.3436	0.3507	0.3577	0.3646	0.3716	0.3784	0.3853	0.3920	0.3988
1.5	0.4055	0.4121	0.4187	0.4253	0.4318	0.4383	0.4447	0.4511	0.4574	0.4637
1.6	0.4700	0.4762	0.4824	0.4886	0.4947	0.5008	0.5068	0.5128	0.5188	0.5247
1.7	0.5306	0.5365	0.5423	0.5481	0.5539	0.5596	0.5653	0.5710	0.5766	0.5822
1.8	0.5878	0.5933	0.5988	0.6043	0.6098	0.6152	0.6206	0.6259	0.6313	0.6366
1.9	0.6419	0.6471	0.6523	0.6575	0.6627	0.6678	0.6729	0.6780	0.6831	0.6881
2.0	0.6931	0.6981	0.7031	0.7080	0.7129	0.7178	0.7227	0.7275	0.7324	0.7372
2.1	0.7419	0.7467	0.7514	0.7561	0.7608	0.7655	0.7701	0.7747	0.7793	0.7839
2.2	0.7885	0.7930	0.7975	0.8020	0.8065	0.8109	0.8154	0.8198	0.8242	0.8286
2.3	0.8329	0.8372	0.8416	0.8459	0.8502	0.8544	0.8587	0.8629	0.8671	0.8713
2.4	0.8755	0.8796	0.8838	0.8879	0.8920	0.8961	0.9002	0.9042	0.9083	0.9123
2.5	0.9163	0.9203	0.9243	0.9282	0.9322	0.9361	0.9400	0.9439	0.9478	0.9517
2.6	0.9555	0.9594	0.9632	0.9670	0.9708	0.9746	0.9783	0.9821	0.9858	0.9895
2.7	0.9933	0.9969	1.0006	1.0043	1.0080	1.0116	1.0152	1.0188	1.0225	1.0260
2.8	1.0296	1.0332	1.0367	1.0403	1.0438	1.0473	1.0508	1.0543	1.0578	1.0613
2.9	1.0647	1.0682	1.0716	1.0750	1.0784	1.0818	1.0852	1.0886	1.0919	1.0953
3.0	1.0986	1.1019	1.1053	1.1086	1.1119	1.1151	1.1184	1.1217	1.1249	1.1282
3.1	1.1314	1.1346	1.1378	1.1410	1.1442	1.1474	1.1506	1.1537	1.1569	1.1600
3.2	1.1632	1.1663	1.1694	1.1725	1.1756	1.1787	1.1817	1.1848	1.1878	1.1909
3.3	1.1939	1.1969	1.2000	1.2030	1.2060	1.2090	1.2119	1.2149	1.2179	1.2208
3.4	1.2238	1.2267	1.2296	1.2326	1.2355	1.2384	1.2413	1.2442	1.2470	1.2499
3.5	1.2528	1.2556	1.2585	1.2613	1.2641	1.2669	1.2698	1.2726	1.2754	1.2782
3.6	1.2809	1.2837	1.2865	1.2892	1.2920	1.2947	1.2975	1.3002	1.3029	1.3056
3.7	1.3083	1.3110	1.3137	1.3164	1.3191	1.3218	1.3244	1.3271	1.3297	1.3324
3.8	1.3350	1.3376	1.3403	1.3429	1.3455	1.3481	1.3507	1.3533	1.3558	1.3584
3.9	1.3610	1.3635	1.3661	1.3686	1.3712	1.3737	1.3762	1.3788	1.3813	1.3838
4.0	1.3863	1.3888	1.3913	1.3938	1.3962	1.3987	1.4012	1.4036	1.4061	1.4085
4.1	1.4110	1.4134	1.4159	1.4183	1.4207	1.4231	1.4255	1.4279	1.4303	1.4327
4.2	1.4351	1.4375	1.4398	1.4422	1.4446	1.4469	1.4493	1.4516	1.4540	1.4563
4.3	1.4586	1.4609	1.4633	1.4656	1.4679	1.4702	1.4725	1.4748	1.4770	1.4793
4.4	1.4816	1.4839	1.4861	1.4884	1.4907	1.4929	1.4951	1.4974	1.4996	1.5019
4.5	1.5041	1.5063	1.5085	1.5107	1.5129	1.5151	1.5173	1.5195	1.5217	1.5239
4.6	1.5261	1.5282	1.5304	1.5326	1.5347	1.5369	1.5390	1.5412	1.5433	1.5454
4.7	1.5476	1.5497	1.5518	1.5539	1.5560	1.5581	1.5602	1.5623	1.5644	1.5665
4.8	1.5686	1.5707	1.5728	1.5748	1.5769	1.5790	1.5810	1.5831	1.5851	1.5872
4.9	1.5892	1.5913	1.5933	1.5953	1.5974	1.5994	1.6014	1.6034	1.6054	1.6074
5.0	1.6094	1.6114	1.6134	1.6154	1.6174	1.6194	1.6214	1.6233	1.6253	1.6273
5.1	1.6292	1.6312	1.6332	1.6351	1.6371	1.6390	1.6409	1.6429	1.6448	1.6467
5.2	1.6487	1.6506	1.6525	1.6544	1.6563	1.6582	1.6601	1.6620	1.6639	1.6658
5.3	1.6677	1.6696	1.6715	1.6734	1.6752	1.6771	1.6790	1.6808	1.6827	1.6845
5.4	1.6864	1.6882	1.6901	1.6919	1.6938	1.6956	1.6974	1.6993	1.7011	1.7029
5.5	1.7047	1.7066	1.7084	1.7102	1.7120	1.7138	1.7156	1.7174	1.7192	1.7210
<i>N</i>	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09

