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THE
MATHEMATICAL
GROUNDWORK
OF
ECONOMICS




AN INTRODUCTORY

TREATISE

BY

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PREFACE

THERE seems to be no book in existence, at least in English, that presents in a coherent form the mathematical treatment of the theory of political economy which has been developed during the past eighty years or more. The more familiar parts of the theory are assumed by writers or indicated in footnotes or appendices, the less familiar must be sought in the treatises or journals in which they appear; the various writers on the mathematical theory have proceeded from different hypotheses and adopted different notations, and students are consequently hindered in the use of this very valuable aid to analysis. Though the simpler applications of mathematics made by competent writers and lecturers can be appreciated by any intelligent readers and students, the more complicated analyses are only within the power of those who have mathematical aptitude, and it is for them that this book is arranged. The actual number of mathematical theorems used is quite small, but among them are some uses of the calculus which do not form part of the usual elementary curriculum, and these are brought together in an appendix.

I have attempted to reduce to a uniform notation, and to present as a properly related whole, the main part of the mathematical methods used by Cournot, Jevons, Pareto, Edgeworth, Marshall, Pigou, and Johnson, so far as these are applied to the fundamental equations of exchange and to the elementary study of taxation. Since I cannot be sure that I have not in some cases misinterpreted these writers, I have not given many detailed references, and must content

myself with this general acknowledgement of indebtedness. I have not intended to advance any new theorems in economics, nor do I claim any originality in mathematical results, for the few theorems which I have not consciously adapted from others may in fact already have been published. Perhaps, however, there is in my analysis a more definite attempt than has been usual to deal equally with the hypotheses of competition and of monopoly, to find a place for incomplete monopoly and to indicate how perfect competition and perfect monopoly are mathematically the extreme cases of a more general conception.

My thanks are due to Professor A. C. Pigou and Dr. H. Dalton for advice on the general contents of the study, and to Mr. L. R. Connor who has devoted much time to correction and verification of the detail.

A. L. B.

March, 1924.

CONTENTS

CHAP.	PAGE
INTRODUCTION	1
I. SIMPLE EXCHANGE OF TWO COMMODITIES.	
§ 1. Marginal utility, indifference curves, offer curves	5
2. Equilibrium of exchange	8
3. The contract curve	9
4. The demand and supply curves	9
5. Elasticity of demand	10
6. Money prices	12
7. The utility surface	14
APPENDUM : Independent, complementary, and alternative utility	14
II. MULTIPLE EXCHANGE.	
§ 1. Notation	19
2. Equations of equilibrium for perfect competition	20
3. Equations of equilibrium for monopoly	22
4. Aggregate demand and supply	25
NOTE : On universal monopoly	26
III. PRODUCTION.	
§ 1. Factors of production	28
2. The law of substitution. Joint demand for factors	28
3. The supply curve	30
4. The integral supply curve	31
5. Elasticity of supply	32
6. Increasing, constant, and diminishing return	33
7. Marginal supply prices	34
8. Several manufacturers, one commodity	36
9. Alternative demand for factors ; several commodities	38

CHAP.	PAGE
IV. SUPPLY AND DEMAND OF THE FACTORS OF PRODUCTION.	
§ 1. Disutility. Labour	40
2. Capital	42
3. Equation of supply	43
4. Equation of demand	43
5. The share of the factors	44
NOTATION	46
V. GENERAL EQUATIONS OF SUPPLY AND DEMAND IN A STATIONARY POPULATION.	
§ 1. Interdependence of equations	47
2. Supply equations	48
3. Demand equations	50
4. Combination of supply and demand	51
5. Stability of equilibrium	53
VI. APPLICATIONS OF THE GENERAL EQUATIONS.	
§ 1. The inclination of the demand curve	54
2. The case of alternative demand	56
3. Demand and supply of one commodity; prices and quantities under competition and under monopoly	58
4. Various questions of monopoly and combination : I. Manufacturers. II. Factors. III. Bilateral. IV. Consumers	60
5. Joint and composite demand and supply. Derived demand	65
VII. SURPLUS VALUE, RENT AND TAXATION.	
§ 1. Producers' surplus	69
2. Economic rent	70
3. Taxation, in the case of competition	72
4. Taxation in the case of producer's monopoly	75
APPENDIX. SUMMARY OF THE MATHEMATICAL IDEAS AND FORMULAE USED	78
INDEX	97

APPENDIX

SUMMARY OF THE MATHEMATICAL IDEAS AND FORMULAE USED

THE following notes are only likely to be useful to those who have at some time studied the elements of the calculus in an ordinary course. Only a very limited region of the calculus is used in ordinary economic reasoning, but in some respects it is of a kind to which prominence is not given in the usual mathematical training, while much attention is devoted to other aspects of its use, in physics, &c. It has therefore seemed worth while to trace the theory of the calculus from the beginning up to the theorems and methods used in the text, to enable readers to refresh their memories about the particular results wanted and to become used to the notation adopted. The definitions and proofs are not rigid in the mathematical sense, and any careful reader will detect numerous lacunae.

The results may, however, be accepted as true in the sense and with the limitations used in the text, and complete proofs can readily be found by those who have sensitive mathematical consciences.

Functions.

If two variables x and y are so related that y is determinate when x is given, y is said to be an (explicit) function of x . This relationship is written $y=f(x)$; but since several functions may be involved in the same problem, variants of f (e.g. $F, \phi...$) or other letters ($\chi, U...$) are used also to express the functions.

If two or more variables $x, y, z...$ determine another variable, u , then $u=f(x, y, z...)$.

If x and y are connected by any equation such as

$$x+y+8=0, \quad x^2+2y^2-7=0, \quad \sin(x+y)-3=0,$$

the relationship may be written generally as

$$f(x, y) = 0.$$

f is then said to be an *implicit* function.

