Closure for the Sraffa System: A Theory of Value, Distribution and Growth

Rajas Parchure

Gokhale Institute of Politics and Economics,
Pune 411 004, India

I Introductory

The purpose of this paper is to pursue the implications of closing, in the sense of rendering determinate, the Sraffa equations of production, prices and distribution [Sraffa (1960)] by adding to them demand equations of the type estimated in Stone's (1954) linear expenditure system. The motivation for the inquiry is this: it seems only natural that the closure for a system of production and price equations that have been formulated on objectivist foundations should be sought in a system of demand equations that has been designed to estimate observed consumer behaviour because it holds the promise of formulating a general equilibrium theory with empirical foundations.[1]

II Supply and Demand

Consider an economy that is described by the Sraffa system;

$$\begin{aligned}
&(A_{11}P_1 + A_{21}P_{21} + ... + A_{n1}P_n)(1+r) + wL_1 = P_1B_{1s}\\
&(A_{12}P_1 + A_{22}P_2 + ... + A_{n2}P_n)(1+r) + wL_2 = P_2B_{2s} \qquad \text{... (1)}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(A_{1n}P_1 + A_{2n}P_2 + ... + A_{nn}P_n)(1+r) + wL_n = P_nB_{ns}
\end{aligned}$$

where A_{ji} are the inputs, L_i the labour requirements, B_{is} the outputs produced, P_i the prices and w and r the wage rate and the rate of profit respectively. One of the prices *a'lá'* Ricardo Walras or the wage *a'lá'* Smith-Keynes may be fixed as the measure of value. Let us suppose w = 1 and measure all prices and values in units of "labour commanded".[2]

The capital stock, gross national product and net national product of the economy in (1) measured in units of labour commanded are:

$$K = \sum_i \sum_j A_{ji}P_j \qquad \text{... 2(a)}$$

$$G = K(1+r) + L = \sum P_iB_{is} \qquad \text{... 2(b)}$$

$$Y = rK + L = \sum P_iF_i \qquad \text{... 2(c)}$$

where $F_i = B_{is} - A_i, A_i = \sum_j A_{ji}, \quad L = \sum_i L_i$. If a commodity serves purely as a final consumption good $A_i = 0$ and $F_i = B_{is}$. The net national income is divided in a two-fold way, between wages and profits and between workers and capitalists. If *s* is the proportion of capital stock K owned by capitalists, the incomes of the two classes are,

$$Y_k = srK \qquad \ldots 3(a)$$

$$Y_L = (1-s)rK \qquad \ldots 3(b)$$

A viable economy is one that replaces the capital stock and uses the net national income for final consumption and new investment. Final consumption behaviour will be described by the linear expenditure system which has the properties of homogeneity, additivity and symmetry that are supposed to be desirable from the standpoint of consumption theory. [Stone (1954), Deaton and Muellbauer (1980)]. Accordingly, the consumption demand equation of capitalists and workers are,

$$\begin{aligned} P_i B_{iK} &= P_i B_{iKo} + a_i \, (Y_k - \sum P_i B_{iKo}) \\ P_i B_{iL} &= P_i B_{iLo} + b_i \, (Y_L - \sum P_i B_{iLo}) \end{aligned} \qquad \ldots (4)$$

where B_{iKo}, B_{iLo} are the "committed" quantities and the second terms represent "supernumerary" quantities that reflect the tastes and preferences of the two classes.[3] Equations (4) are subject to the restrictions that $\Sigma a_i = a < 1$ $(a_i \geq 0)$ and $\Sigma\, b_i = b < 1$ $(b_i \geq 0)$ and are applicable only to the goods that serve for purposes of final consumption.

The saving by the two classes are,

$$\begin{aligned} S_K &= (1-a)\,(Y_K - B_{KO}) \\ S_L &= (1-b)\,(Y_L - B_{LO}) \end{aligned} \qquad \ldots (5)$$

where $B_{KO} = \sum P_i B_{iKO}, B_{LO} = \sum P_i B_{iLO}$. The long-run ownership ratio s is determined by Pasinetti's (1962) equation,

$$\frac{s}{1-s} = \frac{S_K}{S_L} = \frac{(1-a)(Y_K - B_{KO})}{(1-b)(Y_L - B_{LO})} \qquad \ldots (6)$$

Substituting from (3) into (6) gives a quadratic equation to determines,

$$[(b-a)rK)]\, s^2 + [(a-b)rK + (1-b))\,(L - B_{LO}) - (1-a)\,B_{KO}]\, s + (1-a)\,B_{KO} = 0 \ldots (7)$$

whose greater positive root (lower than unity) is the required solution. Given *s* and equation (4), it becomes possible to determine the quantities demanded for final consumption,

$$B_{id} = B_{iK} + B_{iL} = B_{iKO} + B_{iLO} + \frac{a_i\,(Y_K - B_{KO}) + b_i\,(Y_L - B_{LO})}{P_i} \qquad \ldots (8)$$

The demand equations for the capital goods must be derived by considering the manner in which saving is allocated between the industries and further between the capital goods that constitute the capital clock of each industry. It will be assumed that production takes place by means of a fixed-coefficients constant-returns-to-scale technology. This assumption implies that the allocation of saving will be made by a simple proportionate rule; the new investment in industry i will be

$$\Delta K_i = (\frac{K_i}{K}) S = (\frac{\sum_j A_{ji} P_j}{K}) S$$

And within each industry there will be a second round of allocation between the individual capital goods A_{ji} which too is on proportionate basis,

$$(\Delta A_{ji}) P_j = (\frac{A_{ji} P_j}{K_L}) \Delta K_i$$

$$= (\frac{A_{ji} P_j}{K_i}) (\frac{K_i}{K}) S$$

Adding up the investment demands for capital good j across all the industries gives the aggregate new investment demand commodity j,

$$\Delta A_{jd} = \Sigma_i \Delta A_{ji} \qquad \ldots (9)$$

Thus the aggregate demand for a commodity is the sum of the economy-wide replacement demand, new investment demand and final consumption demand (if it serves in a dual capacity).

$$B_{jd} = A_{jd} + \Delta A_{jd} + B_{jF} \qquad \ldots (10)$$

If it is purely a capital good $B_{jF} = 0$ and if it is purely a consumption good $B_{jd} = B_{jF}$ Starting from an arbitrary supply situation B_{is} in equation (1) we have come around in a full circle through the size and distribution of income, the allocation to income to consumption and saving and their allocation to the individual consumption and capital goods to the corresponding demand situation in equations (8) and (10). It is only natural to inquire as to how the equilibrium with supplies equal to demands is to be brought about.