$\log_e .1 = .6974-3$ $\log_e .01 = .3948-5$ $\log_e .001 = .0922-7$

1.00 — Four-Place Natural Logarithms — 5.59

Table 3

5.50 — Four-Place Natural Logarithms — 10.09

<i>N</i>	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
5.5	1.7047	1.7066	1.7084	1.7102	1.7120	1.7138	1.7156	1.7174	1.7192	1.7210
5.6	1.7228	1.7246	1.7263	1.7281	1.7299	1.7317	1.7334	1.7352	1.7370	1.7387
5.7	1.7405	1.7422	1.7440	1.7457	1.7475	1.7492	1.7509	1.7527	1.7544	1.7561
5.8	1.7579	1.7596	1.7613	1.7630	1.7647	1.7664	1.7681	1.7699	1.7716	1.7733
5.9	1.7750	1.7766	1.7783	1.7800	1.7817	1.7834	1.7851	1.7867	1.7884	1.7901
6.0	1.7918	1.7934	1.7951	1.7967	1.7984	1.8001	1.8017	1.8034	1.8050	1.8066
6.1	1.8083	1.8099	1.8116	1.8132	1.8148	1.8165	1.8181	1.8197	1.8213	1.8229
6.2	1.8245	1.8262	1.8278	1.8294	1.8310	1.8326	1.8342	1.8358	1.8374	1.8390
6.3	1.8405	1.8421	1.8437	1.8453	1.8469	1.8485	1.8500	1.8516	1.8532	1.8547
6.4	1.8563	1.8579	1.8594	1.8610	1.8625	1.8641	1.8656	1.8672	1.8687	1.8703
6.5	1.8718	1.8733	1.8749	1.8764	1.8779	1.8795	1.8810	1.8825	1.8840	1.8856
6.6	1.8871	1.8886	1.8901	1.8916	1.8931	1.8946	1.8961	1.8976	1.8991	1.9006
6.7	1.9021	1.9036	1.9051	1.9066	1.9081	1.9095	1.9110	1.9125	1.9140	1.9155
6.8	1.9169	1.9184	1.9199	1.9213	1.9228	1.9242	1.9257	1.9272	1.9286	1.9301
6.9	1.9315	1.9330	1.9344	1.9359	1.9373	1.9387	1.9402	1.9416	1.9430	1.9445
7.0	1.9459	1.9473	1.9488	1.9502	1.9516	1.9530	1.9544	1.9559	1.9573	1.9587
7.1	1.9601	1.9615	1.9629	1.9643	1.9657	1.9671	1.9685	1.9699	1.9713	1.9727
7.2	1.9741	1.9755	1.9769	1.9782	1.9796	1.9810	1.9824	1.9838	1.9851	1.9865
7.3	1.9879	1.9892	1.9906	1.9920	1.9933	1.9947	1.9961	1.9974	1.9988	2.0001
7.4	2.0015	2.0028	2.0042	2.0055	2.0069	2.0082	2.0096	2.0109	2.0122	2.0136
7.5	2.0149	2.0162	2.0176	2.0189	2.0202	2.0215	2.0229	2.0242	2.0255	2.0268
7.6	2.0281	2.0295	2.0308	2.0321	2.0334	2.0347	2.0360	2.0373	2.0386	2.0399
7.7	2.0412	2.0425	2.0438	2.0451	2.0464	2.0477	2.0490	2.0503	2.0516	2.0528
7.8	2.0541	2.0554	2.0567	2.0580	2.0592	2.0605	2.0618	2.0631	2.0643	2.0656
7.9	2.0669	2.0681	2.0694	2.0707	2.0719	2.0732	2.0744	2.0757	2.0769	2.0782
8.0	2.0794	2.0807	2.0819	2.0832	2.0844	2.0857	2.0869	2.0882	2.0894	2.0906