It is often not necessary to know the form of the function nor to be able to evaluate it. Important relations can be established and results obtained from the mere knowledge that certain quantities determine others.

The function contains numbers and often constants (generally written a, b, c, \dots), that is quantities which remain unchanged while x, y, \dots vary. It is necessary to know these numbers and constants if the function is to be evaluated numerically.

$f(x_1)$ means the value of $f(x)$ when the particular value x_1 is given to the general variable x .

$f(x)$ is said to be continuous over the range $x = a$ to $x = b$, when x can take all values from a to b , to each of which there is a real finite value of $f(x)$, and when, if x makes a finite change, the change in $f(x)$ is also finite. This may be explained by saying that a continuous function can be graphically represented by a line drawn without the pen leaving the paper or marking a sharp angle. The definition here given is only a preliminary or popular one, but it is sufficient for the sequel.

Derived functions or differential coefficients.

Let the values of y corresponding to a range of values of x be plotted on squared paper, so that when $x = OM$, $y = MP$, and as x increases from OM to ON , P moves along a curve (or straight line) to Q . The line PQ is the *graph* of the function; $y = f(x)$ is the *equation* of the curve (Figure A, p. 81).

The point P is written (x, y) . x and y are the *co-ordinates* of P ; x is the *abscissa*; y the *ordinate*; OX, OY are the *axes of reference*.

Let the co-ordinates of Q be $x + h$ and $y + k$, so that (if PL is parallel to OX and meets NQ in L) $MN = h$, $LQ = k$.

Draw PT to touch the curve at P , and join PQ and produce it.

Then

$$\tan QPL = LQ \div PL = k/h = (y + k - y)/h = (f(x + h) - f(x))/h.$$

Now let Q approach P along the curve. The chord PQ rotates about P , till as Q reaches P it coincides with PT , and the angle QPL becomes the angle $TPL = \theta$, say.

Tan θ is the limit of $(f(x+h)-f(x))/h$, when h approaches, and finally becomes, zero. This result is written

$$\tan \theta = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} = D_x y = f'(x),$$

each of these expressions being a convenient way of writing the process and result briefly.

For example, the graph in Figure A represents

$$y = f(x) = 1 + 7x - x^2.$$

$$\begin{aligned} \text{Tan } \theta &= D_x y = f'(x) \\ &= \lim_{h \rightarrow 0} \frac{\{1 + 7(x+h) - (x+h)^2\} - \{1 + 7x - x^2\}}{h} \\ &= \lim_{h \rightarrow 0} \frac{7h - 2hx - h^2}{h} = 7 - 2x. \end{aligned}$$

Thus when $x = 2$ (and $y = 11$), the point P in the figure,

$$f'(x) = 7 - 4 = 3.$$

The tangent at P rises 3 units vertically to 1 unit horizontally. The *gradient* is 3.

$f'(x)$ is the rate of increase of $f(x)$ per unit change of x at the point x .

$f'(x)$ is called the *derived function*, the *derivative*, the *differential coefficient* or the *gradient* of $f(x)$.

When $f'(x)$ is positive the curve rises to the right. Where $f'(x)$ is zero ($x = 3\frac{1}{2}$ in the figure) the curve ceases to rise. When $f'(x)$ is negative ($x > 3\frac{1}{2}$) the curve falls.

The *maximum* of $f(x)$ is when $f'(x) = 0$, if (as in this case) $f'(x)$ changes from positive to negative as x increases through the maximal position, that is if at this point the curve is concave to OX (and above it).

If now we take the curve

$$y = x^2 - 7x + 15, \quad f'(x) = 2x - 7. \quad (\text{Figure B.})$$

$$f'(x) = 0, \quad \text{when } x = 3\frac{1}{2}.$$

$$f'(x) < 0, \quad \text{when } x < 3\frac{1}{2}. \quad f'(x) > 0, \quad \text{when } x > 3\frac{1}{2}.$$

$$f(x) \text{ is a minimum when } x = 3\frac{1}{2}.$$

* Formerly this expression was written $\frac{dy}{dx}$. Since this suggests a fraction and not the result of a process, the form here used is to be preferred.

The *minimum* of $f(x)$ is when $f'(x) = 0$ and the curve is convex to OX (and above it).

These results are general. The first test for the presence of a maximum or minimum is that $f'(x) = 0$. To decide whether this gives a maximum or gives a minimum it is necessary to

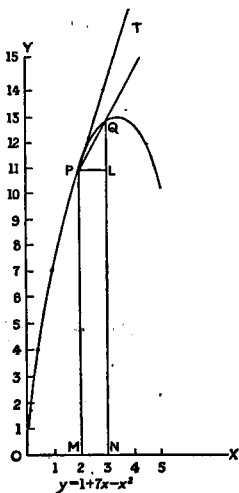


FIGURE A.

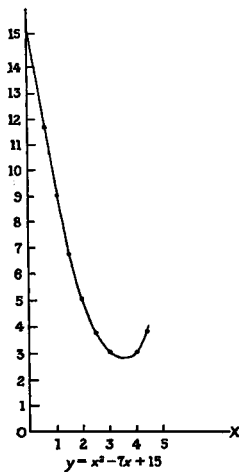


FIGURE B.

know the *sign* of $f'(x)$ for values of x to the left and right of the maximal position, unless (as is very often the case) we know *a priori* which to expect.

Successive differentiation. Expansions.

The process of differentiation can of course be applied to the derived function. We thus obtain the *second derivative*, and so on successively.

Thus in the first example taken,

$$f'(x) = 7 - 2x.$$

The second derivative

$$D^2_{xy} = f''(x) = \lim_{h \rightarrow 0} \frac{f(7-2(x+h)) - (7-2x)}{h} = -2.$$

If $f''(x)$ is negative, $f'(x)$ if positive is becoming less as x increases, and if negative is becoming numerically greater negatively.

