

## Chapter II: Closed Sraffa System with Stock-Flow Variables

6. If money cannot appear without making standard assumptions disappear, it would only be plausible by *making money appear with making standard theory disappear and modifying the standard assumptions to make a new theory appear*. This is the aim of the following chapter- to provide an alternative to the standard mainstream neoclassical theory. The basic system described here can be extended in various logical manners depending on the purpose the reader aims to take it. In a true sense, we would aim to extend the agrarian Sraffa system<sup>36</sup> to non-agrarian industries as well in the first place. The production system in its natural form seems to be an agrarian system where production of commodities is happening by means of production. However, in actual industrial set-ups, the production has several important components as stocks or inventories of commodities, fixed capital and circulating capital. However, the aim of this synthesis is to provide for the role of money in the economy. This role can only be provided for by foregoing the standard theory since as discussed, attempts to make money appear without making standard theory disappear have turned futile. Hence, we take the course of Sraffa's economics in order to demonstrate the synthesis. Money, it should be noted, is a balance sheet item. Money is normally and, at most times, is a stock concept which facilitates the *flow* in the economy. Therefore, the Sraffa model needs to be adequately expanded and made accommodative so as to incorporate money. We can, in this chapter, clearly see that the basic Sraffa model can be extended to incorporate such concepts of stock and flow variables. Stock variables would be regarded as the fixed capital in the economy and the regular activities of the enterprise would be the flow variables in such an economy. The stocks primarily could be conceived to consist of machinery, factory shed, stock of raw materials and the like. The regular payments to like the provision of seeds for production, uniforms for labour, administrative expenses, rent, rates and electricity et al would ideally form the flow variables in the economy. The flow

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<sup>36</sup> Refer annexure to this chapter for a detailed discussion on the theory and development of Sraffa's economics. The annexure contains notes on the basic Sraffa system of book one of this 1960 Production of Commodities by Means of Commodities

variables are required over the entire production period in the sense that since these are regular payments, they need to be made often. The stocks form the capital in the system and as such, like every other form of capital, command a rate of profits. We would, for simplicity sake, assume the rates of profits across all industries to be uniform- preserving the Sraffian tradition. This model of continuous industrial production carries over all the essential properties of the Sraffa system. When the production process is properly primed, the flow of purchases of inputs is exactly matched to the flow of output and its sale so that capital is needed only to finance the holding of stocks. A model that contains stocks and flows can be represented as under:

$$\begin{aligned}
 (S_{11}p_1 + S_{21}p_2 + \dots + Sk_1B_t)r + (A_{11}p_1 + A_{21}p_2 + \dots + Ak_1B_t) + wL_1 &= p_1B_1 \\
 (S_{12}p_1 + S_{22}p_2 + \dots + Sk_2B_t)r + (A_{12}p_1 + A_{22}p_2 + \dots + Ak_2B_t) + wL_2 &= p_2B_2 \\
 \dots\dots\dots \\
 (S_{1n+m}p_1 + S_{2n+m}p_2 + \dots + Sk_{n+m}B_t)r + (A_{1n+m}p_1 + A_{2n+m}p_2 + \dots + Ak_{n+m}B_t) + wL_{n+m} &= p_{n+m}B_{n+m}
 \end{aligned}$$

In the above model, the stocks are represented by the matrix  $S$  and the flows are represented by the matrix  $A$ . We represent outputs using the standard  $B$  matrix and the usual variables prices ( $p$ ), rate of profits ( $r$ ) and the wage rate,  $w$  remain the same from the standard Sraffa model. We aim to use the same notation throughout the course of this thesis. It would be prudent to note the major changes in the model incorporating the stock-flow model over the standard Sraffa model. There are no changes in the parsimony of the model; however, the interpretation of standard variables changes. Every industry in particular would have its own rate of profit, though in equilibrium, these rates converge to a uniform rate through flight of capital (stocks) from one industry to another. This equality of rate of profits still remains the same in the current set-up as well. However, in terms of a standard Sraffa model, the own rate of growth of individual commodities was

defined as  $\sum_{i=1}^{m+n} \sum_{j=1}^{m+n} p_i B_i - p_i A_{ij} / \sum_{i=1}^{m+n} \sum_{j=1}^{m-n} p_i A_{ij}$ . However, this definition of own rate

of growth changes in the stock-flow model and takes the form:

$$\left( \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} p_i B_i - p_i A_{ij} \right) / \left( \sum_{i=1}^{m+n} \sum_{j=1}^{m-n} p_i S_{ij} \right).$$