III Equilibrium

To bring about the equilibrium the quantities supplied must be changed in the direction of the quantities demanded. However, any attempt to do so would immediately cause a change in the size and distribution of income and consequently a change in the quantities demanded themselves. Thus, the procedure to find the equilibrium must be iterative and convergent.

The first step in the procedure is to clear a difficulty with respect to the demands and supplies of commodities that serve as capital goods. The quantities of capital goods available for new investment in equations (1) are $F_i = B_{is} - A_i$. In a situation of disequilibrium these imply different "own rates of growth" for each capital good.

$$g_i = \frac{F}{A_i} \qquad \ldots (11)$$

Capital goods showing low own rates will be in excess demand and those with high own rates will be in excess supply. But the assumption of a fixed-coefficient constant-returns technology implies that a situation in which some capital goods are in excess supply and others in excess demand will result in wastages and/or bottlenecks that market forces must eliminate. In effect these market forces should bring about a situation in the capital goods industries in which the gross outputs B_{is}, net of final consumption demand B_{iF} if any, are such as to meet the replacement demand requirements from all industries in the economy and leave a surplus that equalizes all the own rates of growth $g_i = g$; i.e.

$$\frac{B_{is}X_i - B_{iF}}{\Sigma A_{ij}X_i + A_{iC}} = (1 + \mathrm{g}) \qquad \ldots (12)$$

where X_i are the scale factors (or "process intensities"), $\Sigma A_{ij}X_i$ is the replacement demand for capital good i from all capital goods industries and A_{iC} is the replacement demand for purely consumption goods industries. The equations in (12) are special form of the demand equations in (10) for the case of a uniform rate of growth $\Delta A_{jd} / A_{jd} = g$. If we suppose that there are m commodities that serve as capital goods equation (12) can be written as a system of equations,

$$\begin{aligned}
&(A_{11}X_1 + A_{12}X_2 + \ldots + A_{1m}X_m + A_{1C})(1+g) + B_{1F} = B_{1S}X_1 \\
&(A_{21}X_1 + A_{22}X_2 + \ldots + A_{2m}X_m + A_{2c})(1+g) + B_{2F} = B_{2s}X_2 \qquad \ldots 13(a) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&(A_{m1}X_1 + A_{m2}X_2 + \ldots + A_{mm}X_m + A_{mc})(1+g) + B_{mF} = B_{ms}X_m
\end{aligned}$$

containing m equations in m + 1 unknowns, viz. the scale factors, $X_1, \ldots X_m$ and the rate of growth g. To these we may add the equation

$$L_1X_1 + L_2X_2 + \ldots + L_mX_m = \sum_{i=1}^{m} L_i = L_k \qquad \ldots 13(b)$$

which states that the primary factor, labour, that is employed in the capital goods industries will be reallocated between those industries to ensure that the outputs of capital goods $B_{is}X_i$ are in line with the replacement, new investment and final consumption demands for them.

The resemblance of the system of equations (13) with Sraffa's standard system is striking. In fact, the standard system represents an extreme version of (13) in which the outputs of consumption goods are zero. By itself the system (13) stands in a relationship of duality to the system of price equations (1)[5].

The next step in the procedure is to equalize the supplies of the consumption goods to the demands for them. This is done by plugging in the quantities demanded of consumption goods that serve in dual capacity into the B_{iF} appearing in (13) and for the pure consumption goods by finding their input requirements A_{iC} and substituting them in (13). What is needed is a link between the primal price system (1) and the dual output system (13) since quantities demanded can be calculated only when supply prices are known. This link is given by the equation defining the relationship of the rate of profit and the rate of growth.

$$g = \frac{(1-a)\left[srK - B_{KO}\right] + (1-b)\left[L + (1-s)rK - B_{LO}\right]}{k} \qquad \ldots (14)$$

better read as

$$r = \frac{gK - (1-b)(L - B_{LO}) + (1-a)B_{KO} + (1-b)B_{LO}}{(1-a)sK + (1-b)(1-s)K} \qquad \ldots (15)$$

The precise procedure is codified in the algorithm presented below.

IV Algorithm

The primary unknowns of the system are the n prices, n industrial outputs, the ownership share s, the rate of profit r and the growth rate g, i.e. 2n + 3 in all. Suppose that of the n goods, k are purely capital goods, d serve in dual capacity and c are purely consumption goods (k + d +c = n). It will be supposed that there is at least one good that serves purely as a capital good and at least one good that serves purely as consumption good. Then, on the side of equations there are n price equations (1), 1 ownership share equation (7), k + d + 1 equations in (13), n-k consumption demand equations in (4) and the growth-profit equation (15), a total of 2n + 3 equations. However, one of these will be dependent since, by Walras's Law, if n-1 markets are cleared the n^{th} must be, so there are only 2n + 2 independent equations. The required equation is to fix the total quantity of employment in economy L_E,

$$\sum_{i=1}^{n} L_i = L_E \qquad \ldots (16)$$

Since all the equations are *non-linear*, the equilibrium must be determined by an iterative algorithm. A convenient algorithm is as follows:

Step 1 : Start with an arbitrary value of r < R and determine the prices in equations (1), K from 2(a) and B_{KO} and B_{LO} in equations (4). The lower bound $r_{\min}$ is explained in the following section.

Step 2 : Solve the owernship ratio s from equation (7) with value of L fixed at L_E in (16) and find Y_K, Y_L in (3) and B_{id} in (8).

Step 3 : For the consumption goods industries, the bisection method may be used to set the new levels of outputs;

$$B_{iF}^{(1)} = \frac{B_{is} + B_{id}}{2}$$

Calculate their input requirements so as to calculate the aggregate requirement for each capital good A_{ic} .

Step 4: Feed the value of A_{ic} and $B_{iF}^{(1)}$ in equation (13) and solve for g and X_i.

Step 5: Obtain a new trial value of the rate of profit r^1 from equation (16) corresponding to the values of g, S, K, B_{KO}, B_{LO} found in the current iteration.

Step 6: Multiply the price equations of k + d capital goods in 1 by X_i and set the production levels of the pure consumption goods at $B_{iF}^{(1)}$ to obtain the new price system and find the prices corresponding to r^1.