A little consideration will show that if $f''(x)$ is negative the curve is concave to OX (if above it), and if $f''(x)$ is positive the curve is convex.

The complete test for a maximum (if $f''(x)$ is not zero) is that $f'(x) = 0$ and $f''(x)$ is negative, and for a minimum that $f'(x) = 0$ and $f''(x)$ is positive.

In the adjacent Figure (C) of a convex curve, PT is the tangent at P and meets the ordinate of a neighbouring point Q at T . PL is parallel to OX .

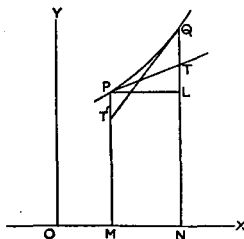


FIGURE C.

Write

$$\delta x = h = MN, \quad \delta y = k = NQ.$$

δx and δy are small finite increments or 'infinitesimals' of x and y .

$$LT = PL \tan LPT = h f'(x).$$

$$\delta y = NQ - MP$$

$$= f(x+h) - f(x) = LQ$$

$$= LT + TQ = f'(x) \cdot \delta x + TQ.$$

TQ , the departure of the curve from its tangent, diminishes as Q approaches P .

We shall immediately give an informal proof that TQ is comparable with h^2 , i.e. with $(\delta x)^2$. Assuming this we have

$$\delta y = f'(x) \cdot \delta x + \text{a quantity involving } (\delta x)^2. \quad \dots \text{ Formula 1.}$$

$$\therefore \frac{\delta y}{\delta x} = f'(x) + \text{a quantity involving } \delta x, \text{ and in the limit,}$$

$$\text{when } h \text{ is zero, } \frac{\delta y}{\delta x} = D_{xy}.$$

To obtain a rough proof of the proposition just used, draw the tangent at Q to meet MP at T' . The gradient of this tangent

is $f'(x+h)$. In the case where Q is above T , it is evident (the curve being continuous and h small) that QT' cuts PL between P and L and therefore $QL < hf'(x+h)$. Hence

$$hf''(x) < f(x+h) - f(x) < hf'(x+h),$$

and

$$f(x+h) - f(x) = hf''(x+ch),$$

where $x+ch$ is some value intermediate between x and $x+h$, and continuity is assumed.

The same result is obtained if the curve is concave, and this proposition is true for all continuous functions.

Hence similarly

$$f''(x+ch) - f''(x) = chf'''(x+c_1h)$$

where c_1 is intermediate between 1 and c .

Combining these results, we have

$$\delta y = f(x+h) - f(x) = hf'(x) + ch^2f''(x+c_1h),$$

where c and c_1 are proper fractions, and $h = \delta x$.

A change in y is therefore obtained approximately by multiplying the change in x by the first derived function, the equation being the more exact the smaller the change in x .

This result is fundamental in a considerable part of the application to Economics.

A rough examination of the general expansion of $f(x+h)$ can be obtained as follows.

Take x as fixed, say x_0 , and h as variable. Write

$$f(x_0+h) = F(h).$$

Thus in Figure C let

$$OM = x_0, \quad MP = f(x_0), \quad NQ = f(x_0+h) = F(h).$$

Suppose that $F(h)$ is expansible in ascending powers of h with all the terms finite and the series convergent, i. e. tends to a unique finite limit when the number of terms is increased indefinitely.

Write $F(h) = a_0 + a_1h + a_2h^2 + a_3h^3 + a_4h^4 + \dots$, where a_0, a_1, \dots are constants to be determined.

Differentiate successively with regard to h .

$$F'(h) = a_1 + 2a_2h + 3a_3h^2 + 4a_4h^3 + \dots$$

$$F''(h) = 2a_2 + 3 \cdot 2a_3h + 4 \cdot 3a_4h^2 + \dots$$

$$F'''(h) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4h + \dots$$

In each of these equations take the case where $h = 0$.

$$a_0 = F(0), \quad a_1 = F'(0), \quad a_2 = \frac{1}{2} F''(0),$$

$$a_3 = \frac{1}{2 \cdot 3} F'''(0), \quad \dots a_r = \frac{1}{r!} F^{(r)}(0), \dots,$$

where $F'(0)$ is the result of writing $h = 0$ after $F(h)$ is differentiated and so on.

Then $F'(0)$ is the gradient of the curve PQ at P and therefore is the same as $f'(x_0)$, that is the result of writing $x = x_0$ in the derivative of $f(x)$. Similarly $F''(0) = f''(x_0)$ and so on.

We have then

$$f(x_0 + h) = F(h) = f(x_0) + hf'(x_0) + \frac{1}{2} h^2 f''(x_0) + \dots \frac{1}{r!} h^r f^{(r)}(x_0) + \dots \quad \text{Formula 2,}$$

the process being continued as far as we please.

This is Taylor's Series.

In the functions used in the text it is generally the case that the successive terms become rapidly smaller over the part of the curves that are considered in the neighbourhood of equilibrium. Such an assumption is much more hazardous when larger changes are considered, as in the cases of taxation and monopoly (pp. 60 and 75 seq).

Standard derivatives and rules of differentiation.

The following are standard derived functions, as shown in any text-book on the calculus :

$$D_x (x^n) = nx^{n-1},$$

where n is any positive or negative integer or fraction ;

$$\text{e.g.} \quad D_x \sqrt{x} = \frac{1}{2} x^{-\frac{1}{2}}.$$

$$D_x (a^x) = a^x \cdot \log_a a. \quad D_x (e^x) = e^x.$$

$$D_x (\log_a x) = \frac{1}{x} \cdot \log_a e. \quad D_x (\log_e x) = \frac{1}{x}.$$

$D_x (\sin x) = \cos x, \quad D_x (\cos x) = -\sin x, \quad D_x (\tan x) = \sec^2 x,$
where x is the radian measure of the angle.