8.1	2.0919	2.0931	2.0943	2.0956	2.0968	2.0980	2.0992	2.1005	2.1017	2.1029
8.2	2.1041	2.1054	2.1066	2.1078	2.1090	2.1102	2.1114	2.1126	2.1138	2.1150
8.3	2.1163	2.1175	2.1187	2.1199	2.1211	2.1223	2.1235	2.1247	2.1258	2.1270
8.4	2.1282	2.1294	2.1306	2.1318	2.1330	2.1342	2.1353	2.1365	2.1377	2.1389
8.5	2.1401	2.1412	2.1424	2.1436	2.1448	2.1459	2.1471	2.1483	2.1494	2.1506
8.6	2.1518	2.1529	2.1541	2.1552	2.1564	2.1576	2.1587	2.1599	2.1610	2.1622
8.7	2.1633	2.1645	2.1656	2.1668	2.1679	2.1691	2.1702	2.1713	2.1725	2.1736
8.8	2.1748	2.1759	2.1770	2.1782	2.1793	2.1804	2.1815	2.1827	2.1838	2.1849
8.9	2.1861	2.1872	2.1883	2.1894	2.1905	2.1917	2.1928	2.1939	2.1950	2.1961
9.0	2.1972	2.1983	2.1994	2.2006	2.2017	2.2028	2.2039	2.2050	2.2061	2.2072
9.1	2.2083	2.2094	2.2105	2.2116	2.2127	2.2138	2.2148	2.2159	2.2170	2.2181
9.2	2.2192	2.2203	2.2214	2.2225	2.2235	2.2246	2.2257	2.2268	2.2279	2.2289
9.3	2.2300	2.2311	2.2322	2.2332	2.2343	2.2354	2.2364	2.2375	2.2386	2.2396
9.4	2.2407	2.2418	2.2428	2.2439	2.2450	2.2460	2.2471	2.2481	2.2492	2.2502
9.5	2.2513	2.2523	2.2534	2.2544	2.2555	2.2565	2.2576	2.2586	2.2597	2.2607
9.6	2.2618	2.2628	2.2638	2.2649	2.2659	2.2670	2.2680	2.2690	2.2701	2.2711
9.7	2.2721	2.2732	2.2742	2.2752	2.2762	2.2773	2.2783	2.2793	2.2803	2.2814
9.8	2.2824	2.2834	2.2844	2.2854	2.2865	2.2875	2.2885	2.2895	2.2905	2.2915
9.9	2.2925	2.2935	2.2946	2.2956	2.2966	2.2976	2.2986	2.2996	2.3006	2.3016
10.0	2.3026	2.3036	2.3046	2.3056	2.3066	2.3076	2.3086	2.3096	2.3106	2.3115
<i>N</i>	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09

$\log_e .0001 = .7897-10$ $\log_e .00001 = .4871-12$ $\log_e .000001 = .1845-14$

5.50 — Four-Place Natural Logarithms — 10.09

TABLE 4

Areas of the Normal Probability Curve

$$\int_0^u \phi(t) dt$$

<i>u</i>	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4773	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4983	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4989	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.5000
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
<i>u</i>	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09