It is important enough to understand that the profits only on a stock component form a part of the valuation of the commodities. The production relations do not allow for the sales revenue to cover up the capital or the stock parts. In fact, the stocks are perennially present. This system has  $n$  equations in  $n+2$  unknowns- the  $n$  prices and the 2 distributive variables. Either one of the prices can be fixed as *numeraire* so that there would be  $n+1$  unknowns. The uniform rate of profits that prevails across industries is

$$r = \frac{p_i B_i - \sum_j A_{ji} p_j - w L_i}{\sum_j S_{ji} p_j}$$

This rate is assumed to prevail across all industries. However, the system still moves with Sraffa's *one degree of freedom*. In order to eliminate this degree of freedom so as to enable us determine the unknowns of the system, we would require an additional equation. The Sraffa system is silent on the consumption demand equations. In order to provide necessary number of equations to determine as many unknowns, it would only be relevant to introduce consumption demand equations to suitably close the Sraffa system. We would resort to the demand equations in order to appropriately *close* the system. We would use the Engel-Stone-Geary type of functions<sup>37</sup> to do so. The functions would be suitably

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<sup>37</sup> Stone-Geary Function

The Stone-Geary function is often used to model problems involving subsistence levels of consumption. In these cases, a certain minimal level of some good has to be consumed, irrespective of its price or the consumer's income.

The Stone-Geary uses the natural log function to model utility. The sum of all the proportions of the goods consumed must equal 1. In the problem below, the subsistence levels of A and B are  $\alpha$  and  $\beta$ . The term I is income, and  $p_k$  {k=a,b} are the prices of A and B.

The Lagrangean and the First-Order conditions are:

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$$L = \gamma \ln(A - \alpha) + (1 - \gamma) \ln(B - \beta) + \lambda(I - p_a A - p_b B)$$

$$L_A = \frac{\gamma}{A - \alpha} - p_a \lambda = 0$$

$$L_B = \frac{1 - \gamma}{B - \beta} - p_b \lambda = 0$$

$$L_\lambda = I - p_a A - p_b B = 0$$

Use the first 2, FO conditions to eliminate the Lagrangean Multiplier.

$$\frac{\gamma/(A - \alpha)}{(1 - \gamma)/(B - \beta)} = \frac{p_a \lambda}{p_b \lambda}$$

$$\Rightarrow \frac{\gamma}{1 - \gamma} \frac{B - \beta}{A - \alpha} = \frac{p_a}{p_b}$$

$$\Rightarrow p_a(1 - \gamma)(A - \alpha) = p_b \gamma(B - \beta)$$

$$\Rightarrow A - \alpha = \frac{p_b \gamma(B - \beta)}{p_a(1 - \gamma)}$$

$$\Rightarrow A = \frac{p_b}{p_a} \frac{\gamma}{1 - \gamma} (B - \beta) + \alpha \quad B = \frac{p_a}{p_b} \frac{1 - \gamma}{\gamma} (A - \alpha) + \beta$$

Substitute into the third, FO condition.

$$I = p_a A + p_b \left( \frac{p_a}{p_b} \frac{1 - \gamma}{\gamma} (A - \alpha) + \beta \right)$$

$$\Rightarrow I = p_a A + \frac{p_b}{p_b} \frac{1 - \gamma}{\gamma} p_a (A - \alpha) + p_b \beta$$

$$\Rightarrow I - p_b \beta = p_a A + \left( \frac{1 - \gamma}{\gamma} p_a (A - \alpha) \right)$$

$$\Rightarrow I - p_b \beta = p_a A + (1 - \gamma) \left( \frac{p_a A}{\gamma} - \frac{p_a \alpha}{\gamma} \right)$$

$$\Rightarrow I - p_b \beta = p_a A + \frac{p_a A}{\gamma} - \frac{p_a \alpha}{\gamma} - p_a A + p_a A$$