Step 7: Repeat steps 1 to 6 until $B_{is} - B_{id} \approx 0$ and $X_i \approx 1$. Ascertain the primary unknowns and the secondary ones including the GNP, NNP, real wage rate(s), capital output ratio, etc.

V Viable Solutions

The system of equations gives an economically meaningful solution if the coefficients satisfy two sets of conditions; conditions relating to technology and conditions relating to the postulated consumption behaviour. The former set of conditions is well known viz, the matrix of technical coefficients augmented to include the fixed quantities of consumption B_{io} must satisfy the Hawkins-Simon (1949) conditions. Of course, for practical purposes the conditions given by Solow (1952) viz. $B_{is} \geq \Sigma A_{ij} + B_{io}$ with strict inequality holding for some *i* would usually suffice.

The second set of conditions pertain to the consumption behaviour described by the linear expenditure system. Consider equation (7) which solves for the ownership share whose value must lie between 0 and 1. It is well worth the while to deconstruct it piece by piece. Suppose for a moment that $B_{KO} = B_{LO} = 0$, a < 1, b = 1 i.e. workers do not save. Then the equation gives two solutions $s = 0$ and $s = 1$ of which the greater $s = 1$ (capitalists own the entire capital stock) is the relevant solution. Next, suppose $B_{1KO} = B_{LO} = 0$ but a, b < 1 in which case the root

$$s = 1 - \frac{(1-b)L}{(b-a)rK}$$

is the solution provided s lies between 0 and 1, i.e. provided b > a and (b-a) > (1-b)L/rK. The former is the well-known Kaldor-Pasinetti stability condition [Kaldor (1956), Pasinetti (1962)]. If $B_{KO} = 0$, $B_{LO} > 0$, a, b < 1, there is no problem except that $L > B_{LO}$ is an additional condition that must be met. In other words if $B_{LO} > 0$, there is a lower bound to the wage rate B_{LO}/L and meaningful solutions are obtained only if $w > B_{LO}/L$ (i.e. $1 > B_{LO}/L$ since we have chosen the wage rate as numeraire).

Incidentally, in all the cases considered so far the Cambridge equation (called the Pasinetti paradox by Samuelson and Modigliani ( ) is found to hold. Substituting

$$s = 1 - \frac{(1-b)(L-B_{LO})}{(b-a)rK} = 1-e \qquad \ldots (17)$$

into equation (15) gives, for $B_{LO} = 0$, the Cambridge equation

$$r = \frac{g}{1-a} \qquad \ldots (18)$$

If $B_{KO} > 0$ additional restrictions must be placed. To begin with, there must now be a lower bound for the rate of profit below which it cannot lie since $srK > B_{KO,}$ which give the value of r_{min} in Step 1 of the algorithm. Next, the discriminant of equation (7) must be positive for the equation to have real solutions. Dividing equation (7) by the leading coefficient gives

$$s^2 + (-1 + e - f)s + f = 0 \qquad \ldots (19)$$

The discriminant $D = (-1 + e - f)^2 - 4f > 0$. Further, since the leading coefficient must be positive (Kaldor-Pasinetti condition) and the third coefficient is positive (a < 1, $B_{KO} > 0$), the second coefficient must be negative for the solution of s to be positive, which it will be because e < 1. But then, because the coefficients alternate in sign twice both roots will be positive and the greater root, which is the relevant solution, must be less than unity. The condition for that is e > 0. All in all, the most restrictive of these conditions is that the discriminant be positive and that happens only if the value of B_{KO} in relation to K is sufficiently small.

VI Numerical Illustrations

Consider an economic system that is in the following state

$$\begin{aligned}
&(2P_1 + 5P_2 + 3P_3)(1+r) + 5w = 20P_1 \\
&(5P_1 + 7P_2 + 5P_3)(1+r) + 5w = 25P_2 \\
&(1P_1 + 4P_2 + 1P_3)(1+r) + 10w = 30P_3 \\
&(3P_1 + 2P_2 + 5P_3)(1-r) + 10w = 60P_4 \\
&(2P_1 + 5P_2 + 7P_3)(1+r) + 10w = 50P_5
\end{aligned} \qquad \ldots (20)$$

13 23 21

where commodities 1, 2, 3 are basic goods and 4 and 5 are purely consumption goods, and $L_I = 20$ and $L_E = 40$. The parameters of the linear expenditure system are as follows:

$$B_{4KO} = 0.2 \quad a_4 = 0.05 \quad B_{5KO} = 0.1 \quad a_5 = 0.05$$
$$B_{4LO} = 2 \quad b_4 = 0.7 \quad B_{5LO} = 1 \quad b_5 = 0.2$$

i.e. a = 0.1, b = 0.9. Following the algorithm of section 4, the system gives the solution whose convergent path is reported in Tables 1(a), 1(b) and 1(c) below.

Table 1(a): Conversion of Rate of Profit, Ownership Ratio, Growth and Multipliers

Iteration No.	r	μ	g	X_1	X_2	X_3
0	0.4000	0.7311	0.3201	0.9312	1.2104	0.9291
5	0.3789	0.7050	0.3371	1.0024	0.9992	0.9991
10	0.3856	0.7139	0.3436	1.0001	1.0001	0.9998
20	0.3874	0.7164	0.3454	1.0000	1.0000	0.9999
100	0.3875	0.7165	0.3455	1.0000	1.0000	1.0000

Table 2(a): Conversion of Prices

Iteration No.	P_1	P_2	P_3	P_4	P_5
0	0.8317	0.9801	0.5822	0.3385	0.4979
5	0.8003	0.9383	0.5683	0.3299	0.4827
10	0.8116	0.9533	0.5733	0.3330	0.4881
20	0.8148	0.9575	0.5747	0.3339	0.4897
100	0.8149	0.9577	0.5748	0.3339	0.4897

Table 3(a): Conversion of Outputs

Iteration No.	B_1	B_2	B_3	B_4	B_5
0	20.0000	25.0000	30.0000	60.0000	50.0000
5	19.3056	29.6376	27.7382	95.9599	21.4237
10	19.3566	29.6533	27.6905	96.0659	20.3219
20	19.3601	29.6700	27.6778	95.7557	20.2226
100	19.3602	29.6710	27.6772	95.7373	20.2188

If we start with a different initial value of r = 0.6, the path is different but it converges to the same equilibrium as shown in Table 2