Areas of the Normal Probability Curve

TABLE 5
Student's t-Distribution

Degrees of Freedom	5 per Cent	1 per Cent
1	12.706	63.657
2	4.303	9.925
3	3.182	5.841
4	2.776	4.604
5	2.571	4.032
6	2.447	3.707
7	2.365	3.499
8	2.306	3.355
9	2.262	3.250
10	2.228	3.169
11	2.201	3.106
12	2.179	3.055
13	2.160	3.012
14	2.145	2.977
15	2.131	2.947
16	2.120	2.921
17	2.110	2.898
18	2.101	2.878
19	2.093	2.861
20	2.086	2.845
21	2.080	2.831
22	2.074	2.819
23	2.069	2.807
24	2.064	2.797
25	2.060	2.787
26	2.056	2.779
27	2.052	2.771
28	2.048	2.763
29	2.045	2.756
30	2.042	2.750
40	2.021	2.704
50	2.008	2.678
Degrees of Freedom	5 per Cent	1 per Cent

Student's t-Distribution

TABLE 6

χ^2 Probability Scale

Values of χ^2 Corresponding to Certain Chances of Exceeding χ^2

Degrees of Freedom	P = .90	.70	.50	.30	.20	.10	.05	.02	.01
1	0.02	0.15	0.45	1.07	1.64	2.71	3.84	5.41	6.63
2	0.21	0.71	1.39	2.41	3.22	4.60	5.99	7.82	9.21
3	0.58	1.42	2.37	3.66	4.64	6.25	7.81	9.84	11.34
4	1.06	2.19	3.36	4.88	5.99	7.78	9.49	11.67	13.28
5	1.61	3.00	4.35	6.06	7.29	9.24	11.07	13.39	15.09
6	2.20	3.83	5.35	7.23	8.56	10.64	12.59	15.03	16.81
7	2.83	4.67	6.35	8.38	9.80	12.02	14.07	16.62	18.47
8	3.49	5.53	7.34	9.52	11.03	13.36	15.51	18.17	20.09
9	4.17	6.39	8.34	10.66	12.24	14.68	16.92	19.68	21.67
10	4.86	7.27	9.34	11.78	13.44	15.99	18.31	21.16	23.21
11	5.58	8.15	10.34	12.90	14.63	17.27	19.67	22.62	24.72
12	6.30	9.03	11.34	14.01	15.81	18.55	21.03	24.05	26.22
13	7.04	9.93	12.34	15.12	16.98	19.81	22.36	25.47	27.69
14	7.79	10.82	13.34	16.22	18.15	21.06	23.68	26.87	29.14
15	8.55	11.72	14.34	17.32	19.31	22.31	25.00	28.26	30.58
16	9.31	12.62	15.34	18.42	20.46	23.54	26.30	29.63	32.00
17	10.08	13.53	16.34	19.51	21.61	24.77	27.59	30.99	33.41
18	10.86	14.44	17.34	20.60	22.76	25.99	28.87	32.35	34.80
19	11.65	15.35	18.34	21.69	23.90	27.20	30.14	33.69	36.19
20	12.44	16.27	19.34	22.77	25.04	28.41	31.41	35.02	37.57
21	13.24	17.18	20.34	23.86	26.17	29.61	32.67	36.34	38.93
22	14.04	18.10	21.34	24.94	27.30	30.81	33.92	37.66	40.29
23	14.85	19.02	22.34	26.02	28.43	32.01	35.17	38.97	41.64
24	15.66	19.94	23.34	27.10	29.55	33.20	36.41	40.27	42.98
25	16.47	20.87	24.34	28.17	30.67	34.38	37.65	41.57	44.31
26	17.29	21.79	25.34	29.25	31.79	35.56	38.88	42.86	45.64
27	18.11	22.72	26.34	30.32	32.91	36.74	40.11	44.14	46.96
28	18.94	23.65	27.34	31.39	34.03	37.92	41.34	45.42	48.28
29	19.77	24.58	28.24	32.46	35.14	39.09	42.56	46.69	49.59
30	20.60	25.51	29.34	33.53	36.25	40.26	43.77	47.96	50.89
Degrees of Freedom	P = .90	.70	.50	.30	.20	.10	.05	.02	.01

χ^2 Probability Scale

For larger degrees of freedom, let $t = \sqrt{2\chi^2} - \sqrt{2n - 1}$ where $n =$ degrees of freedom. Then, approximately,

$$P = \frac{1}{2} - \int_0^t \varphi(t) dt$$

and Table 25 may be used.

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