Also the following working rules are easily proved from the definition of a derived function :

$$D_x (af(x)) = a \cdot f'(x) ;$$

e.g. $D_x (3x) = 3, D_x (3x^2) = 3 \times 2x = 6x.$

$$D_x f(ax) = af'(ax) ;$$

e.g. $D_x \sin(ax) = a \cos ax.$

$$D_x (f(x) + a) = f'(x) ;$$

e.g. $D_x (x^2 + 3) = 2x.$

$$D_x f(x+a) = f'(x+a) ;$$

e.g. $D_x (x+a)^2 = 2(x+a),$ for if $f(x) = x^2, f'(x) = 2x.$

These rules may be combined, thus :

$$D_x \{af(bx+c) + d\} = ab \cdot f'(bx+c) ;$$

e.g. $D_x \{2 \sin(3x+4) + 5\} = 2 \times 3 \cos(3x+4) = 6 \cos(3x+4).$

If $f(x)$ and $\phi(x)$ are two functions of x , the following rules can be obtained :

$$D_x \{f(x) \pm \phi(x)\} = f'(x) \pm \phi'(x) ;$$

e.g. $D_x (x^2 + \log_e x) = 2x + 1/x.$

$$D_x \{f(x) \times \phi(x)\} = f'(x) \times \phi(x) + f(x) \times \phi'(x) ;$$

e.g. $D_x (x^2 \sin x) = 2x \sin x + x^2 \cos x.$

$$D_x \{f(x) \div \phi(x)\} = \{f'(x) \times \phi(x) - f(x) \times \phi'(x)\} \div (\phi(x))^2 ;$$

e.g. $D_x (\tan x) = D_x (\sin x \div \cos x)$

$$= \{\cos x \times \cos x - \sin x \times (-\sin x)\} \div \cos^2 x$$

$$= (\cos^2 x + \sin^2 x) \div \cos^2 x = 1 + \tan^2 x = \sec^2 x,$$

as above.

If $y = F(u)$, where $u = f(x)$,

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} = \frac{F(u + \delta u) - F(u)}{\delta u} \times \frac{f(x + \delta x) - f(x)}{\delta x},$$

identically.

In the limit, obtained by diminishing δx and consequently δu and δy also,

$$D_x y = D_u F(u) \times D_x f(x) ;$$

e.g. if F stands for \log_e , f for \sin , and so $u = \sin x$,

$$\begin{aligned} D_x \{ \log_e (\sin x) \} &= D_u (\log_e u) \times D_x (\sin x) \\ &= \frac{1}{u} \times \cos x = \frac{1}{\sin x} \times \cos x = \cot x. \end{aligned}$$

$$D_x \{ \sin (x^2) \} = 2x \cos (x^2).$$

These forms and rules are sufficient for the differentiation of common functions of one variable.

Functions of two or more variables. Partial differentiation.

Let a variable z depend on two other variables x and y , so that $z = f(x, y)$, and let w and y depend on another variable t . Required to connect a change in t with a change in z .

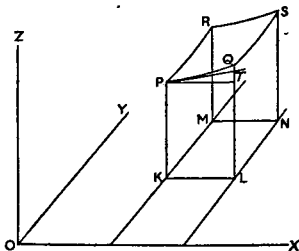


FIGURE D.

To fix ideas, suppose a point to be moving in the plane XOY (Figure D), and at any time t to be at the point $K(x, y)$. Let a vertical $KP(z)$ be erected whose height is $f(x, y)$. Then as the point moves about the plane XOY , P will move always vertically over the point on a surface whose equation is

$$z = f(x, y).$$

Consider movements parallel to OX , i.e. to the plane ZOX .

If the point moves from K to L , y is constant (say y_1) while x varies, and P traces out a plane curve PQ . The gradient at

P of this curve is $D_x f(x, y_1)$, that is the result of differentiating $f(x, y_1)$ where y_1 does not vary. This expression is variously written

$$D_x(z) \text{ (} y \text{ const), } \frac{\partial z}{\partial x}, f'_x(x, y) \text{ (} y \text{ const), } f_x, \text{ and } z_x \quad \text{Formula 3.}$$

f_x is at once the briefest and most convenient of these forms. It means the result of the process of differentiation with respect to x applied to the function, y being kept constant; e.g. if

$$f(x, y) = ax^2 + by^2, \quad f_x = 2ax, \quad f_y = 2by.$$

This quantity f_x is called the *partial derived function* (or derivative or differential coefficient) with respect to x .

If the point P had moved along the tangent at P in the plane of PQ it would have risen hf_x , to T , when x increased to $x+h$, h being KL .

Similarly if we take movements parallel to OY or the plane ZOY , let the point in the plane XOY move from K to M ($KM=k$) and P trace the curve PR . Its initial gradient would be f_y , and if it had moved along the tangent to PR it would have risen kf_y .

Now if h and k are small the heights of Q and R only differ from those obtained at T and the corresponding point under R by quantities involving h^2 and k^2 (by formula 1), which are therefore very small. The rises in the two paths are therefore very nearly hf_x and kf_y .

Further it can be shown (though the complete proof is difficult) that the rise along the path QS , where $KLNM$ is a rectangle and NS is vertical, differs from the rise along PR only by a quantity of the order hk .

If, then, the point in the plane XOY moves from K to N by any path and in consequence a line FS is traced on the surface, the increase of height from P to S differs from $hf_x + kf_y$ by a quantity involving h^2 , k^2 , or hk as factors. Write δz for this increase.

$$\begin{aligned} \delta z = z + \delta z - z &= f(x+h, y+k) - f(x, y) = hf_x + kf_y \\ &= f_x \cdot \delta x + f_y \cdot \delta y, \text{ approximately,} \end{aligned} \quad \text{Formula 4,}$$

where $\delta x, \delta y$ are the increments of x and y .