Table 2: Conversion of Solutions Irrespective of the Initial Value of r

Iteration No.	r	s	g	X_1	X_2	X_3
0	0.6	0.8807	0.3709	0.9326	1.2334	0.9169
5	0.3938	0.7254	0.3517	1.0022	0.9974	1.0005
10	0.3889	0.7184	0.3469	1.0000	0.9997	1.0001
20	0.3876	0.7166	0.3456	0.9999	0.9999	1.0000
100	0.3875	0.7165	0.3455	1.0000	1.0000	1.0000

Likewise for prices and outputs. Thus, in the equilibrium state, the economy looks as follows:

$$(1{,}9360P_1 + 4.8400P_2 + 2.9040P_3)\ (1+r) + 4.8400w\ = 19.3602P_1$$

$$(5.9342P_1 + 9.3078P_2 + 5.9342P_3)(1+r) + 5.9342w\ = 29.6710P_2$$

$$(0.9925P_1 + 3.6902P_2 + 0.9225P_3)(1+r) + 9.2257w\ = 27.6772P_3 \qquad \ldots (21)$$

$$(4.7868P_1 + 3.1912P_2 + 7.9781P_3)(1+r) + 15.9562w = 95.7373P_4$$

$$\frac{(0.8087P_1}{14.3884} + \frac{2.0218P_2}{22.0513} + \frac{2.8306P_3)\ (1+r)}{20.5695} + \frac{4.0437w}{40} = 20.2188P_5$$

The values of the other economic values in equilibrium are as follows: G = 101.9825, Y = 57.31201, K = 44.6703, S = 15.4347, the real wage rates in terms of commodities 4 and 5 are 2.9949 and 2.0420 respectively.

A different structure of consumer tastes and preferences, e.g. $a_4 = 0.03$, $a_4 = 0.07$, $b_4 = 0.4$, $b_5 = 0.5$, determines the equilibrium, r = 0.3251, s = 0.6436, g = 0.2890, K = 42.5675, Y = 53.8400, S = I = 12.3050, G = 96.4076. The prices, outputs and industrial employment are tabulated below:

Table 3: Equilibrium

Commodity	Prices	Outputs	Employment
1	0.7399	17.8762	4.4690
2	0.8579	30.7957	6.1591
3	0.5415	28.1154	9.3718
4	0.3133	59.0145	9.8357
5	0.4533	50.8211	10.1642

In comparison with the earlier situation, the prices of all the commodities (including commodity 5) exhibit a decline chiefly due to a decline in the rate of profit. The rate of profit itself exhibits a decline because of the decline in the rate of growth. And the rate of growth has declined because commodity 5 is capital intensively produced as compared to commodity 4 which results in an increase A_{ic} i.e. the quantities of capital goods required in the consumption goods industries. It may incidentally be noted that the non-substitution theorem does not hold good; changes in the desired composition of consumption do lead to changes in relative prices for an economy with a positive rate of profit. The non-substitution theorem holds good only in a situation of zero rate of profit.

Finally, consider an example incorporating a dual purpose commodity and let commodity 3 be that commodity. Suppose the parameters of the linear expenditure system are as follows:

$$B_{3KO} = 0 \quad a_3 = 0.02 \quad b_{4KO} = 0.2 \quad a_4 = 0.04 \quad B_{5KO} = 0.1 \quad a_5 = 0.04$$

$$B_{3LO} = 0 \quad b_3 = 0.1 \quad B_{4LO} = 2 \quad b_4 = 0.6 \quad B_{5LO} = 1 \quad b_5 = 0.2$$

The equilibrium for the system (19) is r = 0.2490, g = 0.2178, S = 0.4164, K = 34.6306, Y = 48.6240, G = 83.2547, S = I = 7.5449. The prices, outputs and employments shown in Table 4.

Table 4: Equilibrium

Commodity	Prices	Outputs	Employment
1	0.6651	16.2513	4.0628
2	0.7854	25.4243	5.0848
3	0.5085	32.5568	10.8522
4	0.2927	92.9298	15.4883
5	0.4168	22.5585	4.5117

Of the gross output of 32.5568 units of commodity 3, 8.7960 are used for final consumption and 23.7608 for replacement and new investment at the equilibrium rate of growth.

VII Disequilibrium to Equilibrium

The algorithmic method of finding the equilibrium is a formal method, which the economist, so to speak, uses to find the equilibrium of an economy on being supplied with the technological and behavioural data. It is not the economy's method of finding its own equilibrium. Let us now consider this question. It must be recognized at the very outset that the description of the process by which an economy makes the transition from the disequilibrium state to the equilibrium state which is given below is only one of several possible descriptions. Also, it is well to point out that it will be subject to two limitations, one of which is analytic and the other interpretative. The analytical limitation pertains to the use of equation (7) to solve for the ownership share. This equation will have to be discarded and replaced by the simpler formula (16) with $B_{KO} = 0$. The reason is that, in the disequilibrium situation, the term rK in the equation will stand replaced by the sum of the realized profits Π and this value may not always be such as to give a real solution for the ownership share. The alternative of using the equilibrium ownership share solved by using equation (7) and supposing it to remain constant in the disequilibrium states works but it entirely conceals the behaviour of the ownership share in disequilibrium. The interpretative limitation is that the Sraffa system is set in discrete time so that calculations must be made and presented time frame by time frame instead of a continuous moving picture. If the starting point is one of severe disequilibrium and/or one or more capital goods' prices are extremely sensitive to their outputs, situations where the realized rate of profit for some industry $r_i < -1$ may occur in the initial iterations. Such situations should *not* be interpreted literally as meaning that industry i closes down. It should be given a in terms of interpretation continuity viz. that forces would soon operate to cut down the supply and raise the profitability to acceptable levels so that the effect of these forces become *visible* in the next couple of time frames.