Multiply the last equation above through by  $\gamma/p_a$ .

$$\frac{\gamma}{p_a} (I - p_a \alpha - p_b \beta) = A - \alpha$$

$$\Rightarrow A^* = \alpha + \frac{\gamma}{p_a} (I - p_a \alpha - p_b \beta)$$

$$\Rightarrow B^* = \beta + \frac{1 - \gamma}{p_b} (I - p_a \alpha - p_b \beta)$$

Notes

adjusted with respect to the form and the nature; however, the essential properties of reflexivity, transitivity and importantly symmetry and substitutability of the Slutsky matrix would be intact. The required missing degree of freedom between the equations and the variables can be also filled in by considering the composition of commodities which the individual agents desire to purchase; the demand equations for the n commodities. Walras' law dictates that only n-1 of these will be independent, and that we shall use the empirical demand functions that designed are designed by Stone as where is the capital stock, L is the annual labour and the constants alpha and beta are propensities of capitalists and wage earners to consume or spend. This would take the following form

$$p_i B_i = \alpha r k + \beta w L$$

The following points need to be noted. 1) The demand equations of the above type are introduced only for consumption Sraffa commodities. Such commodities would be referred to as consumption goods in the economy. 2) The marginal propensities  $\alpha$  and  $\beta$  would always be given;  $\alpha$  being the MPC of capital owners (capitalists/ industrialists/entrepreneurs) and  $\beta$  being the MPC of wage earners (workers/ households)<sup>38</sup>. With this equation, assuming we have a two commodity system as follows, we can determine the prices by assuming the suitable demand equation for commodity 2.

$$\begin{aligned} (S_{11}p_1 + S_{21}p_2 + \dots + Sk_1B_t)r + (A_{11}p_1 + A_{21}p_2 + \dots + Ak_1B_t) + wL_1 &= p_1B_1 \\ (S_{12}p_1 + S_{22}p_2 + \dots + Sk_2B_t)r + (A_{12}p_1 + A_{22}p_2 + \dots + Ak_2B_t) + wL_2 &= p_2B_2 \\ \alpha r k + \beta w L &= p_1B_1 \end{aligned}$$

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1. Each of the functions of A\* and B\* are the Marshallian demand functions for the Stone-Geary utility.
  2. The first term on the right-hand-side of the equality, is the subsistence consumption. A consumer will always consume this amount irrespective of their budgets or the price.
  3. The term I - pa $\alpha$  - pb $\beta$  is the income the consumer has left over, after the subsistence levels are met. It is in effect, the residual income.
  4. The amount of A and B that this residual income is used to buy, is now negatively influenced by price, and positively influenced by the good's importance. For instance, if  $\gamma$  increases, it implies that good A is relatively more important than B. According to these demand functions, our consumer will purchase less of B and more of A, all other things equal.

<sup>38</sup> The terms in the parenthesis have been used interchangeably in the entire document

Therefore, the prices of both the goods in this system are determinate and all the standard properties of Sraffa system are preserved in the continuous production economy<sup>39</sup>.

