

LIFE CONTINGENCIES

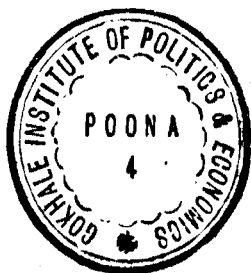
CAMBRIDGE
UNIVERSITY PRESS
LONDON: BENTLEY HOUSE
NEW YORK, TORONTO, BOMBAY
CALCUTTA, MADRAS: MACMILLAN

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LIFE CONTINGENCIES

BY

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CAMBRIDGE

Published for the Institute of Actuaries

AT THE

UNIVERSITY PRESS

1945

<i>First published</i>	1922
<i>Second Edition</i>	1929
<i>Third Edition</i>	1932
<i>Reprinted</i>	1938
"	1941
"	1945

PRINTED IN GREAT BRITAIN

INTRODUCTION

PART II of the Text Book of the Institute of Actuaries, dealing with the Theory of Life Contingencies, was first issued in 1887. This work, for which the Institute must ever be indebted to its distinguished author Mr George King, did more than simplify the progress of the actuarial student to his desired goal; it systematised and co-ordinated the presentation of the complex theory with which it dealt, thus elevating to the status of a definite branch of scientific knowledge a subject which, though fully ripe for such recognition, had up to that time suffered from the disadvantage of comparative inaccessibility.

During the long period which has elapsed since the Text Book was first published considerable changes have been made in the educational course prescribed by the Institute; in particular the value to the actuarial student of a competent knowledge of the elements of the Differential and Integral Calculus has been more fully realised and these subjects are now included in the mathematical course with which the training of the student begins. To secure a consistent educational scheme it has therefore been found necessary to re-arrange the volume hitherto entitled Part II of the Text Book and to bring into greater prominence those mathematical demonstrations which were formerly treated as subjects of optional study. The theoretical basis of these demonstrations is now included in the mathematical text book which is issued as a separate work, and certain chapters of the old Text Book, Part II, are omitted from the present Volume which thus deals exclusively with the theory of actuarial science so far as relates to Life Contingencies.

These are the principal changes introduced, but the opportunity has also been taken to make certain alterations which bring the work more fully into conformity with modern requirements. At the request of the Council the compilation of the new treatise was undertaken by Mr E. F. Spurgeon who brought to his task the indispensable qualifications of long experience as a Tutor and a conspicuous gift of exposition. The resulting Volume is issued by the Council in the belief that it will fully meet the needs of students and promote the attainment of those high professional qualifications which are connoted by the Fellowship of the Institute of Actuaries.

A. W. W.

May 1922.

AUTHOR'S PREFACE

IN the preparation of this Volume and in determining the order in which the various subjects should be dealt with, special attention has been directed to the following considerations, namely,

- (i) That the student should, at an early stage of his work, acquire a sound knowledge of the *principles* of the construction of Mortality Tables, and of the evaluation of Annuities and Assurances, *on the simplest possible basis*.
- (ii) That in applying these principles to the more intricate parts of his work, the student should automatically revise his knowledge of the earlier portions, on a sound grasp of which so much depends.
- (iii) That such information on practical points as will be of assistance to the student in his more advanced studies should, where possible, be given, even though at times these may not be strictly within the limits of Part II of the Institute Examinations.

The Book has been divided into three parts:—Part I deals with Mortality Tables and Single-Life Functions; Part II with Functions involving two or more lives; and Part III with those subjects which could not conveniently be fitted into either of the earlier parts.

In Part I it was essential that the first subject dealt with should be the Mortality Table, and in the second chapter it was deemed advisable to proceed directly to Mortality Tables constructed from Life Assurance Statistics, with particular reference to Select Tables. This enables us, in Chapter III, to deal at once with Single-Life Annuities payable annually, Single-Life Assurances payable at the end of the year of death and Annual Premiums based on Select Tables. In Chapter IV, Premium Conversion Tables are dealt with; and in Chapter V, varying Annuities and Assurances and Premiums for special classes of assurances with reference to many practical points. Chapter VI deals with the values of Single-Life Policies subject to annual Premiums.