Suppose that the economy is equilibrium in equation (20) is disturbed and stands in the state of disequilibrium shown in equation (19). To find the price of the capital goods in the disequilibrium we may use the initial approximation,

$$P_{im} = \frac{A_1(1+g_e)P_{ie}}{B_{is}} \qquad \ldots (21)$$

Where numerator is expenditure on the capital good I evaluated at the equilibrium good rate and equilibrium price and A_1 the economy made replacement demand in disequilibrium and B_{is} is the aggregate supply in disequilibrium. Substituting these

prices in the first 3 equations gives the realized profits Π_I in those industries. Substituting them in the consumption goods price equations 4 and 5 and supposing as first approximation that equilibrium consumption goods prices apply gives the solution for Π_4 and Π_5 of the consumption goods industries. Having found these we proceed to refine the approximate solutions. Calculate the value of s using $\Pi = \Sigma\Pi_i$ in formula (16) and proceed to find the market i.e. demand price of the consumption goods

$$P_{im} = \frac{a_i Y_k + b_i (Y_L - B_{LO})}{B_{iS} - B_{iLO}} \qquad i = 4,5 \qquad \ldots (22)$$

But the profits of industries 4 and 5 will need to be evaluated at the disequilibrium prices and when that is done $\Pi = \Sigma\Pi_i$ will undergo a change, But with the new level of realised profit we must recalculate the price of consumption goods in (22) and so on. This sub loop of calculations of finding the demand prices of the consumption goods for a given (tentative) set of capital goods' prices can be continued until the desired level of accuracy $P_{im}^{t+1} - P_{im}^{t} \approx 0$ is reached and then return to the main loop of correcting the capital goods prices.

The expenditure on the capital goods which was arbitrarily fixed in (21) needs to be corrected. To do so consider the gross profit attributable to each of the capital goods. It is

$$H_i = \sum_{i-1}^{5} A_{ij} \ (1 + r_i) P_{im} \qquad j = 4,5 \qquad \ldots (23)$$

where P_{im}, $(1+r_i)$ and r_j are the going disequilibrium prices of the capital goods, the gross profit rate and the new profit rate. Next consider the gross saving (GS) in the economy, i.e. S + K also evaluated in the disequilibrium. The gross saving will be allocated between the individual capital goods in proportion to their value H_i which gives the expenditure of each capital good i. Hence,

$$P_{im} = \left[\frac{(H_i)(GS)}{\sum H_i} \right] / B_{is} \qquad \ldots (24)$$

gives the next set of disequilibrium prices that represent a closer approximation. Repeat the process until the prices of capital goods are ascertained to the desired level of accuracy, $P_{im}^{t+1} - P_{im}^{t} \approx 0 \quad (i = 1,2,3)$. This gives the disequilibrium prices. The tables 5(a) and 5(b) below depict the convergence of the variables to their disequilibrium levels to levels of accuracy up to the third place of decimals in the case of the disequilibrium described in the example in section 6, equation (19).

Table 5(a): Prices in Disequilibrium

Iteration	P_{1m}	P_{2m}	P_{3m}	P_{4m}	P_{5m}
1	0.7087	1.1780	0.5394	0.5374	0.1902
				0.5335	0.1868
				0.5335	0.1867*
20	1.3236	1.1428	0.1589	0.5343	0.1876
50	1.6968	0.9777	0.0475	0.5352	0.1888
100**	1.8011	0.9381	0.0110	0.5356	0.1893

*Sub-loop for correction of consumption good prices.
** Differences between the current and preceding prices are -0.00079 -0.00028 and 0.00029, i.e. accuracy to third place of decimals*

Table 5(b): Rate of Profit and Ownership Share in Disequilibrium

Iteration	*s*	r_1	r_2	r_3	r_4	r_5
1	0.7059	0.0278	0.6877	0.0376	2,0660	-1.0596
20	0.6642	1.4295	0.5293	-1.8643	2.1283	-1.0650
50	0.6817	2.4345	0.2490	-2.5157	2.0757	-1.0639
100	0.6992	2.7258	0.1808	-2.7375	2.0244	-1.0629

This gives a description of prices and profits in the initial state of disequilibrium. It is important to note that the iterations shown in Tables 5 are *not* processes in time. Their only purpose is to find refined approximations of the values of the variables in disequilibrium. Nor should the value of $r_i < -1$ be interpreted literally to mean that the industries shut down. Forces of accumulation are at work to increase the supplies of the industries that are making larger profits and reduce those of industries slipping into losses. Forces of demand are at work to ensure that the capital goods industries produce outputs that will not be wasted but will conform to the technological needs for them. Thus, consider the transition to the next disequilibrium state.

The gross saving in the economy will be allocated proportionately to the gross profit in each industry,

$$dK_i = \left(\frac{GP_i}{\sum GP_i} \right) (GS) \qquad \ldots (25)$$

And within each industry there will be a second round of allocation of dK_i between the individual capital goods, which too will be made in proportion to the value of the capital good (evaluated at the disequilibrium price) in the capital stock of the industry.

$$\left(dA_{ji}\right) P_{jm} = \left(\frac{A_{ji} P_{jm}}{K} \right) \left(dK_i\right) \qquad \ldots (26)$$

Adding dA_{ji} to A_{ji} in (19) and finding the corresponding outputs gives new levels of activity of the industries. We need, however, to take into account the effects of the demand forces in the capital goods industries. In the example, there are 3 capital goods all of which are basic goods. Consequently, the demand for each capital good would directly depend upon the supplies available of the other two. A convenient

approximation for fixing the new levels of supplies of the capital goods in line with demands for them is to suppose that

$$B_{1S} = A_1\left[(1+g_2)(1+g_3)\right]^{1/2}$$
$$B_{2S} = A_2\left[(1+g_3)(1+g_1)\right]^{1/2}$$
$$B_{3S} = A_3\left[(1+g_1)(1+g_2)\right]^{1/2}$$

i.e. the output of capital good i equals the total use made of it in the economy A, [arrived at after the allocations in (2b)] times the geometric mean of the own rates of growth of the other capital goods industries that make use of good i in their production.
For the example these are,