It is the flows that have to be replaced period on period as the system gravitates to equilibrium. The dual system that would be used to determine the outputs in the system is similar but for the fact that the growth rate is attributed to the stock coefficients and the flow coefficients are determined simultaneously along with the stock coefficients. More so, the dual or the q-system would take the following form:

$$\left( S_{11}x_1 + S_{12}x_2 + \dots S_{1n}x_n + \sum_{j=1}^m S_{1j} \right)g + \left( A_{11}x_1 + A_{12}x_2 + \dots A_{1n}x_n + \sum_{j=1}^m A_{1j} \right) = B_1x_1$$

$$\left( S_{21}x_1 + S_{22}x_2 + \dots S_{2n}x_n + \sum_{j=1}^m S_{2j} \right)g + \left( A_{21}x_1 + A_{22}x_2 + \dots A_{2n}x_n + \sum_{j=1}^m A_{2j} \right) = B_2x_2$$

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$$\left( S_{n1}x_1 + S_{n2}x_2 + \dots S_{nn}x_n + \sum_{j=1}^m S_{nj} \right)g + \left( A_{n1}x_1 + A_{n2}x_2 + \dots A_{nn}x_n + \sum_{j=1}^m A_{nj} \right) = B_nx_n$$

$$L_1x_1 + L_2x_2 + \dots L_nx_n = L$$

More so, the properties in terms of the relation between the rate of profits and the rate of growth would not change much and instances where it changes dramatically would be pointed out; not for the reason of drawing comparisons, but for the cause that this relation would be the important relation between two important variables- rate of profit and rate of growth. The standard system of the dual is being used here. We would aim to close this chapter on the following note: a basic Sraffa model can be extended to incorporate the stock-flow concepts<sup>40</sup> in the economy and we would aim to use such an extended Sraffa model in the following work.

7. Let us take up a numerical example to study the workings of this system. Let us assume the following hypothetical economic system with stocks, flows, labour

<sup>39</sup> Refer end-note to this chapter for more on this

<sup>40</sup> The model with stock-flow has been presented earlier by Prof. Rajas Parchure in his *Pure Theory of Value*. An interested reader may drill the subject further down from here. For my purposes, I assume (and I know as well) that the conclusions presented in the book are accurate.

and output. The model has two capital (capital) and one consumption (consumption) goods industries

$$\begin{aligned}(1p_1 + 2p_2)r + (1p_1 + 2p_2) + 5w &= 20p_1 \\ (3p_1 + 2p_2)r + (1p_1 + 3p_2) + 5w &= 30p_2 \\ (2p_1 + 3p_2)r + (3p_1 + 2p_2) + 10w &= 30p_3\end{aligned}$$

For the last industry, since it is a consumption good, we would have a demand equation specifying its consumption pattern. Let us assume that the workers consume with a propensity of 0.2 out of their wage incomes only (to be precise, let us assume that workers do not have a share in profits). Thus, we have the demand equation

$$0.2(20w) = 30p_3$$

It is important to mention the dual of this system which specifies the output requirements of the individual industries

$$\begin{aligned}(1q_1 + 3q_2 + 2)g + (1q_1 + 1q_2 + 3) &= 20q_1 \\ (2q_1 + 2q_2 + 3)g + (2q_1 + 3q_2 + 2) &= 30q_2 \\ 5q_1 + 5q_2 &= 10\end{aligned}$$

Finally, we have the growth profit equation of the following type

$$r = \frac{g(6p_1 + 7p_2) - 0.8(20w)}{(6p_1 + 7p_2)}$$

Therefore, we have 8 equations in 9 unknowns (3 prices, profit rate, wage rate, growth rate, and the three outputs). We would begin by assuming an arbitrary rate of profits; say 0.20 and let us assume the wage-rate as the numeraire.

Accordingly, we obtain  $p_1 = 0.29$ ,  $p_2 = 0.20$ ,  $p_3 = 0.38$  with  $r=0.20$  and  $w=1$ .