Let δt be the time interval between K and N .

$$\frac{\delta z}{\delta t} = f_x \cdot \frac{\delta x}{\delta t} + f_y \cdot \frac{\delta y}{\delta t} \text{ approximately.}$$

Now proceed to the limit when δt approaches zero, and consequently δx , δy , δz approach zero, and the quantities h^2 , k^2 , hk , &c. which are omitted in Formula 4 vanish. We have

$$D_t z = f_x \cdot D_t x + f_y \cdot D_t y.$$

Thus if $z = ax^2 + by^2$, where $x = \cos t$, $y = \sin t$, $f_x = 2ax$, $f_y = 2by$, $D_t x = -\sin t$, $D_t y = \cos t$, and

$$\begin{aligned} D_t z &= -2ax \sin t + 2by \cos t \\ &= -2a \cos t \sin t + 2b \sin t \cos t = (b-a) \sin 2t. \end{aligned}$$

[This result may also be obtained directly by writing

$$z = a \cos^2 t + b \sin^2 t,$$

but it is not usual that the substitution should be so simple.]

The equation does not depend on the geometrical illustration but is universally true. For example we may take t , which is an independent variable completely at choice, as identical with x , and obtain

$$D_x z = f_x + f_y \cdot D_x y \dots \text{Formula 5.}$$

The result may be generalized to any number of variables, so that if $z = f(x_1, x_2, \dots, x_n)$,

$$D_t z = f_{x_1} \cdot D_t x_1 + f_{x_2} \cdot D_t x_2 + \dots + f_{x_n} \cdot D_t x_n \dots \text{Formula 6,}$$

and $D_{x_1} z = f_{x_1} + f_{x_2} \cdot D_{x_1} x_2 + \dots + f_{x_n} \cdot D_{x_1} x_n \dots \text{Formula 7.}$

e.g. If $z = x_1^2 + x_2 x_3 + x_1 x_3 = f(x_1, x_2, x_3)$,

$$f_{x_1} = 2x_1 + x_3, \quad f_{x_2} = x_3, \quad f_{x_3} = x_2 + x_1,$$

and $D_{x_1} z = 2x_1 + x_3 + x_3 \cdot D_{x_1} x_3 + (x_2 + x_1) \cdot D_{x_1} x_3.$

We cannot evaluate this till we know the relationship between x_2 and x_1 and between x_3 and x_1 .

The formula is commonly used as

$$\delta z = f_{x_1} \cdot \delta x_1 + f_{x_2} \cdot \delta x_2 + \dots + f_{x_n} \cdot \delta x_n \dots \text{Formula 8,}$$

the variable on which x_1, x_2, \dots, x_n depend not being named,

In this form it is very important in Economics.

In words, if a quantity z is dependent on variables x_1, x_2, \dots, x_n , and these variables owing to a common cause have at the same time small increments $\delta x_1, \delta x_2, \dots$, whose squares and products are negligible, then the resulting increment in z is obtained by adding the increments in x_1, x_2, \dots , each multiplied by the partial derivative of z with respect to it computed on the assumption that the other x 's do not vary.

Maxima and minima.

In Figure D (p. 86) z is a maximum or minimum where the tangent plane to the surface on which P moves is horizontal, so that when motion takes place in any direction the point starts along the plane and then falls below it (in the case of a maximum), or rises above it (in the case of a minimum). Where $z = f(x, y)$ and the tangent plane is horizontal, every line in it is horizontal, so that $f_x = 0 = f_y$, since these are the gradients in two of the directions.

More generally, when $z = f(x_1, x_2, \dots, x_n)$, z cannot be a maximum or minimum, unless the effect of an infinitesimal change of any of the x 's is to make $\delta z = 0$. From formula 8 this will be the case if

$$0 = f_{x_1} = f_{x_2} = \dots = f_{x_n} \dots \text{Formula 9.}$$

If we know *a priori*, as is often the case, that there is a maximum or a minimum in the region considered, these equations are sufficient. If not, terms of a higher degree in the increments must be examined.

[e.g. $z = x^2 + y^2 + 2x + 4y = (x+1)^2 + (y+2)^2 - 5$,

is clearly a minimum when $x = -1, y = -2$.

In this case, $f_x = 2x + 2, = 0$ if $x = -1$,

and $f_y = 2y + 4, = 0$ if $y = -2$.

If, however, $z = x^2 - 2xy + 2y^2 + 2x + 4y$,

$$f_x = 2x - 2y + 2, \quad f_y = -2x + 4y + 4,$$

and these are zero if $x = -4, y = -3$.

All we can say without further examination is that, if there is a maximum or minimum, it is at this point.]

It is often the case that $x_1, x_2 \dots$ are not independent, but are connected with each other by one or more equations. The equations $0 = f_{x_1} = f_{x_2} = \dots$ will not then in general be consistent with the connecting equations and the partial derivatives cannot all vanish together. The procedure then is to eliminate as many of the x 's as there are connecting equations and proceed with the remainder taken as independent variables.

$$\begin{aligned} \text{[Thus, if } z &= x^2 + y^2 + 2x + 4y \text{ and } y = x + 2, \\ z &= x^2 + (x+2)^2 + 2x + 4(x+2) = 2x^2 + 10x + 12, \\ D_x z &= 4x + 10, = 0 \text{ if } x = -2.5, \end{aligned}$$

and, since $D_x^2 z = 4$ and is positive, this gives a minimum for z , viz. $z = -\frac{1}{2}$.

This is the solution of the problem of finding the lowest point of the given surface in the vertical plane $y = x + 2$. The minimum of z without any restriction is -5 (p. 89) when $x = -1$, $y = -2$.]