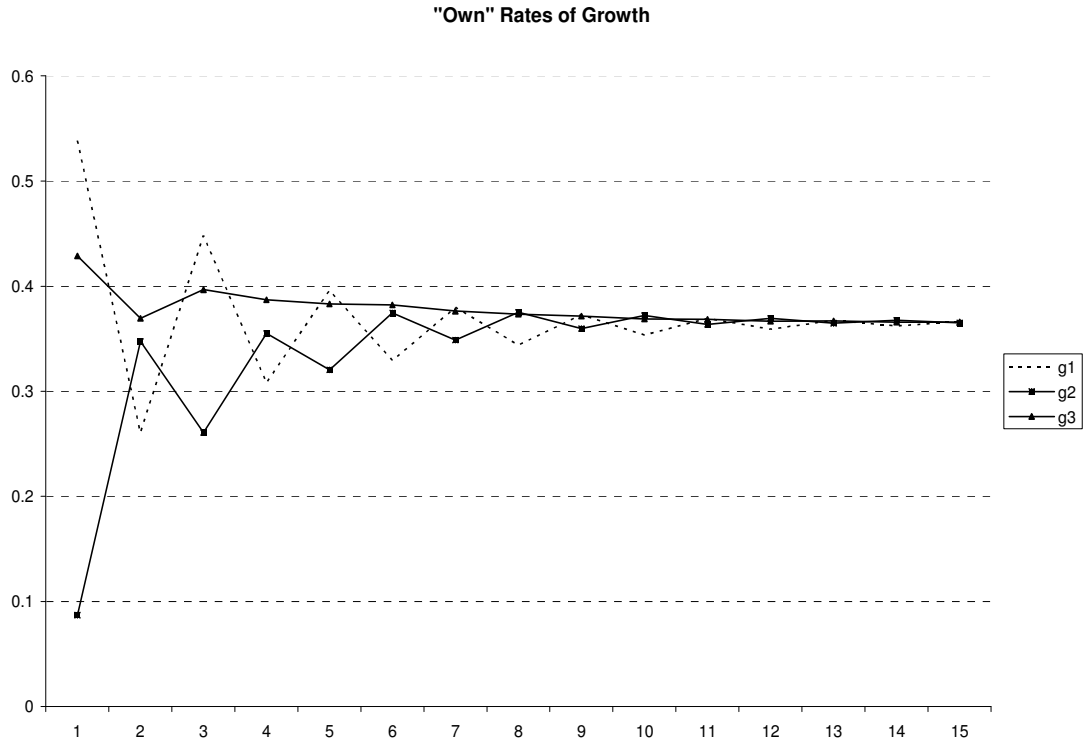
$$B_{1S} = (20.0085)\left[(1.0869)(1.4285)\right]^{1/2} = 25.2713$$
$$B_{2S} = (24.9925)\left[(1.4285)(1.5384)\right]^{1/2} = 36.4024$$
$$B_{3S} = (29.2095)\left[(1.5384)(1.0869)\right]^{1/2} = 37.0285$$

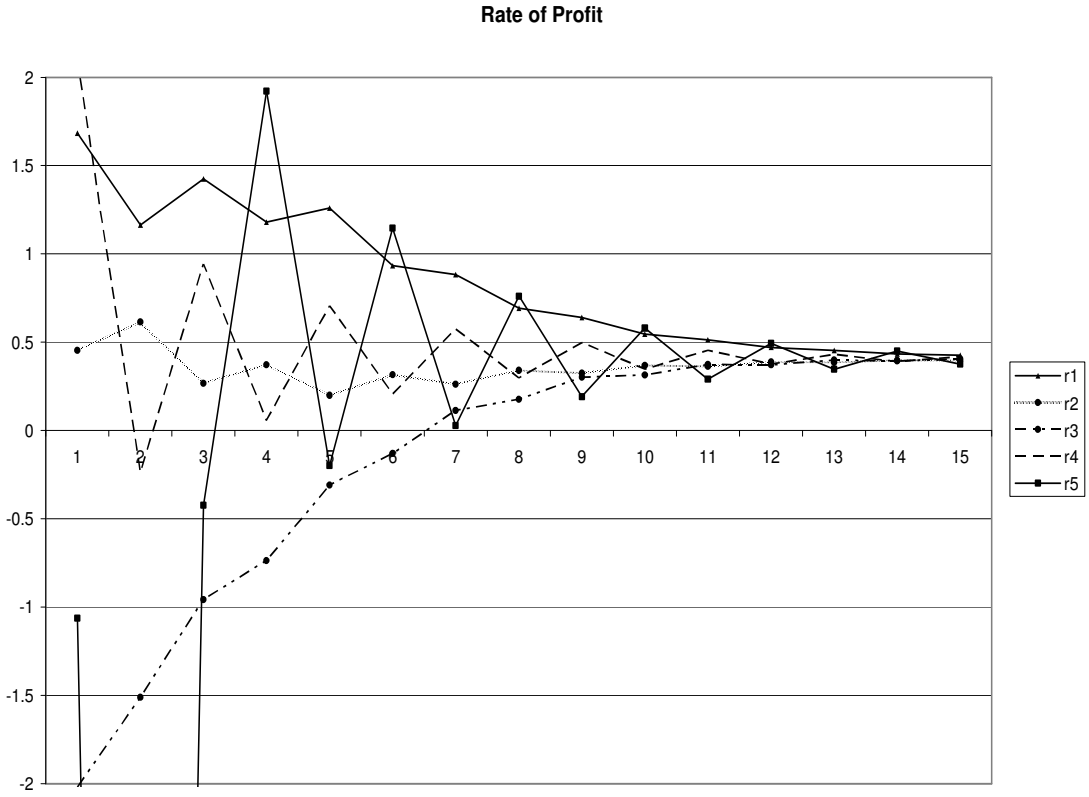
For the consumption goods industries the outputs are $B_{4S} = 181.311$ and $B_{5S} = -3.1633$ corresponding to the allocation made in (26). [For reasons stated earlier the negative output of commodity 5 is a chimera that may be ignored.] The next disequilibrium state obtained is as follows:

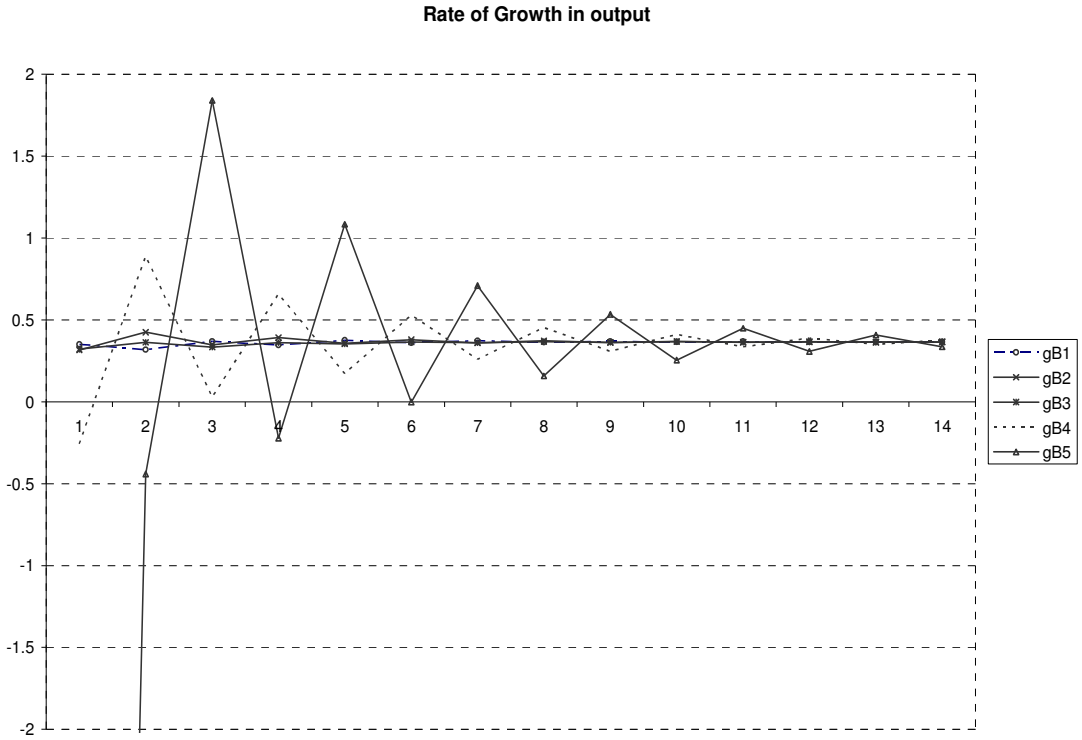
$$\begin{aligned}
(2.5271P_1 + 6.3178P_2 + 3.7906P_3)(1+r) + 6.3178w &= 25.2713P_1 \\
(7.2804P_1 + 10.1926P_2 + 7.2804P_3)(1+r) + 7.2804w &= 36.4024P_2 \\
(1.2342P_1 + 4.9371P_2 + 1.2342P_3)(1+r) + 12.3428w &= 37.0285P_3 \qquad \ldots (27) \\
(9.0655P_1 + 6.0437P_2 + 15.1092P_3)(1+r) + 30.2185w &= 181.311P_4 \\
(-0.1265P_1 - 0.3163P_2 - 0.4428P_3)(1+r) - 0.6326w &= -3.1633P_5
\end{aligned}$$

The growth rate of labour employed $g_L = 0.3895$ may be used to compute the initial market prices of the capital goods which will be subsequently corrected by the procedure in equations (21), (22), (23) and (24). And so on. The path taken by the prices, the rates of profit, the growth rates of the outputs and the ownership share are shown in the graphs 1, 2 and 3 below.

Graphs


"Own" Rates of Growth
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
g1
g2
g3


Rate of Profit
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
r1
r2
r3
r4
r5



The foregoing represents one of the several possible accounts that can be given of the transition from disequilibrium to equilibrium. It would be possible to give an alternative account in terms of output adjustments with prices fixed at the equilibrium supply prices and/or partly in terms of flexible prices.

VIII Public Goods

This section illustrates how the model can be extended to apply to the provision of public goods provided by the government. Thus, suppose that the government acquires materials and labour, A_{G1}, A_{G2} ,.... A_{Gn} and L_G and incurs an expenditure $X_G = A_{G1}P_1 + A_{G2}P_2 + \ldots + A_{Gn}Pn + wL_G$ in order to make available public services to the society. The expenditure is financed by tax revenues. It is supposed, to begin with, that only profits are taxed the rate t. Then

$$trK = \Sigma A_{Gi} P_i + wL_G \qquad \ldots (28)$$

gives the balanced budget equation. This adds one equation to the system and one unknown, viz. the tax rate.

The system of equations must now stand modified. The disposable incomes of the capitalists and workers will now be,

$$Y_{Kd} = (1-t)\mu rK$$
$$Y_{Ld} = L_E + (1-t)(1-\mu)rK$$

where $L_E = \sum L_i + L_G$. The total saving by capitalists and workers are (we shall supposing $B_{KO} = 0$ to avoid the complex roots problem).

$$S_K = (1-a)(1-t)(srK)$$
$$S_L = (1-b)\left[(1-t)(1-s)rK + L_E - B_{LO}\right]$$

The ownership share is,

$$s = 1 - \frac{(1-b)(L-B_{LO})}{(1-t)(b-a)rK}$$

The growth profit relation is,

$$r = \frac{g}{(1-a)(1-t)} \qquad \ldots (29)$$

Finally, the quantities of capital goods acquired by government will be added to B_{iF} in the dual system 13(a) to solve the growth rate and outputs of capital goods. The acquisitions of purely consumption goods will appear in the demand equations (8). Consider the example in (19) and suppose $B_{KO} = 0$. Suppose the government acquires the following quantities of the 5 commodities, $A_{1G} = 0.2, A_{2G} = 0.1, A_{3G} = 0.1,$ $A_{4G} = 1, A_{5G} = 0.5$ and employs 1 unit of labour so that $L_E = 41$. The solution obtained is g = 0.3382, r = 0.4171, s = 0.7175 and t = 0.0988.

Table 6: Solution with Tax on Profits Alone

Commodity	Prices	Outputs	Employment
1	0.8560	19.4513	4.8628
2	1.0124	29.6242	5.9248
3	0.5930	27.6369	9.2123
4	0.3451	95.6282	15.9380
5	0.5096	20.3098	4.0619

The net national product is 60.5433 and the tax revenue and public expenditure are 1.9317.

If both profits and wages are taxed at a uniform rate t, the ownership share s is as in (16) and the growth-profit relation in (29) [Steedman (1972)]. The equilibrium for the economy (with $B_{KO} = 0$) is r = 0.3883, g = 0.3381, t = 0.0324 and s = 0.7136. The prices, outputs and employment are shown in Table 7.

Table 7: Equilibrium with Tax on Wages and Profit

Commodity	Prices	Outputs	Employment
1	0.8159	19.4470	4.8617
2	0.9591	29.6275	5.9255
3	0.5752	27.6382	9.2127
4	0.3342	92.5246	15.9207
5	0.4902	20.3961	4.0792

The net national product is 58.3727 and the tax revenue and public expenditure is 1.8960.