From this, we can immediately determine  $B_3 = 10.40$  using the demand equation and hence, we would adjust the equation of the third industry to this effect

$$(0.69p_1 + 1.04p_2)r + (1.04p_1 + 0.69p_2) + 3.46w = 10.40p_3$$

and hence, the dual changes as well to

$$\begin{aligned}(1q_1 + 3q_2 + 0.69)g + (1q_1 + 1q_2 + 1.04) &= 20q_1 \\ (2q_1 + 2q_2 + 1.04)g + (2q_1 + 3q_2 + 0.69) &= 30q_2 \\ 5q_1 + 5q_2 &= 10\end{aligned}$$

Using the growth profit relation, we have  $g=6.01$ . and thus, the output multipliers are 1.67 and 0.44. With these, we obtain the new production equations for industry 1 and 2 respectively as under.

$$0.67((1p_1 + 2p_2)r + (1p_1 + 2p_2) + 5w = 20p_1)$$

$$0.44((3p_1 + 2p_2)r + (1p_1 + 3p_2) + 5w = 30p_2)$$

$$(0.69p_1 + 1.04p_2)r + (1.04p_1 + 0.69p_2) + 3.46w = 10.40p_3$$

And so on, the process would continue until all demand for commodity 3 equates the supply at the going prices. Thus, we obtain the solutions in the augmented Sraffa model to incorporate stock-flows using the following algorithm

- Start with an arbitrary rate of profits and a numeraire, in this case the wages
- Determine the prices and demand for consumption goods at the going prices of capital goods industries
- Adjust the price equations for the consumption goods to reflect the new demand by multiplying the entire equations by the ratio  $B^i_{new} / B^i_{old}$
- Determine the growth rate using the growth-profit relationship and formulate the dual of the problem
- Determine the output multipliers and create new set of *capital* price equations for capital industries
- Repeat the process until  $B^i_{new} - B^i_{old} = 0$

In the following chapter, we would adapt this base model of Sraffa stock-flows to find a place for money and determine the properties of the system so developed



End Note to Chapter 2

Properties of the Extended Sraffa Model<sup>41</sup>

Writing the original equations for unit outputs in matrix notation

$$SP + AP + wL = P \dots\dots\dots I$$

whose solution for the price vector is

$$P = [I - A - rS]^{-1} wL$$

$$= [I - (I - A)^{-1} Sr]^{-1} (I - A)^{-1} wL$$

which is positive if the Hawkins-Simon conditions (Hawkins and Simon 1949) that the principal minors of  $I-A$  be positive hold. It is easy to see that the principal minors of  $I - (I - A)^{-1} Sr$  will be positive if  $(I - A)^{-1}$  is positive. Consider the price system at  $w=0$

$$P = [(I - A)]^{-1} (S Pr)$$

with  $\lambda = 1/r$  the above expression is

$$[\lambda I - (I - A)^{-1} S]P = 0$$

If  $(I - A)^{-1}$  is positive  $(I - A)^{-1} S$  must be non-negative. So by Perron-Frobenius theorem, it has a dominant positive eigenvalue  $\lambda_d$  with which is associated a non-negative vector  $X_d$ . The maximum rate of profit,  $R$ , is the reciprocal of  $\lambda_d$ . Thus for values  $r < R$  the matrix  $I - (I - A)^{-1} Sr$  has positive minors so that the solution of prices must be strictly positive. We can also write the equation I in the following form by multiplying it by the eigenvector  $X_d$ :

$$X_d S Pr + X_d AP + X_d wL = X_d P \dots\dots\dots II$$

At  $w=0$

$$r = R = \frac{X_d (I - A)P}{X_d SP}$$

If  $X_d (I - A)P$ , the standard national product is set equal to unity and further if

$X_d L = 1$  then substituting this in equation II gives

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<sup>41</sup> Based entirely on a working paper developed by Dr. Rajas Parchure (2008), "The Sraffa System for Continuous Industrial Production", *GIFE working paper series*, November

$$r=R(1-w)$$

which is the linear wage-profit frontier.

Also the dual of the system in terms of unit output is

$$S^T X + A^T X + F = X$$

where F denotes the final consumption. At F=0 the economy has maximum rate of growth  $G=R$ . So for  $g < R$

$$\begin{aligned} X &= [I - A^T - gS^T]^{-1} F \\ &= [I - gS^T (I - A^T)^{-1}]^{-1} (I - A^T)^{-1} F \end{aligned}$$

which are gross outputs required to satisfy final consumption vector F. The similarity of the above to the Leontief's dynamic system is at once apparent