The process of partial differentiation can be carried on successively. Thus, if $z = f(x, y)$, $f_{xx} = D_x(f_x)$, y const., is the second partial derived function of z with respect to x . It will measure the change of gradient of the curve PQ (Figure D, p. 86). Similarly f_{yy} measures the change of gradient of the curve PR . f_{xy} means $D_y(f_x)$, x const.; it can be shown, but not easily, that the same result is obtained from $D_x(f_y)$, y const., so that $f_{xy} = f_{yx}$. This measures the change in the gradient of the tangent parallel to the plane ZOY due to a movement of the section in the direction OY .

The more complete statement of the equation to which

$$\delta z = f_{x_1} \cdot \delta x_1 + f_{x_2} \cdot \delta x_2 + \dots$$

is an approximation, is

$$\begin{aligned} \delta z &= f_{x_1} \cdot \delta x_1 + f_{x_2} \cdot \delta x_2 + \dots \\ &+ \frac{1}{2} \{ f_{x_1 x_1} (\delta x_1)^2 + f_{x_2 x_2} (\delta x_2)^2 + \dots \\ &\quad + 2 f_{x_1 x_2} \delta x_1 \cdot \delta x_2 + \dots \} \quad \text{Formula 10,} \\ &+ \text{terms involving cubes and higher powers of } \delta x, \end{aligned}$$

where all possible squares and products are included in $\{ \}$.

An expansion by this formula is used on pp. 17-18 above.

An investigation of the complete formula can be made on the lines of that on pp. 83-4 and formula 2, as follows.

Write

$$\begin{aligned} f(x_0 + h, y_0 + k) &= F(h, k) \\ &= a + b_1 h + b_2 k + c_1 h^2 + c_2 hk + c_3 k^2 + d_1 h^3 + d_2 h^2 k \\ &\quad + d_3 h k^2 + d_4 k^3 + \dots \end{aligned}$$

Differentiate successively with respect to h and to k .

$$F_h = b_1 + 2c_1 h + c_2 k + 3d_1 h^2 + 2d_2 hk + d_3 k^2 + \dots$$

$$F_{hh} = 2c_1 + 3 \cdot 2d_1 h + 2d_2 k + \dots$$

$$F_{hk} = c_2 + 2d_2 h + 2d_3 k + \dots$$

Take the case in each equation where $h = 0 = k$,

$a = F(0, 0)$, $b_1 = F_h$, $c_1 = \frac{1}{2}F_{hh}$, $c_2 = F_{hk}$, and similarly $b_2 = F_k$, $c_3 = \frac{1}{2}F_{kk}$, in each case 0 being written for h and k after differentiation.

But then (as on p. 84) F_h = the gradient at P of the curve PQ (Figure D, p. 86) $= f_{x_0}$, $F_k = f_{y_0}$, and similarly $F_{hh} = f_{x_0 x_0}$, &c.

$$\begin{aligned} \therefore \delta z &= f(x_0 + h, y_0 + k) - f(x_0, y_0) \\ &= hf_{x_0} + kf_{y_0} + \frac{1}{2}(h^2 f_{x_0 x_0} + 2hk f_{x_0 y_0} + k^2 f_{y_0 y_0}) \\ &\quad + \text{terms involving cubes of } h, \text{ \&c.} \end{aligned} \quad \text{Formula 11.}$$

This result can easily be extended to any number of variables.

The above analysis is not a proof, but a determination of coefficients on the hypothesis that an expansion of this kind is possible.

With two variables $f(x, y)$ is a maximum or minimum at (x_0, y_0) only if $f_{x_0} = 0 = f_{y_0}$ and the complex term involving squares is of the same sign for all variations; this is the case if $f_{x_0 x_0} \times f_{y_0 y_0} > (f_{x_0 y_0})^2$. Given this condition, $f(x_0, y_0)$ is a maximum or minimum according as $f_{x_0 x_0}$ is negative or positive.

Tangents.

It is often necessary to determine $D_x y$ when we are given $f(x, y) = 0$. $f(x, y) = 0$ is the equation of a plane curve and $D_x y$ is its gradient at any point (x, y) .

Write $z = f(x, y)$.

Then $\delta z = f_x \cdot \delta x + f_y \cdot \delta y$ and $D_x z = f_x + f_y \cdot D_x y$ (pp. 87-8, formulae 4 and 5).

But since $z = f(x, y)$ is always zero, z is invariable, δz is zero, and $D_x z$ is zero.

$$\therefore 0 = f_x + f_y \cdot D_x y, \text{ or } D_x y = -f_x/f_y.$$

The tangent at P , which we will call (x_1, y_1) (see Figure A, p. 81), is a line through (x_1, y_1) with gradient $D_x y$, and its equation is therefore

$$y - y_1 = (x - x_1) \tan TPL = (x - x_1) \cdot D_x y,$$

that is $(x - x_1) \cdot f_{x_1} + (y - y_1) \cdot f_{y_1} = 0$. . Formula 12, where f_{x_1}, f_{y_1} are the results of writing $x = x_1, y = y_1$ in the partial derivatives of $f(x, y)$.

Thus, if $f(x, y) = ax^2 + 2hxy + by^2 - c = 0$,

$$f_x = 2ax + 2hy, \quad f_y = 2hx + 2by,$$

and the tangent at a point (x_1, y_1) on the curve is

$$(x - x_1)(2ax_1 + 2hy_1) + (y - y_1)(2hx_1 + 2by_1) = 0,$$

that is $x(ax_1 + hy_1) + y(hx_1 + by_1) = ax_1^2 + 2hx_1y_1 + by_1^2$
 $= c.$

Notice that we can write an equation for $D_x y$ at once from such a curve as $ax^2 + 2hxy + by^2 - c = 0$, thus

$$2ax + 2hy + D_x y(2hx + 2by) = 0.$$

Integration.