Finally, consider the case of an excise tax levied on the outputs of all the private goods industries. The budget equation will be:

$$t \sum P_i B_i = X_G$$

The price system (1) stands modified to solve for the post-tax prices of commodities $P_i/1-t$ and it is supposed that government procurement takes place at these post- tax prices. The equilibrium is r = 0.3757, g = 0.3382, t = 0.0182 and s = 0.7158. The prices, outputs and industrial employment is tabulated below:

Table 8: Equilibrium with Excise Tax

Commodity	Prices	Outputs	Employment
1	0.8490	19.4489	4.8622
2	1.0007	22.6260	5.9252
3	0.5938	27.6376	9.2125
4	0.3453	95.5710	15.9285
5	0.5080	20.3575	4.0715

The net national product at factor cost is 58.4792, the tax revenue and public expenditure is 1.9286 and the net national product at market prices is 60.4078.

Implicit in the discussion made above is the fact that economic theory can endogenously determine only one tax rate on a predefined tax base for a given level of public expenditure. The subject of several tax rates on several tax bases belongs to the political economy of taxation.

IX Discussion

This paper has investigated a model of general equilibrium that emphasizes technological and behavioural economic relationships having objectivist and empirical foundations. The model works with postulates that have an empirical content and can result in inferences that can be tested against national income data. All that is needed to implement the model are the input-output matrix (preferably will as many coefficients in physical form as is possible) and the linear expenditure system, which follows the aggregation scheme of the input-output matrix. The tricky part is obtaining the classwise linear expenditure system On the theoretical side it may be used to obtain a perspective on several controversies in economic theory including in particular the debate on capital theory [Harcourt (1972), Bharadwaj and Schefold (1990)], the controversy over the Cambridge equation [Steedman (1972), Samuelson and Modigliani (1966), Meade (1963), Pasinetti (1974)] and possibilities of closing the Sraffa system [Dobb (1973), Panico (1988) Pivetti (1991)].

Notes

(1) Stone (1954) hinted that "The special form [of the linear expenditure system] is equivalent to the treatment of households in a closed Leontief model" [Brackets mine]. In practice the Leontief model is not closed that way. It is closed by supposing fixed quantities of consumption by households.

(2) Sraffa (1960), Chapter V, recommended that, "the quantity of labour that can be purchased by the standard net product" be used as the *numeraire*.

(3) To interpret the "committed" quantities as "subsistence" requirements would be stretching things too far. After all the committed quantities in equation (4) are merely intercepts of linear functions fitted to data that in reality are generated by non-linear Engels consumption-income relations. But the consumption behaviour described in equation 4(b) fits exactly Sraffa's conception of the wage, "we have up to this point regarded wages as consisting of the necessary subsistence of workers… .We must now take into account the other aspect of wages since, besides the ever-present element of subsistence, they may include a share in the surplus product". [Sraffa (1960), Chapter II].

(4) Linear expenditure systems are local linear approximations of essentially non-linear Engels functions. The non-linearies are better captured by the Almost Ideal Demand System [Deaton and Muellbauer (1980)].

(5) The connection between Sraffa's price system and the Post-Keynesian model of growth and distribution has been repeatedly emphasized by Sraffa scholars. [See Kurz & Salvadori (1995), Chapter].

(6) As Nell (1999) has put it, "In general, the interdependence of basics tends to pull growth rates together".

(7) The "own rates of profit" r_i in disequilibrium may be interpreted, by analogy to Sraffa's (1932) "natural rates of interest", as being the sum of the equilibrium rate of profit and the premium on the commodity in the forward market.

$$r_i = r + \frac{P_{iF} - P_{im}}{P_{im}}$$

In a situation of disequilibrium the own rates are different for different commodities. Commodities in excess supply will have own rates greater than the equilibrium rate and vice versa. Accordingly, their (spot) market prices will stand below their equilibrium prices. If the market expects that the disequilibrium will be corrected in the near future, i.e. the supply of the commodity is expected to decline, the forward price will stand above the spot prices showing a premium. *Vice versa* for commodities in excess demand. In equilibrium the own rates of all commodities are equalized to one another and to the equilibrium rate of profit so that $P_{iF} = P_{im}$.

References

Bharadwaj K. and B. Schefold (1990), "Essays on Piero Sraffa, Critical Perspectives on the Revival of Classical Theory", London: Unwin Hyman.

Deaton, A., J. Muellbauer (1980), "An Almost Ideal Demand System", The American Economic Review, Vol. 70, No. 3, pp. 312-326.

Dobb M. (1973), "Theories of Value and Distribution since Adam Smith", Cambridge: Cambridge University Press.

Harcourt G. (1972), "Some Cambridge Controvenics in the Theory of Capital", Cambridge: Cambridge University Press.

Hawkins D. and H. Simon (1949), "Note: Some Conditions for Macroeconomic Stability", Econometrica, Vol.17, pp. 245-248.

Kaldor N. (1956), "Alternative Theories of Distribution", Review of Economic Studies, Vol. 23, No. 2, pp. 83-100.

Kurz H.D. & N. Salvadori (1995), "Theory of Production: A Long Period Analysis", Cambridge University Press: New York.

Meade J.E. (1963), "The Rate of Profit in a Growing Economy", Economic Journal, Vol. 73, No. 361, pp. 665-674.

Nell E.J. (1999), "Wicksell after Sraffa: Capital Arbitrage and Normal Rates of Growth Interest and Profits" in Mongiovi G. and Petri F. eds. (1999), Value Distribution and Capital, Essays in Honour of Pierangelo Garegonani, Rutledge: London and New York.

Panico C. (1988), "Interest and Profit in the Theories of Value and Distribution", London Macmillan.

Pasinetti (1962), "Rate of Profit and Income Distribution in Relation to the Rate of Economic Growth", Review of Economic Studies, 29(4), pp. 267-279.

Pasinetti (1974), "Growth and Income Distribution: Essays in Economic Theory", Cambridge: Cambridge University Press.

Pivetti M. (1991), "An Essay on Money and Distribution", London: Macmillan.

Samuelson P. and F. Modigliani (1966), "The Pasinetti Paradox in Neoclassical and More General Models", Review of Economic Studies, Vol. 33, No. 3, pp. 269-301.

Solow R. M. (1952), "On the Structure of Linear Models", Econometrica, Vol. 20, No. 1, pp. 29-46.

Sraffa P. (1960), "Production of Commodities by Means of Commodities", Cambridge, Cambridge University Press.

Stone J.R.N. (1954), "Linear Expenditure Systems and Demand Analysis: An Application to the Pattern of British Demand", Economic Journal, Vol. 64, No. 255, pp. 511-527.

Streedman (1972), "The State and the Outcome of the Pasinetti Process", Economic Journal, 82, December, pp. 1387-1395.