It will be seen that by taking the subjects in this order the student will, by the end of Chapter VI, have acquired a knowledge of all the principles involved in the calculation of probabilities and in the evaluation of Annuities and Assurances, on the basis of *Single Life Annual Functions*, all the later work being an extension or modification of the matters dealt with in these Chapters. As a result of this arrangement, we are able in a single chapter (Chapter VII) to dispose of Annuities and Premiums payable more frequently than once a year, together with the values of policies subject to such premiums. In Chapters VIII and IX Assurances payable at the moment of death and Complete Single-Life Annuities are discussed.

Chapter X, dealing with Life Office Valuations, has been included in order to avoid the difficulty which Part II students have met with in the past, namely, that they have acquired a knowledge of Policy Values only to find that in their later work Policy Values, as such, are seldom used.

In order to prepare for subjects included in Part II of the Book it now becomes necessary, in Chapter XI, to consider Makeham's Law of Mortality, and in Chapter XII to deal with statistical applications of the Mortality Table and the Expectation of Life.

In Part II, when considering Joint-Life and Contingent Probabilities, Joint-Life and Last-Survivor Annuities and Assurances, and Contingent Assurances, the student will be merely extending the principles studied in the first few chapters.

Part III consists of two chapters only. In Chapter XX (Construction of Tables) the advantage of the Arithmometer has been emphasised and it has been considered necessary again to include a plate of a machine showing a particular calculation, the description of the machine being taken word for word from the old Text Book. In Chapter XXI the question of Tables involving two or more causes of decrement has been introduced in the simplest possible manner.

At the end of the Volume it was considered advisable to include, firstly, a Mortality Table based on population data (The English Life Table, No. 8), secondly, a Select Life Table (The O^(NM)) and thirdly, the H^M Table (Makeham Graduation) because of its value for instructional purposes. The Tables of Monetary Functions on the basis of the English Life Table, No. 8, which have not previously

been published, have been supplied entirely by the Prudential Assurance Company; the Tables based on the $O^{(NM)}$ experience have been taken from those published by the Institute of Actuaries and Faculty of Actuaries jointly; and those based on the H^M experience from the old Text Book Part II.

It has been a distinct advantage to have a great part of the work already mapped out by Mr George King in the old Text Book, and to have been able to make use of his collection of symbols and expressions in such chapters as that relating to Compound Survivorship Annuities and Assurances. It would have added very materially to the work of preparation of the present volume had these not been available.

In conclusion I wish to express my high appreciation of the very valuable assistance rendered to me by Messrs W. W. Williamson, F.I.A., and C. C. Barrett, F.I.A., both in criticising my manuscript and in reading the proofs. If this book prove of the value to students and to the profession which I hope, their exceedingly helpful criticisms and suggestions will have contributed thereto in no small degree.

E. F. S.

AUTHOR'S PREFACE TO SECOND EDITION

IN the Second Edition of this book the plan adopted in the First Edition has been retained.

The principal alterations are, the transfer from Chapter IV 13 to Chapter III 26 of the remarks concerning the comparison of premiums calculated on the basis of select and aggregate tables respectively, the revision of Chapter VI 44 and 45 and the addition of paragraph 14 to Chapter XXI.

Advantage has, of course, been taken of the opportunity afforded of correcting errors and misprints which appeared in the First Edition, and some slight changes in wording have also been made.

E. F. S.

AUTHOR'S PREFACE TO THIRD EDITION

THE principal alterations in the Third Edition occur in Chapters V and X. In Chapter V, paragraph 34 (Options) has been redrafted in order to introduce more "practical" formulae, and paragraph 36 (Deferred Assurances with Return of Premiums) has been slightly altered for a similar reason. In Chapter X, paragraphs 3 to 11 have been redrafted in order to bring out more clearly, for the benefit of students, the actual bases of the formulae employed in practice; no alteration in principle is, however, involved.

Various suggestions have been received that extracts from papers and notes published during the last few years should be included in the Third Edition. These have been carefully considered, but as only slight variations of the application of principles already dealt with in this book were involved, it was not deemed advisable to increase the size of the work by their inclusion.

The Author desires to express his very great appreciation of the assistance he has received in the preparation of the Third Edition from Messrs W. F. Gardner, F.I.A., and G. A. Hosking, F.I.A., who have, indeed, undertaken practically the whole of the task of examining the suggestions received, in addition to themselves making recommendations concerning various points which had come before them in the course of their tutorial work.

E. F. S.

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