Integration is the process of finding the original function when the derived function is given, and is the reverse of differentiation.

The symbol \int signifies integration, and is defined by

$$\int f'(x) \cdot dx = f(x) + C,$$

where C (any constant) is introduced, since evidently

$$D_x \{f(x) + C\} = f'(x).$$

Thus $\int x^{n-1} dx = \frac{1}{n} x^n + C$, since $D_x \left(\frac{1}{n} x^n\right) = x^{n-1}$.

The most important use of integration in the present connexion is in its relationship to areas.

Write $f'(x) = F(x)$.

Let CD be the graph of $y = F(x)$ from $x = a$ (OA) to $x = b$ (OB) (Figure E).

Divide AB into n equal parts

$$\Delta N_1, N_1N_2, \dots \text{ each} = \delta x = (b-a)/n.$$

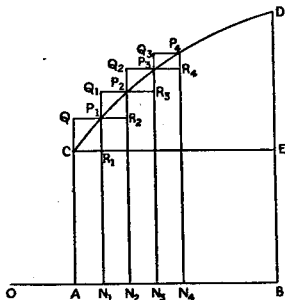


FIGURE E.

Let N_1P_1, N_2P_2, \dots be ordinates, and complete the rectangles as in the figure.

Take the case of a curve that rises from C to D ; other cases can readily be handled in the same way.

Let S, S' be the areas of the rectilinear figures

$$ACR_1P_1R_2P_2, \dots, D, \text{ and } AQP_1Q_1P_2, \dots, D.$$

Then the curvilinear area ACP_1P_2, \dots, D is intermediate between S and S' . $S' - S =$ sum of such areas as QR_1, Q_1R_2, \dots , and approximately $= \delta x \times ED$, where CE is parallel to AB . When n is large and therefore δx is small, this difference is negligible as compared with S , and S may be identified as the area of the curve.

Take n so large that $(\delta x)^2$ can be neglected.

Then from p. 82, formula 1,

$$f(a + \delta x) - f(a) = f'(a) \cdot \delta x = F(a) \cdot \delta x = AC \cdot AN_1$$

$$f(a + 2\delta x) - f(a + \delta x) = F(a + \delta x) \cdot \delta x = N_1 P_1 \cdot N_1 N_2$$

$$f(a + n \cdot \delta x) - f(a + (n-1) \delta x) = F(a + (n-1) \delta x) \cdot \delta x.$$

Adding we have, since $b = a + n\delta x$,

$$\begin{aligned} f(b) - f(a) &= \text{sum of such areas as } AC R_1 N, N_1 P_1 R_2 N_2 \\ &= S \text{ with sufficient approximation} \\ &= \text{area of curve.} \end{aligned}$$

It is not difficult to verify that this final equation is absolutely true, when we suppose n indefinitely increased.

The area of the curve is the limit of the sum of the rectangles $F(x) \cdot \delta x$ from $x = a$ to $x = b$, when n is definitely increased,

$$= \text{limit of } \sum_a^b F(x) \cdot \delta x \text{ and this is written } \int_a^b F(x) \cdot dx.$$

The whole process is then summarized as

$$\text{area of curve} = \int_a^b F(x) dx = \int_a^b f'(x) dx = f(b) - f(a) \quad \text{Formula 13.}$$

Thus the area from OX to the curve $y = x^2$ is for any value of x

$$\int_0^x x^2 dx = \frac{1}{3} x^3 - \frac{1}{3} \cdot 0 = \frac{1}{3} x^3.$$

Note on elimination.

Two linear equations

$$a_1 x + b_1 y + c_1 = 0, \quad a_2 x + b_2 y + c_2 = 0$$

give one pair of values of x and y , viz.

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1}.$$

Or we can eliminate y and obtain one equation for x ,

$$(a_1 b_2 - a_2 b_1) x + c_1 b_2 - c_2 b_1 = 0.$$

From two equations involving three quantities x, y, z ,

$$a_1 x + b_1 y + c_1 z + d_1 = 0, \quad a_2 x + b_2 y + c_2 z + d_2 = 0$$

we can eliminate one (x), and obtain a relation between the others,

$$(a_1c_2 - a_2c_1)x + (b_1c_2 - b_2c_1)y + d_1c_2 - c_1d_2 = 0.$$

Or we can say, from the first equation,

$$x = -\frac{1}{c_1}(a_1x + b_1y + d_1),$$

and when this value of x is written in the second equation we have

$$c_1(a_2x + b_2y + d_2) - c_2(a_1x + b_1y + d_1) = 0.$$

From this it can be seen that, if we have n linear equations connecting n quantities, we can determine the quantities separately, and that, if there are more than n quantities, we can eliminate $\overline{n-1}$ of them and obtain one equation involving the remainder; the procedure being virtually to solve for $\overline{n-1}$ selected quantities from $\overline{n-1}$ of the equations and substitute the results in the first equation.

With linear equations, if the quantities a, b, c, \dots and n are given the solution is only a matter of patience. When we have the same problem involving squares, products, or other functions of x, y, \dots , the procedure is the same essentially, though it is not always possible to carry it out by simple methods.

Thus suppose we have three equations involving four quantities

$$f_1(u, v, x, y) = 0, \quad f_2(u, v, x, y) = 0, \quad f_3(u, v, x, y) = 0.$$

Solve the third as an equation in y , obtaining

$$y = F(u, v, x).$$

Put this value in the first and second, obtaining

$$F_1(u, v, x) = 0, \quad F_2(u, v, x) = 0.$$

Solve the last equation for x , obtaining $x = \phi(u, v)$ and put this value in $F_1(u, v, x) = 0$. We have then one equation involving u and v only, x and y being eliminated.

e.g. Eliminate x and y from the equations

$$u^2 + v^2 + x^2 = 20, \quad u^2 + 2v^2 + y^2 = 30, \quad u + x + y = 10.$$

From the second and third equations

$$\begin{aligned} u^2 + 2v^2 + (10 - u - x)^2 &= 30 \\ x &= 10 - u \pm \sqrt{30 - u^2 - 2v^2}. \end{aligned}$$

Then from the first

$$u^2 + v^2 + (10 - u \pm \sqrt{30 - n^2 - 2v^2})^2 = 20,$$

which reduces to

$$5u^4 + v^4 + 6u^2v^2 - 120u^3 - 120uv^2 + 900u^2 + 580v^2 - 2000u + 100 = 0.$$

Thus the actual solution rapidly becomes laborious in quite simple cases.

When there are as many (n) equations as variables, and $n-1$ variables are eliminated, the remaining equation in one variable is not generally linear and there may be several real roots, each giving a set of simultaneous values for the variables. The equations are then said to have multiple solutions, and some further knowledge is necessary to know which is appropriate to the problem.

INDEX

A

Aggregate demand, 25.
 Aggregate supply, 25.
 Alternative demand, 38, 56, 66.
 Alternative factors, 31.
 Alternative supply, 66.
 Alternative utility, 15, 17, 18, 56.

B

Barter, 5.
 Bargaining locus, 8.
 Bilateral monopoly, 62.

C

Capital, 42.
 Commodities, 5.
 Commodity equations, 21, 48, 50, 58.
 Competition, 20; equilibrium in competition, 20, 58.
 Complementary utility, 15, 16, 18, 56.
 Composite demand, 65-6.
 Composite supply, 65-6.
 Constant return, 33, 36, 59, 69.
 Consumers' combination, 62-5.
 Consumers' goods, 65.
 Consumers' surplus, 77.
 Consumption, 6, 19.
 Contract curve, 9, 10, 13.
 Cost of production, 29, 31, 49.

D

Decreasing return, 34, 35, 37, 59, 63, 69, 73 seq.
 Demand, 10; aggregate, 25; composite, 65, 66; derived, 68; elasticity of, 10; indirect, 65, 68; joint, 28, 65, 66, 67.
 Demand curve, 10, 13; inclination of, 55.
 Derivatives, 80; partial, 86, standard, 84-5.
 Derived demand, 68.
 Derived function, 79; second, 81; partial, 87.

Differential coefficient, 79.
 Differentiation, 79; partial, 86; rules of, 84; successive, 81.
 Diminishing return, *v.* decreasing.
 Duality, 40; marginal, 41, 43, 49.
 Duopoly, 38.

E

Edgeworth, Prof. F. Y., 8.
 Efficiency of money, 32.
 Elasticity of demand, 10; of demand for factors, 44; of supply, 32.
 Elimination, 94 seq.
 Equations of demand, 50; for factors, 43; of equilibrium, 20-2, 51, 58; of supply, 43, 48.
 Equilibrium, 8, 20, 51, 58, 59; stability of, 53.
 Exchange, simple, 5; multiple, 19.
 Expansions, 81.
 Expenditure, 51.

F

Factors of production, 28; demand for, 43; share of, 44; supply of, 43, 48.
 Functions, 78; derived, 79.

G

Goods, 1; consumers', 65; producers', 65.

I

Income, 51.
 Increasing return, 33-6, 59, 63, 69, 73 seq.
 Indifference curves, 6, 8, 10.
 Indirect demand, 65, 68.
 Independent utility, 15, 17, 18, 55.
 Integral supply curve, 31.
 Integration, 32.
 Interdependence, 47, 52.

- J**
- Jevons*, 9; *Johnson, W. E.*, 32, 57.
 Joint demand, 65, 66, 67; for factors, 28.
 Joint supply, 65; use of factors, 31, utility, 15, 18.
- L**
- Labour, 40; disutility of, 40.
 Land, 42; rent of, 70.
- M**
- Margin of cultivation, 71.
 Marginal supply price, 34, 69.
 Marginal utility, 9; of money, 12, 21, 43, 55.
 Market, 8, 20.
Marshall, Dr. A., 9, 39, 65.
 Maximizing equations, 21, 58.
 Maximum, 80, 89; effect of small change on, 24.
 Minimum, 80, 89.
 Money prices, 12.
 Monopoly, 22-5; bilateral, 62; universal, 26.
 Monopoly, equilibrium under, 59.
 Monopoly of all commodities, 60; of factors of production, 61.
- N**
- Notation, 46.
- O**
- Offer curve, 7, 8, 10, 13, 31.
- P**
- Partial derived function, 87; differentiation, 86.
 Personal equations, 21, 51, 58.
Pigou, Prof. A. C., 34, 37, 45.
 Prices, 8, 12, 19, 29.
- Producers' goods, 65; surplus, 69.
 Production, 28, 49; joint, 31.
 Production function, 29, 36, 48.
 Profits, 37, 59, 69, 70, 72, 74.
- R**
- Rent, economic, 70.
 Rules of differentiation, 84 seq.
- S**
- Satisfaction, 1, 13.
 Saving, 51.
 Second derivatives, 81.
 Share of factors, 44.
 Stability of equilibrium, 87, 53.
 Substitution, law of, 28, 49.
 Successive differentiation, 81.
 Supply, 10; aggregate, 25; composite, 65-6; curve of, 10, 13, 30; elasticity of, 32; joint, 65.
 Surplus, consumers', 77; producers', 69.
- T**
- Tangents, 91.
 Taxation, 72 seq.; maximum receipts, 74, 76; monopoly, 75-6; yield of, 72 seq.
 Taylor's series, 84.
- U**
- Utility, 3, 6, 64; equations, 21, 50, 58; function, 3, 6; surface, 14, 15 seq. See marginal utility.
- V**
- Value in exchange, 3.
- W**
- Work, 40.